Alternative construction of the closed form of the Green's function for the wavized Maxwell fish-eye problem

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Abstract

In the recent paper [J. Phys. A 44 (2011) 065203], we have arrived at the closed-form expression for the Green's function for the partial differential operator describing propagation of a scalar wave in an N-dimensional ($N \geqslant 2$) Maxwell fish-eye medium. The derivation has been based on unique transformation properties of the fish-eye wave equation under the hyperspherical inversion. In this communication, we arrive at the same expression for the fish-eye Green's function following a different route. The alternative derivation we present here exploits the fact that there is a close mathematical relationship, through the stereographic projection, between the wavized fish-eye problem in $\mathbb{R}^N$ and the problem of propagation of scalar waves over the surface of the N-dimensional hypersphere.

Key words: Maxwell's fish-eye problem; Green's function; scalar wave optics; gradient-index (GRIN) optics

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In the recent paper [1], we have constructed the closed-form expression for the Green's function for the partial differential operator describing propagation of a scalar wave in an N-dimensional ($N \geqslant 2$) Maxwell fish-eye medium. Our considerations, inspired by an earlier work of Demkov and Ostrovsky [2], have been based on the use of unique transformation properties of the scalar fish-eye wave equation under the hyperspherical inversion. In this communication, we show it is possible to arrive at the same representation of the fish-eye Green's function proceeding along a different but, we believe, equally elegant route. The reasoning we present below is conceptually rooted in the brilliant observation made several decades ago by Carathéodory [3], who pointed out, in the context of geometrical optics, that the remarkable properties of the Maxwell fish-eye are related to the one-to-one stereographic-projection correspondence between propagation in that medium and the free motion on the sphere (cf also Refs. [4, 5]).

To begin, we observe that the fish-eye Green's function in $\mathbb{R}^N$, $N \geqslant 2$, solves the inhomogeneous partial differential equation

$$\left[\boldsymbol{\nabla}^2_{\mathbb{R}^N} + \frac{4\nu(\nu+1)\rho^2}{(r^2+\rho^2)^2}\right] G_\nu(\boldsymbol{r},\boldsymbol{r}') = \delta^{(N)}(\boldsymbol{r}-\boldsymbol{r}'), \tag{1}$$

where $\boldsymbol{\nabla}^2_{\mathbb{R}^N}$ is the Laplace operator in $\mathbb{R}^N$ with respect to coordinates of the observation point $\boldsymbol{r}$, $\boldsymbol{r}'$ is the point where the unit delta source is located, $\rho > 0$ and $\nu \in \mathbb{C}$. After introducing the hyperspherical coordinates $\{r, \Omega_{N-1}\}$, with $r = |\boldsymbol{r}|$ and with Ω_{N-1} standing collectively for $N-1$

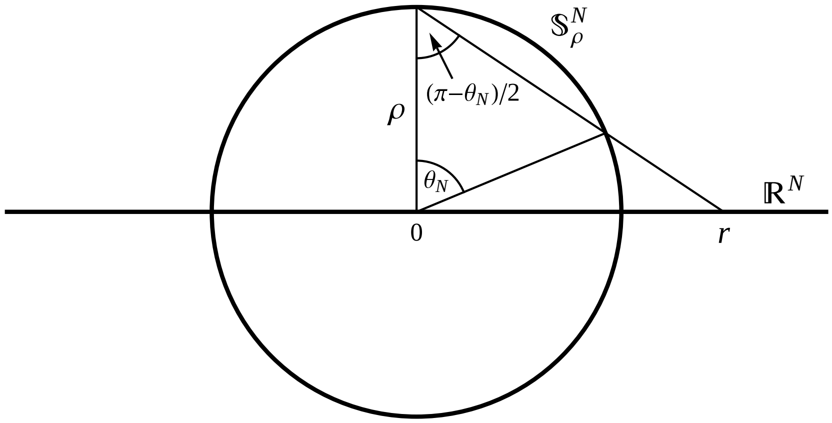

Figure 1: The transformation (3) is the inverse stereographic projection of the space $\mathbb{R}^N$ onto the hypersphere $\mathbb{S}_\rho^N$ of radius ρ.

angles characterizing the orientation of the radius vector $\boldsymbol{r}$ (and similarly for $\boldsymbol{r}'$), Eq. (1) casts into the form

$$\left[\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\boldsymbol{\nabla}^2_{\mathbb{S}^{N-1}} + \frac{4\nu(\nu+1)\rho^2}{(r^2+\rho^2)^2}\right] G_\nu(\boldsymbol{r},\boldsymbol{r}') = \frac{\delta(r-r')\delta^{(N-1)}(\Omega_{N-1}-\Omega'_{N-1})}{r^{(N-1)/2}r'^{(N-1)/2}}, \quad (2)$$

where $\boldsymbol{\nabla}^2_{\mathbb{S}^{N-1}}$ is the Laplace–Beltrami operator on the unit hypersphere $\mathbb{S}^{N-1}$. Now we make the most crucial step in our reasoning and switch from the radial variables r and r' to the angular variables θ_N and θ'_N, according to

$$\cot\frac{\theta_N}{2} = \frac{r}{\rho}, \qquad \cot\frac{\theta'_N}{2} = \frac{r'}{\rho} \qquad (0 \leqslant \theta_N, \theta'_N \leqslant \pi), \quad (3)$$

the angular coordinate ensembles Ω_{N-1} and Ω'_{N-1} remaining unchanged. The geometrical meaning of the transformation (3) becomes obvious after a glance at Fig. 1: this is the inverse stereographic projection of the space $\mathbb{R}^N$ onto the hypersphere $\mathbb{S}_\rho^N$ of radius ρ, the space to be projected being the equatorial hyperplane of the hypersphere. Since, in view of Eq. (3) and of the well-know properties of the Dirac delta, it holds that

$$\delta(r-r') = \frac{2}{\rho}\sin\frac{\theta_N}{2}\sin\frac{\theta'_N}{2}\,\delta(\theta_N-\theta'_N), \quad (4)$$

the transformation in question changes Eq. (2) into

$$\left[\frac{\partial^2}{\partial\theta_N^2} + \left(\cot\frac{\theta_N}{2} - \frac{N-1}{\sin\theta_N}\right)\frac{\partial}{\partial\theta_N} + \frac{\boldsymbol{\nabla}^2_{\mathbb{S}^{N-1}}}{\sin^2\theta_N} + \nu(\nu+1)\right] G_\nu(\boldsymbol{r},\boldsymbol{r}')$$
$$= \frac{1}{2\rho^{N-2}}\frac{\delta(\theta_N-\theta'_N)\delta^{(N-1)}(\Omega_{N-1}-\Omega'_{N-1})}{\sin\frac{\theta_N}{2}\cot^{(N-1)/2}\frac{\theta_N}{2}\sin\frac{\theta'_N}{2}\cot^{(N-1)/2}\frac{\theta'_N}{2}}. \quad (5)$$

Both the differential operator on the left-hand side and the multiplier of the deltas on the right-hand side of Eq. (5) look complicated. However, a remarkable simplification is achieved after one replaces the Green's function $G_\nu(\boldsymbol{r},\boldsymbol{r}')$ by the function $\mathcal{G}_{\nu-N/2+1}(\Omega_N,\Omega'_N)$, the two being related by

$$G_\nu(\boldsymbol{r},\boldsymbol{r}') = \left(\frac{2}{\rho}\right)^{N-2}\sin^{N-2}\frac{\theta_N}{2}\sin^{N-2}\frac{\theta'_N}{2}\,\mathcal{G}_{\nu-N/2+1}(\Omega_N,\Omega'_N). \quad (6)$$

Here, Ω_N stands for the set $\{\theta_N, \Omega_{N-1}\}$ (and similarly for Ω'_N); the reason for attaching the particular subscript to $\mathcal{G}$ will become clear shortly. Insertion of Eq. (6) into Eq. (5), followed by some obvious rearrangements, results in

$$\left[\frac{\partial^2}{\partial\theta_N^2} + (N-1)\cot\theta_N \frac{\partial}{\partial\theta_N} + \frac{\boldsymbol{\nabla}^2_{\mathbb{S}^{N-1}}}{\sin^2\theta_N} + \left(\nu - \frac{N}{2} + 1\right)\left(\nu + \frac{N}{2}\right)\right]\mathcal{G}_{\nu-N/2+1}(\Omega_N, \Omega'_N) = \frac{\delta(\theta_N - \theta'_N)\delta^{(N-1)}(\Omega_{N-1} - \Omega'_{N-1})}{\sin^{(N-1)/2}\theta_N \sin^{(N-1)/2}\theta'_N}. \quad (7)$$

The first three terms in the square bracket on the left-hand side of Eq. (7) are immediately recognized to form the Laplace–Beltrami operator on the unit hypersphere $\mathbb{S}^N$:

$$\frac{\partial^2}{\partial\theta_N^2} + (N-1)\cot\theta_N \frac{\partial}{\partial\theta_N} + \frac{\boldsymbol{\nabla}^2_{\mathbb{S}^{N-1}}}{\sin^2\theta_N} \equiv \boldsymbol{\nabla}^2_{\mathbb{S}^N} \qquad (N \geqslant 2), \quad (8)$$

while the expression on the right-hand side of Eq. (7) is simply the Dirac delta on $\mathbb{S}^N$:

$$\frac{\delta(\theta_N - \theta'_N)\delta^{(N-1)}(\Omega_{N-1} - \Omega'_{N-1})}{\sin^{(N-1)/2}\theta_N \sin^{(N-1)/2}\theta'_N} = \delta^{(N)}(\Omega_N - \Omega'_N). \quad (9)$$

Hence, with the definition

$$\lambda = \nu - \frac{N}{2} + 1, \quad (10)$$

Eq. (7) may be rewritten compactly as

$$\left[\boldsymbol{\nabla}^2_{\mathbb{S}^N} + \lambda(\lambda + N - 1)\right]\mathcal{G}_\lambda(\Omega_N, \Omega'_N) = \delta^{(N)}(\Omega_N - \Omega'_N). \quad (11)$$

This is the equation defining the Green's function for the Helmholtz operator on the hypersphere $\mathbb{S}^N$; it has been studied by us in Ref. [6]. There, it has been shown that the solution to Eq. (11) is

$$\mathcal{G}_\lambda(\Omega_N, \Omega'_N) = \frac{\pi C_\lambda^{(N-1)/2}\left(-\cos\angle(\Omega_N, \Omega'_N)\right)}{(N-1)S_N \sin(\pi\lambda)}, \quad (12)$$

where $C_\lambda^\alpha(\xi)$ is the Gegenbauer function, $\angle(\Omega_N, \Omega'_N)$ is the angle between the directions Ω_N and Ω'_N, while

$$S_N = \frac{2\pi^{(N+1)/2}}{\Gamma\left(\frac{N+1}{2}\right)} \quad (13)$$

is the area of $\mathbb{S}^N$. Hence, on invoking Eq. (6), we see that the closed-form representation of the fish-eye Green's function in $\mathbb{R}^N$ is

$$G_\nu(\boldsymbol{r}, \boldsymbol{r}') = \frac{2^{N-4}\Gamma\left(\frac{N-1}{2}\right)}{\rho^{N-2}\pi^{(N-1)/2}\sin\left[\pi\left(\frac{N}{2} - \nu\right)\right]} \sin^{N-2}\frac{\theta_N}{2} \sin^{N-2}\frac{\theta'_N}{2}\, C_{\nu-N/2+1}^{(N-1)/2}\left(-\cos\angle(\Omega_N, \Omega'_N)\right). \quad (14)$$

To accomplish the task fully, we have to express the right-hand side of Eq. (14) in terms of the radius vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ instead of the hyperangles Ω_N and Ω'_N. To this end, at first we observe that the cosine of the angle $\angle(\Omega_N, \Omega'_N)$ may be written as

$$\cos\angle(\Omega_N, \Omega'_N) = \cos\theta_N \cos\theta'_N + \sin\theta_N \sin\theta'_N \cos\angle(\Omega_{N-1}, \Omega'_{N-1}). \quad (15)$$

However, from Eq. (3) it follows that

$$\cos\theta_N = \frac{\cot^2\frac{\theta_N}{2} - 1}{\cot^2\frac{\theta_N}{2} + 1} = \frac{r^2 - \rho^2}{r^2 + \rho^2} \quad (16)$$

and

$$\sin\theta_N = \frac{2\cot\frac{\theta_N}{2}}{\cot^2\frac{\theta_N}{2} + 1} = \frac{2\rho r}{r^2 + \rho^2} \quad (17)$$

(and similarly for $\cos\theta'_N$ and $\sin\theta'_N$), so that

$$\begin{aligned}\cos\angle(\Omega_N,\Omega'_N) &= 1-\frac{2\rho^2[r^2+r'^{\,2}-2rr'\cos\angle(\Omega_{N-1},\Omega'_{N-1})]}{(r^2+\rho^2)(r'^{\,2}+\rho^2)} \\ &= 1-\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^{\,2}+\rho^2)}.\end{aligned} \tag{18}$$

Furthermore, invoking Eq. (3) again, we see that

$$\sin\frac{\theta_N}{2}=\frac{1}{\sqrt{\cot^2\frac{\theta_N}{2}+1}}=\frac{\rho}{\sqrt{r^2+\rho^2}} \tag{19}$$

(and similarly for $\sin\frac{\theta'_N}{2}$). Plugging Eqs. (18) and (19) into Eq. (14), we eventually arrive at

$$G_\nu(\boldsymbol{r},\boldsymbol{r}')=\frac{2^{N-4}\Gamma\left(\frac{N-1}{2}\right)}{\pi^{(N-1)/2}\sin\left[\pi\left(\frac{N}{2}-\nu\right)\right]}\frac{\rho^{N-2}C^{(N-1)/2}_{\nu-N/2+1}\left(-1+\frac{2\rho^2(\boldsymbol{r}-\boldsymbol{r}')^2}{(r^2+\rho^2)(r'^{\,2}+\rho^2)}\right)}{(r^2+\rho^2)^{N/2-1}(r'^{\,2}+\rho^2)^{N/2-1}}. \tag{20}$$

This representation of the fish-eye Green's function in $\mathbb{R}^N$ is identical with the one found by us in Ref. [1, Eq. (3.42)] using the hyperspherical inversion technique.

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