Immunity of information encoded in decoherence-free subspaces to particle loss

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We demonstrate that for an ensemble of qudits, subjected to collective decoherence in the form of perfectly correlated random SU(d) unitaries, quantum superpositions stored in the decoherence free subspace are fully immune against the removal of one particle. This provides a feasible scheme to protect quantum information encoded in the polarization state of a sequence of photons against both collective depolarization and one photon loss, and can be demonstrated with photon quadruplets using currently available technology.

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Quantum systems are powerful yet fragile carriers of information. The ability to create and manipulate superposition states offers verifiably secure cryptography [1], reduces the complexity of certain computational problems [2], and enables novel communication protocols [3]. However, in practical settings one needs to protect the quantum states carrying information against decoherence, i.e. uncontrolled interactions with the environment. This is accomplished by building redundancy into the physical implementation. Compared to the classical case, this task is much more challenging [4] due to limitations in handling quantum information, exemplified strikingly by the no-cloning theorem [5].

When an ensemble of elementary quantum systems decoheres through symmetric coupling with the environment, one can identify collective states that remain invariant in the course of evolution. These states span a so-called *decoherence-free subspace* (DFS) that is effectively decoupled from the interaction with the environment [6]. More generally, it is possible to identify subspaces that can be formally decomposed into a tensor product of two subsystems, one of which "absorbs" decoherence, while the second one, named a *noiseless* or *decoherence-free subsystem*, remains intact [7].

In this communication we consider the DFS for an ensemble of n qudits, i.e. elementary d-level systems, composed of states $|\Psi\rangle$ that are invariant with respect to an arbitrary perfectly correlated SU(d) transformation:

$$\hat{U}^{\otimes n}|\Psi\rangle = |\Psi\rangle, \qquad \hat{U} \in \mathrm{SU}(d). \tag{1}$$

We show that this DFS features an additional degree of robustness, namely that the stored quantum information is immune to the loss of one of the qudits. This result, applied to the polarization state of single photons with $d = 2$, offers *combined* protection against two common optical decoherence mechanisms: photon loss [8] due to spurious reflections, residual absorption, scattering, etc. as well as the collective rotation of the polarization that occurs inevitably in optical fibers used for long-haul transmission [9, 10]. Consequently, we provide here rigorous foundations to a speculation presented in Ref. [11] that DFS-based quantum cryptography can be made tolerant also to photon loss. We describe here a proposal for a proof-of-principle experiment based on currently available photonic technologies that demonstrates the robustness of DFS encoding. It is worth noting that another physical realization of the qubit case can be also an ensemble of spin-$\frac{1}{2}$ particles [12] coupled identically to a varying magnetic field.

Because of two relevant physical realizations, we will first discuss in more detail the qubit case, i.e. when $d = 2$. The complete Hilbert space of an ensemble of n qubits, each described by a space $\mathcal{H}_{1/2}$, can be subjected to Clebsch-Gordan decomposition which yields [13]

$$(\mathcal{H}_{1/2})^{\otimes n} = \bigoplus_{j=(n \bmod 2)/2}^{n/2} \mathbb{C}^{K_n^j} \otimes \mathcal{H}_j, \tag{2}$$

where the direct sum is taken with the step of one and K_n^j are multiplicities of spin-j Hilbert spaces $\mathcal{H}_j$, given explicitly by

$$K_n^j = \frac{2j+1}{n/2+j+1}\binom{n}{n/2+j}. \tag{3}$$

The action of $\hat{U}^{\otimes n}$ affects only $\mathcal{H}_j$ in Eq. (2), leaving $\mathbb{C}^{K_n^j}$ unchanged. In particular, for an even number of n qubits in the ensemble, the *singlet subspace* corresponding to $j = 0$ is free from decoherence. Furthermore, removing one particle from that ensemble maps any initial state from the singlet subspace onto a certain state from the *doublet subspace* $\mathbb{C}^{K_{n-1}^{1/2}} \otimes \mathcal{H}_{1/2}$. Because $K_{n-1}^{1/2} = K_n^0$, it is plausible that the quantum superposition will end up entirely in the decoherence-free subsystem $\mathbb{C}^{K_{n-1}^{1/2}}$ where it will remain protected from collective depolarization. We will now demonstrate that this is indeed the case.

Let us first inspect in more detail the structure of the singlet space. Useful insights are provided by the construction of a complete set of states in this subspace in the form of products of two-qubit singlet states. For a

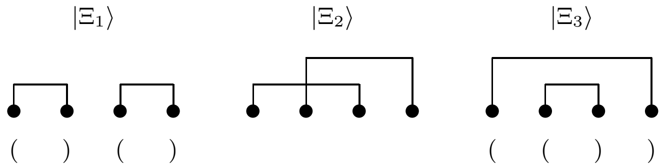

FIG. 1: Diagrams depicting singlet pair product states for 4 qubits. The qubits are represented as dots with connections identifying pairs that form singlet states. States belonging to the Dyck basis are labelled with corresponding Dyck words, i.e. closed and correctly nested strings of '(' and ')' parentheses. Generally, Dyck basis states are those and only those which can be represented by diagrams with no crossing connections.

permutation $\sigma\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ we define a *singlet pair product state* (SPPS) $|s_\sigma\rangle$ according to

$$|s_\sigma\rangle = \bigotimes_{k=1}^{n/2} |\psi^-\rangle_{\sigma_{2k-1}\sigma_{2k}}, \tag{4}$$

where $|\psi^-\rangle_{ij} = (|01\rangle_{ij} - |10\rangle_{ij})/\sqrt{2}$ is the singlet state of qubits i and j. In the simplest non-trivial case of $n = 4$ qubits we have three SPPSs that are not equivalent by a sign change, shown schematically in Fig. 1:

$$\begin{aligned} |\Xi_1\rangle &= |\psi^-\rangle_{12}|\psi^-\rangle_{34}, \\ |\Xi_2\rangle &= |\psi^-\rangle_{13}|\psi^-\rangle_{42}, \\ |\Xi_3\rangle &= |\psi^-\rangle_{14}|\psi^-\rangle_{23}. \end{aligned} \tag{5}$$

As the the singlet space in this case is two-dimensional, these SPPSs are linearly dependent. This occurs also for more qubits: the number of non-equivalent SPPSs is $n!/(2^{n/2}(n/2)!)$ which exceeds the dimension of the singlet subspace K_n^0 given in Eq. (3). It can be shown [14] that a complete set of linearly independent vectors can be obtained by selecting such SPPSs that for each two factors $|\psi^-\rangle_{i_1 j_1}$ and $|\psi^-\rangle_{i_2 j_2}$ with $i_1 < i_2$ we have either $i_1 < j_1 < i_2 < j_2$ or $i_1 < i_2 < j_2 < j_1$. States satisfying this condition can be identified one-to-one with strings composed of left and right parentheses that form correctly nested closed expressions, known in the theory of formal languages as Dyck words [15]. This correspondence is depicted schematically for $n = 4$ qubits in Fig. 1. We will therefore refer to the basis selected from SPPSs using the condition specified above as the *Dyck basis*.

Let us now consider the removal of one of the particles, which can be assumed without loss of generality to be the first one. Any vector $|s_\mu\rangle$ from the Dyck basis can be written as

$$|s_\mu\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_1 |d_\mu^{(0)}\rangle_{\bar{1}} + |1\rangle_1 |d_\mu^{(1)}\rangle_{\bar{1}}\right), \tag{6}$$

where $|\cdot\rangle_1$ and $|\cdot\rangle_{\bar{1}}$ denote respectively the states of the first qubit and the remaining qubits $2,\ldots,n$. The states $|d_\mu^{(i)}\rangle_{\bar{1}} = \sqrt{2}\,{}_1\langle i|s_\mu\rangle$, $i = 0,1$, appearing in the above decomposition have several properties. First, any two states $|d_\mu^{(0)}\rangle_{\bar{1}}$ and $|d_\nu^{(1)}\rangle_{\bar{1}}$ have different numbers of 0s and 1s and therefore satisfy the orthogonality condition

$${}_{\bar{1}}\langle d_\mu^{(0)}|d_\nu^{(1)}\rangle_{\bar{1}} = 0. \tag{7}$$

Second, we have ${}_{\bar{1}}\langle d_\mu^{(0)}|d_\nu^{(0)}\rangle_{\bar{1}} = {}_{\bar{1}}\langle d_\mu^{(1)}|d_\nu^{(1)}\rangle_{\bar{1}}$, which follows from the fact that interchanging all 0s with 1s transforms $|d_\mu^{(0)}\rangle_{\bar{1}}$ into $(-1)^{n/2}|d_\mu^{(1)}\rangle_{\bar{1}}$ and vice versa. Furthermore, calculating $\langle s_\mu|s_\nu\rangle$ from Eq. (6) using the preceding observations leads to

$${}_{\bar{1}}\langle d_\mu^{(0)}|d_\nu^{(0)}\rangle_{\bar{1}} = {}_{\bar{1}}\langle d_\mu^{(1)}|d_\nu^{(1)}\rangle_{\bar{1}} = \langle s_\mu|s_\nu\rangle. \tag{8}$$

Suppose now that we have prepared an even number of n qubits in an arbitrary state from the singlet subspace, which can be written in the Dyck basis as $|\psi\rangle = \sum_\mu \alpha_\mu |s_\mu\rangle$. After removal of the first qubit the state of the remaining qubits is described by a density matrix

$$\hat{\varrho}_{\bar{1}} = \frac{1}{2}\sum_{\mu\nu} \alpha_\mu \alpha_\nu^* |d_\mu^{(0)}\rangle_{\bar{1}}\langle d_\nu^{(0)}| + \frac{1}{2}\sum_{\mu\nu} \alpha_\mu \alpha_\nu^* |d_\mu^{(1)}\rangle_{\bar{1}}\langle d_\nu^{(1)}|. \tag{9}$$

Owing to Eq. (7) the two components of the sum occupy orthogonal subspaces. In each of the subspaces the superposition is fully preserved, as the scalar products between any $|d_\mu^{(i)}\rangle_{\bar{1}}$ and $|d_\nu^{(i)}\rangle_{\bar{1}}$ with the same i are equal to that between $|s_\mu\rangle$ and $|s_\nu\rangle$ according to Eq. (8). Furthermore, pairs of states $|d_\mu^{(0)}\rangle_{\bar{1}}$ and $|d_\mu^{(1)}\rangle_{\bar{1}}$ transform as doublet states, since they are tensor products of an SPPS of $n-2$ qubits and either $|1\rangle$ or $-|0\rangle$ for the qubit that was initially paired with the removed particle. Thus unitary transformations $\hat{U}^{\otimes n}$ do not mix states $|d_\mu^{(i)}\rangle_{\bar{1}}$ with different indices μ. Consequently, $\hat{\varrho}_{\bar{1}}$ given in Eq. (9) can be represented as a tensor product of the original logical qubit with a fully mixed state of a two-level subsystem, corresponding respectively to $\mathbb{C}^{K_n^{1/2}}$ and $\mathcal{H}_{1/2}$ in Eq. (2).

The above result can be generalized to an ensemble of n qudits, i.e. d-dimensional systems, subjected to collective decoherence of the form $\hat{U}^{\otimes n}$, where $\hat{U}$ is an arbitrary SU(d) transformation. In this case, a DFS satisfying Eq. (1) exists only when n is a multiple of d, which follows from the structure of the Young tableux for irreducible representations of tensor products of the $SU(d)$ group [16].

As before, for concreteness we will consider removal of the first qudit. Let us consider arbitrary two states $|\Psi\rangle$ and $|\Phi\rangle$ from the DFS and expand them in the form analogous to Eq. (6):

$$|\Psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1} |i\rangle_1 |\Psi^{(i)}\rangle_{\bar{1}}, \quad |\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1} |i\rangle_1 |\Phi^{(i)}\rangle_{\bar{1}} \tag{10}$$

where $|i\rangle_1, i = 0,\ldots,d-1$ is an orthonormal basis in the space of the first qudit, and $|\Psi^{(i)}\rangle_{\bar{1}} = \sqrt{d}\,{}_1\langle i|\Psi\rangle$ and $|\Phi^{(i)}\rangle_{\bar{1}} = \sqrt{d}\,{}_1\langle i|\Phi\rangle$ are states of the remaining $n-1$ qudits. We will first show that the following general property holds:

$$ {}_{\bar{1}}\langle\Phi^{(i)}|\Psi^{(j)}\rangle_{\bar{1}} = \delta_{ij}\langle\Phi|\Psi\rangle. \tag{11} $$

Let us note that Eqs. (7) and (8) are its particular cases. As we will see, this property guarantees that the loss of one particle does not destroy the quantum information encoded in the DFS.

In order to show that for $i \neq j$ the states $|\Phi^{(i)}\rangle$ and $|\Psi^{(j)}\rangle$ are orthogonal as implied by Eq. (11), let us consider the action of a diagonal unitary operator $\hat{V}^{\otimes n}$, where $\hat{V} = \mathrm{diag}(e^{i\phi_0}, \ldots, e^{i\phi_{d-1}})$ with arbitrary phases $\phi_0, \ldots, \phi_{d-1}$ that sum up to zero. Invariance of $|\Phi^{(i)}\rangle_{\bar{1}}$ and $|\Psi^{(j)}\rangle_{\bar{1}}$ under $\hat{V}^{\otimes n}$ implies that in the basis formed by tensor products of states $|0\rangle, \cdots, |d-1\rangle$ they are composed only from terms that have exactly n/d particles in each of these d states. Consequently, projecting the first qudit on orthogonal states $|i\rangle_1$ and $|j\rangle_1$ leaves the remaining qudits in distinguishable states.

In order to verify the case when $i = j$ in Eq. (11) it is convenient to use the transformation of states $|\Psi^{(i)}\rangle_{\bar{1}}$ under the action of $\hat{U}^{\otimes(n-1)}$. In order to derive this transformation, let us rewrite the invariance condition from Eq. (1) to the form $\hat{U}^\dagger \otimes \mathbb{1}^{\otimes(n-1)}|\Psi\rangle = \mathbb{1} \otimes \hat{U}^{\otimes(n-1)}|\Psi\rangle$ and project the first qudit onto $\sqrt{d}\,{}_1\langle i|$. This yields the identity:

$$ \hat{U}^{\otimes(n-1)}|\Psi^{(i)}\rangle_{\bar{1}} = \sqrt{d}\left({}_1\langle i|\hat{U}^\dagger\right)|\Psi\rangle = \sum_{j=0}^{d-1}\left(\langle j|\hat{U}|i\rangle\right)^*|\Psi^{(j)}\rangle_{\bar{1}} \tag{12} $$

Let us now specialize this result to a special unitary transformation that cyclically shifts the labelling of the basis states:

$$ \hat{W} = (-1)^{d-1}\sum_{i=0}^{d-1}|i+1\rangle\langle i|, \tag{13} $$

where the addition $i+1$ is understood to be modulo d. Using this $\hat{W}$ in Eq. (12) implies that $|\Psi^{(i+1)}\rangle = (-1)^{d-1}\hat{W}^{\otimes(n-1)}|\Psi^{(i)}\rangle$, i.e. $|\Psi^{(i)}\rangle$ and $|\Psi^{(i+1)}\rangle$ are related by a unitary that is independent of $|\Psi\rangle$. This means that $\langle\Phi^{(i+1)}|\Psi^{(i+1)}\rangle = \langle\Phi^{(i)}|\Psi^{(i)}\rangle$. This fact combined with expanding the scalar product $\langle\Phi|\Psi\rangle$ using Eq. (10) completes the proof of Eq. (11).

With Eq. (11) in hand, further steps are straightforward. A removal of the first qudit maps a state $|\Psi\rangle$ onto a statistical mixture

$$ \hat{\varrho}_{\bar{1}} = \mathrm{Tr}_1\left(|\Psi\rangle\langle\Psi|\right) = \frac{1}{d}\sum_{i=0}^{d-1}|\Psi^{(i)}\rangle_{\bar{1}}\langle\Psi^{(i)}|. \tag{14} $$

Eq. (11) implies that analogously to the SU(2) case the components with different i occupy orthogonal subspaces. Within each subspace the state is fully preserved, which follows from applying Eq. (11) to pairs of states from an arbitrary basis in the DFS. The final step is to show that the state $\hat{\varrho}_{\bar{1}}$ is invariant with respect to $\hat{U}^{\otimes(n-1)}$. This follows from the fact that both the initial state $|\Psi\rangle$ and the procedure of tracing out a particle are invariant with respect to SU(d) transformations. Explicitly, the invariance of $\hat{\varrho}_{\bar{1}}$ can be verified with a calculation based on Eq. (12):

$$ \begin{aligned} \hat{U}^{\otimes(n-1)}\hat{\varrho}_{\bar{1}}(\hat{U}^\dagger)^{\otimes(n-1)} &= \sum_{i=0}^{d-1}\left({}_1\langle i|\hat{U}^\dagger\right)|\Psi\rangle\langle\Psi|\left(\hat{U}|i\rangle_1\right) \\ &= \mathrm{Tr}_1\left(|\Psi\rangle\langle\Psi|\right) = \hat{\varrho}_{\bar{1}}. \end{aligned} \tag{15} $$

Thus the encoded state is fully preserved.

As DFS states of photon quadruplets can be generated using parametric down-conversion [10, 17], we will close the paper with a proposal for an feasible experiment that demonstrates the robustness of DFS encoding. Let us consider four-photon states $|\Xi_k\rangle$, $k = 1, 2, 3$, defined in Eq. (5) as well as their orthogonal complements in the two-dimensional DFS, which we will denote as $|\Xi_k^\perp\rangle$. The index k corresponds to three non-equivalent orderings of the photons and it can be changed by suitable rerouting of the photons. As demonstrated in [10], the states $|\Xi_1\rangle$ and $|\Xi_1^\perp\rangle$ can be discriminated unambiguously by detecting polarizations in the horizontal-vertical basis $|0\rangle, |1\rangle$ for photons 12 and in the diagonal basis $(|0\rangle \pm |1\rangle)/\sqrt{2}$ for photons 34. Restricted to the DFS subspace, this strategy yields the standard projective measurement.

It is easy to check that the above individual measurement no longer works if one of the photons is missing. It turns out that this problem can be solved by resorting to collective measurements. Suppose that we interfere photon pairs 12 and 34 on two separate balanced beam splitters, playing the role linear-optics Bell state analyzers [18]. The state $|\Xi_1\rangle$ will yield exactly one photon in each output port of each beam splitter. In contrast, because the orthogonal state $|\Xi_1^\perp\rangle$ can be written as [7]:

$$ |\Xi_1^\perp\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle_{12}|11\rangle_{34} + |11\rangle_{12}|00\rangle_{34} - |\psi^+\rangle_{12}|\psi^+\rangle_{34}\right), \tag{16} $$

where $|\psi^+\rangle_{ij} = (|01\rangle_{ij} + |10\rangle_{ij})/\sqrt{2}$, it will always produce two photons at the same output port for each of the two beam splitters. If one photon is lost, the states $|\Xi_1\rangle$ and $|\Xi_1^\perp\rangle$ will still give distinguishable outcomes: registering two photons at a single output unambiguously heralds $|\Xi_1^\perp\rangle$, while registering a photon pair at two different outputs of the same beam splitters detects $|\Xi_1\rangle$. The third photon will emerge separately from the second beam splitter. This detection scheme is summarized in Fig. 2.

The scalar products between any two the states $|\Xi_k\rangle$ and $|\Xi_l\rangle$ with $k \neq l$ are equal to $\langle\Xi_k|\Xi_l\rangle = -\frac{1}{2}$. In the Bloch representation of the two-dimensional DFS, they form a regular triangle inscribed into a great circle on

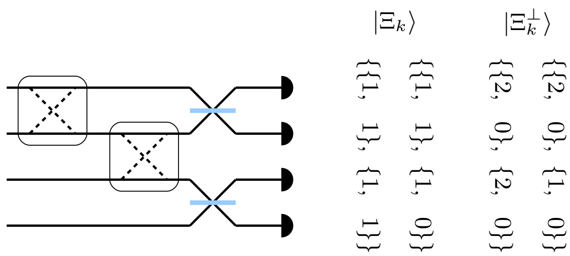

FIG. 2: An experimental scheme for loss-tolerant detection of a logical qubit encoded in four photons. The projection basis $|\Xi_k\rangle$, $|\Xi_k^\perp\rangle$, where $k = 1, 2, 3$, is selected by a suitable rerouting of input photons. Pairs of photons are interfered on two balanced beam splitters and photon numbers are counted at their outputs. Combinations of outcomes for individual detectors that correspond to unambiguous identification of $|\Xi_k\rangle$ and $|\Xi_k^\perp\rangle$ are indicated with photon numbers in curly brackets. The ordering within both inner and outer brackets does not matter.

the Bloch sphere, constituting a so-called *trine* that warrants cryptographic security [11, 19]. To generate a key, the sender Alice prepares photon quadruplets in one of randomly selected states $|\Xi_1\rangle$, $|\Xi_2\rangle$, or $|\Xi_3\rangle$. The ability to perform a projection onto any pair of orthogonal states $|\Xi_k\rangle, |\Xi_k^\perp\rangle$ enables the receiving party Bob to tell, in the case when an outcome $|\Xi_k^\perp\rangle$ is obtained, which state has definitely not been prepared by Alice. Such correlations between Alice's preparations and Bob's outcomes can be distilled into a secure key.

We have shown that the projective measurement onto $|\Xi_k\rangle, |\Xi_k^\perp\rangle$ can be implemented in a way that tolerates the loss of one photon. In a cryptographic setting, the crucial issue is to ensure that an eavesdropper Eve does does not map the state of intercepted photons outside the DFS, which would enable eavesdropping attacks beyond those already studied [11, 19]. To verify that this is not the case, Bob could perform in principle a full quantum state reconstruction on some of the transmissions, which however would be resource consuming. We conjecture that a sufficient strategy to detect such an attack would be: (i) to detect polarizations of photons emerging after the beam splitters; (ii) for a subset of transmissions to count directly received photons to ensure that no multiphoton states in individual time bins occur; (iii) for another subset of transmissions to apply before the beam splitters random and uncorrelated transformations $\hat{U} \otimes \hat{U}$ and $\hat{U}' \otimes \hat{U}'$ and check that states $|\Xi_k\rangle$ always yield the correct outcome when Bob used the matching basis for his measurement.

Concluding, we have shown that DFS encoding is immune to removing one particle. Unfortunately, this property does not seem to generalize in a straightforward manner to the loss of more particles. For example, when two qubits are removed from a Dyck basis state, the result will either lie in the singlet subspace of $n-2$ qubits, or will be a mixture of singlet and $j = 1$ states. Nevertheless, our result shows how to protect information in the few-photon regime from both collective depolarization and the first-order effects of linear attenuation. We have proposed an experimental demonstration of this combined protection which can provide a robust quantum cryptography protocol.

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