From quantum A_N (Calogero) to H_4 (rational) model

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Abstract

A brief and incomplete review of known integrable and (quasi)-exactly-solvable quantum models with rational (meromorphic in Cartesian coordinates) potentials is given. All of them are characterized by (i) a discrete symmetry of the Hamiltonian, (ii) a number of polynomial eigenfunctions, (iii) a factorization property for eigenfunctions, and admit (iv) the separation of the radial coordinate and, hence, the existence of the 2nd order integral, (v) an algebraic form in invariants of a discrete symmetry group (in space of orbits).

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In this Talk we will make an attempt to overview our constructive knowledge about (quasi)-exactly-solvable potentials having a form of a meromorphic function in Cartesian coordinates. All these models have a discrete group of symmetry, admit separation of variable(s), possess an (in)finite set of polynomial eigenfunctions. They have an infinite discrete spectrum which is linear in the quantum numbers. All of them are characterized by the presence of a hidden (Lie) algebraic structure. Each of them is a type of isospectral deformation of the isotropic harmonic oscillator.

Let us consider the Hamiltonian = the Schrödinger operator

$$\mathcal{H} \ = \ -\Delta \ + \ V(x) \ , \qquad x \in R^d \ . \tag{1}$$

A problem of quantum mechanics is to solve the Schrödinger equation

$$\mathcal{H}\Psi(x) \ = \ E\Psi(x) \quad , \quad \Psi(x) \in L^2(R^d) \tag{2}$$

finding the spectrum (the energies E and eigenfunctions Ψ). Since the Hamiltonian is an infinite-dimensional matrix, solving the Schrödinger equation is equivalent to diagonalizing the infinite-dimensional matrix. It is transcendental problem, the characteristic polynomial is of infinite order and it has infinitely-many roots. Usually, we do not know how to make such a diagonalizing exactly (explicitly) but we can ask: *Do models exist for which the roots (energies), some of the them or all, can be found explicitly (exactly)?* Such models do exist and we call them *solvable*. If all energies are known they are called *Exactly-Solvable* (ES), if only some number of them is known we call them *Quasi-Exactly-Solvable* (QES). Surprisingly, all such models I am familiar with are provided by integrable systems. The Hamiltonians of these models are of the form

$$\mathcal{H}_{ES} = -\frac{1}{2}\Delta + \omega^2 r^2 + \frac{W(\Omega)}{r^2} \tag{3}$$

in the exactly-solvable case and

$$\mathcal{H}_{QES} = -\frac{1}{2}\Delta + \tilde{\omega}_k^2 r^2 + \frac{W(\Omega) + \Gamma}{r^2} + ar^6 + br^4 \ , \tag{4}$$

in the quasi-exactly-solvable case, where $\omega, \tilde{\omega}_k, \Gamma$ are parameters, $W(\Omega)$ is a function on unit sphere and r is the radial coordinate. In both cases there exists the integral

$$\mathcal{F} \ = \ \frac{1}{2}\,\mathcal{L}^2 + W(\Omega) \ , \tag{5}$$

where $\mathcal{L}$ is the angular momentum operator, due to the separation of variables in spherical coordinates.

Now we consider some examples among which are known so far.

Case $O(N)$

The Hamiltonian reads

$$\mathcal{H}_{O(N)} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial {x_i}^2} + \omega^2 {x_i}^2\right) + \frac{\nu(\nu-1)}{\sum_{i=1}^{N} {x_i}^2}\ , \qquad (6)$$

or, in spherical coordinates,

$$\mathcal{H}_{O(N)} = -\frac{1}{2r^N}\frac{\partial}{\partial r}\left(r^N\frac{\partial}{\partial r}\right) + \frac{1}{2}\omega^2 r^2 + \frac{\mathcal{F}+\nu(\nu-1)}{r^2}\ , \qquad (7)$$

$$\mathcal{F}\ =\ \frac{1}{2}\,\mathcal{L}^2\ . \qquad (8)$$

The Hamiltonian (6) is $O(N)$ symmetric. It describes a spherical-symmetric harmonic oscillator with a generalized centrifugal potential. Needless to say that the Hamiltonian $\mathcal{H}_{O(N)}$ and $\mathcal{F}$ commute,

$$[\mathcal{H}_{O(N)}, \mathcal{F}]\ =\ 0\ . \qquad (9)$$

Thus, $\mathcal{F}$ has common eigenfunctions with the Hamiltonian $\mathcal{H}_{O(N)}$. The spectrum can be immediately found explicitly, and all eigenfunctions are of the type

$$P_n(r^2) r^{\tilde{\ell}} Y_{\{\ell\}}(\Omega) e^{-\frac{\omega r^2}{2}}\ ,$$

where $Y_{\{\ell\}}(\Omega)$ is a N-dimensional spherical harmonics, $\mathcal{F}Y_{\{\ell\}}(\Omega) = \gamma Y_{\{\ell\}}(\Omega)$. The Hamiltonian (6) describes an N-dimensional harmonic oscillator with generalized centrifugal term. Substituting in (7) the operator $\mathcal{F}$ by its eigenvalue γ and gauging away $\Psi_0 = r^{\tilde{\ell}} e^{-\frac{\omega r^2}{2}}$ we arrive at the Laguerre operator

$$h_{O(N)} \equiv (\Psi_0)^{-1}(\mathcal{H}_{O(N)} - E_0)\Psi_0|_{r^2=t}\ =\ -2t\partial_t^2 + (2\omega - 1 - \frac{N}{2} - \tilde{\ell})\partial_t \qquad (10)$$

where E_0 is the lowest energy and the parameter $\tilde{\ell}$ is chosen in such a way as to remove singular term $\propto \frac{1}{r^2}$ in the potential in (7). (10) is the algebraic form of the Hamiltonian (7). The gauge-rotated Hamiltonian $h_{O(N)}$ (10) is $sl(2)$-Lie-algebraic (see below), it has infinitely-many finite-dimensional invariant subspaces in polynomials $\mathcal{P}_n, n = 0, 1, \ldots$ forming the infinite flag (see below), its eigenfunctions $P_n(r^2 = t)$ are nothing but the associated Laguerre polynomials.

By adding to $h_{O(N)}$ (10) the operator

$$\delta h^{(qes)} = 4(at^2 - \gamma)\frac{\partial}{\partial t} - 4akt + 2\omega k \ , \tag{11}$$

we get the operator $h_{O(N)} + \delta h^{(qes)}$ which has a single finite-dimensional invariant subspace

$$\mathcal{P}_k = \langle t^p | 0 \leq p \leq k \rangle \ ,$$

of the dimension $(k+1)$. Hence, this operator is quasi-exactly-solvable. Making the change of variable $t = r^2$ and gauge rotation with $\tilde{\Psi}_0 = t^{\hat{\gamma}} e^{-\frac{\omega t}{2} - \frac{at^2}{4}}$ we arrive at the $O(N)$-symmetric QES Hamiltonian [2]

$$\mathcal{H}_{O(N)} = -\frac{1}{2r^N}\frac{\partial}{\partial r}\left(r^N \frac{\partial}{\partial r}\right) + a^2 r^6 + 2a\omega r^4 + \frac{1}{2}\tilde{\omega}^2 r^2 + \frac{\mathcal{F} + \Gamma}{r^2} \ , \tag{12}$$

where $\hat{\gamma}, \Gamma, \tilde{\omega}$ are parameters and γ is replaced by the operator $\mathcal{F}$. In (12) a finite number of the eigenfunctions is of the form

$$P_k(r^2) r^{2\hat{\gamma}} Y_{\{\ell\}}(\Omega) e^{-\frac{\omega r^2}{2} - \frac{at^2}{4}} \ ,$$

they can be found algebraically. It is worth noting that at $a = 0$ the operator $h_{O(N)} + \delta h^{(qes)}$ remains exactly-solvable, it preserves the infinite flag of polynomials $\mathcal{P}$ and the emerging Hamiltonian has a form of (7).

Case $\mathbb{Z}_2^N$

The Hamiltonian reads

$$\mathcal{H}_{\mathbb{Z}_2^N} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial {x_i}^2} + \omega^2 {x_i}^2\right) + \frac{1}{2}\sum_{i=1}^{N}\frac{\nu_i(\nu_i - 1)}{{x_i}^2} \ , \tag{13}$$

or, in spherical coordinates,

$$\mathcal{H}_{\mathbb{Z}_2^N} = -\frac{1}{2r^N}\frac{\partial}{\partial r}\left(r^N \frac{\partial}{\partial r}\right) + \frac{1}{2}\omega^2 r^2 + \frac{\mathcal{F} \ + \ W_{\mathbb{Z}_N}(\Omega)}{r^2} \ , \tag{14}$$

where

$$W_{\mathbb{Z}_2^N}(\Omega) = \frac{1}{2}\sum_{i=1}^{N}\nu_i(\nu_i - 1)\left(\frac{r}{x_i}\right)^2 \ ,$$

and $\mathcal{F}$ is given by (8). The Hamiltonian (13) is $\mathbb{Z}_2^N$ symmetric. It defines the so-called Smorodinsky-Winternitz integrable system [1] which is in reality the maximally-superintegrable (there exist $(2N-1)$ integrals including the Hamiltonian) and exactly-solvable. Gauging away in (13) the ground state, $\Psi_0 = \prod_{i=1}^N (x_i^2)^{\frac{\nu_i}{2}} \exp{(-\frac{\omega x_i^2}{2})}$, and changing variables to $t_i = x_i^2$ we arrive at the algebraic form. Also it admits QES extension. The system described by the Hamiltonian (13) at $\nu_i = \nu$ is a particular case of the BC_N-rational system (see below).

Case A_{N-1}

This is the celebrated Calogero Model (A_{N-1} Rational model) which was found in [3]. It describes N identical particles on a line (see Fig.1) with singular pairwise interaction.

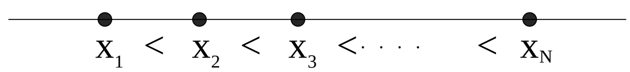

FIG. 1: N-body Calogero model

The Hamiltonian is

$$\mathcal{H}_{\rm Cal} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial {x_i}^2} + \omega^2 {x_i}^2\right) + \nu(\nu-1)\sum_{i>j}^{N}\frac{1}{(x_i - x_j)^2}\ , \tag{15}$$

where the singular part of the potential can be written as

$$\sum_{i>j}^{N}\frac{1}{(x_i-x_j)^2} \ = \ \frac{W_{A_{N-1}}(\Omega)}{r^2}\ ,\ W_{A_{N-1}}(\Omega) = \sum_{i=1}^{N}\left(\frac{1}{\frac{x_i}{r} - \frac{x_j}{r}}\right)^2\ , \tag{16}$$

Here r is the radial coordinate in the space of relative coordinates (see below for a definition) and $W_{A_{N-1}}(\Omega)$ is a function on the unit sphere.

Symmetry: S_n (permutations $x_i \to x_j$) plus $\mathbb{Z}_2$ (all $x_i \to -x_i$). The ground state of the Hamiltonian (15) reads

$$\Psi_0(x) = \prod_{i<j} |x_i - x_j|^{\nu} e^{-\frac{\omega}{2}\sum x_i^2}\ . \tag{17}$$

Let us make the gauge rotation

$$h_{\rm Cal} = 2\Psi_0^{-1}\left(\mathcal{H}_{\rm Cal} - E_0\right)\,\Psi_0\ ,$$

and introduce center-of-mass variables

$$Y = \sum x_i\ ,\ y_i = x_i - \frac{1}{N}Y\ ,\ i = 1,\ldots,N\ ,$$

and then permutationally-symmetric, translationally-invariant variables

$$(x_1, x_2, \ldots x_N) \to \big(Y, t_n(x) = \sigma_n(y(x))|\ n = 2, 3 \ldots N\big)\ ,$$

where

$$\sigma_k(x) = \sum_{i_1<i_2<\ldots<i_k} x_{i_1} x_{i_2} \ldots x_{i_k}\ ,\ \sigma_k(-x) = (-)^k \sigma_k(x)\ ,$$

are elementary symmetric polynomials, and

$$t_1 = 0\ ,\ t_2 \sim \sum_{i<j} (x_i - x_j)^2 = r^2\ ,$$

hence, the variable t_2, which plays fundamental role, is defined by radius in space of relative coordinates. After the center-of-mass separation, the gauge rotated Hamiltonian takes the algebraic form [4]

$$h_{\rm Cal}\ =\ \mathcal{A}_{ij}(t) \frac{\partial^2}{\partial t_i \partial t_j} + \mathcal{B}_i(t) \frac{\partial}{\partial t_i}\ , \tag{18}$$

where

$$\mathcal{A}_{ij}\ =\ \frac{(N-i+1)(1-j)}{N} t_{i-1}\, t_{j-1} + \sum_{l \geq \max(1,j-i)} (2l - j + i) t_{i+l-1} t_{j-l-1}\ ,$$

$$\mathcal{B}_i\ =\ \frac{1}{N}(1 + \nu N)(N - i + 2)(N - i + 1)\, t_{i-2} + 2\omega\,(i-1)\, t_i\ .$$

Eigenvalues of (18) are

$$\epsilon_{\{p\}}\ =\ 2\omega \sum_{i=2}^{N} (i-1)\ p_i\ ,$$

hence, the spectrum is linear in the quantum numbers $p_{2,3,\ldots,N} = 0, 1, \ldots$, it corresponds to *anisotropic* harmonic oscillator with frequency ratios $1 : 2 : 3 : \ldots : (N-1)$.

It is easy to check that the gauge-rotated Hamiltonian $h_{\rm Cal}$ has infinitely many finite-dimensional invariant subspaces

$$\mathcal{P}_n^{(N-1)}\ =\ \langle {t_2}^{p_2} {t_3}^{p_3} \ldots {t_N}^{p_N} |\ 0 \leq \Sigma p_i \leq n \rangle\ .$$

where $n = 0, 1, 2, \ldots$. As a function of n the spaces $\mathcal{P}_n^{(N-1)}$ form the infinite flag (see below).

Remark.

gl_{d+1}-algebra (acting in R^d), for the Young tableaux as a row $(n, \underbrace{0, 0, \ldots 0}_{d-1})$, has a form

$$
\begin{aligned}
\mathcal{J}_i^- &= \frac{\partial}{\partial t_i}, \qquad\qquad i = 1, 2 \ldots d\ , \\
\mathcal{J}_{ij}{}^0 &= t_i \frac{\partial}{\partial t_j}, \qquad i, j = 1, 2 \ldots d\ , \\
\mathcal{J}^0 &= \sum_{i=1}^{d} t_i \frac{\partial}{\partial t_i} - n\,, \\
\mathcal{J}_i^+ &= t_i \mathcal{J}^0 = t_i \left(\sum_{j=1}^{d} t_j \frac{\partial}{\partial t_j} - n \right), \quad i = 1, 2 \ldots d\ .
\end{aligned} \tag{19}
$$

The total number of generators is $(d{+}1)^2$. If $n = 0, 1, 2 \ldots$, the finite-dimensional irreps occur

$$
\mathcal{P}_n^{(d)} \ = \ \langle t_1{}^{p_1} t_2{}^{p_2} \ldots t_d{}^{p_d} |\ 0 \le \Sigma p_i \le n \rangle\ ,
$$

for which there is a property

$$
\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_n \subset \ldots \mathcal{P}\ .
$$

Such a nested construction is called *infinite flag (filtration)* $\mathcal{P}$. It is worth noting that the flag $\mathcal{P}^{(d)}$ is made out of finite-dimensional irreducible representation spaces $\mathcal{P}_n^{(d)}$ of the algebra gl_{d+1} taken in realization (19). It is evident that **any operator made out of generators (19) has finite-dimensional invariant subspace which is finite-dimensional irreducible representation space.**

It seems evident that the Hamiltonian (18) has to have a representation as a second order polynomial in generators (19) at $d = N-1$ acting in R^{N-1},

$$
h_{\rm Cal} \ = \ Pol_2(\mathcal{J}_i^-\ ,\ \mathcal{J}_{ij}{}^0)\ ,
$$

where the raising generators $\mathcal{J}_i^+$ are absent. Thus, $gl(N-1)$ (or, strictly speaking, its maximal affine subalgebra) is the hidden algebra of N-body Calogero model. The eigenfunctions of N-body Calogero model are elements of the flag of polynomials $\mathcal{P}^{(N-1)}$. Each subspace $\mathcal{P}_n^{(N-1)}$ is represented by the Newton polytope (pyramid). It contains C_{n+N-1}^{N-1} eigenfunctions, which is equal to the volume of the Newton polytope.

Making the gauge rotation of the integral (5) with $W_{A_{N-1}}(\Omega)$ given by (16)

$$
f_{\rm Cal} \ = \ \Psi_0^{-1}\, (\mathcal{F}_{\rm Cal} - F_0)\, \Psi_0\ , \tag{20}
$$

where F_0 is the lowest eigenvalue of the integral, $\mathcal{F}_{\rm Cal}\Psi_0 = F_0\Psi_0$, the integral gets the algebraic form,

$$f_{\rm Cal} \ = \ f_{ij}(t)\frac{\partial^2}{\partial t_i \partial t_j} + g_i(t)\frac{\partial}{\partial t_i} \ ,$$

where f_{ij} is 2nd degree polynomial in t, $f_{2j} = 0$, and g_i is 1st degree polynomial in t, $g_2 = 0$. It also can be rewritten as the second degree polynomial in the $gl(N-1)$ generators,

$$f_{\rm Cal} \ = \ Pol_2(\mathcal{J}_i^- \ , \ \mathcal{J}_{ij}{}^0) \ .$$

$sl(2)$-Quasi-Exactly-Solvable generalization of the Calogero model

By adding to $h_{\rm Cal}$ (18), the operator

$$\delta h^{(qes)} \ = \ 4(at_2^2 - \gamma)\frac{\partial}{\partial t_2} - 4akt_2 + 2\omega k \ , \qquad (21)$$

we get the operator $h_{\rm Cal} + \delta h^{(qes)}$ having finite-dimensional invariant subspace

$$\mathcal{P}_k = \langle t_2^p | 0 \le p \le k \rangle \ .$$

By making a gauge rotation of $h_{\rm Cal} + \delta h^{(qes)}$ and changing of variables to Cartesian one we arrive at the Hamiltonian [5]

$$\mathcal{H}^{\rm (qes)}_{Cal} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial {x_i}^2} + \omega^2 x_i^2\right) + \nu(\nu-1)\sum_{j<i}^{N}\frac{1}{(x_i-x_j)^2} +$$
$$\frac{2\gamma\left[\gamma - 2n(1+\nu+\nu n)+3\right]}{r^2} +$$
$$+ \ a^2 r^6 \ + \ 2a\omega r^4 \ - \ a\left[2k + 2n(1+\nu+\nu n) - \gamma - 1\right] r^2 \ . \qquad (22)$$

For the Hamiltonian, $(k+1)$ eigenfunctions are of the form

$$\Psi_k^{\rm (qes)}(x) = \prod_{i<j}^{n} |x_i - x_j|^\nu (r^2)^\gamma P_k(r^2) \ \exp\left[-\frac{\omega}{2}\sum_{i=1}^{n} x_i^2 - \frac{a}{4}r^4\right] \ = \ (r^2)^\gamma P_k(r^2)\exp(-\frac{a}{4}r^4)\Psi_0 \ ,$$

where Ψ_0 is given by (17), P_k is a polynomial of degree k in $r^2 = \sum_{i<j}(x_i - x_j)^2 = t_2$. All remaining eigenfunctions can be represented in the same form but P_k's are not polynomials anymore being functions depending on all variables x_i. It is worth noting that at $a = 0$ the operator $h_{Cal} + \delta h^{(qes)}$ remains exactly-solvable, it preserves the flag of polynomials $\mathcal{P}^{(N-1)}$ and the emerging Hamiltonian has a form of (15) with the extra term $\frac{\Gamma}{r^2}$ in the potential. Its ground state eigenfunction is $(r^2)^\gamma\Psi_0$. It is the exactly-solvable generalization of the Calogero model (15) with the Weyl group $W(A_{N-1})$ as the discrete symmetry group,

$$\mathcal{H}_{\rm W(A_{N-1})} \ = \ \mathcal{H}_{\rm Cal} + \frac{\Gamma}{r^2} \ .$$

Case: Hamiltonian Reduction Method

(for review and references see e.g. Olshanetsky-Perelomov [6])

In this method a family of integrable and exactly-solvable Hamiltonians associated with Weyl (Coxeter) symmetry was found with the Calogero model as one of its representatives. The idea of the method is beautiful and sufficient transparent,

- Take a simple group G,
- Define the Laplace-Beltrami (invariant) operator on its symmetric space (free/harmonic oscillator motion)
- Radial part of Laplace-Beltrami operator is the Olshanetsky-Perelomov Hamiltonian relevant from physical point of view. The emerging Hamiltonian is the Weyl-symmetric, it can be associated with root system, it is integrable with integrals given by the invariant operators of higher than two orders with a property of solvability.

Rational case:

This case appears when the coordinates of the symmetric space are introduced in such a way that the zero-curvature surface occurs. Emerging Calogero-Moser-Sutherland-Olshanetsky-Perelomov Hamiltonian in Cartesian coordinates has the form,

$$\mathcal{H} = \frac{1}{2}\sum_{k=1}^{N}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2\right] \ + \frac{1}{2}\sum_{\alpha\in R_+} \nu_{|\alpha|}(\nu_{|\alpha|}-1)\frac{|\,\alpha|^2}{(\alpha\cdot x)^2}\ , \tag{23}$$

where R_+ is a set of positive roots, x is a position vector and $\nu_{|\alpha|}$ are coupling constants (parameters) which depend on the root length. If roots of the same length, then $\nu_{|\alpha|}$ have to be equal, if all roots are of the same length like for A_n, then all $\nu_{|\alpha|} = \nu$. In the Hamiltonian Reduction the parameters $\nu_{|\alpha|}$ take a set of discrete values, however, they can be generalized to any real value without loosing a property of integrability as well as of solvability with the only constraint of the existence of L^2-solutions of the corresponding Schrödinger equation. Configuration space for (23) is the Weyl chamber. Ground state wave function is written explicitly,

$$\Psi_0(y) \ = \ \prod_{\alpha\in R_+} |(\alpha\cdot x)|^{\nu_{|\alpha|}}\, e^{-\omega x^2/2}\ . \tag{24}$$

The Hamiltonian (23) is completely-integrable: there exists a commutative algebra of integrals (including the Hamiltonian) of dimension which is equal to the dimension of the configuration space (for integrals, see Oshima [7] with explicit forms of those). For each Hamiltonian (23) after separation of center-of-mass coordinate (if applicable) the radial coordinate (in the space of relative coordinates) can be also separated. It gives rise to the existence of one more integral of the second order (5). Hence, the Hamiltonian (23) is super-integrable. The Hamiltonian (23) is invariant with respect to the Weyl (Coxeter) group transformation, which is the discrete symmetry group of the corresponding root space.

The Hamiltonian (23) has a hidden (Lie)-algebraic structure. In order to reveal it we need to

- Gauge away the ground state eigenfunction making *similarity transformation* $(\Psi_0)^{-1}\,(\mathcal{H}-E_0)\Psi_0\ =\ h$
- Consider the Hamiltonian in the space of orbits of Weyl (Coxeter) group by taking the *Weyl (Coxeter) polynomial invariants as new coordinates*), these invariants are

$$t_a^{(\Omega)}(x) = \sum_{\alpha\in\Omega} (\alpha\cdot x)^a\ ,$$

  where a's are the *degrees* of the Weyl (Coxeter) group, Ω is an orbit.

The invariants t are defined ambiguously, up to invariants of lower degrees, they depend on chosen orbit.

Case BC_N

The BC_N-Rational model is defined by the Hamiltonian,

$$\mathcal{H}_{BC_N}\ =\ -\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial {x_i}^2}-\omega^2 x_i^2\right)\ +$$

$$\nu(\nu-1)\sum_{i<j}\left[\frac{1}{(x_i-x_j)^2}+\frac{1}{(x_i+x_j)^2}\right]\ +\ \frac{\nu_2(\nu_2-1)}{2}\sum_{i=1}^{N}\frac{1}{x_i^2}\ , \qquad (25)$$

where ω,ν,ν_2 are parameters. If $\nu\ =\ 0$, the Hamiltonian (25) is reduced to (13). The symmetry of the system is $S_N\oplus(\mathbb{Z}_2)^N$ (permutations $x_i\to x_j$ and $x_i\to -x_i$).

The ground state function for (25) reads

$$\Psi_0=\left[\prod_{i<j}|x_i-x_j|^{\nu}|x_i+x_j|^{\nu}\prod_{i=1}^{N}|x_i|^{\nu_2}\right]e^{-\frac{\omega}{2}\sum_{i=1}^{N}x_i^2}\ , \qquad (26)$$

(cf.(24)). Making the gauge rotation

$$h_{BC_N} = (\Psi_0)^{-1} \left(\mathcal{H}_{BC_N} - E_0\right) \Psi_0 \ ,$$

and changing variables

$$(x_1, x_2, \ldots x_N) \to \big(\sigma_k(x^2)| \ k = 1, 2, \ldots, N\big) \ ,$$

where

$$\sigma_k(x^2) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_k}^2 \ ,$$
$$\sigma_1(x^2) = x_1^2 + x_2^2 + \ldots + x_N^2 \ = \ r^2 \ ,$$

where r is radius, we arrive at [8]

$$h_{BC_N} = \mathcal{A}_{ij}(\sigma) \frac{\partial^2}{\partial\sigma_i \partial\sigma_j} + \mathcal{B}_i(\sigma) \frac{\partial}{\partial\sigma_i} \ , \tag{27}$$

with coefficients

$$\mathcal{A}_{ij} \ = \ -2 \sum_{l \geq 0} (2l + 1 + j - i) \, \sigma_{i-l-1} \, \sigma_{j+l} \ ,$$
$$\mathcal{B}_i \ = \ [1 + \nu_2 + 2\nu(N - i)] \, (N - i + 1) \, \sigma_{i-1} + 2 \, \omega \, i \, \sigma_i \ .$$

This is the algebraic form of the BC_N Hamiltonian. Assuming polynomiality of the eigenfunctions we find the eigenvalues:

$$\epsilon_n \ = \ 2\omega \sum_{i=1}^{N} i \ n_i \ ,$$

hence, the spectrum is equidistant, linear in the quantum numbers and corresponds to *anisotropic* harmonic oscillator with frequency ratios $1 : 2 : 3 : ... : N$. The Hamiltonian h_{BC_N} has infinitely many finite-dimensional invariant subspaces of the form

$$\mathcal{P}_n^{(N)} \ = \ \langle {\sigma_1}^{p_1} {\sigma_2}^{p_2} \ldots {\sigma_N}^{p_N} | \ 0 \leq \Sigma p_i \leq n \rangle \ ,$$

where $n = 0, 1, 2, \ldots$. They naturally form the flag $\mathcal{P}^{(N)}$. The Hamiltonian can be immediately rewritten in terms of generators (19) as a polynomial of the second degree,

$$h_{\mathrm{BC_N}} \ = \ Pol_2(\mathcal{J}_i^- \ , \ {\mathcal{J}_{ij}}^0) \ ,$$

where the raising generators $\mathcal{J}_i^+$ are absent. Hence, $gl(N)$ is the hidden algebra of BC_N rational model, the same algebra as for A_{N+1}-rational model. The eigenfunctions of BC_N-rational model are elements of the flag of polynomials $\mathcal{P}^{(N)}$. Each subspace $\mathcal{P}_n^{(N)}$ contains C_{n+N}^N eigenfunctions (volume of the Newton polytope (pyramid) $\mathcal{P}_n^{(N)}$).

The BC_N Hamiltonian admits 2nd order integral as result of separation of radial variable

$$\mathcal{H}_{BC_N} \ = \ -\frac{1}{2r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right)+\omega^2 r^2+\frac{1}{2r^2}\underbrace{(-\Delta_\Omega^{(N-1)}+\mathcal{W}(\Omega))}_{\mathcal{F}_{BC_N}}\ . \tag{28}$$

Evidently, the commutator

$$[\mathcal{H}_{BC_N}\ ,\ \mathcal{F}_{BC_N}]\ =\ 0\ .$$

Gauge-rotated integral

$$f_{BC_N}\ =\ \Psi_0^{-1}\left(\mathcal{F}_{BC_N}-F_0\right)\Psi_0\ ,$$

where $\mathcal{F}_{BC_N}\Psi_0=F_0\Psi_0$, takes the algebraic form in t-coordinates,

$$f_{BC_N}\ =\ f_{ij}(t)\frac{\partial^2}{\partial t_i\partial t_j}+g_i(t)\frac{\partial}{\partial t_i}\ ,$$

where f_{ij} is 2nd degree polynomial, $f_{1j}=0$, and g_i is 1st degree polynomial, $g_1=0$,

$$f_{BC_N}\ =\ Pol_2(\mathcal{J}_i^-\ ,\ \mathcal{J}_{ij}{}^0)\ ,$$

in terms of the $gl(N)$ generators. It is worth mentioning that the commutator of $[h,f]$ vanishes only in the realization (19), otherwise,

$$[h_{BC_N}(\mathcal{J}),f_{BC_N}(\mathcal{J})]\ \neq\ 0\ .$$

$sl(2)$-Quasi-Exactly-Solvable generalization

By adding to h_{BC_N} the operator

$$\delta h^{(qes)}=4(a\sigma_1^2-\gamma)\frac{\partial}{\partial\sigma_1}-4ak\sigma_1+2\omega k\ ,$$

which is the similar to one for Calogero model, we get the operator $h_{BC_N}+\delta h^{(qes)}$ which has the finite-dimensional invariant subspace

$$\mathcal{P}_k=\langle\sigma_1^p|0\le p\le k\rangle\ .$$

Making a gauge rotation of $h_{BC_N} + \delta h^{(qes)}$ and changing the variables σ's back to Cartesian one the Hamiltonian becomes

$$
\begin{aligned}
\mathcal{H}^{(\rm qes)}_{BC_N} \ &= \ -\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial {x_i}^2} - \omega^2 x_i^2\right) \ + \\
\nu(\nu-1)\sum_{i<j}&\left[\frac{1}{(x_i-x_j)^2}+\frac{1}{(x_i+x_j)^2}\right] + \ \frac{\nu_2(\nu_2-1)}{2}\sum_{i=1}^{N}\frac{1}{x_i^2} \ + \\
&\frac{2\gamma\left[\gamma - 2N(1+2\nu(N-1)+\nu_2)+3\right]}{r^2} \ + \\
a^2 r^6 \ + \ 2a\omega r^4 \ &- \ a\left[2k+2N(1+2\nu(N-1)+\nu_2)-\gamma-1\right] r^2 \ ,
\end{aligned}
\tag{29}
$$

for which $(k+1)$ eigenfunctions are of the form

$$
\Psi^{(\rm qes)}_k(x) = \prod_{i<j}^{n} |x_i^2 - x_j^2|^{\nu} \prod_{i=1}^{n} |x_i|^{\nu_2} (r^2)^{\gamma} P_k(r^2) \ \exp\left[-\frac{\omega r^2}{2} - \frac{a}{4} r^4\right] \ ,
$$

where P_k is a polynomial of degree k in $r^2 = \sum_{i=1}^N x_i^2$.

It is worth noting that at $a=0$ the operator $h_{BC_N} + \delta h^{(qes)}$ remains exactly-solvable, it preserves the flag of polynomials $\mathcal{P}^{(N)}$ and the emerging Hamiltonian has a form of (25) with the extra term $\frac{\Gamma}{r^2}$ in the potential. Its ground state eigenfunction is $(r^2)^{\gamma}\Psi_0$. It is the exactly-solvable generalization of the BC_N-Rational model (25) with the Weyl group $\rm W(BC_N)$ as the discrete symmetry group,

$$
\mathcal{H}_{\rm W(BC_N)} \ = \ \mathcal{H}_{BC_N} + \frac{\Gamma}{r^2} \ .
$$

Now we are in a position to draw an intermediate conclusion about A_N and BC_N rational models.

- Both A_N- and BC_N- rational (and trigonometric) models possess **algebraic** forms associated with preservation of the **same** flag of polynomials $\mathcal{P}^{(N)}$. The flag is invariant wrt linear transformations in space of orbits $t \mapsto t + A$. It preserves the algebraic form of Hamiltonian.

- Their Hamiltonians (as well as higher integrals) can be written in the Lie-algebraic form

$$
h = Pol_2(\mathcal{J}(b \subset gl^{(*)}_{N+1})) \ ,
$$

where Pol_2 is a polynomial of 2nd degree in generators $\mathcal{J}$ of the maximal affine subalgebra of the algebra b of the algebra gl_{N+1} in realization $(*)$. Hence, gl_{N+1} is their **hidden algebra**. From this viewpoint all four models are different faces of a **single** model.

- *Supersymmetric A_N- and BC_N- rational (and trigonometric) models possess* **algebraic** *forms, preserve the* **same** *flag of (super)polynomials and their* **hidden algebra** *is the superalgebra $gl(N+1|N)$ (see [8]).*

In a connection to flags of polynomials we introduce a notion *'characteristic vector'*. Let us consider a flag made out of "triangular" linear space of polynomials

$$\mathcal{P}^{(d)}_{n,\vec{f}} \ = \ \langle x_1^{p_1} x_2^{p_2} \ldots x_d^{p_d} | 0 \le f_1 p_1 + f_2 p_2 + \ldots + f_d p_d \le n \rangle \ ,$$

where the "grades" f's are positive integer numbers and $n = 0, 1, 2, \ldots$. In lattice space $\mathcal{P}^{(d)}_{n,\vec{f}}$ defines a Newton pyramid.

DEFINITION. Characteristic vector is a vector with components f_i :

$$\vec{f} = (f_1, f_2, \ldots f_d) \ .$$

From geometrical point of view $\vec{f}$ is normal vector to the base of the Newton pyramid. The characteristic vector for flag $\mathcal{P}^{(d)}$ is,

$$\vec{f_0} \ = \ \underbrace{(1, 1, \ldots 1)}_{d} \ .$$

Case G_2

Take the Hamiltonian

$$\mathcal{H}_{G_2} \ = \ \frac{1}{2}\sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial {x_i}^2} \ + \ \omega^2 x_i^2\right) + \nu(\nu-1)\sum_{i<j}^{3}\frac{1}{(x_i - x_j)^2} \ + $$
$$3\mu(\mu-1)\sum_{k<l,k,l\neq m}^{3}\frac{1}{(x_k + x_l - 2x_m)^2} \ , \qquad (30)$$

where ω, ν, μ are parameters. It describes the Wolfes model of three-body interacting system [9] or, in the Hamiltonian reduction nomenclature, G_2 -rational model. The symmetry of the model is dihedral group D_6. The ground state function is

$$\Psi_0 = \prod_{i<j}^{3} |x_i - x_j|^{\nu} \prod_{k<l,\ k,l\neq m}^{3} |x_i + x_j - 2x_k|^{\mu} e^{-\frac{1}{2}\omega \sum_{i=1}^{3} x_i^2} \ .$$

Making the gauge rotation

$$h_{G_2} = (\Psi_0)^{-1} (\mathcal{H}_{G_2} - E)\, \Psi_0 \ ,$$

and changing variables

$$Y = \sum x_i \ , \ y_i = x_i - \frac{1}{3} Y \ , \ i = 1,2,3 \ ,$$

$$(x_1, x_2, x_3) \to \big(Y,\ \lambda_1,\ \lambda_2\big) \ ,$$

where

$$\lambda_1 = -y_1^2 - y_2^2 - y_1 y_2 \sim -r^2 \quad , \quad \lambda_2 = [y_1 y_2 (y_1 + y_2)]^2 \ ,$$

and separating the center-of-mass coordinate we arrive at

$$h_{\mathrm{G}_2} \ = \ \lambda_1 \partial^2_{\lambda_1 \lambda_1} \ + \ 6\lambda_2 \partial^2_{\lambda_1 \lambda_2} \ - \ \frac{4}{3} \lambda_1^2 \lambda_2 \partial^2_{\lambda_2 \lambda_2}$$

$$+ \ \big\{2\omega\lambda_1 + 2[1 + 3(\mu + \nu)]\big\} \partial_{\lambda_1} + \ \big[6\omega\lambda_2 - \frac{4}{3}(1 + 2\mu)\lambda_1^2\big] \partial_{\lambda_2} \ ,$$

which is the algebraic form of the Wolfes model. The eigenvalues of h_{G_2} are

$$\epsilon_{\{p\}} \ = \ 2\omega(p_1 + 3p_2) \ .$$

It coincides to the spectrum of *anisotropic* harmonic oscillator with frequency ratio $1:3$.

Separating the center-of-mass in (30) and introducing the polar coordinates (ϱ, φ) in the space of relative coordinates we arrive at the Hamiltonian

$$\tilde{\mathcal{H}}_{G_2}(\varrho, \varphi; \nu, \mu) \ = \ -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{9\nu(\nu-1)}{r^2 \cos^2 3\varphi} + \frac{9\mu(\mu-1)}{r^2 \sin^2 3\varphi} \ . \tag{31}$$

It is evident that the integral of motion which appears due to separation of variables in polar coordinates (cf.(5)) has the form

$$\mathcal{F} \ = \ -\partial_\varphi^2 + \frac{9\nu(\nu-1)}{\cos^2 3\varphi} + \frac{9\mu(\mu-1)}{\sin^2 3\varphi} \ . \tag{32}$$

It is evident that after gauge rotation with Ψ_0 and change of variables to (λ_1, λ_2) the integral $\mathcal{F}$ takes algebraic form.

The Hamiltonian h_{G_2} has infinitely many finite-dimensional invariant subspaces

$$\mathcal{P}^{(2)}_{n,(1,2)} \ = \ \langle {\lambda_1}^{p_1} {\lambda_2}^{p_2} |\ 0 \le p_1 + 2p_2 \le n \rangle \ , \ n = 0, 1, 2, \dots \ ,$$

hence the flag $\mathcal{P}^{(2)}_{(1,2)}$ with the characteristic vector $\vec{f} = (1,2)$ is preserved by h_{G_2}. The eigenfunctions of h_{G_2} are are elements of the flag of polynomials $\mathcal{P}^{(2)}_{(1,2)}$. Each subspace $\mathcal{P}^{(2)}_{n,(1,2)} - \mathcal{P}^{(2)}_{n-1,(1,2)}$ contains $\sim n$ eigenfunctions which is equal to length of the Newton line $\mathcal{L}_n = \langle {\lambda_1}^{p_1} {\lambda_2}^{p_2} | p_1 + 2p_2 = n \rangle$.

A natural question to ask: *What about hidden algebra?* Namely: *Does algebra exist for which $\mathcal{P}^{(2)}_{n,(1,2)}$ is the space of (irreducible) representation?* Surprisingly, this algebra exists and it is, in fact, known.

Let us consider the Lie algebra spanned by seven generators

$$
\begin{aligned}
J^1 &= \partial_t \ , \\
J^2_n = t\partial_t - \frac{n}{3} \ , \ J^3_n &= 2u\partial_u - \frac{n}{3} \ , \\
J^4_n &= t^2\partial_t + 2tu\partial_u - nt \ , \\
R_i = t^i\partial_u \ , \ i = 0,1,2 \ , &\quad L \equiv (R_0, R_1, R_2) \ .
\end{aligned}
\tag{33}
$$

It is non-semi-simple algebra $gl(2,\mathbf{R}) \ltimes \mathcal{R}^{(2)}$ (S. Lie, [10] at $n = 0$ and A. González-Lopéz et al, [11] at $n \neq 0$ (Case 24)). If the parameter n in (33) is a non-negative integer, it has

$$
\mathcal{P}^{(2)}_n = (t^p u^q | 0 \leq (p + 2q) \leq n) \ ,
$$

as common (reducible) invariant subspace. By adding

$$
T^{(2)}_0 = u\partial_t^2 \ ,
$$

to $gl(2,\mathbf{R}) \ltimes \mathcal{R}^{(2)}$ (see (33)), the action on $\mathcal{P}^{(2)}_{n,(1,2)}$ gets irreducible. Multiple commutators of J^4_n with $T^{(2)}_0$ generate new operators acting on $\mathcal{P}^{(2)}_{n,(1,2)}$,

$$
T^{(2)}_i \equiv \underbrace{[J^4,[J^4,[\ldots J^4, T^{(2)}_0]\ldots]}_{i} = u\partial_t^{2-i} J_0(J_0+1)\ldots(J_0+i-1) \ , \ i = 0,1,2 \ ,
$$

where $J_0 = t\partial_t + 2u\partial_u - n$, and all of them are of degree 2. These new generators have a property of nilpotency,

$$
T^{(2)}_i = 0 \ , \ i > 2 \ ,
$$

and commutativity:

$$
[T^{(2)}_i, T^{(2)}_j] = 0 \ , \quad i,j = 0,1,2 \ , \quad \mathfrak{L} \equiv (T^{(2)}_0, T^{(2)}_1, T^{(2)}_2) \ . \tag{34}
$$

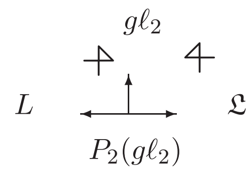

FIG. 2: Triangular diagram relating the subalgebras L, $\mathfrak{L}$ and $g\ell_2$. $P_2(g\ell_2)$ is a polynomial of the 2nd degree in $g\ell_2$ generators. It is a generalization of Gauss decomposition for semi-simple algebras.

(33) plus (34) span a linear space with a property of decomposition:
$g^{(2)} \doteq L \rtimes (gl_2 \oplus J_0) \ltimes \mathfrak{L}$ (see Fig.2).

Eventually, *infinite-dimensional, eleven-generated algebra (by (33) and* J_0 *plus (34), so that the eight generators are the 1st order and three generators are of the 2nd order differential operators)* occurs. The Hamiltonian h_{G_2} can be rewritten in terms of the generators (33), (34) with the absence of the highest weight generator J^4_n,

$$h_{G_2} = (J^2 + 3J^3)J^1 \;-\; \frac{2}{3}J^3 R_2 \;+\; 2[3(\mu+\nu)+1]J^1$$

$$+2\omega J^2 + 3\omega J^3 - \frac{4}{3}(1+2\mu)R_2 \ ,$$

where $J^{2,3} = J_0^{2,3}$. Hence, $gl(2,\mathbf{R}) \ltimes \mathcal{R}^{(2)}$ is the hidden algebra of the Wolfes model.

(i) G_2 Hamiltonian admits two mutually-**non**-commuting integrals: of 2nd order as the result of the separation of radial variable r^2 (see (32)) and of the 6th order. If $\omega = 0$ the latter integral degenerates to the 3rd order integral (the square root can be calculated in closed form).

(ii) Both integrals after gauge rotation with Ψ_0 take in variables $\lambda_{1,2}$ the algebraic form. Both preserve the same flag $\mathcal{P}^{(2)}_{(1,2)}$.

(iii) Both integrals can be rewritten in term of generators of the algebra $g^{(2)}$: integral of 2nd order in terms of $gl(2,\mathbf{R}) \ltimes \mathcal{R}^{(2)}$ generators only and while one of the 6th order contains generators from $\mathfrak{L}$ as well [13].

$sl(2)$-Quasi-Exactly-Solvable generalization

By adding to h_{G_2}, the operator (the same as for Calogero and BC_N models)

$$\delta h^{(qes)} = 4(a\lambda_1^2 - \gamma)\frac{\partial}{\partial \lambda_1} - 4ak\lambda_1 + 2\omega k \ ,$$

we get the operator $h_{G_2} + \delta h^{(qes)}$ having single finite-dimensional invariant subspace

$$\mathcal{P}_k = \langle \lambda_1^p | 0 \le p \le k \rangle \ .$$

Making a gauge rotation of $h_{G_2} + \delta h^{(qes)}$, changing of variables $(Y, \lambda_{1,2})$ back to Cartesian coordinates and adding the center-of-mass the Hamiltonian becomes

$$\mathcal{H}_{G_2}^{(qes)} \ = \ -\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial^2}{\partial {x_i}^2} - \omega^2 x_i^2\right) \ +$$

$$\nu(\nu-1)\sum_{i<j}^{3}\frac{1}{(x_i - x_j)^2} + 3\mu(\mu-1)\sum_{i<l,\ i,l\neq m}^{3}\frac{1}{(x_i + x_l - 2x_m)^2} \ +$$

$$\frac{4\gamma(\gamma+3\mu+3\nu)}{r^2} \ + \ a^2 r^6 \ + \ 2a\omega r^4 \ + \ 2a\left[2k - 3(\mu+\nu) - 2(\gamma+1)\right] r^2 \ , \qquad (35)$$

for which $(k+1)$ eigenfunctions are of the form

$$\Psi_k^{(\mathrm{qes})} =$$

$$\prod_{i<j}^{3} |x_i - x_j|^{\nu} \prod_{i<j;i,j\neq p}^{3} |x_i + x_j - 2x_p|^{\mu} (r^2)^{\gamma} P_k(r^2) \exp\left[-\frac{\omega}{2}\sum_{i=1}^{3} x_i^2 - \frac{a}{4} r^4\right] \ ,$$

where P_k is a polynomial of degree k in r^2 .

It is worth noting that at $a = 0$ the operator $h_{G_2} + \delta h^{(qes)}$ remains exactly-solvable, it preserves the flag of polynomials $\mathcal{P}^{(2)}_{(1,2)}$ and the emerging Hamiltonian has a form of (30) with the extra term $\frac{\Gamma}{r^2}$ in the potential. Its ground state eigenfunction is $(r^2)^{\gamma}\Psi_0$. It is the exactly-solvable generalization of the G_2-Rational model (30) with the Weyl group $\mathrm{W(G_2)}$ as the discrete symmetry group,

$$\mathcal{H}_{\mathrm{W(G_2)}} \ = \ \mathcal{H}_{G_2} + \frac{\Gamma}{r^2} \ .$$

Cases F_4 and $E_{6,7,8}$

In some details these four cases are described in [12].

Case $I_2(k)$

In some details this case is described in [13]. It is worth noting that the Hamiltonian Reduction nomenclature is assigned to this case the parameter k takes any real value. Discrete symmetry group D_{2k} of the Hamiltonian appears for integer k.

Case H_3

The H_3 rational Hamiltonian reads

$$\mathcal{H}_{H_3} = \frac{1}{2}\sum_{k=1}^{3}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + \frac{\nu(\nu-1)}{x_k^2}\right] +$$
$$2\nu(\nu-1)\sum_{\{i,j,k\}}\sum_{\mu_{1,2}=0,1}\frac{1}{[x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k]^2}\ , \qquad (36)$$

where $\{i,j,k\} = \{1,2,3\}$ and all even permutations, ω, ν are parameters and

$$\varphi_\pm = \frac{1 \pm \sqrt{5}}{2}\ ,$$

the golden ratio and its algebraic conjugate. Symmetry of the Hamiltonian (36) is H_3 Coxeter group (full symmetry group of the icosahedron). It has the order 120. In total, the Hamiltonian (36) is symmetric with respect to the transformation

$$x_i \longleftrightarrow x_j\ ,$$
$$\varphi_+ \longleftrightarrow \varphi_-\ .$$

The ground state is given by

$$\Psi_0 = \Delta_1^\nu \Delta_2^\nu \exp\left(-\frac{\omega}{2}\sum_{k=1}^{3} x_k^2\right)\ , \quad E_0 = \frac{3}{2}\omega(1+10\nu)\ ,$$

where

$$\Delta_1 = \prod_{k=1}^{3} x_k\ ,$$
$$\Delta_2 = \prod_{\{i,j,k\}}\prod_{\mu_{1,2}=0,1}[x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k]\ .$$

Making the gauge rotation

$$h_{H_3} = -2(\Psi_0)^{-1}(\mathcal{H}_{H_3} - E_0)(\Psi_0)\ ,$$

we arrive at new spectral problem

$$h_{H_3}\phi(x) = -2\epsilon\phi(x)\ .$$

After changing variables ($x_{1,2,3} \to \tau_{1,2,3}$):

$$
\begin{aligned}
\tau_1 =& x_1^2 + x_2^2 + x_3^2 = r^2 \ , \\
\tau_2 =& -\frac{3}{10}\,(x_1^6 + x_2^6 + x_3^6) + \frac{3}{10}(2 - 5\varphi_+)\,(x_1^2 x_2^4 + x_2^2 x_3^4 + x_3^2 x_1^4) \\
& + \frac{3}{10}(2 - 5\varphi_-)\,(x_1^2 x_3^4 + x_2^2 x_1^4 + x_3^2 x_2^4) - \frac{39}{5} \ , \\
\tau_3 =& \frac{2}{125}\,(x_1^{10} + x_2^{10} + x_3^{10}) + \frac{2}{25}(1 + 5\varphi_-)\,(x_1^8 x_2^2 + x_2^8 x_3^2 + x_3^8 x_1^2) \\
& + \frac{2}{25}(1 + 5\varphi_+)\,(x_1^8 x_3^2 + x_2^8 x_1^2 + x_3^8 x_2^2) \\
& + \frac{4}{25}(1 - 5\varphi_-)\,(x_1^6 x_2^4 + x_2^6 x_3^4 + x_3^6 x_1^4) \\
& + \frac{4}{25}(1 - 5\varphi_+)\,(x_1^6 x_3^4 + x_2^6 x_1^4 + x_3^6 x_2^4) \\
& - \frac{112}{25}\,(x_1^6 x_2^2 x_3^2 + x_2^6 x_3^2 x_1^2 + x_3^6 x_1^2 x_2^2) \\
& + \frac{212}{25}\,(x_1^2 x_2^4 x_3^4 + x_2^2 x_3^4 x_1^4 + x_3^2 x_1^4 x_2^4) \ ,
\end{aligned}
$$

in the gauge-rotated Hamiltonian, it emerges in the algebraic form [14]

$$
h_{H_3} = \sum_{i,j=1}^{3} A_{ij} \frac{\partial^2}{\partial\tau_i \partial\tau_j} + \sum_{j=1}^{3} B_j \frac{\partial}{\partial\tau_j} \ ,
$$

where

$$
A_{11} = 4\tau_1 \ , \ A_{12} = 12\tau_2 \ , \ A_{13} = 20\tau_3 \ ,
$$

$$
A_{22} = -\frac{48}{5}\tau_1^2\tau_2 + \frac{45}{2}\tau_3 \ , \ A_{23} = \frac{16}{15}\tau_1\tau_2^2 - 24\tau_1^2\tau_3 \ , \ A_{33} = -\frac{64}{3}\tau_1\tau_2\tau_3 + \frac{128}{45}\tau_2^3 \ ,
$$

$$
B_1 = 6 + 60\nu - 4\omega\tau_1 \ , \ B_2 = -\frac{48}{5}(1 + 5\nu)\tau_1^2 - 12\omega\tau_2 \ ,
$$

$$
B_3 = -\frac{64}{15}(2 + 5\nu)\tau_1\tau_2 - 20\omega\tau_3 \ ,
$$

which is amazingly simple comparing the quite complicated and lengthy form of the original Hamiltonian (36). The Hamiltonian h_{H_3} preserves infinitely-many spaces

$$
\mathcal{P}_n^{(1,2,3)} = \langle \tau_1^{n_1}\tau_2^{n_2}\tau_3^{n_3} | 0 \le n_1 + 2n_2 + 3n_3 \le n \rangle \ , \quad n \in \mathbf{N} \ ,
$$

with characteristic vector is (1,2,3), they form an infinite flag. The spectrum of h_{H_3} is given by

$$
\epsilon_{p_1,p_2,p_3} = 2\omega(p_1 + 3p_2 + 5p_3) \ , \quad p_i = 0, 1, 2, \dots \ ,
$$

with degeneracy $p_1 + 3p_2 + 5p_3 =$ integer. It corresponds to the anisotropic harmonic oscillator with frequency ratios 1:3:5 . Eigenfunctions $\phi_{n,i}$ of h_{H_3} are elements of $\mathcal{P}_n^{(1,2,3)}$, The number of eigenfunctions in $\mathcal{P}_n^{(1,2,3)}$ is maximal possible - it is equal to dimension of $\mathcal{P}_n^{(1,2,3)}$.

The space $\mathcal{P}_n^{(1,2,3)}$ is finite-dimensional representation space of a Lie algebra of differential operators which we call the $h^{(3)}$ algebra. It is infinite-dimensional but finitely generated algebra of differential operators with 30 generating elements of $1^{\rm st}$ (14), $2^{\rm nd}$ (10) and $3^{\rm rd}$ (5) orders, respectively, plus one of zeroth order. They span 5 + 5 Abelian subalgebras [16] and one Cartan type algebra (for details see [14]). The Hamiltonian h_{H_3} can be rewritten in terms of the generators of the $h^{(3)}$-algebra.

By adding to h_{H_3} (36) the operator (11) in the variable τ_1 (of the same type as for Calogero, BC_N and G_2 models) we get the operator $h_{H_3} + \delta h^{(qes)}$ which has the finite-dimensional invariant subspace

$$\mathcal{P}_k = \langle {\tau_1}^p | 0 \leq p \leq k \rangle \ ,$$

of the dimension $(k+1)$. Hence, this operator is quasi-exactly-solvable. Making the gauge rotation of this operator and changing variables τ back to Cartesian ones we arrive at the quasi-exactly-solvable Hamiltonian of a similar type as for Calogero, BC_N and G_2 models [14]. By adding to h_{H_3} (36) the operator $4\gamma\frac{\partial}{\partial\tau_1}$ we preserve the property of exact-solvability. This operator preserves the flag $\mathcal{P}^{(1,2,3)}$ and the emerging Hamiltonian has a form of (36) with the extra term $\frac{\Gamma}{r^2}$ in the potential. Its ground state eigenfunction is $(r^2)^\gamma\Psi_0$. It is the exactly-solvable generalization of the H_3-Rational model (36) with the Coxeter group H_3 as the discrete symmetry group,

$$\mathcal{H}_{\mathrm{H}_3} \ = \ \mathcal{H}_{H_3} + \frac{\Gamma}{r^2} \ .$$

Case H_4

The H_4 rational Hamiltonian reads

$$\mathcal{H}_{H_4} = \frac{1}{2}\sum_{k=1}^{4}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + \frac{\nu(\nu-1)}{x_k^2}\right] + 2\nu(\nu-1)\sum_{\mu_{2,3,4}=0,1}\frac{1}{[x_1+(-1)^{\mu_2}x_2+(-1)^{\mu_3}x_3+(-1)^{\mu_4}x_4]^2} + \qquad (37)$$
$$2\nu(\nu-1)\sum_{\{i,j,k,l\}}\sum_{\mu_{1,2}=0,1}\frac{1}{[x_i+(-1)^{\mu_1}\varphi_+ x_j+(-1)^{\mu_2}\varphi_- x_k + 0\cdot x_l]^2} \ ,$$

where $\{i,j,k,l\} = \{1,2,3,4\}$ and all even permutations, ω, ν are parameters and

$$\varphi_{\pm} = \frac{1 \pm \sqrt{5}}{2} \ ,$$

the golden ratio and it algebraic conjugate. Symmetry of the Hamiltonian (37) is H_4 Coxeter group (the symmetry group of the *600-cell*). It has order 14400. In total, the Hamiltonian (37) is symmetric with respect to the transformation

$$x_i \longleftrightarrow x_j \ , \ \varphi_+ \longleftrightarrow \varphi_- \ .$$

The ground state function and its eigenvalue are

$$\Psi_0 = \Delta_1^{\nu}\Delta_2^{\nu}\Delta_3^{\nu} \exp\left(-\frac{\omega}{2}\sum_{k=1}^{4} x_k^2\right) \ , \quad E_0 = 2\omega(1+30\nu) \ ,$$

where

$$\Delta_1 = \prod_{k=1}^{4} x_k \ ,$$
$$\Delta_2 = \prod_{\mu_{2,3,4}=0,1} [x_1 + (-1)^{\mu_2}x_2 + (-1)^{\mu_3}x_3 + (-1)^{\mu_4}x_4] \ ,$$
$$\Delta_3 = \prod_{\{i,j,k,l\}} \prod_{\mu_{1,2}=0,1} [x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k + 0 \cdot x_l] \ .$$

Making a gauge rotation of the Hamiltonian

$$h_{H_4} = -2(\Psi_0)^{-1}(\mathcal{H}_{H_4} - E_0)(\Psi_0) \ ,$$

and introducing new variables $\tau_{1,2,3,4}$ as invariant wrt H_4 Coxeter group polynomials in x of degrees 2, 12, 20, 30 (degrees of H_4), we arrive at the Hamiltonian in the algebraic form [15]

$$h_{H_4} = \sum_{i,j=1}^{4} A_{ij}\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{j=1}^{4} B_j\frac{\partial}{\partial\tau_j} \ , \tag{38}$$

where

$$A_{11} = 4\ \tau_1 \ , \ A_{12} = 24\ \tau_2 \ , \ A_{13} = 40\ \tau_3 \ , \ A_{14} = 60\ \tau_4 \ ,$$

$$A_{22} = 88\ \tau_1\tau_3 + 8\ \tau_1^5\tau_2 \ , \ A_{23} = -4\ \tau_1^3\tau_2^2 + 24\ \tau_1^5\tau_3 - 8\ \tau_4 \ ,$$

$$A_{24} = 10\ \tau_1^2\tau_2^3 + 60\ \tau_1^4\tau_2\tau_3 + 40\ \tau_1^5\tau_4 - 600\ \tau_3^2 \ ,$$

$$A_{33} = -\frac{38}{3}\ \tau_1\tau_2^3 + 28\ \tau_1^3\tau_2\tau_3 - \frac{8}{3}\ \tau_1^4\tau_4 \ ,$$

$$A_{34} = 210\ \tau_1^2\tau_2^2\tau_3 + 60\ \tau_1^3\tau_2\tau_4 - 180\ \tau_1^4\tau_3^2 + 30\ \tau_2^4\ ,$$

$$A_{44} = -2175\ \tau_1\tau_2^3\tau_3 - 450\tau_1^2\tau_2^2\tau_4 - 1350\ \tau_1^3\tau_2\tau_3^2 - 600\ \tau_1^4\tau_3\tau_4\ ,$$

$$B_1 = 8(1+30\nu) - 4\omega\tau_1\ ,\ B_2 = 12(1+10\nu)\ \tau_1^5 - 24\omega\tau_2\ ,$$

$$B_3 = 20(1+6\nu)\ \tau_1^3\tau_2 - 40\omega\tau_3\ ,\ B_4 = 15(1-30\nu)\ \tau_1^2\tau_2^2 - 450(1+2\nu)\ \tau_1^4\tau_3 - 60\omega\tau_4\ ,$$

which is amazingly simple comparing the very complicated and lengthy form of the original Hamiltonian (37). It is easy to check that the algebraic operator h_{H_4} preserves infinitely-many finite-dimensional invariant subspaces

$$\mathcal{P}_n^{(1,5,8,12)} = \langle \tau_1^{n_1}\tau_2^{n_2}\tau_3^{n_3}\tau_4^{n_4} | 0 \le n_1 + 5n_2 + 8n_3 + 12n_4 \le n \rangle\ , \quad n \in \mathbf{N}\ ,$$

all of them with the same characteristic vector (1, 5, 8, 12), they form the infinite flag. The spectrum of the Hamiltonian h_{H_4} (38) has a form

$$\epsilon_{k_1,k_2,k_3,k_4} = 2\omega(k_1 + 6k_2 + 10k_3 + 15k_4)\ , \quad k_i = 0,1,2,\dots\ ,$$

with degeneracy $k_1 + 6k_2 + 10k_3 + 15k_4 =$ integer. It corresponds to the anisotropic harmonic oscillator with frequency ratios 1:6:10:15 . Eigenfunctions $\phi_{n,i}$ of h_{H_4} are elements of $\mathcal{P}_n^{(1,5,8,12)}$. The number of eigenfunctions in $\mathcal{P}_n^{(1,5,8,12)}$ is equal to dimension of $\mathcal{P}_n^{(1,5,8,12)}$.

By adding to h_{H_4} (37) the operator (11) in the variable τ_1 (of the same type as for Calogero, BC_N and G_2, H_3 models) we get the operator $h_{H_4} + \delta h^{(qes)}$ which has the finite-dimensional invariant subspace

$$\mathcal{P}_k = \langle {\tau_1}^p | 0 \le p \le k \rangle\ ,$$

of dimension $(k+1)$. Hence, this operator is quasi-exactly-solvable. Making the gauge rotation of this operator and changing variables τ back to Cartesian ones we arrive at the quasi-exactly-solvable Hamiltonian of a similar type as for Calogero, BC_N and G_2 models [15]. By adding to h_{H_4} (37) the operator $4\gamma\frac{\partial}{\partial\tau_1}$ we preserve the property of exact-solvability. This operator preserves the flag $\mathcal{P}^{(1,5,8,12)}$ and the emerging Hamiltonian has a form of (37) with the extra term $\frac{\Gamma}{r^2}$ in the potential. Its ground state eigenfunction is $(r^2)^\gamma\Psi_0$. It is the exactly-solvable generalization of the H_4-Rational model (37) with the Coxeter group H_4 as the discrete symmetry group,

$$\mathcal{H}_{\mathrm{H}_4} \ = \ \mathcal{H}_{H_4} + \frac{\Gamma}{r^2}\ .$$

Conclusions

- For rational Hamiltonians for all classical A_N, BC_N and exceptional root spaces $G_2, F_4, E_{6,7,8}$ (also trigonometric) and non-crystallographic $H_{3,4}, I_2(k)$ there exists an algebraic form after gauging away the ground state eigenfunction and changing variables to symmetric (invariant) variables. Their eigenfunctions are polynomials in these variables. They are orthogonal with the squared ground state eigenfunction as the weight factor.

- Their hidden algebras are $gl(N)$ for the case of classical A_N, BC_N and **new** infinite-dimensional but finite-generated algebras of differential operators for all other cases. All these algebras have finite-dimensional invariant subspace(s) in polynomials.

- Generating elements of any such hidden algebra can be grouped into even number of (conjugated) Abelian algebras L_i, $\mathfrak{L}_i$ and one Lie algebra B. They obey a (generalized) Gauss decomposition (see Fig.3). A description of all these algebras will be given elsewhere.

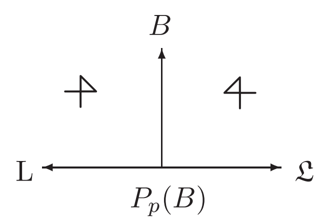

FIG. 3: Triangular diagram relating the subalgebras L, $\mathfrak{L}$ and B. $P_p(B)$ is a polynomial of the pth degree in B generators. It is a generalization of Gauss decomposition for semi-simple algebras where $p = 1$.

General view ((quasi)-exact-solvability)

There are two solvable potentials in $1D$ in $[0, \infty)$ generalized to D:

★ ES-case

$$\omega^2 r^2 + \frac{\gamma}{r^2} \qquad \to \qquad \omega^2 r^2 + \frac{\gamma(\Omega)}{r^2}$$

(a generalization by replacing the symmetry $O(N)$ by its discrete subgroup of symmetry given by Weyl(Coxeter) group);

★

QES-case

$$\omega^2 r^2 + \frac{\gamma}{r^2} + ar^6 + br^4 \qquad \to \qquad \tilde{\omega}^2 r^2 + \frac{\tilde{\gamma}(\Omega)}{r^2} + ar^6 + br^4$$

(a generalization by replacing the symmetry $O(N)$ by its discrete subgroup of symmetry given by Weyl(Coxeter) group).

Concluding I must emphasize that the algebraic nature of the considered systems was revealed when,

Invariants of the discrete group of symmetry of a system are taken as variables (space of orbits).

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[1] N.W. Evans, *Group theory of the Smorodinsky-Winternitz system*, *J. Math. Phys.* **32**, 3369-3375 (1991)

[2] A.V. Turbiner, *Quasi-Exactly-Solvable Problems and the $SL(2,R)$ Group*, *Comm.Math.Phys.* **118**, 467-474 (1988)

[3] F. Calogero, *Solution of a three-body problem in one dimension*, *J. Math. Phys.* **10**, 2191–2196 (1969)

[4] W. Rühl and A. V. Turbiner, *Exact solvability of the Calogero and Sutherland models*, *Mod. Phys. Lett.* **A10** (1995) 2213–2222
`hep-th/9506105`

[5] A.M. Minzoni, M. Rosenbaum and A. Turbiner, *Quasi-Exactly-Solvable Many-Body Problems*, *Mod. Phys. Lett.* **A11** (1996) 1977-1984
`hep-th/9606092`

[6] M.A. Olshanetsky and A.M. Perelomov, *Quantum integrable systems related to Lie algebras*, *Phys. Repts.* **94** (1983) 313.

[7] T. Oshima, *Completely integrable systems associated with classical root systems*, *SIGMA* **3**, 061 (2007)

[8] L. Brink, A. Turbiner and N. Wyllard, *Hidden Algebras of the (super) Calogero and Sutherland models*,
*Journ. Math. Phys.***39** (1998) 1285-1315
`hep-th/9705219`

[9] J. Wolfes, *On the three-body linear problem with three-body interaction*,
J.Math.Phys. **15** (1974) 1420-1424

[10] S. Lie, *Gruppenregister*, vol. 5 (B.G. Teubner, Leipzig, 1924), pp. 767-773

[11] A. González-Lopéz, N. Kamran and P.J. Olver, *Quasi-Exactly-Solvable Lie Algebras of the first order differential operators in Two Complex Variables*, *J.Phys.***A24** (1991) 3995-4008; *Lie algebras of differential operators in two complex variables*, IMA preprint series # 617 (February 1990), *American J. Math.* **114** (1992) 1163-1185

[12] K.G. Boreskov, A.V. Turbiner and J.C. Lopez Vieyra, *Solvability of the Hamiltonians related to exceptional root spaces: rational case*,
Comm.Math.Phys. **260** (2005) 17-44, `hep-th/0407204` pp.51 (full version)

[13] F. Tremblay, A.V. Turbiner and P. Winternitz, *An infinite family of solvable and integrable quantum systems on a plane*,
Journal of Phys. **A42** (2009) 242001
`math-ph arXiv:0904.0738`

[14] M.A.G. Garcia and A.V. Turbiner, *The quantum H_3 Integrable System*,
Int.Journ.Mod.Phys. **A25**, 5567-5594 (2010)
`math-ph arXiv:1007.0737`

[15] M.A.G. Garcia and A.V. Turbiner, *The quantum H_4 Integrable System*,
Mod.Phys.Lett. **A26**, 433-447 (2011)
`math-ph arXiv:1011.2127`

[16] These subalgebras can be divided into pairs. In every pair the elements of different subalgebras are related via a certain conjugation.