

# Time domain radiation and absorption by subwavelength sources

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## Abstract

Radiation by elementary sources is a basic problem in wave physics. Although computations of radiated fields in the time domain are textbook examples, the emitted energy flux is usually computed on average, and terms that do not contribute to the time-averaged flux are disregarded. We show that the time-domain energy flux emitted by classical subwavelength sources exhibits remarkable features. In particular, a subtle trade-off between source emission and absorption underlies the mechanism of radiation. We discuss some implications for subwavelength focusing and imaging.

PACS numbers: 03.50.-z,42.25.-p,43.20.+g

arXiv:1107.4237v2 [physics.class-ph] 22 Jul 2011

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Any textbook on wave physics or field theory contains a chapter on radiation by elementary sources [1]. Although it is easy to find expressions of radiated fields in the time domain, when one comes to energy considerations, the computations are usually performed on time average [2–5]. Terms that do not contribute to the time-averaged energy flux are disregarded. In this Letter, we study the radiation of subwavelength sources from an energy point of view in the time-domain. We derive the expression of the energy flux for both electromagnetic and acoustic radiation. We show that there is a subtle trade-off between emission of energy and subsequent reabsorption by the source, the difference between emission and reabsorption giving the amount of energy that is irreversibly radiated to the far field. This result suggests a novel point of view on near-field radiation, that applies to any classical subwavelength source. We discuss some implications for subwavelength focusing using time reversal with active sources [6, 7], as well as potential impact on near-field imaging in optics [8] or acoustics [9].

The propagation of electromagnetic waves generated by a spatially localized source in an otherwise homogeneous medium is described by the following equation [2, 3]

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}(\mathbf{r}, t) + \nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) = \mathbf{S}_{em}(\mathbf{r}, t) \quad (1)$$

where  $\mathbf{E}(\mathbf{r}, t)$  is the electric field at point  $\mathbf{r}$  and time  $t$ , and  $c$  is the speed of light in the medium. The source term  $\mathbf{S}_{em}(\mathbf{r}, t)$  is often written in the form  $\mathbf{S}_{em}(\mathbf{r}, t) = -\mu_0 (\partial/\partial t)\mathbf{j}(\mathbf{r}, t)$ , where  $\mathbf{j}(\mathbf{r}, t)$  is the electric current density and  $\mu_0$  the vacuum magnetic permeability. The electromagnetic energy current is given by the Poynting vector  $\mathbf{\Pi}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$ , where  $\mathbf{E}(\mathbf{r}, t)$  is the retarded solution of Eq. (1) and  $\mathbf{H}(\mathbf{r}, t)$  the associated magnetic field. The energy flux  $\phi_{em}(R, t)$  across a sphere with radius  $R$  centered at the origin is  $\phi_{em}(R, t) = \int_{\text{sphere}} \mathbf{\Pi}(\mathbf{r}, t) \cdot \mathbf{u} d^2r$ , where  $\mathbf{u} = \mathbf{r}/|\mathbf{r}|$ .

For acoustic waves in the linear regime, the acoustic pressure field  $p(\mathbf{r}, t)$  generated by a spatially localized source in a homogeneous medium obeys [4, 5]:

$$\frac{1}{c_s^2} \frac{\partial^2 p}{\partial t^2}(\mathbf{r}, t) - \nabla^2 p(\mathbf{r}, t) = S_{ac}(\mathbf{r}, t) \quad (2)$$

where  $c_s$  is the acoustic velocity in the medium and  $S_{ac}(\mathbf{r}, t)$  the source term. The acoustic energy current is  $\mathbf{q}(\mathbf{r}, t) = p(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t)$ ,  $p(\mathbf{r}, t)$  being the retarded acoustic pressure field solution of Eq. (2) and  $\mathbf{v}(\mathbf{r}, t)$  the associated acoustic velocity field. The energy flux follows from  $\phi_{ac}(R, t) = \int_{\text{sphere}} \mathbf{q}(\mathbf{r}, t) \cdot \mathbf{u} d^2r$ .

In this Letter we study the radiation produced by sources of size much smaller than the characteristic length of the wavefield, that will be denoted by “subwavelength sources”. In the case of electromagnetic waves, we use a point electric dipole model, with dipole moment  $\mathbf{p}(t) = f(t)\mathbf{p}_0$ ,  $f(t)$  being the dimensionless time-domain amplitude and  $\mathbf{p}_0$  a time-independent vector accounting for the source polarization. This model describes, e.g., a dipole moment  $\mathbf{p}(t) = q_e\mathbf{L}(t)$  corresponding to an oscillating charge  $q_e$  with oscillation amplitude  $\mathbf{L}(t)$  much smaller than all other relevant characteristic lengths [3]. For a dipole centered at  $\mathbf{r} = 0$ , the electromagnetic source term reads:

$$\mathbf{S}_{em}(\mathbf{r}, t) = -\mu_0 \frac{d^2\mathbf{p}(t)}{dt^2} \delta(\mathbf{r}) \quad (3)$$

where  $\delta(\mathbf{r})$  is the three-dimensional Dirac delta function. In the case of acoustic waves, we use a point mass source model describing a radially oscillating sphere with radius  $a(t) = a_0 + \xi(t)$ , in the limit of vanishingly small radius [5]. For a source centered at  $\mathbf{r} = 0$ , the acoustic source term reads:

$$S_{ac}(\mathbf{r}, t) = \rho_0 s_0 \frac{d^2\xi(t)}{dt^2} \delta(\mathbf{r}) \quad (4)$$

where  $\rho_0$  is the mass density of the unperturbed homogeneous medium and  $s_0 = 4\pi a_0^2$ . For the sake of formal similarity with the electromagnetic case, we will write  $\xi(t) = f(t)\xi_0$  with  $\xi_0$  a time-independent length driving the acoustic source strength.

The time-domain solutions of Eqs. (1) and (2) with the source terms given by Eqs. (3) and (4) can be found in textbooks on electromagnetic and acoustic waves propagation [2–5]. From the field expressions, the energy flux across a sphere with radius  $R$  can be deduced after tedious but straightforward algebra. In the case of electromagnetic waves, one obtains:

$$\begin{aligned} \phi_{em}(R, t) = & \frac{\mu_0 \mathbf{p}_0^2}{6\pi c} \left[ \frac{1}{2} \left( \frac{c}{R} \right)^3 \frac{df^2}{dt} \Big|_{\tau} + \frac{1}{2} \left( \frac{c}{R} \right)^2 \frac{d^2f^2}{dt^2} \Big|_{\tau} \right. \\ & \left. + \left( \frac{c}{R} \right) \frac{d}{dt} \left( \frac{df}{dt} \Big|_{\tau} \right)^2 + \left( \frac{d^2f}{dt^2} \Big|_{\tau} \right)^2 \right]. \end{aligned} \quad (5)$$

For acoustic waves, the explicit calculation of the energy flux leads to:

$$\phi_{ac}(R, t) = \frac{\rho_0 s_0^2 \xi_0^2}{4\pi c_s} \left[ \frac{1}{2} \left( \frac{c_s}{R} \right) \frac{d}{dt} \left( \frac{df}{dt} \Big|_{\tau} \right)^2 + \left( \frac{d^2f}{dt^2} \Big|_{\tau} \right)^2 \right]. \quad (6)$$

In Eqs. (5) and (6), derivatives of the source amplitude  $f(t)$  have to be taken at retarded time  $\tau = t - R/c$  and  $\tau = t - R/c_s$ , respectively. Although their derivation is a rather

simple exercise, we will see that these expressions bring to light fundamental aspects of the mechanism of radiation by subwavelength sources that, to our knowledge, have not been discussed so far.

From a qualitative point of view, the structure of Eqs. (5) and (6) deserves several comments. The far-field limit, obtained for  $R \rightarrow \infty$ , leads in both cases to an energy flux proportional to the square of the second derivative of the source amplitude, in agreement with a well-established result in classical wave theory [1]. For a monochromatic source oscillating at a frequency  $\omega$ , with  $f(t) = \sin(\omega t)$ , this far-field term is the only one that survives a time-averaging of Eqs. (5) and (6). The far-field behavior is extensively discussed in textbooks, both for monochromatic and pulse sources. Nevertheless the time-domain electromagnetic and acoustic energy fluxes contain additional near-field terms whose amplitude depend on the distance  $R$  to the source. The first near-field term scales as  $R^{-1}$  and is identical in Eqs. (5) and (6), except for a factor of two, while additional terms scaling as  $R^{-2}$  and  $R^{-3}$  appear only in the expression for the electromagnetic case. These near-field contributions exhibit peculiar properties that induce unexpected behaviors of the time-domain energy flux. A first remarkable result is that the time-dependent amplitudes of the near-field terms in Eqs. (5) and (6) read as first-order derivatives of functions that are positive (squares) and that recover their initial values after a finite time interval (the pulse duration, or the period for monochromatic excitation). As a result, these amplitudes necessarily change sign during their time evolution, meaning that the near-field terms lead alternatively to outgoing or incoming contributions to the energy flux. Conversely, the far-field term only contributes to an outgoing energy flux.

In order to study the behavior of the time-domain energy flux on a quantitative basis, we can specify the source amplitude function  $f(t)$ . For monochromatic excitation at frequency  $\omega$ , the temporal profile of the sources is simply  $f(t) = \sin(\omega t)$ . For pulse excitation,  $f(t)$  is a bounded function of time of finite duration. Defining  $t = 0$  as the onset of the source excitation, and  $T$  as the pulse duration, a convenient pulse function is  $f(t) = \exp[2T^2/(t(t-T))]$  for  $t \in ]0, T[$  and  $f(t) = 0$  otherwise. Such a function is regular enough so that all the derivatives involved in Eqs. (5) and (6) are finite. The temporal shape of the source amplitude  $f(t)$  and the shape of the associated far field amplitude are shown in Fig. 1. With the average period in the far-field being on the order of  $T/2$  (right panel in Fig. 1), the condition of subwavelength sources is  $|\mathbf{L}(t)| \ll cT$  (electromagnetic case) or  $a(t) \ll c_s T$

(acoustic case). In the following, for sake of brevity and since the velocities play the same role in the electromagnetic and acoustic cases, both  $c$  and  $c_s$  are referred to as  $c$ .

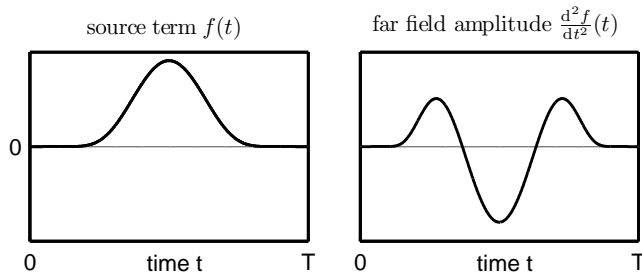


FIG. 1: Time evolution of the source amplitude  $f(t)$  (left) and of its second derivative  $d^2f(t)/dt^2$  (right) that represents the time-dependence of the far-field amplitude for both electromagnetic and acoustic waves.

The knowledge of  $f(t)$  and its derivatives allows us to plot the time evolution of  $\phi_{em}(R, t)$  and  $\phi_{ac}(R, t)$  for different observation distances, covering the near-field, the intermediate and the far-field regimes. We show in Fig. 2 the time evolution of the energy flux in the electromagnetic (top) and acoustic (bottom) situations, and for four different distances. In the far field ( $R \gg cT$ ), the energy flux is always positive and describes the radiated energy flowing irreversibly from the source. In the near field ( $R \ll cT$ ), a completely different behavior is observed. The energy flux oscillates, and takes negative values on some time intervals. This means that part of the energy that has flowed outside the sphere of radius  $R$  at a given time flows back into the sphere at subsequent times. In other words, part of the energy that has been stored in the wavefield is then reabsorbed by the source. This shows that in order to radiate a given amount of energy, a subwavelength source needs to store in the field a larger amount of energy at some time, part of it being reabsorbed afterwards. This is a non-intuitive feature of time-domain radiation by subwavelength sources.

Another peculiar behavior is that the energy flux exhibits a slight sign inversion even at times  $t > T$ , i.e., after the source has become inactive (see the insets in Fig. 2). This sign inversion does not correspond to reabsorption in the source, but to a small part of the energy flowing back and forth through the sphere of radius  $R$ . This “anomaly” becomes insignificant (although non strictly zero) in the far field since it is due to the contribution of terms in the energy flux that decay as  $R^{-1}$  or faster.

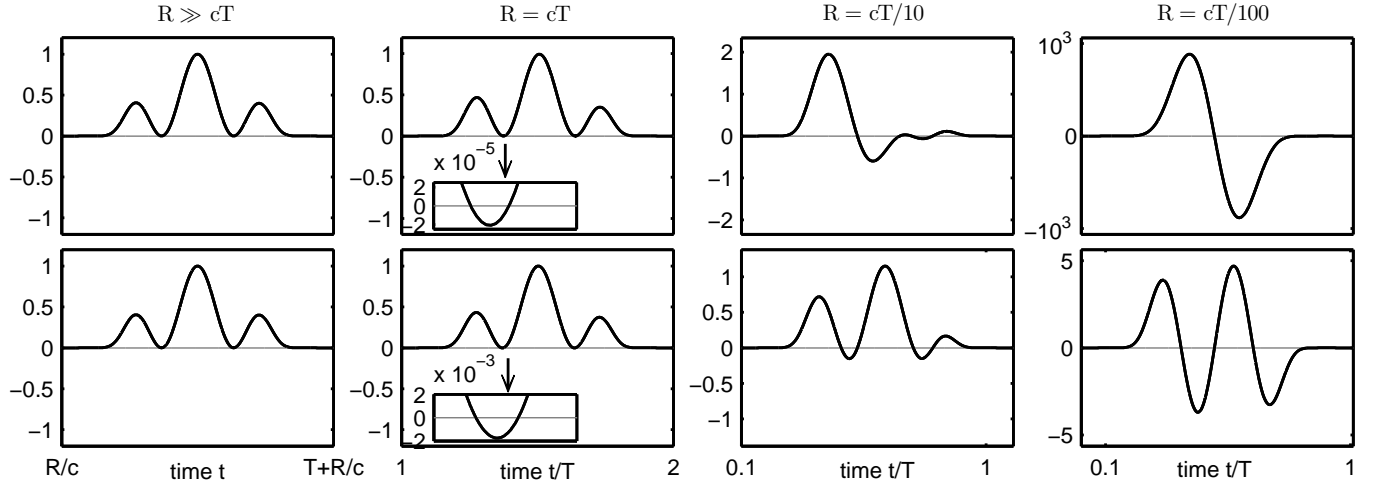


FIG. 2: Time evolution of the electromagnetic and acoustic energy flux  $\phi_{em}(R, t)$  and  $\phi_{ac}(R, t)$  for four different distance regimes. Far-field regime  $R \gg cT$ , limit of the source free regime  $R = cT$ , near-field regimes  $R < cT$  and  $R \ll cT$ . For  $R = cT$ , the insets show the sign inversion of the energy flux.

To get deeper insight into the energy exchange between the source and the field, we introduce  $U_x(R, t)$  defined as the energy stored *outside* the sphere with radius  $R$  at time  $t$  in the electromagnetic or acoustic field (the subscript “ $x$ ” stands for  $em$  or  $ac$ ). It reads:

$$U_x(R, t) = \int_0^t \phi_x(R, t') dt' . \quad (7)$$

The time evolution of  $U_{em}(R, t)$  is shown in Fig. 3 for the same distance regimes as in Fig. 2. Although not shown for the sake of brevity, the same behavior is observed for acoustic waves. As expected from the behavior of the energy flux, we see that  $U_{em}(R, t)$  is not a monotonic function of time  $t$  except in the far field. The increase and decrease of the field energy is chiefly due to the mechanism of emission followed by reabsorption by the source that we discussed previously. In addition, the weak anomaly that is also observed after the source has been switched off is also visible in the time evolution of the field energy (inset in Fig. 3).

The non-monotonic behavior of the time evolution of the energy stored in the field can be characterized by splitting  $U_x(R, t)$  into  $U_x(R, t) = U_x^\infty + \Delta U_x(R, t)$ . The first term  $U_x^\infty = \int_0^\infty \phi_x(R, t) dt$  corresponds to the overall time-averaged energy eventually radiated irreversibly through the sphere of radius  $R$  to the far field, and is independent on  $R$ . The second term describes the time variations of the energy stored in the field beyond the distance

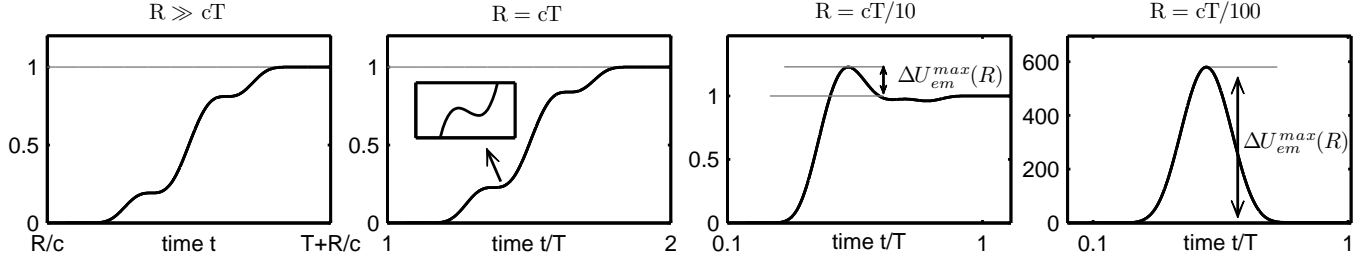


FIG. 3: Time evolution of the electromagnetic energy  $U_{em}(R, t)$  stored outside the sphere of radius  $R$  at time  $t$ , for the same distance regimes as in Fig. 2. The inset shows the dip due to the sign inversion of the energy flux.

$R$ , and either increases or decreases  $U_x(R, t)$  with respect to the asymptotic value  $U_x^\infty$ . This dynamic behavior is fully described by the curves in Fig. 3. It is also interesting to have a look at the distance dependence in the near field of the maximum value of the energy stored in the field  $\Delta U_x^{max}(R) = \max\{\Delta U_x(R, t)\}$ . Conserving only the dominant terms as  $R \rightarrow 0$  in Eqs. (5) and (6), it is easy to show that  $\Delta U_{em}^{max}(R) \sim R^{-3}$  and  $\Delta U_{ac}^{max}(R) \sim R^{-1}$ . Therefore for a quasi point source, the energy transiently stored in the field becomes arbitrarily large at short distance. Although it is known that on average, near-field terms correspond to non-radiating energy [4, 8], our work shows that this non radiated energy is dynamically exchanged between the field and the source. This subtle dynamic process is hidden in the first-place when computation are restricted to time-averaged values.

The peculiar dynamics of the energy exchange between a subwavelength classical source and the radiated field has certainly many implications, some of which are underlined here. There are direct consequences in the context of wave focusing below the diffraction limit. One strategy to reach this goal is to use a time reversal sequence, by emitting with a subwavelength source, and then reversing both the far field recorded on a closed cavity and the source. This ensures a full time reversal of the field, including near-field components [6, 10]. Time reversal below the diffraction limit in a homogeneous medium with this approach has been demonstrated experimentally in acoustics, using an active time-reversed source (denoted as a sink) placed at the focal point [7]. Our work shows that a subwavelength source radiates in the time domain by a process that, from the flow of energy point of view, involves both emission and absorption. In the time reversed case, the time domain evolution

of the energy in the field is given by the curves in Fig. 3 read backwards. Therefore, in a time reversed experiment such as in Ref. [7], the subwavelength sink actually absorbs the focused wave by a process that necessarily involves at some stage emission of energy into the field. For a sink of vanishingly small size, the transient energy that has to be stored in the field becomes arbitrarily large. Another interesting aspect is that the difference between a source and a sink really makes sense when one considers the overall energy balance, obtained after time integration in a pulsed experiment or time averaging in a monochromatic process. Other strategies based on scattering in structured media have been put forward in order to focus waves on subwavelength regions using time reversal [11, 12] or wavefront shaping [13]. Although these focusing techniques do not rely on the use of an active source or sink, they take advantage of near-field radiation by secondary sources (scattering), in which the time-domain energy exchange put forward in this Letter might also be relevant. The concept of subwavelength source is also at the root of many superresolved imaging techniques based on near-field interactions in optics [8, 14], in acoustics [9], or based on the location of subwavelength emitters [15, 16]. The result in this Letter might invite to revise the usual point of view on the resolution limit, by thinking the radiation by subwavelength primary or secondary sources as a dynamic process involving a substantial transient energy storage in the near field. Finally, the concept of coherent perfect absorber (perfect sink) has been put forward recently and might also take advantage of a full analysis in the time domain [17].

In summary, we have derived the expression of the time-domain energy flux radiated by a subwavelength source for electromagnetic and acoustic radiation. We have shown that the radiation of energy is a subtle dynamic process that involves both emission and absorption by the source. We have discussed implications for subwavelength focusing and imaging. Since the results holds for both electromagnetic and acoustic waves, we believe that they underly a universal process of radiation by subwavelength sources. In the case of electromagnetic waves emitted by a single classical dipole emitter, a giant transient storage of electromagnetic energy is necessary in order to radiate a (much smaller part) in the far field. It would be interesting to clarify the way quantum theory handles this point in the computation of spontaneous emission by a single atom.

We acknowledge A.C. Boccara, J.J. Sáenz and A. Sentenac for helpful discussions. This



work was supported by the Agence Nationale de la Recherche (grant JCJC07-195015).

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