Controllability of Complex Networks with Nonlinear Dynamics

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The controllability of large linear network systems has been addressed recently [Liu et al. Nature (London), 473, 167 (2011)]. We investigate the controllability of complex-network systems with nonlinear dynamics by introducing and exploiting the concept of "local effective network" (LEN). We find that the minimum number of driver nodes to achieve full control of the system is determined by the structural properties of the LENs. Strikingly, nonlinear dynamics can significantly enhance the network controllability as compared with linear dynamics. Interestingly, for one-dimensional nonlinear nodal dynamics, any bidirectional network system can be fully controlled by a single driver node, regardless of the network topology. Our results imply that real-world networks may be more controllable than predicted for linear network systems, due to the ubiquity of nonlinear dynamics in nature.

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The ability to control complex networks has been deemed as the ultimate proof of understanding of these systems [1]. In the past decade, various issues have been explored, such as the control of synchronized networks [2–4] and communication networks [5], mitigation and suppression of catastrophic dynamics, e.g., cascading failures, on complex networks [6–8]. In recent years a general framework for determining the network controllability based on control and graph theories has been investigated [1, 9–11], leading to quantitative understanding of the effect of the network structure on its controllability. For example, a fairly recent work [1] revealed that the ability to control a complex network toward any desired states, as measured by the minimum number of driver nodes, is determined by the maximum matching in the network. The framework, however, was established for weighted and directed networks but for *linear*, time-invariant dynamics. In the real world nonlinear dynamics are ubiquitous. It is thus of significant interest to investigate the controllability of complex network systems with nonlinear dynamics.

In this Letter, we address the issue of controllability of nonlinear network systems by proposing and exploiting the concept of "local effective network" (LEN) associated with locally desired state, which is a mapping of the linearized dynamical system around any such state. We find that the structural properties of LEN determine the minimum number of driver nodes to fully control the system. A striking result is that nonlinear dynamics can considerably enhance the network controllability as compared with linear dynamics. For example, in the case of one-dimensional nonlinear nodal dynamics, we can show that any bidirectionally interacting network system can be fully controlled by a single driver node, regardless of the network topology. We note that in the framework of linear dynamics [1], there are limitation on the network controllability, depending on the particular network structure. Our results imply that nonlinear dynamics can make the network significantly more controllable. Indeed, many network systems in nature appear well under control, and we speculate that nonlinear dynamics may play an important role in achieving the ominous system stabilities across many different scales.

To be concrete but without loss of generality, we consider coupled nonlinear oscillator networks described by [12] $\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - \sum_{j=1}^N L_{ij}\mathbf{H}(\mathbf{x}_j)$, where $\mathbf{x}_i$ denotes the d-dimensional state variable of node i, $\mathbf{H}:\mathbb{R}^d \to \mathbb{R}^d$ denotes the coupling function of oscillators, and L_{ij} are the elements of the Laplacian matrix $\mathbf{L}$, which satisfy $L_{ij} = -A_{ij}$ if there is a direct link from j to i with weight A_{ij} (otherwise 0) for $i \neq j$ and $L_{ii} = -\sum_{j=1,j\neq i}^N L_{ij}$. If there are N nodes in the network, the phase-space dimension of the whole system is $Nd \gg 1$. Now suppose in the high-dimensional phase space, the system starts from an initial state and one wishes to bring the system to a final desired state via control. The key to achieve this goal is a "stepwise control" method via linearizing the nonlinear system about finite number of local desired states. Application of control perturbations thus generates a controlled path between the initial and final states. Along such a path, in general a local desired state is sufficiently close to the current state. Let $\mathbf{x}^l = [\mathbf{x}_1^l, \mathbf{x}_2^l, \cdots, \mathbf{x}_N^l]^T$ be such a state. The dynamical system can then be linearized about $\mathbf{x}^l$, as shown in Fig. 1. Letting ξ_i be a small control perturbation, we can write $\mathbf{x}_i = \mathbf{x}_i^l + \xi_i$. The variational equations about $\mathbf{x}_i^l$ are

$$\dot{\xi} = [D\mathbf{F}(\mathbf{x}^l) - \mathbf{L} \otimes D\mathbf{H}(\mathbf{x}^l)] \cdot \xi, \tag{1}$$

where $\xi = [\xi_1, \xi_2, \ldots, \xi_N]^T$ denotes the deviation vector, $D\mathbf{F}(\mathbf{x}^l) = \mathrm{diag}[D\mathbf{F}_1(\mathbf{x}_1^l), D\mathbf{F}_2(\mathbf{x}_2^l), \cdots, D\mathbf{F}_N(\mathbf{x}_N^l)]$ ($D\mathbf{F}_i$'s are $d \times d$ Jacobian matrices of $\mathbf{F}_i$), $\otimes$ denotes direct product, and $D\mathbf{H}$ is the Jacobian matrix of the coupling function $\mathbf{H}$.

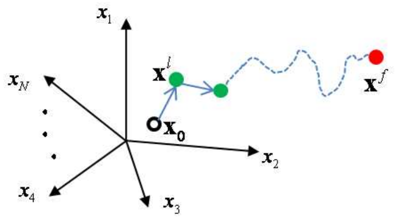

FIG. 1: (Color online.) In the high-dimensional phase space, $\mathbf{x}_0$ is the initial state, $\mathbf{x}^l$ is a local desired state and $\mathbf{x}^f$ is the final desired state. If $\mathbf{x}^l$ is sufficiently close to $\mathbf{x}_0$, the system at $\mathbf{x}_0$ can be linearized about $\mathbf{x}^l$. The controlled path toward the final desired state $\mathbf{x}^f$ is composed of a finite number of controlled segments about the local states.

To better explain our control framework, we explore the system's controllability for one-dimensional nodal and coupling dynamics. For linear coupling function $\mathbf{H}(\mathbf{x})$, the corresponding Jacobian matrix $D\mathbf{H}$ is a constant matrix, due to the fact that $[\mathbf{H}(\mathbf{x})]_k = \delta_{mk}\mathbf{x}_n$ if nth component of $\mathbf{x}$ influences the evolution of mth component through the coupling, where $\delta_{mk} = 1$ if $m = k$ and zero otherwise, for $m, n = 1, \cdots, d$. If $d = 1$, we have $D\mathbf{H} = 1$. As a result, the linearized equation can be simplified to $\dot{\xi} = [D\mathbf{F}(\mathbf{x}^l) - \mathbf{L}] \cdot \xi$, where $D\mathbf{F}(\mathbf{x}^l)$ is a constant because the local controlled state $\mathbf{x}^l$ is a time-invariant state. The dynamical system described by the corresponding variational equations is time invariant and linear as well. Denoting the matrix $[D\mathbf{F}(\mathbf{x}^l) \ - \mathbf{L}]$ by $\mathbf{G}$, we obtain the following control system:

$$\dot{\xi}(t) = \mathbf{G}\xi(t) + \mathbf{B}\mathbf{u}(t), \quad (2)$$

where $\mathbf{B}$ is the $N \times M$ input matrix and M is the number of driver nodes. The time-dependent input vector $\mathbf{u}(t)$ is employed to control the system. The controllability of the system is characterized by the minimum number of driver nodes (there are input signals on them), say N_D, for fully controlled system dynamics.

Kalman's controllability rank condition [13, 14] stipulates that the system described by Eq. (2) can be controlled from any initial state to any desired state in finite time, if and only if the $N \times NM$ controllability matrix $\mathbf{C}$ has full rank, i.e.,

$$\text{rank}(\mathbf{C}) \equiv \text{rank}\big[\mathbf{B}, \mathbf{G}\mathbf{B}, \mathbf{G}^2\mathbf{B}, \cdots, \mathbf{G}^{N-1}\mathbf{B}\big] = N, \quad (3)$$

The structural controllability can then be defined by identifying the minimum number N_D of driver nodes to satisfy the full rank condition (3). Although condition (3) is necessary and sufficient for achieving full control of the network system, it is not feasible to apply to large and weighted complex networks due to lack of exact link weights and high computational complexity. Liu et al. [1] proposed an efficient scheme to assess the controllability based solely on the network structure. In particular, they proved that the minimum number N_D of driver nodes is one if there is a perfect matching in the network. Otherwise N_D equals the number of unmatched nodes who should act as driver nodes: $N_D = \max\{N - N_M, 1\}$, where N_M is the maximum matching, which for directed network is defined as the maximum set of links that do not share starting or ending nodes. A node is matched if it is the ending node of a link that belongs to the set of maximum matching. The definition of N_D is equivalent to the statement that a network can be fully controlled if and only if each unmatched node is directly controlled and there are directed paths from the driver nodes to all matched nodes [15, 16]. A star graph as an example is shown in Fig. 2(a), where $N - 2$ leaf nodes must be controlled to satisfy the full rank condition, because the maximum number of matched nodes is 2.

It is noteworthy that N_D has been used for quantifying canonical linear network systems, where the matrix $\mathbf{G}$ is the adjacency matrix characterizing all connections together with their weights. In the linearized system (2), $\mathbf{G}$ is no longer the adjacency matrix, but it is the Laplacian matrix with $D\mathbf{F}$ added to the diagonal elements. Under nonlinear dynamics, the key issue is whether the maximum matching method can still be applied to the linearized system. The answer is affirmative, which we can prove by defining an LEN of the matrix G, as shown in Fig. 2. The diagonal elements of $\mathbf{G}$ can be treated as self-loops, each starting from a node to itself. The weight of the loop is the value of $D\mathbf{F}_i(\mathbf{x}_i^l) - L_{ii}$, which can be either positive or negative. Since condition (3) is sufficient and necessary for any $\mathbf{G}$ and $\mathbf{B}$, and the maximum matching method based on Lin's theory [15] has been proved to be valid for any weighted networks (positive or negative weights), LEN can be employed to identify N_D of the linearized system. In this framework, determining the controllability of the nonlinear system is equivalent to studying the maximum matching and directed spanning tree associated with LEN of matrix $\mathbf{G}$.

For clarity of analysis, we consider a bidirectional network system described by Eq. (2) but without the term $D\mathbf{F}(\mathbf{x}^l)$. In this case, LEN is defined exclusively by the Laplacian matrix, and it can be proved straightforwardly that the network is perfectly matched and no inaccessible nodes so that $N_D = 1$. This is because, in this case, perfect matching can be realized by choosing all self-loops such that they do not share starting or ending nodes but they cover all nodes in the network (no dilation induced by self-loops according to Lin's structural controllability theorem [15]). Moreover, since we can always find a directed spanning tree to cover all nodes starting from an arbitrary node in a bidirectional network, there exist directed paths from an arbitrary node to all matched nodes, as shown in Fig. 2(b). Therefore, the network system is structurally controllable by a single input at an arbitrary node. However, the system is not strongly structurally controllable, because of the condition $\text{rank}(\mathbf{C}) < N$ for uniform link weights as a consequence of the correlation of elements in $\mathbf{L}$. Here, strongly structural controllability is defined by the condition $\text{rank}(\mathbf{C}) = N$, regardless of any combinations of link weights. We note, however, that any correlation in $\mathbf{L}$ can be broken by imposing small difference among the link weights, making the controllability matrix full rank.

For the linearized system (2) from network with nonlin-

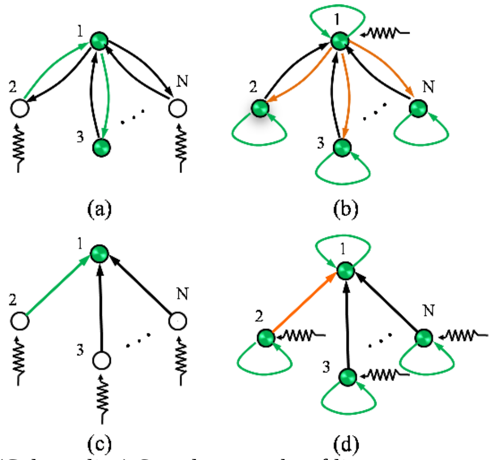

FIG. 2: (Color online) Sample networks of linear systems and LENs of nonlinear systems under minimum number of outside controllers for full control: (a) bidirectional star graph, (b) bidirectional star graph with self-loops, (c) inversely directed star graph, and (d) inversely directed star graph with self-loops. Directed links belong to maximum matching and the matched nodes are marked by green. Directed paths from the driver node to all other nodes are marked by orange. Minimum number of outside controllers are applied in each graph to fully control the system. The configuration of picking the driver nodes and maximum matching are not unique but the minimum number of controllers is.

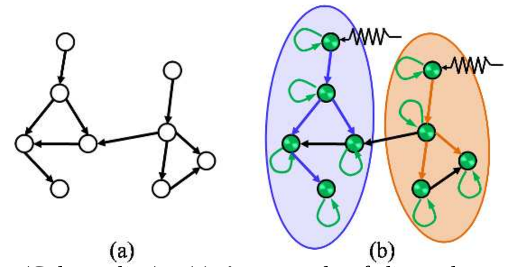

FIG. 3: (Color online). (a) An example of directed network and (b) LEN of the network in (a). In (b), due to the existence of self-loops, all nodes are matched, marked by green color. There are at least two non-overlapping spanning trees, marked by blue and orange colors, respectively. Two driver nodes are needed to fully control the network. Note that the spanning trees are not exclusive, but N_D is fixed for different spanning trees.

ear nodal dynamics, since the local controlled state $\mathbf{x}_i^l$ can be arbitrarily chosen as well as the possible different nodal dynamics, $D\mathbf{F}_i(\mathbf{x}_i^l)$ for distinct nodes are independent of each other so that they can remove the correlation between the diagonal and non-diagonal elements in $\mathbf{L}$, making the system strongly structurally controllable by controlling only a single node. The controllability in terms of maximum matching can be determined by taking a star graph with 3 nodes as an example, as shown in Fig. 2(b). For this example, the linearized system is described by

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \dot{\xi}_3(t) \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & 0 \\ g_{31} & 0 & g_{33} \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} u(t), \quad (4)$$

and the controllability matrix by Eq. (3) is

$$\mathbf{C} = b_1 \begin{bmatrix} 1 & g_{11} & g_{11}^2 + g_{12}g_{21} + g_{13}g_{31} \\ 0 & g_{21} & g_{21}(g_{11} + g_{22}) \\ 0 & g_{31} & g_{31}(g_{11} + g_{33}) \end{bmatrix}. \quad (5)$$

The diagonal elements g_{11}, g_{22} and g_{33} of matrix $\mathbf{G}$ contain independent variable $D\mathbf{F}_i(\mathbf{x}_i^l)$, guaranteeing full rank for the controllability matrix. This is consistent with (i) perfect matching of LEN by selecting all self-loops, and (ii) existence of a directed spanning tree.

Once the variational states ξ are controlled to approach zero values, the original nonlinear coupled system moves to the local desired state. The next local desired state can then be chosen. The final desired state can be reached by repeating this process. Due to the strongly structural controllability at each step, the whole controlling process is also strongly structurally controllable.

Strongly structural controllability holds if the matrix $\mathbf{L}$ in the system equation is replaced by the adjacency matrix $\mathbf{A}$. The variational equation about a local desired state $\mathbf{x}_i^l$ for one-dimensional system reads $\dot{\xi} = [D\mathbf{F}(\mathbf{x}^l) + \mathbf{A}] \cdot \xi$. There are also self-loops at each node in LEN, the weights of which as determined by $D\mathbf{F}(\mathbf{x}^l)$ are independent of each other. *The system is thus fully controllable by one controller at any node, regardless of the network structure*. However, for the correspondent linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, due to the lack of self-loops, perfect matching cannot be realized in general, and more than one driver node may be required to realize full control. We thus see that network systems with nonlinear dynamics are more controllable than with linear dynamics.

We now address the controllability of directed networks. Unlike the case of bidirectional networks, a directed network may not be structurally controllable from single driver node because of lack of directed spanning trees, despite self-loops induced by nonlinear dynamics. As shown in Fig. 2(c) for a reverse star network, in the absence of self-loops, we need at least to control nodes 2 to N, because the maximum matching number is 1. While in the presence of independent self-loops [Fig. 2(d)], we still need to perturb all leaf nodes to fully control the system. Take nodes 1 to 3 as an example. The controllability matrix with controller imposed at nodes 2 and 3 is

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & g_{12}b_2 & g_{12}b_3 & b_2c_1 & b_3c_2 \\ b_2 & 0 & g_{22}b_2 & 0 & b_2g_{22}^2 & 0 \\ 0 & b_3 & 0 & g_{33}b_3 & 0 & b_3g_{33}^2 \end{bmatrix}, \quad (6)$$

where $c_1 = g_{12}(g_{11} + g_{12})$ and $c_2 = g_{13}(g_{11} + g_{33})$. We can find that only when $b_2 \neq 0$ and $b_3 \neq 0$, the controllability matrix has full rank. This means that at least two driver nodes (2 and 3) are needed to fully control the system. Thus all leaf nodes in the reverse star network need to be controlled since they cannot access each other through directed paths, despite perfect matching of the network [15]. The controllability of directed networks can be determined by

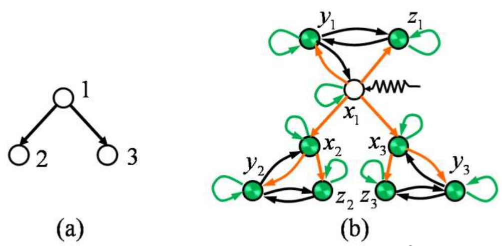

FIG. 4: (Color online). (a) A directed star graph of three nodes, each represented by a Lorenz oscillator. (b) An LEN of the network in (a). Each node in (a) becomes a cluster composed of three nodes in (b), each of which stands for a component variable x_i, y_i or z_i. The structure of the clusters is determined by $D\mathbf{F}$ and possible links among clusters exist between x_i and x_j for $i, j = 1, 2, 3$. Due to self-loops, all nodes are matched, marked by green color. There exists a single spanning tree rooted at x_i. The spanning tree is marked by orange color. Single driver node x_1 can fully control the coupled oscillator network system.

the minimum number of non-overlapping spanning-tree subgraphs in the network. For any two such subgraphs S_i and S_j, we have $S_i \cap S_j = \varnothing$ $(i, j = 1, \cdots, N_s)$, where N_s is the number of directed spanning tree subgraphs. We thus have $N_D = \min\{N_s\}$. In each spanning tree subgraph, once the root node is under control, the subgraph can be fully controlled, because it is perfectly matched by picking all self-loops and there are direct paths from the root to all other nodes. An example is shown in Fig. 3, in which all nodes are matched by picking their self-loops and there are at least two directed spanning trees without overlapping. Therefore, we need to control at least two nodes to realize full control of a directed network system.

Our LEN-based method can be generalized to network systems with high-dimensional nonlinear nodal dynamics. In such a case, the variational equations are given by Eq. (1), where $D\mathbf{H}$ is of dimension d. The matrix $D\mathbf{F}(\mathbf{x}^l) - \mathbf{L} \otimes D\mathbf{H}(\mathbf{x}^l)$ is then of dimension Nd. The relevant LEN thus consists of Nd nodes, and the controllability is determined by the high-dimensional LEN. Take the coupled Lorenz oscillator $[\dot{x}_i = \sigma(y_i - x_i) - c\sum_{j=1}^{N} L_{ij}x_j, \dot{y}_i = x_i(\rho - z_i) - y_i, \dot{z}_i = x_i y_i - \beta z_i]$ as an example. The Jacobian matrix of node i's self-dynamics about the local desired state $\mathbf{x}_i^l = (x_i^l, y_i^l, z_i^l)$ is

$$D\mathbf{F}_i(\mathbf{x}_i^l) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z_i^l & -1 & -x_i^l \\ y_i^l & x_i^l & -\beta \end{pmatrix}. \quad (7)$$

An LEN of a star graph with three nodes is shown in Fig. 4. Every original node is split into a cluster composed of three nodes, each of which stands for a variable component. The connection among the three nodes is determined by $D\mathbf{F}_i(\mathbf{x}_i^l)$ [Eq. (7)], which is the the adjacency matrix. For Lorenz oscillators, the only missing link in $D\mathbf{F}_i(\mathbf{x}_i^l)$ is $[D\mathbf{F}_i]_{13} = 0$, which means there is no link from z_i to x_i. The connection among clusters is determined by the coupling term $-\mathbf{L} \otimes D\mathbf{H}(\mathbf{x}^l)$. For instance, if oscillators are coupled by the x variables, Fig. 4(b) is the LEN of the star graph in Fig. 4(a). Controllability of system in Fig. 4(a) is determined by the conditions $S_i \cap S_j = \varnothing$ $(i, j = 1, \cdots, N_s)$ and $N_D = \min\{N_s\}$ with respect to the LEN in Fig. 4(b). Here, all nodes are matched and there is a single spanning tree rooted at x_i, so that the system is structurally controllable by a single input, but not strongly structurally controllable, because of the point $(x_i^l, y_i^l, z_i^l) = (0, 0, 0)$. At this point, in the cluster of $D\mathbf{F}_i(\mathbf{x}_i^l)$, there are no links between z and other variable components. In this way, there must then exist more than one spanning tree, and thus more than one controller are need to achieve full control.

In conclusion, we have proposed the idea of LEN (local effective network) to analyze the controllability of network systems with nonlinear dynamics. The minimum number of driver nodes required for full control of the system is determined by the structural properties of the LEN, in particular self-loops. For one-dimensional nonlinear nodal dynamics, we proved that bidirectional network can be fully controlled by a single driver node, regardless of network topology. For directed networks with nonlinear dynamics, N_D is equal to the minimum number of spanning trees. The generalization to network systems with high-dimensional nodal dynamics has been also discussed. Our main result is that nonlinear dynamics can significantly enhance the controllability of complex network systems.

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