Some new integrable systems constructed from the bi-Hamiltonian systems with pure differential Hamiltonian operators

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Abstract When both Hamiltonian operators of a bi-Hamiltonian system are pure differential operators, we show that the generalized Kupershmidt deformation (GKD) developed from the Kupershmidt deformation in [10] offers an useful way to construct new integrable system starting from the bi-Hamiltonian system. We construct some new integrable systems by means of the generalized Kupershmidt deformation in the cases of Harry Dym hierarchy, classical Boussinesq hierarchy and coupled KdV hierarchy. We show that the GKD of Harry Dym equation, GKD of classical Boussinesq equation and GKD of coupled KdV equation are equivalent to the new integrable Rosochatius deformations of these soliton equations with self-consistent sources. We present the Lax Pair for these new systems. Therefore the generalized Kupershmidt deformation provides a new way to construct new integrable systems from bi-Hamiltonian systems and also offers a new approach to obtain the Rosochatius deformation of soliton equation with self-consistent sources.

Keywords: Kupershmidt deformation; bi-Hamiltonian systems; Rosochatius deformation; soliton equation with self-consistent sources

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1 Introduction

In recent years the integrable deformations of soliton hierarchies attracted a lot of attention. Among them, the Rosochatius deformations of integrable systems have important physical applications[1]-[6]. Wojciechowski and Kubo et al. researched the Rosochatius deformed Garnier system and Rosochatius deformed Jacobi system, respectively[1, 2]. Zhou generalize the Rosochatius method to study the integrable Rosochatius deformations of some explicit constrained flows of soliton equations [3, 4]. In [5, 6], we proposed a systematic method to generalize the integrable Rosochatius deformations of finite-dimensional integrable systems to integrable Rosochatius deformations of soliton equations with self-consistent sources.

On another hand it is known that one can construct a new integrable system starting from a bi-Hamiltonian system. Fuchssteiner and Fokas showed [7] that compatible symplectic structures lead in natural way to hereditary symmetries, which provides a method to construct a hierarchy of exactly solvable evolution equations. Olver and Rosenau [8] demonstrated that most integrable bi-Hamiltonian systems are governed by a compatible trio of Hamiltonian structures, and their recombination leads to integrable hierarchies of nonlinear equations.

For the following KdV6 equation or nonholonomic deformation of KdV equation [9]

$$u_t = u_{xxx} + 6uu_x - \omega_x, \tag{1a}$$

$$\omega_{xxx} + 4u\omega_x + 2u_x\omega = 0, \tag{1b}$$

many authors studied it and established a lot of integrable properties such as zero-curvature representation, bi-Hamiltonian structure, conversed quantities, multisolitons and so on[10]-[15]. In particular, Kupershmidt found [10] that (1) can be converted into

$$u_t = J(\frac{\delta H_3}{\delta u}) - J(\omega), \tag{2a}$$

$$K(\omega) = 0, \tag{2b}$$

where

$$J = \partial = \partial_x,\ K = \partial^3 + 2(u\partial + \partial u) \tag{3}$$

are the two standard Hamiltonian operators of the KdV hierarchy and $H_3 = u^3 - \frac{u_x^2}{2}$. In general, for a bi-Hamiltonian system

$$u_{t_n} = J(\frac{\delta H_{n+1}}{\delta u}) = K(\frac{\delta H_n}{\delta u}), \tag{4}$$

the ansatz (2) provides a nonholonomic deformation of bi-Hamiltonian systems[10]:

$$u_{t_n} = J(\frac{\delta H_{n+1}}{\delta u}) - J(\omega),$$
$$K(\omega) = 0 \tag{5}$$

which is called as Kupershmidt deformation of bi-Hamiltonian systems. This deformation is conjectured to preserve integrability and the conjecture is verified in a few representative cases in [10]. In [11]-[13], the nonholonomic deformations of mKdV equation, DNLS equation and KdV-type equations were studied. In [14], Zhou constructed the mixed soliton hierarchy and show that the nonholonomic deformation of soliton equations are some special numbers of the mixed soliton hierarchy.

In [15], we showed that the Kupershmidt deformation (2) of KdV equation is equivalent to the Rosochatius deformation of KdV equation with self-consistent sources, and presented the bi-Hamiltonian structure for the Kupershmidt deformation (2). The conjecture is then proved in [16] that the Kupershmidt deformation of a bi-Hamiltonian system is itself bi-Hamiltonian. Based on the Kupershmidt deformation (5), we proposed the generalized Kupershmidt deformation for integrable bi-Hamiltonian systems in [17], which provides a new way to construct new integrable system from a bi-Hamiltonian system. Also we made the conjecture that the generalized Kupershmidt deformation for integrable bi-Hamiltonian systems preserves integrability.

In present paper, when both Hamiltonian operators of bi-Hamiltonian system are pure differential operators, we show that the generalized Kupershmidt deformation (GKD) offers an useful way to construct new integrable systems starting from bi-Hamiltonian systems. By means of the generalized Kupershmidt deformation (GKD), we construct some new systems from some integrable bi-Hamiltonian systems in the cases of Harry Dym hierarchy, classical Boussinesq hierarchy and coupled KdV hierarchy. Then we show that these new systems can be converted into the Rosochatius deformation of soliton equations with self-consistent sources, and their Lax pair can be found by using the method in [5, 6, 18]. This indicates that the generalized Kupershmidt deformation is equivalent to integrable Rosochatius deformation of soliton equations with self-consistent sources. Therefore the conjecture on integrability of the generalized Kupershmidt deformation is verified in these cases. This implies that the generalized Kupershmidt deformation provides a new way to construct new integrable systems from bi-Hamiltonian systems and also offers a new approach to construct the Rosochatius deformation of soliton equations with self-consistent sources in a different way from the method proposed in [5, 6]. However it remains to study how to construct new integrable system from the bi-Hamiltonian systems in which the Hamiltonian operators are not pure differential operators, for example in the case of mKdV hierarchy. In section 2, we construct the generalized Kupershmidt deformation (GKD) of the Harry Dym hierarchy and its Lax pair. We show that the GKD of Harry Dym equation is equivalent to the integrable Rosochatius deformation of Harry Dym equation with self-consistent sources. Section 3 is devoted to convert the GKD of classical Boussinesq equation into the integrable Rosochatius deformation of classical Boussinesq equation with self-consistent sources and present its Lax pair. Section 4 treats the GKD of coupled KdV equation, the new coupled KdV equation with self-consistent sources and it's Lax pair are obtained. The last section presents the conclusion.

2 The generalized Kupershmidt deformation of Harry Dym hierarchy

Consider a hierarchy of soliton equations with bi-Hamiltonian structure

$$u_{t_n} = J(\frac{\delta H_{n+1}}{\delta u}) = K(\frac{\delta H_n}{\delta u}), \tag{6}$$

where J and K are two standard Hamiltonian operators. The associated spectral problem reads

$$L\phi = \lambda\phi. \tag{7}$$

Assume that for N distinct real eigenvalues λ_j, we have

$$L\varphi_j = \lambda_j\varphi_j,\ \ j = 1, 2, \cdots, N,$$

and

$$\frac{\delta\lambda_j}{\delta u} = f(\varphi_j).$$

Based on the Kupershmidt deformation (5), we first generalize Kupershmidt deformation as follows

$$u_{t_n} = J(\frac{\delta H_{n+1}}{\delta u}) - J(\sum_{j=1}^{N}\omega_j), \tag{8a}$$

$$(\gamma_j J - \mu_j K)(\omega_j) = 0,\ \ j = 1, 2, \cdots, N, \tag{8b}$$

where γ_j and μ_j are constants, which also gives to nonholonomic deformation of bi-Hamiltonian systems (6) similar to the integrable KdV6's type of noholonomic deformation of soliton equations. Furthermore, observe that ω_j in (8a) is at the same position as $\frac{\delta H_{n+1}}{\delta u}$ and the eigenvalues λ_j are also the conserved quantity for (6) as H_n, it is reasonable to take $\omega_j = \frac{\delta\lambda_j}{\delta u}$ and this setting is compatible with (8b). So we finally propose the generalized Kupershmidt deformation for a bi-Hamiltonian systems in [17] as follows

$$u_{t_n} = J(\frac{\delta H_{n+1}}{\delta u} - \sum_{j=1}^{N} \frac{\delta\lambda_j}{\delta u}), \tag{9a}$$

$$(\gamma_j J - \mu_j K)(\frac{\delta\lambda_j}{\delta u}) = 0,\ j = 1, 2, \cdots, N. \tag{9b}$$

This deformation is also conjectured to preserve integrability and the conjecture is verified in a few representative cases in [17]. In present paper, for some other specific cases, we will further show that (9) gives rise to the Rosochatius deformation of soliton equation with self-consistent sources or soliton equation with self-consistent sources. Following the procedure in [5,6,19], it is easy to find the Lax representation for the Rosochatius deformation of soliton equation with self-consistent sources, which implies the integrability of the Rosochatius deformation of soliton equation with self-consistent sources or the integrability of the generalized Kupershmidt deformation of soliton equation.

We now consider the eigenvalue problem[19]

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ \lambda u & 0 \end{pmatrix}. \tag{10}$$

the associated Harry Dym hierarchy reads[20]

$$u_{t_n} = J(\frac{\delta H_{n+1}}{\delta u}) = K(\frac{\delta H_n}{\delta u}), \tag{11}$$

where $J = \partial^3$ and $K = u\partial + \partial u$ are two standard Hamiltonian operators, $H_0 = -\int u dx$, $H_{-1} = \int 2u^{\frac{1}{2}} dx$, $H_{-2} = \int \frac{1}{8} u^{-\frac{5}{2}} u_x^2 dx$, $H_{-3} = \int \frac{1}{16}(\frac{35}{16} u^{-\frac{11}{2}} u_x^4 - u^{-\frac{7}{2}} u_{xx}^2) dx, \cdots$.

When $n = -2$, (11) gives the Harry Dym equation

$$u_t = (u^{-\frac{1}{2}})_{xxx}. \tag{12}$$

Assume that for N distinct real λ_j, we have

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ \lambda_j u & 0 \end{pmatrix} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}. \tag{13}$$

Its easy to find that

$$\frac{\delta\lambda_j}{\delta u} = -\lambda_j \psi_{1j}^2. \tag{14}$$

Take $\gamma_j = 1,\ \mu_j = \frac{1}{2}\lambda_j,\ (j = 1, \cdots, N)$ and $n = -2$, the generalized Kupershmidt deformation (8) gives rise to the following new generalized Harry Dym equation

$$u_t = (u^{-\frac{1}{2}})_{xxx} - \sum_{j=1}^{N} \omega_{jxxx}, \tag{15a}$$

$$\omega_{jxxx} - \lambda_j u \omega_{jx} - \frac{1}{2}\lambda_j u_x \omega_j = 0,\ j = 1, 2, \cdots, N. \tag{15b}$$

By replacing ω_j by (14), (15b) yields

$$\psi_{1j}(\psi_{1jxx} - \lambda_j u \psi_{1j})_x + 3\psi_{1jx}(\psi_{1jxx} - \lambda_j u \psi_{1j}) = 0,$$

which immediately leads to

$$\psi_{1jxx} - \lambda_j u \psi_{1j} = \frac{\mu_j}{\psi_{1j}^3},$$

where $\mu_j,\ j = 1, 2, \cdots, N$ are integral constants. Therefore (9) gives rise to the following new generalized Kupershmidt deformation of Harry Dym equation(GKDHDE)

$$u_t = (u^{-\frac{1}{2}})_{xxx} + \sum_{j=1}^{N} \lambda_j (\psi_{1j}^2)_{xxx}, \tag{16a}$$

$$\psi_{1jxx} + (u - \lambda_j)\psi_{1j} = \frac{\mu_j}{\psi_{1j}^3},\ j = 1, 2, \cdots, N \tag{16b}$$

which can be regarded as the Rosochatius deformation of Harry Dym equation with self-consistent sources (RD-HDESCS). When $\mu_j = 0,\ j = 1, \cdots, N$, (16) reduces to the Harry Dym equation with self-consistent sources(HDESCS). In the following, we derive the Lax pair for (16). Fist we derive the Lax pair for HDESCS.

Setting $\psi_1 = \psi,\ \psi_{2x} = \lambda u \psi_1$ and comparing to the Harry Dym equation, we can assume the Lax representation of the HDESCS has the form

$$\psi_{xx} = \lambda u \psi, \tag{17a}$$

$$\psi_t = -\frac{1}{2} B_x \psi + B \psi_x, \tag{17b}$$

$$B = -2u^{-\frac{1}{2}}\lambda + \sum_{j=1}^{N} \frac{\alpha_j f(\psi_j)}{\lambda - \lambda_j} + \sum_{j=1}^{N} \beta_j f(\psi_j), \tag{17c}$$

where $f(\psi_j)$ is undetermined function of ψ_j. The compatibility condition of (17a) and (17b) gives

$$u_t \lambda = LB + (2B_x u + B u_x)\lambda, \tag{18}$$

where $L=-\frac{1}{2}\partial^3$. Then (17c) and (18) yields

$$u_t\lambda=(u^{-\frac{1}{2}})_{xxx}\lambda+\sum_{j=1}^{N}\frac{\alpha_j}{\lambda-\lambda_j}[-\frac{1}{2}f^{'''}\psi_{jx}^3+(-\frac{3}{2}\lambda_j uf^{''}\psi_j+\frac{3}{2}\lambda_j uf')\psi_{jx}$$

$$+\lambda_j u_x(f-\frac{1}{2}\psi_j f')]+\sum_{j=1}^{N}\beta_j[-\frac{3}{2}u\psi_j\psi_{jx}f^{''}-\frac{1}{2}u_x\psi_j f^{'}-\frac{1}{2}u\psi_{jx}f^{'}+2u\psi_{jx}f^{'}+u_x f]\lambda$$

$$+\sum_{j=1}^{N}[\beta_j\lambda_j(-\frac{3}{2}\psi_j\psi_{jx}f^{''}-\frac{1}{2}u_x\psi_j f^{'}-\frac{1}{2}u\psi_{jx}f^{'})+2u\lambda_j\alpha_j\psi_{jx}f^{'}+\lambda_j\alpha_j u_x f] \tag{19}$$

Here f' denotes the partial derivative of the function f with respect to the variable ψ_j. In order to determine f, α_j and β_j, we compare the coefficients of $\frac{1}{\lambda-\lambda_j}$, λ and λ^0, respectively. We first observe the coefficients of $\frac{1}{\lambda-\lambda_j}$. The coefficients of ψ_{jx}^3 gives

$$f^{'''}=0, \tag{20}$$

The coefficients of ψ_{jx} gives

$$f^{''}\psi_j-f'=0, \tag{21}$$

The other terms gives

$$\frac{1}{2}f'\psi_j-f=0. \tag{22}$$

From (20), (21) and (22) we obtain $f=\psi_j^2$. Substituting $f=\psi_j^2$ into the coefficients of λ gives

$$u_t=(u^{-\frac{1}{2}})_{xxx}+\sum_{j=1}^{N}\beta_j(4u\psi_j\psi_{jx}+u_x\psi_j^2).$$

Comparing with the HDESCS we can determine

$$\beta_j=-2\lambda_j^2.$$

Substituting $f=\psi_j^2$ and $\beta_j=-2\lambda_j^2$ into the coefficients of λ^0 gives

$$\sum_{j=1}^{N}[(2\lambda_j^3+\lambda_j\alpha_j)(4u\psi_j\psi_{jx}+u_x\psi_j^2)]=0$$

which gives

$$\alpha_j=-2\lambda_j^2.$$

Thus we obtained the Lax pair of the HDESCS

$$\psi_{xx} = \lambda u \psi, \tag{23a}$$

$$\begin{aligned}\psi_t = &(-\frac{1}{2}u^{-\frac{3}{2}}u_x\lambda - 2\sum_{j=1}^{N}\lambda_j^2\psi_j\psi_{jx} - 2\sum_{j=1}^{N}\frac{\lambda_j^2\psi_j\psi_{jx}}{\lambda-\lambda_j})\psi \\ &+ (-2u^{-\frac{1}{2}}\lambda + 2\sum_{j=1}^{N}\lambda_j^2\psi_j^2 + 2\sum_{j=1}^{N}\frac{\lambda_j^2\psi_j^2}{\lambda-\lambda_j})\psi_x,\end{aligned} \tag{23b}$$

which equivalent to (10) and

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = V\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\ \psi_1 = \psi,\ \psi_2 = \psi_{1x} \tag{24}$$

with

$$V = \begin{pmatrix} -\frac{1}{2}u_x u^{-\frac{3}{2}}\lambda & -2u^{-\frac{1}{2}}\lambda \\ -2u^{\frac{1}{2}}\lambda^2 - \frac{1}{2}(u_{xx}u - \frac{3}{2}u_x^2)u^{-\frac{5}{2}}\lambda & \frac{1}{2}u_x u^{-\frac{3}{2}}\lambda \end{pmatrix} + 2\sum_{j=1}^{N}\begin{pmatrix} 0 & 0 \\ u\psi_{1j}^2\lambda_j^2\lambda & 0\end{pmatrix}$$

$$-2\sum_{j=1}^{N}\frac{\lambda_j^2\lambda}{\lambda-\lambda_j}\begin{pmatrix}\psi_{1j}\psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 & -\psi_{1j}\psi_{2j}\end{pmatrix}.$$

Finally, as proposed in [5, 6], the above formula induces the Lax pair (10) and (24) for RD-HDESCS (16) with

$$V = \begin{pmatrix} -\frac{1}{2}u_x u^{-\frac{3}{2}}\lambda & -2u^{-\frac{1}{2}}\lambda \\ -2u^{\frac{1}{2}}\lambda^2 - \frac{1}{2}(u_{xx}u - \frac{3}{2}u_x^2)u^{-\frac{5}{2}}\lambda & \frac{1}{2}u_x u^{-\frac{3}{2}}\lambda \end{pmatrix} + 2\sum_{j=1}^{N}\begin{pmatrix} 0 & 0 \\ u\psi_{1j}^2\lambda_j^2\lambda & 0\end{pmatrix}$$

$$-2\sum_{j=1}^{N}\frac{\lambda_j^2\lambda}{\lambda-\lambda_j}\begin{pmatrix}\psi_{1j}\psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 + \frac{\mu_j}{\psi_{1j}^2} & -\psi_{1j}\psi_{2j}\end{pmatrix}.$$

3 The generalized Kupershmidt deformation of the classical Boussinesq hierarchy

For the classical Boussinesq eigenvalue problem [21]

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad U = \begin{pmatrix} 0 & 1 \\ -\lambda^2 + \lambda v + u - \frac{1}{4}v^2 & 0\end{pmatrix}, \tag{25}$$

the associated classical Boussinesq hierarchy reads

$$\begin{pmatrix}u\\ v\end{pmatrix}_{t_n} = J\begin{pmatrix} b_{n+1} \\ b_{n+2} - \frac{1}{2}vb_{n+1}\end{pmatrix} = J\begin{pmatrix}\frac{\delta H_{n+1}}{\delta u} \\ \frac{\delta H_{n+1}}{\delta v}\end{pmatrix} = K\begin{pmatrix}\frac{\delta H_n}{\delta u} \\ \frac{\delta H_n}{\delta v}\end{pmatrix}$$

where

$$J=\begin{pmatrix} 0 & 2\partial \\ 2\partial & 0 \end{pmatrix},\ K=\begin{pmatrix} -\frac{1}{2}\partial^3+u\partial+\partial u & v\partial \\ \partial v & 2\partial \end{pmatrix},$$

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \end{pmatrix}=L\begin{pmatrix} b_{n+1} \\ b_n \end{pmatrix},$$

$$L=\begin{pmatrix} v-\frac{1}{2}\partial^{-1}v_x & -\frac{1}{4}\partial^2-\frac{1}{2}\partial^{-1}(u-\frac{1}{4}v^2)_x+u-\frac{1}{4}v^2 \\ 1 & 0 \end{pmatrix},$$

$$b_0=b_1=0,\ b_2=2,\ n=1,2,\cdots.$$

For N distinct real λ_j, from the spectral problem

$$\psi_{1jx}=\psi_{2j},\ \psi_{2jx}=(-\lambda_j^2+\lambda_j v+u-\frac{1}{4}v^2)\psi_{1j}$$

we have

$$\frac{\delta\lambda_j}{\delta u}=-\psi_{1j}^2,\ \frac{\delta\lambda_j}{\delta v}=(-\lambda_j+\frac{1}{2}v)\psi_{1j}^2. \tag{26}$$

When take $\gamma_j=-\lambda_j$, $\mu_j=-1$, and $n=4$, the generalized Kupershmidt deformation (8) gives rise to the following new generalized classical Boussinesq equation

$$u_t=-\frac{3}{4}vv_{xxx}-\frac{1}{2}u_{xxx}+\frac{3}{2}v^2u_x+3uvv_x+3uu_x-\frac{3}{2}v_xv_{xx}$$
$$-2\sum_{j=1}^{N}\omega_{1jx}, \tag{27a}$$

$$v_t=-\frac{1}{2}v_{xxx}+3vu_x+\frac{3}{2}v^2v_x+3uv_x-2\sum_{j=1}^{N}\omega_{2jx}, \tag{27b}$$

$$-\frac{1}{2}\omega_{1jxxx}+2u\omega_{1jx}+u_x\omega_{1j}+v\omega_{2jx}-2\lambda_j\omega_{2jx}=0, \tag{27c}$$

$$v_x\omega_{1j}+v\omega_{1jx}+2\omega_{2jx}-2\lambda_j\omega_{1jx}=0,\ j=1,2,\cdots,N. \tag{27d}$$

The generalized Kupershmidt deformation of the classical Boussinesq hierarchy is constructed from (9) as follows

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n}=J(\begin{pmatrix} \frac{\delta H_{n+1}}{\delta u} \\ \frac{\delta H_{n+1}}{\delta v} \end{pmatrix}-\sum_{j=1}^{N}\begin{pmatrix} \frac{\delta\lambda_j}{\delta u} \\ \frac{\delta\lambda_j}{\delta v} \end{pmatrix}), \tag{28a}$$

$$(K-\lambda_jJ)\begin{pmatrix} \frac{\delta\lambda_j}{\delta u} \\ \frac{\delta\lambda_j}{\delta v} \end{pmatrix}=0,\ j=1,2,\cdots,N. \tag{28b}$$

By substituting (26), (28b) yield

$$\psi_{1j}(\psi_{2jxx}-u\psi_{1j}-\lambda_jv\psi_{1j}+\frac{1}{4}v^2\psi_{1j}+\lambda_j^2\psi_{1j})_x$$

$$+3\psi_{1jx}(\psi_{2jxx} - u\psi_{1j} - \lambda_j v\psi_{1j} + \frac{1}{4}v^2\psi_{1j} + \lambda_j^2\psi_{1j}) = 0,$$

which leads to

$$\psi_{2jxx} = (-\lambda_j^2 + \lambda_j v + u - \frac{1}{4}v^2)\psi_{1j} + \frac{\mu_j}{\psi_{1j}^3},\ j = 1, 2, \cdots, N.$$

Take $n = 4$, the generalized Kupershmidt deformation of classical Boussinesq equation (28) gives rise to the following new system

$$u_t = -\frac{3}{4}vv_{xxx} - \frac{1}{2}u_{xxx} + \frac{3}{2}v^2u_x + 3uvv_x + 3uu_x - \frac{3}{2}v_xv_{xx}$$
$$- \sum_{j=1}^{N}(-4\lambda_j\psi_{1j}\psi_{1jx} + v_x\psi_{1j}^2 + 2v\psi_{1j}\psi_{1jx}), \tag{29a}$$

$$v_t = -\frac{1}{2}v_{xxx} + 3vu_x + \frac{3}{2}v^2v_x + 3uv_x + 4\sum_{j=1}^{N}\psi_{1j}\psi_{1jx}, \tag{29b}$$

$$\psi_{2jxx} = (-\lambda_j^2 + \lambda_j v + u - \frac{1}{4}v^2)\psi_{1j} + \frac{\mu_j}{\psi_{1j}^3},\ j = 1, 2, \cdots, N \tag{29c}$$

which is regarded as the Rosochatius deformation of classical Boussinesq equation with self-consistent sources. Following the procedure in [5, 6, 18], we can find that Eq.(29) has the Lax representation (10) and (24) with

$$U = \begin{pmatrix} 0 & 1 \\ -\lambda^2 + \lambda v + u - \frac{1}{4}v^2 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} -\frac{1}{2}v_x\lambda - \frac{1}{2}vv_x - \frac{1}{2}u_x & 2\lambda^2 + v\lambda + \frac{1}{2}v^2 + u \\ F & \frac{1}{2}v_x\lambda + \frac{1}{2}vv_x + \frac{1}{2}u_x \end{pmatrix}$$
$$+ \sum_{j=1}^{N}\begin{pmatrix} 0 & 0 \\ (\lambda_j - v + \lambda)\psi_{1j}^2 & 0 \end{pmatrix} - \sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\begin{pmatrix} \psi_{1j}\psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 + \frac{\mu_j}{\psi_{1j}^2} & -\psi_{1j}\psi_{2j} \end{pmatrix}$$

where

$$F = -2\lambda^4 + v\lambda^3 + u\lambda^2 + (-\frac{1}{2}v_{xx} + \frac{1}{4}v^3 + 2uv)\lambda - \frac{1}{2}v_x^2 - \frac{1}{2}vv_{xx} - \frac{1}{2}u_{xx} + \frac{1}{4}uv^2 + u^2 - \frac{1}{8}v^4.$$

4 The generalized Kupershmidt deformation of the coupled KdV hierarchy

The coupled KdV hierarchy (cKdVH) and its Backlund transformations was derived by Levi[22]. This hierarchy has two important features: firstly the odd sub-hierarchy can be reduced to ordinary KdV hierarchy and secondly its third member resembles the celebrated Hirota-Satsuma system of equations[23, 24].

The coupled KdV eigenvalue problem reads[25, 26]

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad U = \begin{pmatrix} -\frac{1}{2}(\lambda - u) & -v \\ 1 & \frac{1}{2}(\lambda - u) \end{pmatrix}, \tag{30}$$

the associated cKdVH can be written as the bi-Hamiltonian structure

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = J \begin{pmatrix} a_{n+1} \\ -c_{n+1} \end{pmatrix} = J \begin{pmatrix} \frac{\delta H_{n+1}}{\delta u} \\ \frac{\delta H_{n+1}}{\delta v} \end{pmatrix} = K \begin{pmatrix} \frac{\delta H_n}{\delta u} \\ \frac{\delta H_n}{\delta v} \end{pmatrix}$$

where

$$J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \ K = \begin{pmatrix} 2\partial & \partial^2 + \partial u \\ -\partial^2 + u\partial & \partial v + v\partial \end{pmatrix},$$

$$\begin{pmatrix} a_{n+1} \\ c_{n+1} \end{pmatrix} = L \begin{pmatrix} a_n \\ c_n \end{pmatrix}, \ L = \begin{pmatrix} \partial^{-1} u\partial - \partial & \partial^{-1} v\partial + v \\ 2 & \partial + u \end{pmatrix}, \ n = 1, 2, \cdots$$

$$a_0 = \frac{1}{2}, \ b_0 = c_0 = 0, \ a_1 = 0 \ , b_1 = v, \ c_1 = -1, \ldots.$$

For N distinct real λ_j, from the spectral problem

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2}(\lambda_j - u) & -v \\ 1 & \frac{1}{2}(\lambda_j - u) \end{pmatrix} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \tag{31}$$

we have

$$\frac{\delta \lambda_j}{\delta u} = \psi_{1j}\psi_{2j}, \ \frac{\delta \lambda_j}{\delta v} = -\psi_{2j}^2. \tag{32}$$

When take $\gamma_j = -\lambda_j, \ \mu_j = -1$, and $n = 3$ in (8), the new generalized coupled KdV equation is constructed as follows

$$u_t = u_{xxx} + 6u_x v + 6uv_x + 3u_x^2 + 3uu_{xx} + 3u^2 u_x - \sum_{j=1}^{N} \omega_{2jx}, \tag{33a}$$

$$v_t = v_{xxx} + 6uu_x v + 6vv_x + 3u^2 v_x - 3uv_{xx} - \sum_{j=1}^{N} \omega_{1jx}, \tag{33b}$$

$$2\omega_{1jx} + \omega_{2jxx} + u_x \omega_{2j} + u\omega_{2jx} - \lambda_j \omega_{2jx} = 0, \tag{33c}$$

$$-\omega_{1jxx} + u\omega_{1jx} + v_x \omega_{2j} + 2v\omega_{2jx} - \lambda_j \omega_{1jx} = 0, \ j = 1, 2, \cdots, N. \tag{33d}$$

The generalized Kupershmidt deformation of the cKdVH is constructed as follows

$$\left(\begin{array}{c} u \\ v \end{array}\right)_{t_n} = J(\left(\begin{array}{c} \frac{\delta H_{n+1}}{\delta u} \\ \frac{\delta H_{n+1}}{\delta v} \end{array}\right) - \sum_{j=1}^{N} \left(\begin{array}{c} \frac{\delta \lambda_j}{\delta u} \\ \frac{\delta \lambda_j}{\delta v} \end{array}\right)), \tag{34a}$$

$$(K - \lambda_j J) \left(\begin{array}{c} \frac{\delta \lambda_j}{\delta u} \\ \frac{\delta \lambda_j}{\delta v} \end{array}\right) = 0, \ j = 1, 2, \cdots, N. \tag{34b}$$

From (32) and (34b), we obtain

$$\begin{aligned} & 2\psi_{1jx}\psi_{2j} + 2\psi_{1j}\psi_{2jx} - 2\psi_{2jx}^2 - 2\psi_{2j}\psi_{2jxx} - u_x\psi_{2j}^2 \\ & - 2u\psi_{2j}\psi_{2jx} + 2\lambda_j\psi_{2j}\psi_{2jx} = 0, \end{aligned} \tag{35a}$$

$$\begin{aligned} & - \psi_{1jxx}\psi_{2j} - 2\psi_{1jx}\psi_{2jx} - \psi_{1j}\psi_{2jxx} + u(\psi_{1jx}\psi_{2j} + \psi_{1j}\psi_{2jx}) - v_x\psi_{2j}^2 \\ & - 4v\psi_{2j}\psi_{2jx} - \lambda_j(\psi_{1j}\psi_{2jx} + \psi_{1jx}\psi_{2j}) = 0. \end{aligned} \tag{35b}$$

(35a) yields

$$\psi_{2jx} = \psi_{1j} + \frac{1}{2}(\lambda_j - u)\psi_{2j} + \frac{\mu_j}{\psi_{2j}}, \ j = 1, 2, \cdots, N. \tag{36}$$

(35b) yields

$$\begin{aligned} & [\psi_{2j}(\psi_{1jx} + \frac{1}{2}(\lambda_j - u)\psi_{1j}) + v\psi_{2j}]_x + (\frac{\psi_{1j}\mu_j}{\psi_{2j}})_x + 2\psi_{1jx}(\psi_{2jx} - \frac{\mu_j}{\psi_{2j}}) \\ & \qquad + \psi_{2jx}(2v\psi_{2j} + u\psi_{1j} + \lambda_j\psi_{1j}) = 0. \end{aligned} \tag{37}$$

In order to keep (37) hold, we find that $\mu_j (j = 1, \cdots, N)$ equal zero. So (34b) gives

$$\psi_{1jx} = -\frac{1}{2}(\lambda_j - u)\psi_{1j} - v\psi_{2j}, \ \psi_{2jx} = \psi_{1j} + \frac{1}{2}(\lambda_j - u)\psi_{2j}, \ j = 1, 2, \cdots, N.$$

Then the generalized Kupershmidt deformation of coupled KdV equation gives rise to the following system

$$u_t = u_{xxx} + 6u_xv + 6uv_x + 3u_x^2 + 3uu_{xx} + 3u^2u_x + 2\sum_{j=1}^{N}\psi_{2j}\psi_{2j,x}, \tag{38a}$$

$$v_t = v_{xxx} + 6uu_xv + 6vv_x + 3u^2v_x - 3uv_{xx} - \sum_{j=1}^{N}(\psi_{1j}\psi_{2j})_x, \tag{38b}$$

$$\psi_{1j,x} = -\frac{1}{2}(\lambda_j - u)\psi_{1j} - v\psi_{2j}, \ \psi_{2j,x} = \psi_{1j} + \frac{1}{2}(\lambda_j - u)\psi_{2j}, \ j = 1, 2, \cdots, N, \tag{38c}$$

which is called as the coupled KdV equation with self-consistent sources. Following the procedure in [5, 6, 18], we can find the Lax representation (10) and (24) for (38) with

$$U = \begin{pmatrix} -\frac{1}{2}(\lambda - u) & -v \\ 1 & \frac{1}{2}(\lambda - u) \end{pmatrix},$$

$$V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} + \frac{1}{2}\sum_{j=1}^{N} \begin{pmatrix} \psi_{2j}^2 & 0 \\ 0 & -\psi_{2j}^2 \end{pmatrix} - \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}$$

where

$$A = \frac{1}{2}\lambda^3 + v\lambda - 4uv - v_x - u_{xx} - 3uu_x - u^3$$

$$B = v\lambda^2 + (uv - v_x)\lambda + 2v^2 + u^2 v - 2uv_x - u_x v + v_{xx}$$

$$C = -\lambda^2 - u\lambda - u_x - 2v - u^2.$$

5 Conclusion

The main purpose of this paper is to show that for the bi-Hamiltonian systems with both Hamiltonian operators being differential operators, the generalized Kupershmidt deformation (GKD) developed from the Kupershmidt deformation in [10] offers an useful way to construct new integrable systems starting from bi-Hamiltonian systems. We construct some new integrable systems by making use of the generalized Kupershmidt deformation (GKD) of bi-Hamiltonian systems and to verify the conjecture on the integrability of the generalized Kupershmidt deformation in some specific cases. We obtain the new GKD of Harry Dym equation, GKD of the classical Boussinesq equation and GKD of the coupled KdV equation. Then we show that these new systems can be converted into the Rosochatius deformation of soliton equation with self-consistent sources. Furthermore the Lax pairs for the Rosochatius deformation of soliton equation with self-consistent sources can be constructed in a systematic procedure. These imply that the generalized Kupershmidt deformation of bi-Hamiltonian systems provides a new way to construct new integrable systems from bi-Hamiltonian systems and also offers a new approach to obtain the Rosochatius deformation of soliton equation with self-consistent sources. However, when the Hamiltonian operators are not pure differential operators, it remains to study how to construct new integrable systems from bi-Hamiltonian systems by using the generalized Kupershmidt deformation.

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