

Poisson Yang-Baxter maps with binomial Lax matrices

Theodoros E. Kouloukas,

Department of Mathematics,
University of Patras, Patras, GR-265 00
Greece

Vassilios G. Papageorgiou,

Department of Mathematics,
University of Patras, GR-265 00 Patras,
Greece

June 7, 2011

Abstract

A construction of multidimensional parametric Yang-Baxter maps is presented. The corresponding Lax matrices are the symplectic leaves of first degree matrix polynomials equipped with the Sklyanin bracket. These maps are symplectic with respect to the reduced symplectic structure on these leaves and provide examples of integrable mappings. An interesting family of quadrirational symplectic YB maps on $\mathbb{C}^4 \times \mathbb{C}^4$ with 3×3 Lax matrices is also presented.

Contents

1	Introduction	2
2	Yang-Baxter Maps and lax matrices	3
3	Symplectic Yang–Baxter maps associated to binomial 2×2 Lax matrices	5
3.1	Poisson Yang–Baxter maps from matrix re-factorization	5
3.2	Reduction on symplectic leaves	8
3.3	Classification	10
3.4	Degenerate YB maps	12
4	Higher dimensional Yang-Baxter maps	14
4.1	8-dimensional quadrirational symplectic YB maps with 3×3 Lax matrices	16
4.1.1	A 4-parametric symplectic Y-B map	17
4.1.2	The Boussinesq Y-B map ($\alpha_0 = \alpha^3$, $\alpha_1 = 3\alpha^2$, $\alpha_2 = 3\alpha$)	19
4.1.3	The Goncharenko–Veselov map ($\alpha_0 = -\alpha^3$, $\alpha_1 = -\alpha^2$, $\alpha_2 = \alpha$) . . .	19
5	Conclusion	20

1 Introduction

Set theoretical solutions of the quantum Yang-Baxter equation have extensively been studied by many authors after the pioneer work of Drinfeld [5]. Even before that, examples of such solutions appeared in [20] by Sklyanin. Weinstein and Xu [24] proposed a construction of such solutions using the dressing action of Poisson Lie groups [18]. This was generalized later in [11], in order to construct solutions on any group that acts on itself and the action satisfies a compatibility condition. The algebraic aspects of the Yang-Baxter equation were developed by Etingof, Schedler and Soloviev [6].

Veselov [22, 23] connected the set theoretical solutions of the quantum Yang-Baxter equations with integrable mappings. More specifically, he proved that for such a solution, that admits a Lax matrix, there is a hierarchy of commuting transfer maps which preserve the spectrum of the corresponding monodromy matrix. Furthermore he proposed the shorter term ‘Yang Baxter maps’ for the set theoretical solutions of the quantum Yang-Baxter equation.

Yang-Baxter maps are closely related with integrable equations on quad-graphs. This is due to the *multidimensional consistency property* of these equations, introduced in [4, 12], which in a way seems to be equivalent with the Yang-Baxter property. An explicit classification of equations on quad-graphs with fields in \mathbb{C} that satisfy the 3-dimensional consistency property and of the Yang-Baxter maps on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is given in [1] and [2] respectively (see also [14]). Higher dimensional Yang-Baxter maps are obtained from multi-field integrable lattice equations through symmetry reduction [15, 16].

Loop groups equipped with the Sklyanin bracket provide a natural framework in order to derive Yang-Baxter maps with polynomial Lax matrices. In [17] one of the most fundamental examples of a parametric Yang-Baxter map, Adler’s map, is given by Hamiltonian reduction of the loop group $LGL_2(\mathbb{R})$. Based on these ideas, a construction of Poisson parametric Yang-Baxter maps with first degree polynomial 2×2 Lax matrices was presented by the authors [9] from a re-factorization procedure guided by the conservation of the Casimir functions under the maps. By considering a complete set of Casimir functions, symplectic multiparametric Yang-Baxter maps were derived with explicit formulae in terms of matrix operations.

The purpose of this work is to generalize the method of [9] in order to derive symplectic Yang-Baxter maps with Lax matrices that are obtained by reduction on symplectic leaves of binomial matrices.

The necessary definitions and notation about YB maps and Lax matrices, are given in section 2. Section 3 contains the main theory of the construction of symplectic Yang-Baxter maps associated to 2×2 Lax matrices. This is generalized in higher dimensions in section 4 using further assumptions. A general re-factorization formula of $n \times n$ binomial matrices is presented. A reduction procedure of 3×3 binomial matrices to four dimensional symplectic leaves, provides a family of quad-rational, symplectic YB maps on $\mathbb{C}^4 \times \mathbb{C}^4$. Finally we conclude in section 5 by giving some comments and perspectives for future work.

2 Yang-Baxter Maps and lax matrices

Let \mathcal{X} be any set. A map $R : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$, $R : (x, y) \mapsto (u(x, y), v(x, y))$, that satisfies the *Yang-Baxter equation* :

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23} \quad (1)$$

is called *Yang-Baxter Map (YB)* [22]. Here by R_{ij} for $i, j = 1, \dots, 3$, we denote the map that acts as R on the i and j factor of $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ and identically on the others i.e.

$$\begin{aligned} R_{12}(x, y, z) &= (u(x, y), v(x, y), z), \\ R_{13}(x, y, z) &= (u(x, z), y, v(x, z)), \\ R_{23}(x, y, z) &= (x, u(y, z), v(y, z)), \end{aligned}$$

for $x, y, z \in \mathcal{X}$. From our point of view, we consider that the set \mathcal{X} has the structure of an algebraic variety. The YB map R is called *non-degenerate* if the maps $u(\cdot, y) : \mathcal{X} \rightarrow \mathcal{X}$ and $v(x, \cdot) : \mathcal{X} \rightarrow \mathcal{X}$ are bijective maps and *quadrirational* [2] if they are rational bijective maps.

Parametric YB maps appear in the study of integrable equations on quad-graphs. A *parametric YB map* is a YB map:

$$R : ((x, \alpha), (y, \beta)) \mapsto ((u, \alpha), (v, \beta)) = ((u(x, \alpha, y, \beta), \alpha), (v(x, \alpha, y, \beta), \beta)) \quad (2)$$

where $x, y \in \mathcal{X}$ and the parameters $\alpha, \beta \in \mathbb{C}^n$. We usually keep the parameters separately and denote $R(x, \alpha, y, \beta)$ by $R_{\alpha, \beta}(x, y)$. According to [21] a *Lax Matrix* for the YB map (2) is a matrix $L(x, \alpha, \zeta)$ that depends on the point x , the parameter α and a spectral parameter ζ (we usually denote it just by $L(x; \alpha)$), such that

$$L(u; \alpha)L(v; \beta) = L(y; \beta)L(x; \alpha), \quad (3)$$

for any $\zeta \in \mathbb{C}$. Furthermore if equation (3) is equivalent to $(u, v) = R_{\alpha, \beta}(x, y)$ then we will call $L(x; \alpha)$ *strong Lax matrix*.

A parametric YB map can be represented as a map assigned to the edges of an elementary quadrilateral like in Fig.1.

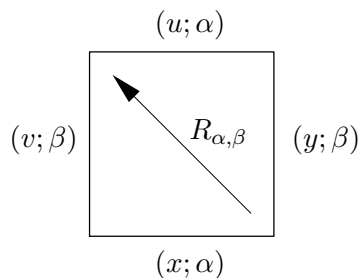


Figure 1: A map assigned to the edges of a quadrilateral

We can also represent the maps $R_{23}R_{13}R_{12}$ and $R_{12}R_{13}R_{23}$ as chains of maps at the faces of a cube like in Fig.2. The first map corresponds to the composition of the down, back, left faces, while the second one to the right, front and upper faces. All the parallel edges

to the x (resp. y, z) axis carry the parameter α (resp. β, γ). If we denote by (x'', y'', z'') and by $(\tilde{x}, \tilde{y}, \tilde{z})$ the corresponding values $R_{23}R_{13}R_{12}(x, y, z)$ and $R_{12}R_{13}R_{23}(x, y, z)$, then Eq.(1) assures that $x'' = \tilde{x}$, $y'' = \tilde{y}$ and $z'' = \tilde{z}$.

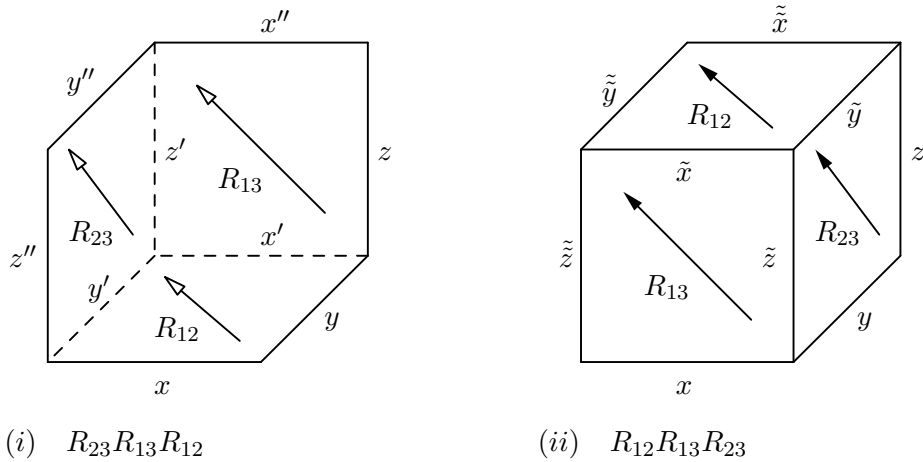


Figure 2: Cubic representation of the Yang–Baxter property

The following proposition [22, 9] gives a sufficient condition for a solution of the Lax equation (3), in order to satisfy the Yang-Baxter property.

Proposition 2.1. *Let $u = u_{\alpha,\beta}(x, y)$, $v = v_{\alpha,\beta}(x, y)$ and $A(x; \alpha)$ a matrix depending on a point x , a parameter α and a spectral parameter ζ , such that $A(u; \alpha)A(v; \beta) = A(y; \beta)A(x; \alpha)$. If the equation*

$$A(\hat{x}; \alpha)A(\hat{y}; \beta)A(\hat{z}; \gamma) = A(x; \alpha)A(y; \beta)A(z; \gamma) \quad (4)$$

implies that $\hat{x} = x$, $\hat{y} = y$ and $\hat{z} = z$, then the map $R_{\alpha,\beta}(x, y) = (u, v)$ is a parametric Yang-Baxter map with Lax matrix $A(x; \alpha)$.

In a more general setting concerning integrable lattices (not necessary YB maps), instead of the notion of a Lax matrix, the notion of a *Lax pair* is more suitable. A Lax pair for a map $\Phi_{\alpha,\beta} : ((x, \alpha), (y, \beta)) \mapsto ((u, \alpha), (v, \beta)) = ((u(x, \alpha, y, \beta), \alpha), (v(x, \alpha, y, \beta), \beta))$ is a pair of matrices L, M depending on a point in \mathcal{X} , a parameter and a spectral parameter ζ such that

$$L(u, \alpha, \zeta)M(v, \beta, \zeta) = M(y, \beta, \zeta)L(x, \alpha, \zeta), \quad (5)$$

for any $\zeta \in \mathbb{C}$. Combinations of Lax pairs can provide solutions of the intertwining Yang-Baxter equation [10].

The dynamical aspects of the Yang-Baxter maps have been extensively investigated in [22] and [23] where commuting transfer maps, that preserve the spectrum of the corresponding monodromy matrices, are introduced for each YB map. These maps are believed to be integrable in the Liouville sense, i.e. symplectic mappings $M^{2n} \rightarrow M^{2n}$ that admit n functionally independent integrals in involution.

3 Symplectic Yang–Baxter maps associated to binomial 2×2 Lax matrices

A general matrix re-factorization procedure provides a way of constructing rational multi-parametric Yang-Baxter maps on $\mathbb{C}^4 \times \mathbb{C}^4$ with 2×2 Lax matrices in the form of first-degree matrix polynomials. These maps are Poisson with respect to the Sklyanin bracket. By reduction on symplectic leaves we derive 4-dimensional symplectic parametric YB maps. The whole procedure generalizes the one presented in [9], where the leading terms of the matrix polynomials were assumed equal.

3.1 Poisson Yang–Baxter maps from matrix re-factorization

We consider the set \mathcal{L}^2 of 2×2 polynomial matrices of the form $L(\zeta) = X - \zeta A$, $\zeta \in \mathbb{C}$ equipped with the Sklyanin bracket [19]:

$$\{L(\zeta) \otimes L(\eta)\} = \left[\frac{r}{\zeta - \eta}, L(\zeta) \otimes L(\eta) \right], \quad (6)$$

where r denotes the permutation matrix: $r(x \otimes y) = y \otimes x$. For

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \text{ and } A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

the brackets between the coordinate functions are given by the antisymmetric Poisson structure matrix :

$$J_A(X) = \begin{pmatrix} 0 & -x_2a_1 + x_1a_2 & x_3a_1 - x_1a_3 & x_3a_2 - x_2a_3 \\ * & 0 & x_4a_1 - x_1a_4 & x_4a_2 - x_2a_4 \\ * & * & 0 & -x_4a_3 + x_3a_4 \\ * & * & * & 0 \end{pmatrix} \quad (7)$$

where $J_A(X)_{ij} = \{x_i - \zeta a_i, x_j - \zeta a_j\}$, for $i, j = 1, \dots, 4$.

There are six linear independent Casimir functions of \mathcal{L}^2 which are the elements a_i , $i = 1, \dots, 4$, of the matrix A and the functions:

$$f_0(X; A) = \det X, \quad f_1(X; A) = a_4x_1 - a_3x_2 - a_2x_3 + a_1x_4,$$

i.e. the coefficients of the polynomial

$$p_X^A(\zeta) := \det(X - \zeta A) = f_2(X; A)\zeta^2 - f_1(X; A)\zeta + f_0(X; A)$$

with $f_2(X; A) = \det A$ (of course $f_2(X; A)$ is also Casimir). For any constant matrix A we denote by i_A the immersion $i_A : X \mapsto X - \zeta A$ and by \mathcal{L}_A^2 the level set

$$\mathcal{L}_A^2 = \{X - \zeta A \mid X \in \text{Mat}(2 \times 2)\}.$$

Furthermore for any pair of matrices $A, B \in GL_2(\mathbb{C})$, we define the matrix functions $\Pi_{A,B}^1, \Pi_{A,B}^2$, with

$$\Pi_{A,B}^1(X, Y) = f_2(X; A)(YA + BX) - f_1(X; A)AB, \quad (8)$$

$$\Pi_{A,B}^2(X, Y) = f_2(X; A)YX - f_0(X; A)AB. \quad (9)$$

Proposition 3.1. (re-factorization) Let A, B be invertible 2×2 matrices, such that $AB = BA$ and $X, Y \in \text{Mat}(2 \times 2)$ with $\det \Pi_{A,B}^1(X, Y) \neq 0$. Then

$$(U - \zeta A)(V - \zeta B) = (Y - \zeta B)(X - \zeta A), \quad (10)$$

and $p_U^A(\zeta) = p_X^A(\zeta)$ (equivalently $p_V^B(\zeta) = p_Y^B(\zeta)$), iff

$$U = U_{A,B}(X, Y) := \Pi_{A,B}^2(X, Y)\Pi_{A,B}^1(X, Y)^{-1}A, \quad (11)$$

$$V = V_{A,B}(X, Y) := A^{-1}(YA + BX - U(X, Y)B). \quad (12)$$

The proof of this proposition is given in [10].

Lemma 1. Let A_i , $i = 1, 2, 3$ be three invertible matrices such that $A_i A_j = A_j A_i$, for $i, j = 1, 2, 3$. Then

$$(X'_1 - \zeta A_1)(X'_2 - \zeta A_2)(X'_3 - \zeta A_3) = (X_1 - \zeta A_1)(X_2 - \zeta A_2)(X_3 - \zeta A_3) \quad (13)$$

and $p_{X'_i}^{A_i}(\zeta) = p_{X_i}^{A_i}(\zeta)$ for every $X_i \in \text{Mat}(2 \times 2)$, $i = 1, 2, 3$ and $\zeta \in \mathbb{C}$, iff $X'_1 = X_1$, $X'_2 = X_2$, $X'_3 = X_3$.

The proof of this lemma can be traced in the appendix of [10].

Proposition 3.2. Let $K : \mathbb{C}^d \rightarrow GL_2(\mathbb{C})$, be a d -parametric family of commuting matrices. For every $\alpha, \beta \in \mathbb{C}^d$ the map

$$\mathcal{R}_{\alpha,\beta}(X, Y) = (U_{K(\alpha),K(\beta)}(X, Y), V_{K(\alpha),K(\beta)}(X, Y)) := (U, V) \quad (14)$$

defined by (11), (12), is a parametric Yang-Baxter map with Lax matrix $L(X; \alpha) = i_{K(\alpha)}(X)$ such that $p_U^{K(\alpha)}(\zeta) = p_X^{K(\alpha)}(\zeta)$ and $p_V^{K(\beta)}(\zeta) = p_Y^{K(\beta)}(\zeta)$.

Proof: For $U = U_{K(\alpha),K(\beta)}(X, Y)$, $V = V_{K(\alpha),K(\beta)}(X, Y)$ and $L(X; \alpha) = i_{K(\alpha)}(X)$, from proposition 3.1 we have that

$$L(U; \alpha)L(V; \beta) = L(Y; \beta)L(X; \alpha)$$

and $p_U^{K(\alpha)}(\zeta) = p_X^{K(\alpha)}(\zeta)$, $p_V^{K(\beta)}(\zeta) = p_Y^{K(\beta)}(\zeta)$. Now, if we set

$$\begin{aligned} R_{\alpha,\beta}^{12}(X, Y, Z) &= (X', Y', Z), \\ R_{\alpha,\gamma}^{13} \circ R_{\alpha,\beta}^{12}(X, Y, Z) &= (X'', Y', Z'), \\ R_{\beta,\gamma}^{23} \circ R_{\alpha,\gamma}^{13} \circ R_{\alpha,\beta}^{12}(X, Y, Z) &= (X'', Y'', Z''), \end{aligned}$$

then $L(Y; \beta)L(X; \alpha) = L(X'; \alpha)L(Y'; \beta)$, and $p_{X'}^{K(\alpha)}(\zeta) = p_X^{K(\alpha)}(\zeta)$, $p_{Y'}^{K(\beta)}(\zeta) = p_Y^{K(\beta)}(\zeta)$. So

$$\begin{aligned} L(Z; \gamma)L(Y; \beta)L(X; \alpha) &= (L(Z; \gamma)L(X'; \alpha))L(Y'; \beta) = L(X''; \alpha)(L(Z'; \gamma)L(Y'; \beta)) \\ &= L(X''; \alpha)L(Y''; \beta)L(Z''; \gamma) \end{aligned}$$

and $p_{X''}^{K(\alpha)}(\zeta) = p_X^{K(\alpha)}(\zeta)$, $p_{Y''}^{K(\beta)}(\zeta) = p_Y^{K(\beta)}(\zeta)$, $p_{Z''}^{K(\gamma)}(\zeta) = p_Z^{K(\gamma)}(\zeta)$.
On the other hand for

$$\begin{aligned} R_{\beta,\gamma}^{23}(X, Y, Z) &= (X, \tilde{Y}, \tilde{Z}), \\ R_{\alpha,\gamma}^{13} \circ R_{\beta,\gamma}^{23}(X, Y, Z) &= (\tilde{X}, \tilde{Y}, \tilde{\tilde{Z}}), \\ R_{\alpha,\beta}^{12} \circ R_{\alpha,\gamma}^{13} \circ R_{\beta,\gamma}^{23}(X, Y, Z) &= (\tilde{\tilde{X}}, \tilde{\tilde{Y}}, \tilde{\tilde{Z}}) \end{aligned}$$

we get $L(Z; \gamma)L(Y; \beta)L(X; \alpha) = L(\tilde{\tilde{X}}; \alpha)L(\tilde{\tilde{Y}}; \beta)L(\tilde{\tilde{Z}}; \gamma)$ and $p_{\tilde{\tilde{X}}}^{K(\alpha)}(\zeta) = p_X^{K(\alpha)}(\zeta)$, $p_{\tilde{\tilde{Y}}}^{K(\beta)}(\zeta) = p_Y^{K(\beta)}(\zeta)$, $p_{\tilde{\tilde{Z}}}^{K(\gamma)}(\zeta) = p_Z^{K(\gamma)}(\zeta)$. So finally we have that

$$\begin{aligned} L(X''; \alpha)L(Y''; \beta)L(Z''; \gamma) &= L(\tilde{\tilde{X}}; \alpha)L(\tilde{\tilde{Y}}; \beta)L(\tilde{\tilde{Z}}; \gamma), \\ p_{X''}^{K(\alpha)}(\zeta) &= p_{\tilde{\tilde{X}}}^{K(\alpha)}(\zeta), p_{Y''}^{K(\beta)}(\zeta) = p_{\tilde{\tilde{Y}}}^{K(\beta)}(\zeta), p_{Z''}^{K(\gamma)}(\zeta) = p_{\tilde{\tilde{Z}}}^{K(\gamma)}(\zeta) \end{aligned}$$

and from lemma 1 we derive $X'' = \tilde{\tilde{X}}$, $Y'' = \tilde{\tilde{Y}}$, $Z'' = \tilde{\tilde{Z}}$, i.e.

$$R_{\beta,\gamma}^{23} \circ R_{\alpha,\gamma}^{13} \circ R_{\alpha,\beta}^{12} = R_{\alpha,\beta}^{12} \circ R_{\alpha,\gamma}^{13} \circ R_{\beta,\gamma}^{23}.$$

We will refer to the Yang-Baxter map of Prop. 3.2 as the *general parametric Yang-Baxter map associated with the function K*. We have to notice that in general the Lax matrix $L(X; \alpha) = i_{K(\alpha)}(X)$ is not a strong Lax matrix. For example by considering $K(\alpha) = B$ for a constant $B \in GL_2(\mathbb{C})$, the equation $i_B(U)i_B(V) = i_B(Y)i_B(X)$ except of the corresponding solution (11),(12), admits also the trivial solution $U = Y$, $V = X$ (elementary involution).

Now we return to the Poisson structure (7). We can extend the Poisson bracket of \mathcal{L}^2 to the Cartesian product $\mathcal{L}^2 \times \mathcal{L}^2$ as follows :

$$\{x_i, x_j\} = J_A(X)_{ij}, \{y_i, y_j\} = J_B(Y)_{ij}, \{x_i, y_j\} = 0, \quad (15)$$

for any $(X - \zeta A, Y - \zeta B) \in \mathcal{L}^2 \times \mathcal{L}^2$ where x_i, x_j, y_i, y_j for $i = 1, \dots, 4$ are the elements of the matrices X, Y respectively.

Proposition 3.3. *The map $\mathcal{R} : \mathcal{L}_{K(\alpha)}^2 \times \mathcal{L}_{K(\beta)}^2 \rightarrow \mathcal{L}_{K(\alpha)}^2 \times \mathcal{L}_{K(\beta)}^2$,*

$$\mathcal{R} : (X - \zeta K(\alpha), Y - \zeta K(\beta)) \mapsto (U_{K(\alpha), K(\beta)}(X, Y) - \zeta K(\alpha), V_{K(\alpha), K(\beta)}(X, Y) - \zeta K(\beta)) \quad (16)$$

is a Poisson map.

Proof: A direct computation of the Poisson brackets of the elements of $U = U_{K(\alpha), K(\beta)}(X, Y)$ and $V = V_{K(\alpha), K(\beta)}(X, Y)$ defined by (11), (12) gives:

$$\{u_i, u_j\} = J_{K_\alpha}(U)_{ij}, \{v_i, v_j\} = J_{K_\beta}(V)_{ij}, \{u_i, v_j\} = 0,$$

for $i = 1, \dots, 4$.

If we consider the permutation map $r : (X, Y) \mapsto (Y, X)$ and the multiplication map $m : (X, Y) \mapsto XY$, then \mathcal{R} is the unique map defined by the commutative diagram:

$$\begin{array}{ccc}
\mathcal{L}_{k(\alpha)}^2 \times \mathcal{L}_{k(\beta)}^2 & \xrightarrow{\mathcal{R}} & \mathcal{L}_{k(\alpha)}^2 \times \mathcal{L}_{k(\beta)}^2 \\
\downarrow r & & \downarrow m \\
\mathcal{L}_{k(\beta)}^2 \times \mathcal{L}_{k(\alpha)}^2 & \xrightarrow{m} & \mathcal{L}_2^2
\end{array}$$

Commutative diagram

Here \mathcal{L}_2^2 denotes the second degree polynomial 2×2 matrices. From proposition 3.3 and the multiplication property of the Sklyanin bracket we conclude that each map of this diagram is Poisson.

3.2 Reduction on symplectic leaves

In the previous section it was pointed out that the matrix A of a generic element

$$X - \zeta A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} - \zeta \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathcal{L}^2,$$

belongs to the center of the Sklyanin algebra. In the four dimensional Poisson submanifold \mathcal{L}_A^2 there are two Casimir functions

$$f_0(X; A) = \det X \quad f_1(X; A) = a_4 x_1 - a_3 x_2 - a_2 x_3 + a_1 x_4.$$

We restrict on the level set of the Casimir functions by solving the system $f_0(X; A) = \alpha_0$, $f_1(X; A) = \alpha_1$ with respect to two elements x_i, x_j of X . So we consider two functions h_A, g_A , defined on an open set $D \subset \mathbb{C}^4$, such that

$$x_i = h(x_k, x_l, \alpha_0, \alpha_1) \text{ and } x_j = g_A(x_k, x_l, \alpha_0, \alpha_1), \quad k, l \notin \{i, j\}. \quad (17)$$

We denote by $pr_{k,l}$ the projection of a matrix to its k, l elements (by ordering the elements of a matrix from one to four as before) and by Pr the map

$$Pr = pr_{k,l} \times pr_{k,l} : (X, Y) \mapsto (pr_{k,l}(X), pr_{k,l}(Y)).$$

By substituting the x_i, x_j to the matrix X we define the parametric matrix $L'_A(x_k, x_l; \alpha_0, \alpha_1)$. For simplicity we renumber $x_k \mapsto x_1, x_l \mapsto x_2$ and we come up to the matrix $L'_A(x_1, x_2; \alpha_0, \alpha_1)$ that satisfies the following equations

$$f_0(L'_A(x_1, x_2; \alpha_0, \alpha_1); A) = \alpha_0 \quad f_1(L'_A(x_1, x_2; \alpha_0, \alpha_1); A) = \alpha_1.$$

The connected components of $\Sigma_A(\alpha_0, \alpha_1) = \{L'_A(x_1, x_2; \alpha_0, \alpha_1) - \zeta A \mid x_1, x_2 \in D \subset \mathbb{C}\}$ are two dimensional symplectic leaves of \mathcal{L}_A^2 .

By the next proposition the general YB map $\mathcal{R}_{\alpha, \beta}$ of Prop. 3.2 is reduced on the symplectic leaves $\Sigma_{K(\alpha)}(\alpha_0, \alpha_1) \times \Sigma_{K(\beta)}(\beta_0, \beta_1)$ of $\mathcal{L}^2 \times \mathcal{L}^2$.

Proposition 3.4. Let $K : \mathbb{C}^d \mapsto GL_2(\mathbb{C})$ be a d -parametric family of commuting matrices. For every $\alpha, \beta \in \mathbb{C}^d$, the map

$$R_{\bar{\alpha}, \bar{\beta}}((x_1, x_2), (y_1, y_2)) = Pr \circ \mathcal{R}_{\alpha, \beta}(L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1), L'_{K(\beta)}(y_1, y_2; \beta_0, \beta_1)), \quad (18)$$

is a non-degenerate symplectic Yang-Baxter map with vector parameters $\bar{\alpha} = (\alpha, \alpha_0, \alpha_1)$, $\bar{\beta} = (\beta, \beta_0, \beta_1) \in V \times \mathbb{C}^2$ and strong Lax matrix

$$L(x_1, x_2; \bar{\alpha}) = i_{K(\alpha)}(L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1)) = L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1) - \zeta K(\alpha). \quad (19)$$

Proof: For $X = L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1)$ and $Y = L'_{K(\beta)}(y_1, y_2; \beta_0, \beta_1)$ we define the matrices $U = U_{K(\alpha), K(\beta)}(X, Y)$, $V = V_{K(\alpha), K(\beta)}(X, Y)$ by (11), (12)

$$(U, V) = \mathcal{R}_{\alpha, \beta}(X, Y) = \mathcal{R}_{\alpha, \beta}(L'_{K(\alpha)}(x_1, x_2; \alpha, \alpha_0, \alpha_1), L'_{K(\beta)}(y_1, y_2; \beta, \beta_0, \beta_1)).$$

Since $f_i(U; K(\alpha)) = f_i(X; K(\alpha)) = \alpha_i$ and $f_i(V; K(\beta)) = f_i(Y; K(\beta)) = \beta_i$ for $i = 0, 1$, then $U = L'_{K(\alpha)}(u_1, u_2; \alpha_0, \alpha_1)$ and $V = L'_{K(\beta)}(v_1, v_2; \beta_0, \beta_1)$. The projection $Pr(U, V)$ gives the corresponding elements $u = (u_1, u_2)$ and $v = (v_1, v_2)$. So the YB property of the map $R_{\bar{\alpha}, \bar{\beta}} : ((x_1, x_2; \bar{\alpha}), (y_1, y_2; \bar{\beta})) \mapsto ((u_1, u_2, \bar{\alpha}), (v_1, v_2, \bar{\beta}))$, is immediately derived from the YB property of the Poisson map $\mathcal{R}_{\alpha, \beta}$. Furthermore proposition 3.2 implies that $i_{K(\alpha)}(U)i_{K(\beta)}(V) = i_{K(\beta)}(Y)i_{K(\alpha)}(X)$, so

$$\begin{aligned} & (L'_{K(\alpha)}(u_1, u_2; \alpha_0, \alpha_1) - \zeta K_\alpha)(L'_{K(\beta)}(v_1, v_2; \beta_0, \beta_1) - \zeta K_\beta) \\ &= (L'_{K(\beta)}(y_1, y_2; \beta_0, \beta_1) - \zeta K_\beta)(L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1) - \zeta K_\alpha) \end{aligned} \quad (20)$$

which means that $L(x_1, x_2; \bar{\alpha}) = L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1) - \zeta K_\alpha$ is a Lax matrix for $R_{\bar{\alpha}, \bar{\beta}}$. Also, from proposition 3.1 we conclude that $L(x_1, x_2; \bar{\alpha})$ is a strong Lax matrix. Finally we notice that equation (20) is directly solvable with respect to $v = (v_1, v_2)$ and $x = (x_1, x_2)$, since

$$\begin{aligned} K_\beta^{-1} L'_{K_\beta}(v; \hat{\beta}) &= (L'_{K_\alpha}(u; \hat{\alpha})K_\beta - L'_{K_\beta}(y; \hat{\beta})K_\alpha)^{-1} L'_{K_\beta}(y; \hat{\beta})K_\beta^{-1} (L'_{K_\alpha}(u; \hat{\alpha})K_\beta - L'_{K_\beta}(y; \hat{\beta})K_\alpha), \\ K_\alpha^{-1} L'_{K_\alpha}(x; \hat{\alpha}) &= (L'_{K_\beta}(y; \hat{\beta})K_\alpha - L'_{K_\alpha}(u; \hat{\alpha})K_\beta)^{-1} L'_{K_\alpha}(u; \hat{\alpha})K_\alpha^{-1} (L'_{K_\beta}(y; \hat{\beta})K_\alpha - L'_{K_\alpha}(u; \hat{\alpha})K_\beta) \end{aligned}$$

for $y = (y_1, y_2)$, $u = (u_1, u_2)$, $\hat{\alpha} = (\alpha_0, \alpha_1)$ and $\hat{\beta} = (\beta_0, \beta_1)$. That proves the non-degeneracy of the YB map (18).

Remark 3.5. From the construction of the Lax matrix $L(x_1, x_2; \bar{\alpha})$ and lemma 1 we can prove that the equation:

$$L(x'_1, x'_2; \bar{\alpha})L(y'_1, y'_2; \bar{\beta})L(z'_1, z'_2; \bar{\gamma}) = L(x_1, x_2; \bar{\alpha})L(y_1, y_2; \bar{\beta})L(z_1, z_2; \bar{\gamma})$$

implies $x' = x$, $y' = y$ and $z' = z$ (without further assumptions). So the YB property of the map (18) can be derived directly from Prop. 2.1.

Remark 3.6. If we set $\alpha_0 = \beta_0 = k$ on the YB map (18) we obtain the parametric YB map $R_{\bar{\alpha}, \bar{\beta}}$ with parameters $\bar{\alpha} = (\alpha, a_1)$, $\bar{\beta} = (\beta, b_1) \in V \times \mathbb{C}$ and Lax matrix $L(x_1, x_2; \alpha, \alpha_1) := L(x_1, x_2; \alpha, k, \alpha_1)$. We have analogous results if we identify any other pair of parameters. If we set $\bar{\alpha} = \bar{\beta}$ then we derive the trivial solution $U = Y$, $V = X$, because this is the only solution of Eq.(10) with $A = B$, $f_0(U; A) = f_0(Y; A)$ and $f_1(U; A) = f_1(Y; A)$.

3.3 Classification

In this section we classify the quadrirational YB maps with 2×2 binomial Lax matrices of our construction. In [9] a classification by Jordan normal forms was given for the case $K(\alpha) = K(\beta) = B$, with B a 2×2 constant matrix. Here we give a more general classification in order to include all the cases that we considered. First we begin by determining the functions K of proposition 3.2. Actually we are going to consider the problem of families of commuting matrices up to conjugation. One can bring one member of the family to its Jordan canonical form and find all matrices commuting with it. From this analysis we conclude that, up to conjugation, there are only two (non-disjoint) families of commuting pairs of matrices

$$I) A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \text{ and } II) A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix}.$$

Since the equation (3) and the YB maps are invariant under conjugation we can restrict to these two general cases of the function $K : \mathbb{C}^2 \rightarrow GL_2(\mathbb{C})$.

The last step towards the classification is to examine the relevance of the choice of variables in the construction of the Lax matrix that we presented in the previous section. In the first case, where $K(\alpha)$ is a matrix of the first family for any $\alpha \in \mathbb{C}^2$, the equations

$$f_0(X; K(\alpha)) = \alpha_0, f_1(X; K(\alpha)) = \alpha_1 \quad (21)$$

are solvable with respect to any pair (x_i, x_j) , for $i, j = 1, \dots, 4$, $i \neq j$, except of the pair (x_2, x_3) , while for a matrix $K(\alpha)$ of the second family the equations are solvable with respect to any pair (x_i, x_j) , $i, j = 1, \dots, 4$, $i \neq j$. Now, let us suppose that, by solving equations (21) in a different way, we have derived two matrices $L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1)$, $M'_{K(\alpha)}(x'_1, x'_2; \alpha_0, \alpha_1)$ such that

$$\begin{aligned} f_0(L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1); K(\alpha)) &= \alpha_0, f_1(L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1); K(\alpha)) = \alpha_1 \text{ and} \\ f_0(M'_{K(\alpha)}(x'_1, x'_2; \alpha_0, \alpha_1); K(\alpha)) &= \alpha_0, f_1(M'_{K(\alpha)}(x'_1, x'_2; \alpha_0, \alpha_1); K(\alpha)) = \alpha_1. \end{aligned}$$

Then there is a local diffeomorphism $\phi_{\bar{\alpha}} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ ($\bar{\alpha} = (\alpha, \alpha_0, \alpha_1) \in \mathbb{C}^4$), such that $\phi_{\bar{\alpha}} : (x_1, x_2) \mapsto (x'_1, x'_2)$ and

$$M'_{K(\alpha)}(\phi_{\bar{\alpha}}(x_1, x_2); \alpha_0, \alpha_1) = L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1).$$

Now if we denote by $R_{\bar{\alpha}, \bar{\beta}}$, $R'_{\bar{\alpha}, \bar{\beta}}$ the parametric YB maps with strong Lax matrices $L(x_1, x_2; \bar{\alpha}) = L'_{K(\alpha)}(x_1, x_2; \alpha_0, \alpha_1) - \zeta K(\alpha)$ and $M(x'_1, x'_2; \bar{\alpha}) = M'_{K(\alpha)}(x'_1, x'_2; \alpha_0, \alpha_1) - \zeta K(\alpha)$ respectively, then

$$(\phi_{\bar{\alpha}} \times \phi_{\bar{\beta}}) \circ R'_{\bar{\alpha}, \bar{\beta}} = R_{\bar{\alpha}, \bar{\beta}} \circ (\phi_{\bar{\alpha}} \times \phi_{\bar{\beta}}). \quad (22)$$

From the above analysis we conclude that every four parametric non-degenerate YB map on $\mathbb{C}^2 \times \mathbb{C}^2$, of proposition 3.4, can be reduced up to equivalence (22) and reparametrization (see also remark 3.6) into one of the following two cases.

Case I

We consider the generic element $X - \zeta K_1(\alpha_1, \alpha_2) \in \mathcal{L}_{K_1(\alpha_1, \alpha_2)}^2$ with

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \text{ and } K_1(\alpha_1, \alpha_2) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}.$$

The Casimir functions in this case are

$$f_1(X; K_1(\alpha_1, \alpha_2)) = \alpha_2 x_1 + \alpha_1 x_4, \quad f_0(X; K_1(\alpha_1, \alpha_2)) = x_1 x_4 - x_2 x_3.$$

By setting $f_0(X; \alpha_1, \alpha_2) = \alpha_3$, $f_1(X; \alpha_1, \alpha_2) = \alpha_4$ and solving with respect to x_3, x_4 , for $\alpha_1, x_2 \neq 0$, we derive the matrix

$$L'_{K_1(\hat{\alpha})}(x_1, x_2; \alpha_3, \alpha_4) = \begin{pmatrix} x_1 & x_2 \\ \frac{x_1(\alpha_4 - \alpha_2 x_1) - \alpha_1 \alpha_3}{\alpha_1 x_2} & \frac{\alpha_4 - \alpha_2 x_1}{\alpha_1} \end{pmatrix} \text{ with } \hat{\alpha} = (\alpha_1, \alpha_2) \quad (23)$$

and the 8-parametric quadrirational YB map of proposition 3.4

$$R_{\bar{\alpha}, \bar{\beta}}^1((x_1, x_2), (y_1, y_2)) = Pr \circ \mathcal{R}_{\hat{\alpha}, \hat{\beta}}^1(L'_{K_1(\hat{\alpha})}(x_1, x_2; \alpha_3, \alpha_4), L'_{K_1(\hat{\beta})}(y_1, y_2; \beta_3, \beta_4)).$$

Here $\mathcal{R}_{\hat{\alpha}, \hat{\beta}}^1$ is the general parametric YB map (14) associated with the function K_1 , the projection $Pr = pr_{1,2} \times pr_{1,2}$ (projections at the elements of the first arrow of a matrix) and the parameters are $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\bar{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)$. According to prop. 3.4, this map admits the strong Lax matrix

$$L_1(x_1, x_2; \bar{\alpha}) = L'_{K_1(\hat{\alpha})}(x_1, x_2; \alpha_3, \alpha_4) - \zeta K_1(\alpha_1, \alpha_2),$$

and for $\alpha_1, \beta_1 \neq 0$ it is a symplectic rational map on $\{(x_1, x_2), (y_1, y_2) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid x_2, y_2 \neq 0\}$, with respect to the reduced symplectic form defined by the brackets:

$$\{x_1, x_2\} = -\alpha_1 x_2, \quad \{y_1, y_2\} = -\beta_1 y_2, \quad \{x_i, y_j\} = 0 \text{ for } i = 1, 2.$$

Case II

For $K_2(\alpha_1, \alpha_2) = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_1 \end{pmatrix}$ we set again $f_0(X; K_2(\alpha_1, \alpha_2)) = \alpha_3$, $f_1(X; K_2(\alpha_1, \alpha_2)) = \alpha_4$ and solve with respect to x_3, x_4 to get

$$L'_{K_2(\hat{\alpha})}(x_1, x_2; \alpha_3, \alpha_4) = \begin{pmatrix} x_1 & x_2 \\ \frac{\alpha_4 x_1 - \alpha_1(x_1^2 + \alpha_3)}{\alpha_1 x_2 - \alpha_2 x_1} & \frac{\alpha_2 \alpha_3 - \alpha_4 x_2 + \alpha_1 x_1 x_2}{\alpha_2 x_1 - \alpha_1 x_2} \end{pmatrix}, \text{ with } \hat{\alpha} = (\alpha_1, \alpha_2) \quad (24)$$

and the corresponding YB map

$$R_{\bar{\alpha}, \bar{\beta}}^2((x_1, x_2), (y_1, y_2)) = Pr \circ \mathcal{R}_{\hat{\alpha}, \hat{\beta}}^2(L'_{K_2(\hat{\alpha})}(x_1, x_2; \alpha_3, \alpha_4), L'_{K_2(\hat{\beta})}(y_1, y_2; \beta_3, \beta_4)),$$

with $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\bar{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)$, $Pr = pr_{1,2} \times pr_{1,2}$ and $\mathcal{R}_{\hat{\alpha}, \hat{\beta}}^2$ the general parametric YB map associated with K_2 . This map admits the strong Lax matrix

$L_2(x_1, x_2; \bar{\alpha}) = L'_{K_2(\hat{\alpha})}(x_1, x_2; \alpha_3, \alpha_4) - \zeta K_2(\alpha_1, \alpha_2)$. The reduced Sklyanin bracket in this case is given by brackets of the coordinates

$$\{x_1, x_2\} = \alpha_2 x_1 - \alpha_1 x_2, \quad \{y_1, y_2\} = \beta_2 y_1 - \beta_1 y_2, \quad \{x_i, y_j\} = 0 \text{ for } i, j = 1, 2.$$

As it was pointed out, YB maps with less parameters can be constructed from these two cases by setting $\alpha_i = \beta_i = k$ for some $i \in \{1, 2\}$. Also, by using appropriate scalings, one can reduce the number of parameters. However, we do not do this here, having in mind degenerate cases in subsection 3.4 below, as well as consideration of continuous limits in the future.

Remark. *If we are interested in real Lax matrices we have to include also the case where $K_3(\alpha_1, \alpha_2) = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}$ and the corresponding YB map of proposition 3.4.*

3.4 Degenerate YB maps

Degenerate YB maps can arise when $K(\alpha)$ is not invertible. A way of constructing degenerate YB maps as limits of the non-degenerate ones was presented in [9] for $K(\alpha) = K(\beta) = \text{Constant}$. We will apply this method here as well for $K(\alpha) \neq K(\beta)$.

We consider a function $K : V \rightarrow GL_2(\mathbb{C})$, $V \subset \mathbb{C}^4$, depending from a parameter ε , such that $K(\alpha, \varepsilon)K(\beta, \varepsilon) = K(\beta, \varepsilon)K(\alpha, \varepsilon)$ and $\lim_{\varepsilon \rightarrow 0} \det K(\alpha, \varepsilon) = 0$ for every $\alpha, \beta \in \mathbb{C}^m$, $m \leq 4$. We construct the corresponding non-degenerate YB map $R_{\bar{\alpha}, \bar{\beta}}(\varepsilon)$ of proposition 3.4. The limit of $R_{\bar{\alpha}, \bar{\beta}}(\varepsilon)$, for $\varepsilon \rightarrow 0$, can lead to a rational degenerate YB map on $\mathbb{C}^2 \times \mathbb{C}^2$. The induced Poisson structure is defined by the limit of the Sklyanin bracket. We apply this construction in the next concrete example.

A generalization of the Adler-Yamilov map

We consider the function $K : \mathbb{C} \rightarrow GL_2(\mathbb{C})$ with $K(\alpha_1) = K_{\alpha_1} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \varepsilon \end{pmatrix}$. The Casimir functions on $\mathcal{L}_{K(\alpha_1)}^2$ are :

$$f_0(X; K(\alpha_1)) = x_{11}x_{22} - x_{12}x_{21}, \quad f_1(X; K(\alpha_1)) = \varepsilon x_{11} + \alpha_1 x_{22}.$$

(Here we denote by x_{ij} the elements of the matrix X). If we set $f_0(X; K(\alpha_1)) = \alpha_2$, $f_1(X; K(\alpha_1)) = \alpha_3$ and solve with respect to x_{11} , x_{22} we have

$$x_{11} = \frac{1}{2\varepsilon}(\alpha_3 - (\alpha_3^2 - 4\alpha_1\varepsilon(\alpha_2 + x_{12}x_{21}))^{1/2}), \quad x_{22} = \frac{1}{2\alpha_1}(\alpha_3 + (\alpha_3^2 - 4\alpha_1\varepsilon(\alpha_2 + x_{12}x_{21}))^{1/2}).$$

By substituting this values to $X - \zeta K(\alpha_1)$ and renaming x_{12} , x_{21} as x_1 and x_2 respectively, we obtain the three-parametric Lax matrix

$$L(x_1, x_2; \bar{\alpha}) = \begin{pmatrix} \frac{\alpha_3 - (\alpha_3^2 - 4\alpha_1\varepsilon(\alpha_2 + x_1x_2))^{1/2}}{2\varepsilon} - \alpha_1\zeta & x_1 \\ x_2 & \frac{\alpha_3 + (\alpha_3^2 - 4\alpha_1\varepsilon(\alpha_2 + x_1x_2))^{1/2}}{2\alpha_1} - \varepsilon\zeta \end{pmatrix} \quad (25)$$

with $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, of the non-degenerate YB map of proposition 3.4

$$R_{\bar{\alpha}, \bar{\beta}}((x_1, x_2), (y_1, y_2)) = ((u_1, u_2), (v_1, v_2)). \quad (26)$$

Here u_1, u_2, v_1, v_2 are the corresponding elements $u_{12}, u_{21}, v_{12}, v_{21}$ of the matrices:

$$\begin{aligned} [u_{ij}] &:= U = (\alpha_1 \varepsilon Y X - \alpha_2 K_{\alpha_1} K_{\beta_1}) ((\alpha_1 \varepsilon (Y K_{\alpha_1} + K_{\beta_1} X) - \alpha_3 K_{\alpha_1} K_{\beta_1})^{-1} K_{\alpha_1} \\ [v_{ij}] &:= V = K_{\alpha_1}^{-1} (Y K_{\alpha_1} + K_{\beta_1} X - U K_{\beta_1}), \end{aligned}$$

for $X = L'_{K(\alpha_1)}(x_1, x_2; \bar{\alpha}) \equiv L(x_1, x_2; \bar{\alpha}) + \zeta K_{\alpha}$ and $Y = L'_{K(\alpha_1)}(y_1, y_2; \bar{\beta}) \equiv L(y_1, y_2; \bar{\beta}) + \zeta K_{\beta}$.

The limit of (26), for $\varepsilon \rightarrow 0$, gives the degenerate 6-parametric Yang-Baxter map $\tilde{R}_{\bar{\alpha}, \bar{\beta}}((x_1, x_2), (y_1, y_2)) = ((\bar{u}_1, \bar{u}_2), (\bar{v}_1, \bar{v}_2))$, where

$$\bar{u}_1 = \frac{\beta_1}{\alpha_1 \beta_3} (\alpha_3 y_1 - Q x_1), \quad \bar{u}_2 = \frac{\alpha_1}{\beta_1} y_2, \quad \bar{v}_1 = \frac{\beta_1}{\alpha_1} x_1, \quad \bar{v}_2 = \frac{\alpha_1}{\beta_1 \alpha_3} (\beta_3 x_2 - Q y_2),$$

$$\text{and } Q = \frac{\alpha_1 \beta_1 (\alpha_2 \beta_3 - \alpha_3 \beta_2)}{\alpha_3 \beta_3 + \alpha_1 \beta_1 x_1 y_2}.$$

This map is symplectic with respect to the symplectic form obtained by taking the limit, for $\varepsilon \rightarrow 0$, of $J_{K_{\alpha}}(L'(x_1, x_2; \bar{\alpha}))$ and $J_{K_{\beta}}(L'(y_1, y_2; \bar{\beta}))$,

$$\{x_1, x_2\} = \alpha_3, \quad \{y_1, y_2\} = \beta_3, \quad \{x_i, y_j\} = 0, \quad (27)$$

and admits the strong Lax matrix

$$M(x_1, x_2; \bar{\alpha}) = \lim_{\varepsilon \rightarrow 0} L(x_1, x_2; \bar{\alpha}) = \begin{pmatrix} \frac{\alpha_1}{\alpha_3} (\alpha_2 + x_1 x_2) - \alpha_1 \zeta & x_1 \\ x_2 & \frac{\alpha_3}{\alpha_1} \end{pmatrix}.$$

If we set $\alpha_3 = \beta_3 = 1$ on the map $\tilde{R}_{\bar{\alpha}, \bar{\beta}}$ we derive the 4-parametric YB map $\tilde{R}_{(\alpha_1, \alpha_2), (\beta_1, \beta_2)}$ with strong Lax matrix $M(x_1, x_2; \alpha_1, \alpha_2, 1)$. The induced symplectic form in this case is the canonical one. Moreover by setting $\alpha_1 = \beta_1 = \alpha_3 = \beta_3 = 1$, $\tilde{R}_{\bar{\alpha}, \bar{\beta}}$ is reduced to the Adler-Yamilov map [3, 9].

According to [13, 10] the monodromy matrix of the 1-periodic ‘staircase’ initial value problem on a quadrilateral lattice is $M_1(x_1, x_2, y_1, y_2) \equiv M(y_1, y_2; \bar{\beta}) M(x_1, x_2; \bar{\alpha})$. The trace of the monodromy matrix gives the two functionally independent integrals :

$$\begin{aligned} J_1(x_1, x_2, y_1, y_2) &= \frac{\alpha_1 \beta_1}{\alpha_3} x_1 x_2 + \frac{\alpha_1 \beta_1}{\beta_3} y_1 y_2 \\ J_2(x_1, x_2, y_1, y_2) &= x_2 y_1 + x_1 y_2 + \frac{\alpha_1 \beta_1}{\alpha_3 \beta_3} (\alpha_2 + x_1 x_2) (\beta_2 + y_1 y_2). \end{aligned}$$

We can verify that these integrals are in involution with respect to (27). So we conclude that the map $\tilde{R}_{\bar{\alpha}, \bar{\beta}}((x_1, x_2), (y_1, y_2)) \mapsto ((\bar{u}_1, \bar{u}_2), (\bar{v}_1, \bar{v}_2))$ is integrable in the Liouville sense. For the Adler-Yamilov map the corresponding integrals are given by setting $\alpha_1 = \beta_1 = \alpha_3 = \beta_3 = 1$ in J_1 and J_2 .

4 Higher dimensional Yang-Baxter maps

In order to generate higher dimensional Yang-Baxter maps we consider the set \mathcal{L}^n of n order polynomial matrices of the form $X - \zeta A$. There are $n(n+1)$ functionally independent Casimir functions on \mathcal{L}^n with respect to the Sklyanin bracket (6), which are again the n^2 elements of A and the n functions f_i , $i = 0, \dots, n-1$, defined as the coefficients of the polynomial $p_X^A(\zeta) = \det(X - \zeta A)$,

$$p_X^A(\zeta) = (-1)^n f_n(X; A) \zeta^n + (-1)^{n-1} f_{n-1}(X; A) \zeta^{n-1} + \dots + (-1) f_1(X; A) \zeta + f_0(X; A)$$

where $f_n(X; A) = \det A$ and $f_0(X; A) = \det X$.

As in the 2×2 case, we consider $K : \mathbb{C}^d \rightarrow GL_n(\mathbb{C})$ a d -parametric family of commuting matrices. Next, for $\alpha \in \mathbb{C}^d$, we denote the value $K(\alpha)$ by K_α and the values of the Casimirs $f_i(X; K(\alpha))$ by $f_i(X; \alpha)$, $i = 0, \dots, n$.

Proposition 4.1. *Let U and V be $n \times n$ matrices that satisfy the following two conditions*

- (i) $f_i(U; \alpha) = f_i(X; \alpha)$ and $f_i(V; \beta) = f_i(Y; \beta)$ for $i = 0, \dots, n-1$,
- (ii) $(U - \zeta K_\alpha)(V - \zeta K_\beta) = (Y - \zeta K_\beta)(X - \zeta K_\alpha)$, *identically in $\zeta \in \mathbb{C}$*

for $X, Y \in Mat(n \times n)$ such that $\det \sum_{i=1}^n (-1)^i f_i(X; \alpha) M_{i-1} \neq 0$. Then

$$U = \left(-f_0(X; \alpha) I - \sum_{i=1}^n (-1)^i f_i(X; \alpha) N_{i-1} \right) \left(\sum_{i=1}^n (-1)^i f_i(X; \alpha) M_{i-1} \right)^{-1} K_\alpha \quad (28)$$

$$V = K_\alpha^{-1} (Y K_\alpha + K_\beta X - U K_\beta), \quad (29)$$

where M_i, N_i are given by:

$$\begin{aligned} M_0 &= I, \quad N_0 = 0, \quad M_1 = (Y K_\alpha + K_\beta X) K_\beta^{-1} K_\alpha^{-1}, \quad N_1 = -Y X K_\beta^{-1} K_\alpha^{-1}, \\ M_i &= M_1 M_{i-1} + N_{i-1}, \quad N_i = N_1 M_{i-1}, \quad \text{for } i = 2, \dots, n. \end{aligned}$$

Proof: Since $f_i(U; \alpha) = f_i(X; \alpha)$, for $i = 1, \dots, n$, then $p_U^{K_\alpha}(\zeta) = p_X^{K_\alpha}(\zeta)$. Cayley-Hamilton theorem states that $p_U^{K_\alpha}(U K_\alpha^{-1}) = p_X^{K_\alpha}(U K_\alpha^{-1}) = 0$. So

$$\sum_{i=1}^n (-1)^i f_i(X; \alpha) (U K_\alpha^{-1})^i = -f_0(X; \alpha) I, \quad i = 1, \dots, n. \quad (30)$$

Furthermore from (ii) we derive the system:

$$UV = YX, \quad UK_\beta + K_\alpha V = YK_\alpha + K_\beta X \quad (31)$$

which implies

$$(U K_\alpha^{-1})^2 = U K_\alpha^{-1} (Y K_\alpha + K_\beta X) K_\beta^{-1} K_\alpha^{-1} - Y X K_\beta^{-1} K_\alpha^{-1}. \quad (32)$$

For simplicity we set $\tilde{U} = UK_\alpha^{-1}$, $M_1 = (YK_\alpha + K_\beta X)K_\beta^{-1}K_\alpha^{-1}$ and $N_1 = -YXK_\beta^{-1}K_\alpha^{-1}$. So equation (32) can be written as $\tilde{U}^2 = \tilde{U}M_1 + N_1$. Also if we set $M_0 = I, N_0 = 0$ and define M_i, N_i from the recurrence relations:

$$M_i = M_1M_{i-1} + N_{i-1}, \quad N_i = N_1M_{i-1} \quad \text{for } i = 1, \dots, n, \quad (33)$$

then we can evaluate the powers of \tilde{U}^k as $\tilde{U}^k = \tilde{U}M_{k-1} + N_{k-1}$ for $k = 1, \dots, n$. So equation (30) becomes:

$$\sum_{i=1}^n (-1)^i f_i(X; \alpha) (\tilde{U}M_{i-1} + N_{i-1}) = -f_0(X; \alpha)I,$$

and finally we have

$$\tilde{U} = \left(-f_0(X; \alpha)I - \sum_{i=1}^n (-1)^i f_i(X; \alpha)N_{i-1} \right) \left(\sum_{i=1}^n (-1)^i f_i(X; \alpha)M_{i-1} \right)^{-1}$$

So $U = \tilde{U}K_\alpha$ and from(31) $V = K_\alpha^{-1}(YK_\alpha + K_\beta X - UK_\beta)$.

Remark 4.2. If we write the first equation of (31) as $UK_\alpha^{-1}K_\alpha V = YX$ and replace $K_\alpha V$ from the second one, we get that

$$UK_\alpha^{-1}(YK_\alpha - UK_\beta) = (YK_\alpha - UK_\beta)K_\alpha^{-1}X.$$

In a similar way we can show that $(UK_\beta - YK_\alpha)K_\beta^{-1}V = YK_\beta^{-1}(UK_\beta - YK_\alpha)$. So if $\det(UK_\beta - YK_\alpha) \neq 0$ (equivalently $\det(K_\alpha V - K_\beta X) \neq 0$ since $UK_\beta - YK_\alpha = K_\alpha V - K_\beta X$) then the matrices UK_α^{-1} , $K_\beta^{-1}V$ are similar with the matrices $K_\alpha^{-1}X$ and YK_β^{-1} respectively, and subsequently $p_U^{K_\alpha}(\zeta) = p_X^{K_\alpha}(\zeta)$, $p_V^{K_\beta}(\zeta) = p_Y^{K_\beta}(\zeta)$. Therefore the condition (i) of proposition 4.1 can be replaced by the assumption $\det(UK_\beta - YK_\alpha) \neq 0$ (equivalently $\det(K_\alpha V - K_\beta X) \neq 0$).

Remark 4.3. Proposition 4.1 holds also if we replace K_α , K_β by two invertible matrices A and B respectively such that $AB = BA$. The reason for restricting to the function K is that we are interested to consider $L(X; \alpha) = X - \zeta K_\alpha$ as a Lax matrix of a YB map, otherwise we would have a Lax pair $L(X; A) = X - \zeta A$, $M(Y; B) = Y - \zeta B$ with $L \neq M$ as in [10].

The Yang-Baxter property of this re-factorization solution, i.e. of the map

$$\mathcal{R}_{\alpha, \beta}(X, Y) \mapsto (U, V),$$

with U, V defined by (28) and (29), is still an open problem. In low dimensions, for certain choices of the function K , this can be checked by direct computation or by proposition 2.1. We conjecture that this is true for any dimension. Anyway, since $f_i(U; \alpha) = f_i(X; \alpha)$ and $f_i(V; \beta) = f_i(Y; \beta)$, the map $\mathcal{R}_{\alpha, \beta}$ can be reduced, as in 2×2 case, to a map on $\mathbb{C}^{n(n-1)} \times \mathbb{C}^{n(n-1)}$ by the restriction to the corresponding level sets of the n Casimir functions f_i , $i = 0, \dots, n-1$. Further reduction on lower dimensional symplectic leaves is also possible.

4.1 8-dimensional quadrirational symplectic YB maps with 3×3 Lax matrices

In the case of \mathcal{L}^3 there exist three Casimir functions, so the map of Prop.4.1 can be reduced to a quadrirational map on $\mathbb{C}^6 \times \mathbb{C}^6$. Further reduction to four dimensional symplectic submanifolds of \mathcal{L}^3 provide maps on $\mathbb{C}^4 \times \mathbb{C}^4$. Next, we demonstrate this procedure for $K_\alpha = K_\beta = I$. Let $L(\zeta) = X - \zeta I$, with $X = [x_{ij}]$, be a generic element of \mathcal{L}_I^3 . In this case the Sklyanin bracket is

$$\{L(\zeta) \otimes L(\eta)\} = \left[\frac{r}{\zeta - \eta}, L(\zeta) \otimes L(\eta) \right]$$

$$= \begin{pmatrix} 0 & -x_{12} & -x_{13} & x_{12} & 0 & 0 & x_{13} & 0 & 0 \\ x_{21} & 0 & 0 & x_{22} - x_{11} & -x_{12} & -x_{13} & x_{23} & 0 & 0 \\ x_{31} & 0 & 0 & x_{32} & 0 & 0 & x_{33} - x_{11} & -x_{12} & -x_{13} \\ -x_{21} & x_{11} - x_{22} & -x_{23} & 0 & x_{12} & 0 & 0 & x_{13} & 0 \\ 0 & x_{21} & 0 & -x_{21} & 0 & -x_{23} & 0 & x_{23} & 0 \\ 0 & x_{31} & 0 & 0 & x_{32} & 0 & -x_{21} & x_{33} - x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{11} - x_{33} & 0 & 0 & x_{12} & 0 & 0 & x_{13} \\ 0 & 0 & x_{21} & -x_{31} & -x_{32} & x_{22} - x_{33} & 0 & 0 & x_{23} \\ 0 & 0 & x_{31} & 0 & 0 & x_{32} & -x_{31} & -x_{32} & 0 \end{pmatrix} \quad (34)$$

Generically the rank of the structure matrix (34) is six. We are interested in finding 4-dimensional symplectic submanifolds of \mathcal{L}_I^3 . For this reason we would like to find conditions such that the rank of the matrix (34) drops down to four.

Let $i_1 < \dots < i_6$, $j_1 < \dots < j_6$, with $i_k, j_k \in \{1, \dots, 9\}$ for $k = 1, \dots, 6$. We denote by $m((i_1, \dots, i_6), (j_1, \dots, j_6))$ the sixth order minor of the matrix (34), consisting of the i_1, \dots, i_6 rows and the j_1, \dots, j_6 columns. Using this notation we prove the next lemma.

Lemma 2. *Consider the system of equations obtained by setting all sixth order minors $m((i_1, \dots, i_6), (j_1, \dots, j_6))$ equal to zero. There is a unique solution of this system with respect to x_{11} , x_{31} , x_{32} , for nonzero x_{13} , x_{23} , namely:*

$$\begin{aligned} x_{11} &= \frac{x_{13}x_{21}}{x_{23}} + x_{22} - \frac{x_{12}x_{23}}{x_{13}}, \quad x_{31} = \frac{x_{21}(x_{12}x_{23} + x_{13}(x_{33} - x_{22}))}{x_{13}x_{23}}, \\ x_{32} &= \frac{x_{12}(x_{12}x_{23} + x_{13}(x_{33} - x_{22}))}{x_{13}^2}. \end{aligned} \quad (35)$$

Substituting these values to $X - \zeta I$ the rank of the Poisson matrix in (34) reduces to four and the Casimirs $f_0(X; I) := \alpha_0$, $f_1(X; I) := \alpha_1$, $f_2(X; I) := \alpha_2$ satisfy

$$4\alpha_0\alpha_2^3 - \alpha_1^2\alpha_2^2 + 4\alpha_1^3 - 18\alpha_0\alpha_1\alpha_2 + 27\alpha_0^2 = 0 \quad (36)$$

Proof: Consider the minors

$$\begin{aligned} m_1 &= m((1, 2, 3, 4, 5, 6), (3, 4, 6, 7, 8, 9)) = -(x_{21}x_{13}^2 - x_{11}x_{23}x_{13} + x_{22}x_{23}x_{13} - x_{12}x_{23}^2)^2, \\ m_2 &= m((1, 2, 3, 4, 6, 7), (3, 4, 5, 6, 8, 9)) = -(x_{23}x_{12}^2 - x_{13}x_{22}x_{12} + x_{13}x_{33}x_{12} - x_{13}^2x_{32})^2, \\ m_3 &= m((1, 2, 3, 5, 6, 9), (1, 2, 3, 5, 6, 9)) = -(x_{12}x_{23}x_{31} - x_{13}x_{21}x_{32})^2. \end{aligned}$$

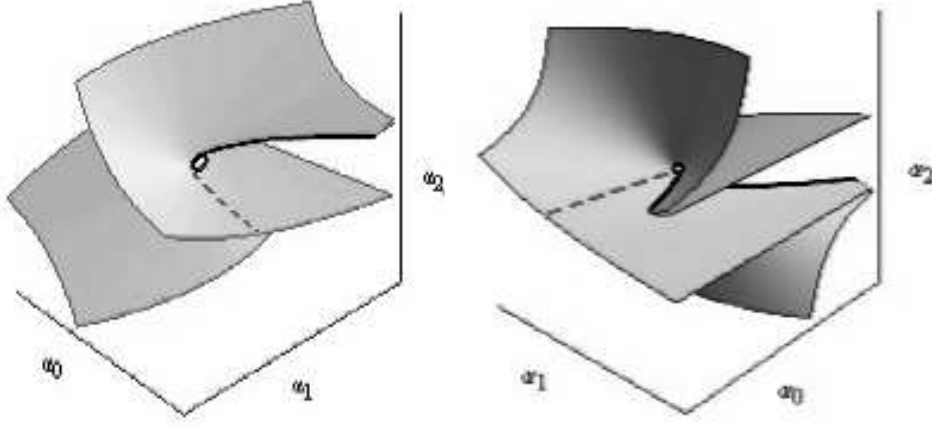


Figure 3: Two views of surface (36) in \mathbb{R}^3 , black curve: $(\alpha^3, 3\alpha^2, 3\alpha)$, dashed curve: $(-\alpha^3, -\alpha^2, \alpha)$

The system $m_1 = m_2 = m_3 = 0$ is linear with respect to x_{11}, x_{31}, x_{32} and for $x_{13}, x_{23} \neq 0$ admits the unique solution (35). Substituting these values to (34) the rank reduces to four and the Casimir functions become:

$$\begin{aligned}
 f_0(X; I) &= \frac{(x_{13}x_{22} - x_{12}x_{23})^2 (x_{21}x_{13}^2 + x_{23}x_{33}x_{13} + x_{12}x_{23}^2)}{x_{13}^3x_{23}} \\
 f_1(X; I) &= \frac{(x_{13}x_{22} - x_{12}x_{23}) (2x_{21}x_{13}^2 + x_{23}(x_{22} + 2x_{33})x_{13} + x_{12}x_{23}^2)}{x_{13}^2x_{23}} \\
 f_2(X; I) &= \frac{x_{13}x_{21}}{x_{23}} + 2x_{22} - \frac{x_{12}x_{23}}{x_{13}} + x_{33}.
 \end{aligned} \tag{37}$$

which satisfy (36).

It is remarkable that two curves on the surface (36) give rise to maps related to the Boussinesq and the matrix KdV equation.

4.1.1 A 4-parametric symplectic Y-B map

If we set the values (35) to X , in order to restrict on the level sets of the Casimir functions of \mathcal{L}_I^3 we set $f_2(X; I) = \alpha_2$, $f_1(X; I) = \alpha_1$ (of course $f_0(X; I)$ will be also constant since (36) must be satisfied) and solve (37) with respect to x_{22} and x_{33} to get

$$x_{22} = \frac{\alpha_2}{3} + \frac{x_{12}x_{23}}{x_{13}} \pm \frac{1}{3}\sqrt{\alpha_2^2 - 3\alpha_1}, \quad x_{33} = \frac{\alpha_2}{3} - \frac{x_{13}x_{21}}{x_{23}} - \frac{x_{12}x_{23}}{x_{13}} \mp \frac{2}{3}\sqrt{\alpha_2^2 - 3\alpha_1}.$$

For simplicity we can change the parameters into $c_1 = \frac{\alpha_2}{3}$ and $c_2 = \pm \frac{1}{3}\sqrt{\alpha_2^2 - 3\alpha_1}$, so $x_{22} = c_1 + c_2 + \frac{x_{12}x_{23}}{x_{13}}$, $x_{33} = c_1 - 2c_2 - \frac{x_{13}x_{21}}{x_{23}} - \frac{x_{12}x_{23}}{x_{13}}$. Substituting these values to (35) and the new x_{ij} to $X - \zeta I$, we obtain the two parametric family of matrices

$$\begin{aligned}
& M(x_{12}, x_{13}, x_{21}, x_{23}; c_1, c_2) \\
&= \begin{pmatrix} \frac{x_{13}x_{21}}{x_{23}} + c_1 + c_2 - \zeta & x_{12} & x_{13} \\ x_{21} & \frac{x_{12}x_{23}}{x_{13}} + c_1 + c_2 - \zeta & x_{23} \\ -\frac{x_{13}x_{21}^2}{x_{23}^2} - \frac{3c_2x_{21}}{x_{23}} - \frac{x_{12}x_{21}}{x_{13}} & -\frac{x_{23}x_{12}^2}{x_{13}^2} - \frac{3c_2x_{12}}{x_{13}} - \frac{x_{21}x_{12}}{x_{23}} & c_1 - 2c_2 - \frac{x_{13}x_{21}}{x_{23}} - \frac{x_{12}x_{23}}{x_{13}} - \zeta \end{pmatrix} \quad (38)
\end{aligned}$$

The reduced Poisson structure is

$$\{x_{12}, x_{21}\} = \frac{x_{12}x_{23}}{x_{13}} - \frac{x_{13}x_{21}}{x_{23}}, \quad \{x_{12}, x_{23}\} = -x_{13}, \quad \{x_{13}, x_{21}\} = x_{23}$$

and $\{x_{12}, x_{13}\} = \{x_{13}, x_{23}\} = \{x_{21}, x_{23}\} = 0$, which defines the symplectic form :

$$\omega = \frac{1}{x_{23}} dx_{13} \wedge dx_{21} - \frac{1}{x_{13}} dx_{12} \wedge dx_{23} + \left(\frac{x_{12}}{x_{13}^2} - \frac{x_{21}}{x_{23}^2} \right) dx_{13} \wedge dx_{23} .$$

We can change to canonical variables by setting

$$x_{13} = X_1, \quad x_{23} = X_2, \quad x_{21} = -x_1 X_2, \quad x_{12} = -x_2 X_1. \quad (39)$$

Then we denote matrix $M(x_{12}, x_{13}, x_{21}, x_{23}; c_1, c_2)$ by

$$L(x_1, x_2, X_1, X_2; c_1, c_2) \equiv L'_I(x_1, x_2, X_1, X_2; \alpha_1, \alpha_2) - \zeta I$$

$$\equiv \begin{pmatrix} c_1 + c_2 - x_1 X_1 - \zeta & -X_1 x_2 & X_1 \\ -x_1 X_2 & c_1 + c_2 - x_2 X_2 - \zeta & X_2 \\ -x_1(x_1 X_1 + x_2 X_2 - 3c_2) & -x_2(x_1 X_1 + x_2 X_2 - 3c_2) & c_1 - 2c_2 + x_1 X_1 + x_2 X_2 - \zeta \end{pmatrix} \quad (40)$$

and the symplectic form ω by the canonical symplectic form $\omega_0 = dx_1 \wedge dX_1 + dx_2 \wedge dX_2$.

From the re-factorization formula (28), (29), for $K_\alpha = K_\beta = I$, $X = L'_I(x_1, x_2, X_1, X_2; \alpha_1, \alpha_2)$ and $Y = L'_I(y_1, y_2, Y_1, Y_2; \beta_1, \beta_2)$, since the Casimir functions on

$$\Sigma_I(\alpha_1, \alpha_2) = \{L(x_1, x_2, X_1, X_2; \alpha_1, \alpha_2) \mid x_1, x_2, X_1, X_2 \in \mathbb{C}\}$$

are

$$f_0(X; I) = (\alpha_1 - 2\alpha_2)(a_1 + a_2)^2, \quad f_1(X; I) = 3(\alpha_1^2 - \alpha_2^2), \quad f_2(X; I) = 3\alpha_1,$$

we obtain the matrices

$$U = (YX(3\alpha_1 I - Y - X) - (\alpha_1 - 2\alpha_2)(a_1 + a_2)^2 I) ((3\alpha_1 I - Y - X)(Y + X) + YX - 3(\alpha_1^2 - \alpha_2^2) I)^{-1},$$

$$V = Y + X - U.$$

If we denote by U_{ij}, V_{ij} the elements of the matrices U and V , we come up to the next proposition.

Proposition 4.4. *The map*

$$R_{((\alpha_1, \alpha_2), (\beta_1, \beta_2))} : ((x_1, x_2, X_1, X_2), (y_1, y_2, Y_1, Y_2)) \mapsto ((u_1, u_2, U_1, U_2), (v_1, v_2, V_1, V_2)) \text{ where}$$

$$\begin{aligned}
U_1 &= U_{13}, \quad U_2 = U_{23}, \quad u_1 = -\frac{U_{21}}{U_{23}}, \quad u_2 = -\frac{U_{12}}{U_{13}}, \\
V_1 &= V_{13}, \quad V_2 = V_{23}, \quad v_1 = -\frac{V_{21}}{V_{23}}, \quad v_2 = -\frac{V_{12}}{V_{13}}
\end{aligned}$$

is a symplectic parametric Yang-Baxter map, with respect to the canonical symplectic form $dx_1 \wedge dX_1 + dx_2 \wedge dX_2 + dy_1 \wedge dY_1 + dy_2 \wedge dY_2$, and admits the strong Lax matrix $L(x_1, x_2, X_1, X_2; \alpha_1, \alpha_2)$.

Proof: The YB property of this map can be checked by direct computation. Moreover $u_i, U_i, v_i, V_i, i = 1, 2$ is the unique solution (proposition 4.1) of the Lax equation:

$$L(u_1, u_2, U_1, U_2; \alpha_1, \alpha_2)L(v_1, v_2, V_1, V_2; \beta_1, \beta_2) = L(y_1, y_2, Y_1, Y_2; \beta_1, \beta_2)L(x_1, x_2, X_1, X_2; \alpha_1, \alpha_2)$$

The explicit formula of the YB map $R_{((\alpha_1, \alpha_2), (\beta_1, \beta_2))}$ of proposition 4.4 is

$$\begin{aligned} (u_1, u_2) &= (y_1, y_2) - \frac{\alpha_1 - \beta_1 - 2(\alpha_2 - \beta_2)}{D} (x_1 - y_1, x_2 - y_2), \\ (v_1, v_2) &= (x_1, x_2) + \frac{\alpha_1 - \beta_1 + \alpha_2 - \beta_2}{D} (x_1 - y_1, x_2 - y_2), \end{aligned}$$

with $D = 2\alpha_2 - \alpha_1 + \beta_1 + \beta_2 + y_1X_1 + y_2X_2 - x_1X_1 - x_2X_2$, and

$$\begin{aligned} U_1 &= \frac{(x_1 - v_1)X_1 + (y_1 - v_1)Y_1}{u_1 - v_1}, & U_2 &= \frac{(x_2 - v_2)X_2 + (y_2 - v_2)Y_2}{u_2 - v_2}, \\ V_1 &= \frac{(x_1 - u_1)X_1 + (y_1 - u_1)Y_1}{v_1 - u_1}, & V_2 &= \frac{(x_2 - u_2)X_2 + (y_2 - u_2)Y_2}{v_2 - u_2}. \end{aligned}$$

We will point out two special cases of this YB map that give rise to Boussinesq and Goncharenko–Veselov maps.

4.1.2 The Boussinesq Y-B map ($\alpha_0 = \alpha^3, \alpha_1 = 3\alpha^2, \alpha_2 = 3\alpha$)

By setting $c_2 = 0, c_1 = \alpha$ to (40) we derive the Lax matrix

$$L_B(x_1, x_2, X_1, X_2; \alpha) = \begin{pmatrix} \alpha - \zeta - x_1X_1 & -X_1x_2 & X_1 \\ -x_1X_2 & \alpha - \zeta - x_2X_2 & X_2 \\ -x_1(x_1X_1 + x_2X_2) & -x_2(x_1X_1 + x_2X_2) & \alpha - \zeta + x_1X_1 + x_2X_2 \end{pmatrix}$$

In this case the Casimir functions on $\Sigma_I(\alpha) = \{L^B(x_1, x_2, X_1, X_2; \alpha) / x_1, x_2, X_1, X_2 \in \mathbb{C}\}$ are

$$f_0(X; I) = \alpha^3, \quad f_1(X; I) = 3\alpha^2, \quad f_2(X; I) = 3\alpha,$$

for $X = L_I^B(x_1, x_2, X_1, X_2; \alpha) \equiv L_B(x_1, x_2, X_1, X_2; \alpha) + \zeta I$. The curve $(\alpha^3, 3\alpha^2, 3\alpha)$ is depicted in fig. 3 with black color.

The corresponding 2-parametric YB map $R_{\alpha, \beta}^B$ with strong Lax matrix $L_B(x_1, x_2, X_1, X_2; c)$ is induced from the YB map $R_{((\alpha_1, \alpha_2), (\beta_1, \beta_2))}$ of proposition 4.1 i.e. $R_{\alpha, \beta}^B = R_{((\alpha, 0), (\beta, 0))}$.

4.1.3 The Goncharenko–Veselov map ($\alpha_0 = -\alpha^3, \alpha_1 = -\alpha^2, \alpha_2 = \alpha$)

In a similar way if we set $c_1 = \frac{\alpha}{3}$ and $c_2 = \frac{2\alpha}{3}$ we obtain the Yang-Baxter map

$$R_{\alpha, \beta}^{GV} = R_{((\frac{\alpha}{3}, \frac{2\alpha}{3}), (\frac{\beta}{3}, \frac{2\beta}{3}))}$$

with strong Lax matrix

$$L_{GV}(\mathbf{x}; \alpha) = \begin{pmatrix} \alpha - \zeta - x_1X_1 & -X_1x_2 & X_1 \\ -x_1X_2 & \alpha - \zeta - x_2X_2 & X_2 \\ -x_1(x_1X_1 + x_2X_2 - 2\alpha) & -x_2(x_1X_1 + x_2X_2 - 2\alpha) & x_1X_1 + x_2X_2 - \alpha - \zeta \end{pmatrix}$$

for $\mathbf{x} = (x_1, x_2, X_1, X_2)$. Here for $X = L_{GV}(\mathbf{x}; \alpha) + \zeta I$, $(f_0(X; I), f_1(X; I), f_2(X; I)) = (-\alpha^3, -\alpha^2, \alpha)$, which is the dashed curve of fig. 3.

Both maps $R_{\alpha, \beta}^B$ and $R_{\alpha, \beta}^{GV}$ are symplectic with respect to the canonical symplectic form $dx_1 \wedge dX_1 + dx_2 \wedge dX_2 + dy_1 \wedge dY_1 + dy_2 \wedge dY_2$.

In [8], Goncharenko and Veselov presented a YB map as interaction of two soliton solutions of the matrix KdV equation and claimed that it admits the Lax matrix of the form:

$$A(\xi, \eta; \lambda) = I + \frac{2\lambda}{\zeta - \lambda} \frac{\xi \otimes \eta}{(\xi, \eta)}, \quad (41)$$

for the n -dimensional vectors ξ and η . Here λ is the YB parameter. Essentially $\xi, \eta \in \mathbb{C}P^{n-1}$ since $\xi \mapsto \mu\xi, \eta \mapsto \nu\eta$ leaves (41) invariant. Even if the case for $n = 2$ is rather trivial, it is quite interesting for higher dimensions.

First we observe that we can multiply the Lax matrix (41) with $\zeta - \lambda$ and change ζ with $-\zeta$ in order to derive an equivalent Lax matrix

$$B(\xi, \eta; \lambda) = \lambda \left(2 \frac{\xi \otimes \eta}{(\xi, \eta)} - I \right) - \zeta I$$

for the same YB map. Now, let $n = 3$, $\xi = (\xi_1, \xi_2, \xi_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$. Considering the affine part of $\mathbb{C}P^2$, we have $\xi = (\xi_1, \xi_2, 1)$, $\eta = (\eta_1, \eta_2, 1)$ and by performing the invertible transformation $(\eta_1, \eta_2, \xi_1, \xi_2,) \mapsto (x_1, x_2, X_1, X_2)$:

$$x_1 = -\eta_1, \quad x_2 = -\eta_2, \quad X_1 = \frac{2\alpha\xi_1}{\xi_1\eta_1 + \xi_2\eta_2 + 1}, \quad X_2 = \frac{2\alpha\xi_2}{\xi_1\eta_1 + \xi_2\eta_2 + 1},$$

the matrix $B(\xi, \eta; \lambda)$ is transformed to the Lax matrix $L_{GV}(\mathbf{x}; -\lambda)$.

5 Conclusion

By generalizing the re-factorization procedure reported in [9], we presented a construction of multidimensional parametric Yang-Baxter maps. The symplectic quadrirational YB maps on $\mathbb{C}^2 \times \mathbb{C}^2$, that was derived in this way, were classified in two cases (three cases for real maps). The re-factorization of 3×3 binomial matrices provided us a family of symplectic YB maps on $\mathbb{C}^4 \times \mathbb{C}^4$ with Lax matrices the four dimensional symplectic leaves of \mathcal{L}_I^3 .

A similar classification procedure with the one presented here for quadrirational YB maps with $n \times n$ binomial Lax matrices, for $n > 2$, is a far more difficult task. The determination of the commuting pairs of invertible $n \times n$ matrices, in addition with the determination of the corresponding symplectic leaves on \mathcal{L}^n , is needed. It would be interesting to investigate this problem for small values of n . Furthermore other re-factorization formulas of higher degree polynomial matrices, guided by the invariance of the Casimir functions of the Sklyanin bracket, could lead to symplectic multidimensional YB maps. The derived maps contain, in general, more than one YB parameters. One can ask if (some of) these parameters are associated to spectral ones, in view of the 3D consistency of the YB maps. This is an interesting question especially with respect to finding invariants of the corresponding transfer maps and is going to be investigated in the future. Other issues deserving further research are initial value problems on lattices connected to the maps reported here, as well as the study of their continuum limits.

acknowledgments TEK acknowledges partial support from the State Scholarships Foundation of Greece. Both authors thank the anonymous referee for useful comments.

References

- [1] V.E. Adler, A.I. Bobenko, Yu.B. Suris, *Classification of integrable equations on quad-graphs. The consistency approach*, Comm. Math. Phys. 233, 2003, 513–543.
- [2] V.E. Adler, A.I. Bobenko, Yu.B. Suris, *Geometry of Yang-Baxter maps: pencils of conics and quadrirational mappings*, Comm. Anal. Geom. 12, 2004, 967–1007.
- [3] V.E. Adler, R.I. Yamilov, *Explicit auto-transformations of integrable chains*, J. Phys. A: Math. Gen. 27, 1994, 477–492.
- [4] A.I. Bobenko, Yu.B. Suris, *Integrable systems on quad-graphs*, Int. Math. Res. Notices, No. 11, 2002, 573–611.
- [5] V.G. Drinfeld, *On some Unsolved Problems in Quantum Group Theory*, Lecture Notes in Math. 1510, 1992, 1–8.
- [6] P. Etingof, T. Schedler, A. Soloviev, *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. 100 no. 2, 1999, 169–209.
- [7] P. Etingof, *Geometric crystals and set-theoretical solutions to the quantum Yang-Baxter equation*, Comm. Algebra, 31 no. 4, 2003, 1961–1973
- [8] V.M. Goncharenko, A.P. Veselov, *Yang-Baxter maps and matrix solitons*, NATO Sci. Ser. II Math. Phys. Chem., 132. New trends in integrability and partial solvability, 2004, 191–197.
- [9] T.E. Kouloukas, V.G. Papageorgiou, *Yang-Baxter maps with first-degree-polynomial 2×2 Lax matrices*, J. Phys. A: Math. Theor. 42, 2009, 404012
- [10] T.E. Kouloukas, V.G. Papageorgiou, *Entwining Yang-Baxter maps and integrable lattices*, arXiv:1006.2145v1.
- [11] J.-H. Lu, M. Yan, Y.-C. Zhu, *On the set-theoretical Yang-Baxter equation*, Duke Math. J. 104, 2000, 1–18.
- [12] F.W. Nijhoff, *Lax pair for the Adler (lattice Krichever-Novikov) system*, Phys. Lett. A, 297, 2002, 49–58.
- [13] V.G. Papageorgiou, F.W. Nijhoff, H.W. Capel, *Integrable mappings and nonlinear integrable lattice equations*, Phys. Lett. A, 147, 1990, 106–114.
- [14] V.G. Papageorgiou, Yu.B. Suris, A.G. Tongas, A.P. Veselov, *On Quadrirational Yang-Baxter Maps*, SIGMA 6, 033, 2010, 9p.

- [15] V.G. Papageorgiou, A.G. Tongas, A.P. Veselov, *Yang-Baxter maps and symmetries of integrable equations on quad-graphs*, J. Math. Phys. 47, 2006, 083502 1–16.
- [16] V.G. Papageorgiou, A.G. Tongas, *Yang-Baxter maps and multi-field integrable lattice equations*, J. Phys. A 40, no. 42, 2007, 12677–12690.
- [17] N. Reshetikhin, A.P. Veselov, *Poisson Lie groups and Hamiltonian theory of the Yang–Baxter maps*, math.QA/0512328, 2005.
- [18] M. Semenov-Tian-Shansky, *Dressing transformations and Poisson group actions*, Publ. RIMS, Kyoto University, 21, 1985, 1237–1260.
- [19] E.K Sklyanin, *Some algebraic structures connected with the Yang-Baxter equation*, Funct. Anal. Appl. 16, No 4, 1983, 263–270.
- [20] E.K Sklyanin, *Classical limits of $SU(2)$ -invariant solutions of the Yang-Baxter equation*, J. Soviet Math. 40, No 1, 1988, 93–107.
- [21] Yu.B. Suris, A.P. Veselov, *Lax matrices for Yang–Baxter maps*, J. Nonlin. Math. Phys. 10, suppl.2, 2003, 223–230.
- [22] A.P. Veselov, *Yang-Baxter maps and integrable dynamics*, Phys. Lett. A, 314, 2003, 214–221.
- [23] A.P. Veselov, *Yang-Baxter maps: dynamical point of view*, Combinatorial Aspects of Integrable Systems (Kyoto, 2004), MSJ Mem. vol 17, 2007, pp 145–67.
- [24] A. Weinstein, P. Xu, 1992 *Classical solutions to the Quantum Yang–Baxter equation*, Commun. Math. Phys. 148, 1992, 309–343.