Spin, Isospin and
Strong Interaction Dynamics

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Abstract:

The structure of spin and isospin is analyzed. Although both spin and isospin are related to the same SU(2) group, they represent different dynamical effects. The Wigner-Racah algebra is used for providing a description of bound states of several Dirac particles in general and of the proton state in particular. Isospin states of the four $\Delta(1232)$ baryons are discussed. The work explains the small contribution of quarks spin to the overall proton spin (the proton spin crisis). It is also proved that the addition of QCD's color is not required for a construction of an antisymmetric state of the $\Delta^{++}(1232)$ baryon.

1. Introduction

The isospin notion has been conceived by W. Heisenberg in 1932 (see [1], p. 106). It aims to construct a mathematical basis that represents the proton-neutron similarity with respect to the strong nuclear force. Both spin and isospin have the same SU(2) group structure. Thus, like spin multiplets of a quantum state, one combines corresponding states of nuclear isobars in an isospin multiplet. For example, the ground state of the ^{14}C, ^{14}O and the $J^\pi = 0^+$ excited state of ^{14}N are members of an isospin triplet. Obviously, one must remember that isospin is a useful *approximation* that neglects proton-neutron differences that are related to their mass and their electromagnetic interactions.

Later developments have shown that the proton-neutron similarity stems from the similarity between the u, d quarks. It follows that the usefulness of isospin symmetry extends to particle physics. For example, the three pions are members of an isospin triplet. Due to historical development, isospin notation takes different form in nuclear and particle physics. Here T and I denote isospin in nuclear and particle physics, respectively. In this work the symbol T is used, mainly because of the following reason. In the case of spin, the symbols J and j denote total and single particle angular momentum operators, respectively. Similarly, the symbols T and t denote the corresponding isospin operators. Thus, due to the same underlying SU(2) group, isospin relations can be readily borrowed from their corresponding spin counterparts. The operators T and t are used in the discussion presented in this work.

This work examines states of electrons and quarks. These particles have spin-1/2 and experimental data are consistent with their elementary pointlike property. Evidently, a theoretical analysis of an elementary pointlike particle is a much simpler task than that of a composite particle. The discussion begins with an examination of relevant properties of electronic states of atoms. The mathematical structure of the

SU(2) group is used later for a corresponding analysis of isospin states.

Two important conclusions are derived from this analysis. First, it is well known that quarks' spin carry only a small fraction of the entire proton's spin [2]. This experimental evidence, which is called the second EMC effect and also the proton spin crisis, is shown here to be an obvious result of the multi-configuration structure of states of more than one Dirac particle. Another result is that the anti-symmetric state of the $\Delta^{++}(1232)$ baryon is well understood and there is no need to introduce a new degree of freedom for its explanation. It means that the historical starting point of the QCD construction has no theoretical basis. (Below, the symbol Δ refers to this isospin quartet of baryons.)

Generally, in order to simplify notation, the specific value of normalization factor is omitted from the expressions. The second and the third sections analyze spin and isospin, respectively. The fourth section provides an explanation for the proton spin crisis. The fifth section explains the antisymmetric structure of the Δ^{++} baryon (without using color). The last section contains concluding remarks.

2. Spin States

A comprehensive discussion of angular momentum can be found in textbooks [3]. In this short work some elements of this theory are mentioned together with a brief explanation. This is done for the purpose of arriving rapidly at the main conclusions. A relativistic notation is used and for this reason the jj coupling [3] takes place.

Let us begin with a discussion of spin and spatial angular momentum. These quantities are dimensionless and this property indicates that they *may* be coupled. Now, the magnetic field depends on space and time. Moreover, the theory must be consistent with the experimental fact where both spatial angular momentum and spin of an electron have the same kind of magnetic field. Thus, it is *required* to construct a relativistically consistent coupling of these quantities. This is the theoretical basis for the well known usage of spin and spatial angular momentum coupling in the analysis of electronic states of atoms.

A motionless free electron is the simplest case and the spin-up electron state is (see [4], p. 10)

$$\psi(x^\mu) = Ce^{-imt} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1}$$

where m denotes the electron's mass.

A second example is the state of an electron bound to a hypothetical pointlike very massive positive charge. Here the electron is bound to a spherically symmetric charge Ze. The general form of a j^π hydrogen atom wave function is (see [5], pp. 926, 927)

$$\psi(r\theta\phi) = \begin{pmatrix} F\mathcal{Y}_{jlm} \\ G\mathcal{Y}_{jl'm} \end{pmatrix}, \tag{2}$$

where $\mathcal{Y}_{jlm}$ denotes the ordinary Y_{lm} coupled with a spin-1/2 to j, $j = l \pm 1/2$, $l' = l \pm 1$, F, G are radial functions and the parity is $(-1)^l$.

By the general laws of electrodynamics, the state must be an eigenfunction of angular momentum and parity. Furthermore, here we have a problem of *one* electron (the source at the origin is treated as an inert object) and indeed, its wave function (2) is an eigenfunction of both angular momentum and parity (see [5], p. 927).

The next problem is a set of n-electrons bound to an attractive positive charge at the origin. (This is a kind of an ideal atom where the source's volume and spin are ignored.) Obviously, the general laws of electrodynamics hold and the system is represented by an eigenfunction of the total angular momentum and parity J^π. Here a single electron is affected by a spherically symmetric attractive field *and* by the repulsive fields of the other electrons. Hence, a single electron does not move in a spherically symmetric field and it *cannot* be represented by a well defined single particle angular momentum and parity.

The general procedure used for solving this problem is to expand the overall state as a sum of configurations. In every configuration, the electrons' single particle angular momentum and parity are well defined. These angular momenta are coupled to the overall angular momentum J and the product of the single particle parity is the parity of the entire system. The role of configurations has already been recognized in the early decades of quantum physics [6]. An application of the first generation of electronic computers has provided a numerical proof of the vital role of finding the correct configuration interaction required for a description of even the simplest case of the ground state of the two electron He atom [7]. The result has proved that several configurations are required for a good description of this state and no configuration dominates the others. This issue plays a very important role in the interpretation of the state of the proton and of the Δ^{++}.

For example, let us write down the 0^+ ground state He_g of the Helium atom as a sum of configurations:

$$\begin{aligned}\psi(\mathrm{He}_g) \;=\; & f_0(r_1)f_0(r_2)\tfrac{1}{2}^+\tfrac{1}{2}^+ + f_1(r_1)f_1(r_2)\tfrac{1}{2}^-\tfrac{1}{2}^- + f_2(r_1)f_2(r_2)\tfrac{3}{2}^-\tfrac{3}{2}^- + \\ & f_3(r_1)f_3(r_2)\tfrac{3}{2}^+\tfrac{3}{2}^+ + f_4(r_1)f_4(r_2)\tfrac{5}{2}^+\tfrac{5}{2}^+ + ... \qquad (3)\end{aligned}$$

Here and below, $f_i(r)$, $g_i(r)$ and $h_i(r)$ denote the two-component Dirac radial wave function (multiplied be the corresponding coefficients). In order to couple to $J=0$ the two single particle j states must be equal and in order to make an even total parity both must have the same parity. These requirements make a severe restriction on acceptable configurations needed for a description of the ground state of the He atom.

Higher two-electron total angular momentum allows a larger number of acceptable configurations. For example, the $J^\pi = 1^-$ state of the He atom can be written as follows:

$$\begin{aligned}
\psi(\mathrm{He}_{1^-}) &= g_0(r_1)h_0(r_2)\tfrac{1}{2}^+\tfrac{1}{2}^- + g_1(r_1)h_1(r_2)\tfrac{1}{2}^+\tfrac{3}{2}^- + g_2(r_1)h_2(r_2)\tfrac{1}{2}^-\tfrac{3}{2}^+ + \\
&\quad g_3(r_1)h_3(r_2)\tfrac{3}{2}^-\tfrac{3}{2}^+ + g_4(r_1)h_4(r_2)\tfrac{3}{2}^-\tfrac{5}{2}^+ + g_5(r_1)h_5(r_2)\tfrac{3}{2}^+\tfrac{5}{2}^- + \\
&\quad g_6(r_1)h_6(r_2)\tfrac{5}{2}^+\tfrac{5}{2}^-... \qquad (4)
\end{aligned}$$

Using the same rules one can apply simple combinatorial calculations and find a larger number of acceptable configurations for a three or more electron atom. The main conclusion of this section is that, unlike a quite common belief, there are only *three* restrictions on configurations required for a good description of a J^π state of more than one Dirac particles:

1. Each configuration must have the total angular momentum J.

2. Each configuration must have the total parity π.

3. Following the Pauli exclusion principle, each configuration should not contain two or more identical single particle quantum states of the same Dirac particle.

These restrictions indicate that a state can be written as a sum of many configurations, each of which has a well defined single particles angular momentum and parity of its Dirac particles.

The mathematical basis of this procedure is as follows. Take the Hilbert subspace made of configurations that satisfy the three requirements mentioned above and calculate the Hamiltonian matrix. A diagonalization of this Hamiltonian yields eigenvalues and eigenstates. These eigenvalues and eigenstates are related to a set of physical states that have the given J^π. As pointed out above, calculations show that for a quite good approximation to a quantum state one needs a not very small number of configurations and that no configuration has a dominant weight. These conclusions will be used later in this work.

3. Isopin States

Spin and isospin are based on the same mathematical group called SU(2). Its three generators are denoted j_x, j_y, j_z. An equivalent basis is (see [1], pp. 357-363)

$$j_+ = j_x + ij_y, \quad j_- = j_x - ij_y, \quad j_z. \tag{5}$$

All the j operators mentioned above commute with the total j^2 operator. For this reason, if one of them operates on a member of a $(2J+1)$ multiplet of an SU(2) irreducible representation then the result belongs to this multiplet. The two $j_\pm$ operators are of a particular importance. Thus, let $\psi_{J,M}$ denote a member of such a multiplet and one finds

$$J_z J_- \psi_{J,M} = (M-1) J_- \psi_{J,M}. \tag{6}$$

This relation means that J_- casts $\psi_{J,M}$ into $\psi_{J,M-1}$

$$J_- \psi_{J,M} = \sqrt{J(J+1) - M(M-1)}\, \psi_{J,M-1}, \tag{7}$$

where the appropriate coefficient is written explicitly. Analogous relations hold for the J_+ operator.

Let us turn to isospin. The required operators are simply obtained by taking the mathematical structure of spin and replacing the total spin operator J and the single particle spin operator j by the corresponding isospin operators T, t. (Here, like in the spin case, M, m denote the eigenvalue of T_z, t_z, respectively.) The issue to be examined is the structure of the isospin multiplet of the four baryons:

$$\Delta^-,\ \Delta^0,\ \Delta^+,\ \Delta^{++}. \tag{8}$$

These $\Delta(1232)$ baryons have the lowest energy of the family of the Δ baryons [8]. The Δ^{++} baryon has three u quarks and $\psi_\Delta(uuu)$ denotes its state. Therefore, its isospin state is $T = 3/2$, $M = 3/2$ and the isospin component of the wave function is symmetric with respect to an exchange of any pair of quark.

Let us examine the operation of T_- on Δ^{++}.

$$T_-\psi_\Delta(uuu) = (t_{1-} + t_{2-} + t_{3-})\psi_\Delta(uuu) = \psi_\Delta(duu) + \psi_\Delta(udu) + \psi_\Delta(uud), \qquad (9)$$

where t_{i-} operates on the ith quark. This is the way how one obtains a yet unnormalized expression for the Δ^+ baryon from that of Δ^{++}. A successive application of T_- yields expressions for every member of the isospin quartet (8).

Now, the Δ^{++} state is symmetric with respect to its quark constituents and the same property holds for the operator $T_- = t_{1-} + t_{2-} + t_{3-}$. Hence, also the Δ^+ is symmetric with respect to its uud quark states. This argument proves that isospin space of *every* member of the baryonic quartet (8) is symmetric. The same result can be obtained from a different argument. Quarks are fermions and their overall state must be antisymmetric with respect to an interchange of any pair of quarks. Now, the isospin operators used above do not affect other coordinates of quarks. It means that for every members of the isospin quartet (8), the entire symmetry of the other coordinates remain antisymmetric and the isospin coordinate is symmetric.

The data confirms the similarity between members of an isospin multiplet. Thus, for example, the mass difference between the Δ^0 and Δ^{++} baryons is less than 3 MeV [8], whereas the mass difference between the Δ multiplet and the nucleons is about 300 MeV. This evidence shows the goodness of the isospin notion, where strong interactions dominate the state of members of an isospin multiplet and the effect of all other interactions can be regarded as a small perturbation.

4. The Proton Spin Crisis

The proton's $J^\pi = 1/2^+$ state is determined by three valence uud quarks. The non-negligible probability of the existence of an additional quark-antiquark pair (see [1], p. 282) indicates that it is a highly relativistic system. The discussion of section 2 holds for the spin-1/2 point-like quarks and the expansion in configurations is a useful approach. Here the three single particle j^π represent the uud quarks, in that order. Evidently, each configuration must satisfy the three requirement written few lines below (4). However, the Pauli exclusion principle of restriction 3 does not hold for the d quark. Thus, in analogy to (3) and (4) one expands the proton's wave function as a sum of terms of specific configurations. A truncated expression for this expansion is shown below:

$$\begin{aligned}\psi(uud) \;\; = \;\; & f_0(r_1)f_0(r_2)h_0(r_3)_{\frac{1}{2}^+\frac{1}{2}^+(0)\frac{1}{2}^+} + f_1(r_1)f_1(r_2)h_1(r_3)_{\frac{1}{2}^-\frac{1}{2}^-(0)\frac{1}{2}^+} + \\ & f_2(r_1)g_2(r_2)h_2(r_3)_{\frac{1}{2}^+\frac{1}{2}^+(1)\frac{1}{2}^+} + f_3(r_1)g_3(r_2)h_3(r_3)_{\frac{1}{2}^-\frac{1}{2}^-(1)\frac{1}{2}^+} + \\ & f_4(r_1)g_4(r_2)h_4(r_3)_{\frac{1}{2}^+\frac{1}{2}^-(0)\frac{1}{2}^-} + f_5(r_1)g_5(r_2)h_5(r_3)_{\frac{1}{2}^+\frac{1}{2}^-(1)\frac{1}{2}^-} + \\ & f_6(r_1)g_6(r_2)h_6(r_3)_{\frac{1}{2}^+\frac{3}{2}^+(1)\frac{1}{2}^+} + f_7(r_1)g_7(r_2)h_7(r_3)_{\frac{1}{2}^-\frac{3}{2}^+(1)\frac{1}{2}^-} + \\ & f_8(r_1)g_8(r_2)h_8(r_3)_{\frac{1}{2}^+\frac{1}{2}^+(1)\frac{3}{2}^+} + f_9(r_1)g_9(r_2)h_9(r_3)_{\frac{1}{2}^-\frac{1}{2}^-(1)\frac{3}{2}^+} + \\ & f_a(r_1)g_a(r_2)h_a(r_3)_{\frac{1}{2}^-\frac{3}{2}^-(1)\frac{1}{2}^+} + f_b(r_1)g_b(r_2)h_b(r_3)_{\frac{1}{2}^+\frac{3}{2}^-(1)\frac{1}{2}^-} + \\ & f_c(r_1)g_c(r_2)h_c(r_3)_{\frac{1}{2}^+\frac{1}{2}^-(1)\frac{3}{2}^-} + \ldots \end{aligned} \tag{10}$$

The symbols $_{0\ldots 9,a,b,c}$ are used for enumerating the terms. Here, like in (3) and (4), $f_i(r)$, $g_i(r)$ and $h_i(r)$ denote the Dirac two-component radial wave function of the uud quarks, respectively (multiplied be the corresponding coefficients). In each term, the number in parentheses indicates how the two angular momenta of the uu quarks are coupled. Below, J_{uu} denotes the value of this quantity.

The following remarks explain the form of these terms. An important issue is the coupling of the two uu quark that abide by the Pauli exclusion principle. For this reason, J_{uu} is given explicitly in each term. Another restriction stems from the rule of angular momentum addition. Thus, for every term, the following relation must hold in order to yield a total spin-1/2 for the proton: $J_{uu} = j_d \pm 1/2$. These rules explain the specific structure of each term of (10) which is described below.

In terms $_{0,1}$ the two spin-1/2 are coupled antisymmetrically to $J_{uu} = 0$ and the two radial function are the same. In terms $_{2,3}$ these spins are coupled symmetrically to $J_{uu} = 1$ and antisymmetry is obtained from the two orthogonal radial functions. In terms $_{4,5}$ the different orbitals of the uu quarks enable antisymmetrization. Thus, the two spin-1/2 functions are coupled to $J_{uu} = 0$ and $J_{uu} = 1$, respectively. The radial functions are not the same because of the different orbitals. In terms $_{6,7}$ the spins are coupled to $J_{uu} = 1$. In terms $_{8,9}$ we have a symmetric angular momentum coupling $J_{uu} = 1$ and the antisymmetry is obtained from the orthogonality of the radial function $f_i(r)$, $g_i(r)$. Terms $_{a,b}$ are analogous to terms $_{6,7}$, respectively. In term $_c$ the different uu orbitals enable antisymmetrization and they are coupled to $J_{uu} = 1$.

A comparison of the expansion of the He atom ground state (3) and that of the proton (10) shows the following points:

1. If the expansion is truncated after the same value of a single particle angular momentum then the number of terms in the proton's expansion is significantly larger.

2. This conclusion is strengthened by the fact that the proton has a non-negligible probability of an additional quark-antiquark pair. An inclusion of this pair increases the number of acceptable configurations.

3. Calculations show that the number of configurations required for the ground

state spin-0 of the two electron He atom is not very small and that there is no single configuration that dominates the state [7]. Now the proton is a spin-1/2 relativistic particle made of three valence quarks. Therefore, it is very reasonable to assume that its wave function takes a multiconfiguration form.

Using angular momentum algebra, one realizes that in most cases an individual quark does not take the proton's spin direction. This is seen on two levels. First, the upper and the lower parts of the quark single particle function have $l = j \pm 1/2$. Furthermore, the relativistic quark state indicates that the coefficients of the upper and the lower part of the Dirac four component function take a similar size. Hence, for the case where $j = l - 1/2$, the Clebsch-Gordan coefficients [3] used for coupling the spatial angular momentum and the spin indicate that the spin of either the upper or the lower Dirac spinor has no definite direction and that the coefficient of the spin down is not smaller than that of the spin up (see [3], p. 519).

Let us turn to the coupling of the quark spins. The 3-quark terms can be divided into two sets having $j_{uu} = 0$ and $j_{uu} > 0$, respectively. For $j_{uu} = 0$ one finds that the single particle $j_d = 1/2$ and this spin is partially parallel to the proton's spin. For cases where $j_{uu} > 0$, the proton's quark spins are coupled in a form where they take both up and down direction so that they practically cancel each other. The additional quark-antiquark pair increases spin direction mixture. It can be concluded that the quark spin contribute a not very large portion of the proton spin and the rest comes from the quark spatial motion. This conclusion is supported by experiment [9].

5. The State of the Δ^{++} Baryon

In textbooks it is argued that without QCD, the state of the Δ^{++} baryon demonstrates a fiasco of the Fermi-Dirac statistics (see [10], p. 5). The argument is based on the claim that the Δ^{++} takes the lowest energy state of the Δ baryons [11] and therefore, its spatial wave function consists of three single particle symmetric s-waves of each of its three uuu quarks. Now the $J^\pi = 3/2^+$ state of the Δ baryons shows that also their spin is symmetric. It means that the Δ^{++} is regarded to have space, spin and isospin symmetric components of its wave function. As stated above, textbooks claim that this outcome contradicts the Fermi-Dirac statistics. However, using the physical issues discussed in this work and the following energy level diagram of the nucleon and the Δ baryons, it is proved that this textbook argument is incorrect.

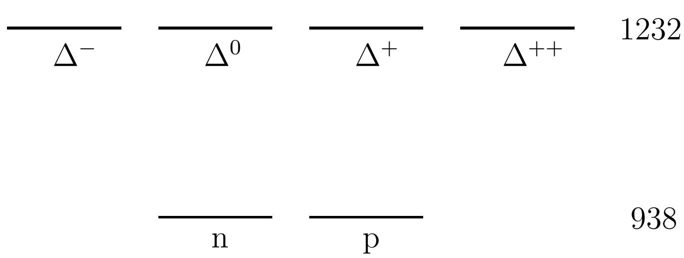

Fig. 1: Energy levels of the nucleon and the Δ isospin multiplets (MeV).

- As explained in section 3, all members of an isospin multiplet have the same symmetry. Hence, if there is a problem with the Fermi-Dirac statistics of the Δ^{++} then the same problem exists with Δ^+ and Δ^0. It follows that if the above mentioned textbook argument is correct then it is certainly incomplete.

- The data described in fig. 1 shows that Δ^+ is an excited state of the proton. Hence, its larger mass is completely understood. Thus, there is no problem with the Fermi-Dirac statistics of the Δ^+ baryon. Analogous relations hold for the neutron and the Δ^0 baryons. Using the identical statistical state of the four Δ baryons (8), one realizes that there is no problem with the Fermi-Dirac statistics of the Δ^{++} and the Δ^- baryons.

- The multi-configuration structure of a bound system of Dirac particles is known for about 50 years [7]. In particular, the multi-configurations structure of all baryons (like in (10)) proves that, contrary to the above mentioned textbook argument (see [10], p. 5), the single particle spatial wave functions of the three u quarks of the Δ^{++} baryon *are not a pure s-wave.*

6. Conclusions

This work uses the Wigner-Racah mathematical structure and proves two very important points. It explains the small contribution of quark's spin to the overall proton spin. Therefore, it eliminates the basis for the proton spin crisis. It also proves that everything is OK with the Fermi-Dirac statistics of the Δ^{++} baryon. It follows that there is no need to introduce the QCD's color degree of freedom in order to build an antisymmetric wave function for this baryon.

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