ON THE DEGREE-CHROMATIC POLYNOMIAL OF A TREE

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ABSTRACT. The degree chromatic polynomial $P_m(G,k)$ of a graph G counts the number of k-colorings in which no vertex has m adjacent vertices of its same color. We prove Humpert and Martin's conjecture on the leading terms of the degree chromatic polynomial of a tree.

Given a graph G, Humpert and Martin defined its m-chromatic polynomial $P_m(G,k)$ to be the number of k-colorings of G such that no vertex has m adjacent vertices of its same color. They proved this is indeed a polynomial. When $m = 1$, we recover the usual chromatic polynomial of the graph $P(G,k)$ [3].

The chromatic polynomial is of the form $P(G,k) = k^n - ek^{n-1} + o(k^{n-1})$, where n is the number of vertices and e the number of edges of G. For $m > 1$ the formula is no longer true, but Humpert and Martin conjectured the following formula which we now prove:

Theorem 1 ([1, 2], Conjecture)**.** *Let T be a tree with n vertices and let m be an integer with $1 < m < n$, then the following equation holds.*

$$P_m(T,k) = k^n - \sum_{v \in V(T)} \binom{d(v)}{m} k^{n-m} + o(k^{n-m})$$

Proof. For a given coloring of T, say vertices v_1 and v_2 are "friends" if they are adjacent and have the same color. For each v, let A_v be the set of colorings such that v has at least m friends. We want to find the number of colorings which are not in any A_v, and we will use the inclusion-exclusion principle:

$$P_m(T,k) = k^n - \sum_{v \in V} |A_v| + \sum_{v_1, v_2 \in V} |A_{v_1} \cap A_{v_2}| - \dots$$

We first show that $|A_v| = \binom{d(v)}{m} k^{n-m} + o(k^{n-m})$. Let $A_v^{(l)}$ be the set of k-colorings such that v has exactly l friends. Then

Date: July 4, 2011.

Key words and phrases. Chromatic polynomial, graph coloring, tree.

$$|A_v| = \sum_{l=m}^{n-1} |A_v^{(l)}| = \sum_{l=m}^{n-1} \binom{d(v)}{l} k(k-1)^{n-l-1} = \binom{d(v)}{m} k^{n-m} + o(k^{n-m})$$

To complete the proof, it is sufficient to see that for any set S of at least 2 vertices $|\bigcap_{v\in S} A_v| = o(k^{n-m})$; clearly we may assume $S = \{v_1, v_2\}$. Consider the following cases:

Case 1 (v_1 and v_2 are not adjacent). Split A_{v_1} into equivalence classes

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \neq v_2$$

Note that each equivalence class C consists of k colorings, at most $\frac{d(v_2)}{m}$ of which are in A_{v_2}. Therefore

$$|A_{v_1} \cap A_{v_2}| = \sum_C |C \cap A_{v_2}| \leq \sum_C \frac{d(v_2)}{m} = \frac{|A_{v_1}|}{k} \cdot \frac{d(v_2)}{m}$$

it follows that $\frac{|A_{v_1} \cap A_{v_2}|}{|A_{v_1}|}$ goes to 0 as k goes to infinity, so $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$.

Case 2 (v_1 and v_2 are adjacent). Let W be the set of adjacent vertices to v_2 other than v_1. They are not adjacent to v_1 as T has no cycles. Split A_{v_1} into equivalence classes.

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \notin W$$

Each equivalence class C consists of $k^{|W|}$ colorings. If v_1 and v_2 are friends in the colorings of C, then a coloring in $|C \cap A_{v_2}|$ must contain at least $m-1$ vertices in W of the same color as v_2. Therefore

$$|C \cap A_{v_2}| = \sum_{l=m-1}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|} k^{|W|-1}$$

Notice that here we are using $m \geq 2$ so that $l \geq 1$. Otherwise, if v_1 and v_2 are not friends in the colorings of C, then

$$|C \cap A_{v_2}| = \sum_{l=m}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|} k^{|W|-1}$$

Therefore

$$\begin{aligned}|A_{v_1} \cap A_{v_2}| = \sum_C |C \cap A_{v_2}| < \sum_C 2^{|W|} k^{|W|-1} \\ = \frac{|A_{v_1}|}{k^{|W|}} \cdot 2^{|W|} k^{|W|-1} = \frac{|A_{v_1}| \cdot 2^{|W|}}{k}\end{aligned}$$

and $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$ follows as in the first case.

This completes the proof of the theorem. □

Acknowledgments. I would like to thank Federico Ardila for bringing this problem to my attention, and for helping me improve the presentation of this note. I would also like to acknowledge the support of the SFSU-Colombia Combinatorics Initiative.

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