ON DUAL EQUIVALENCE AND SCHUR POSITIVITY

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Abstract. We define dual equivalence for any collection of combinatorial objects endowed with a descent set, and we show that giving a dual equivalence establishes the symmetry and Schur positivity of the quasi-symmetric generating function. We give an explicit formula for the Schur expansion of the generating function in terms of distinguished elements of the dual equivalence classes. These concepts and proofs simplify in the ubiquitous case when the collection of objects has a sufficiently nice reading word.

1. Introduction

Symmetric function theory plays an important role in many areas of mathematics including combinatorics, representation theory, and algebraic geometry. Multiplicities of irreducible components, dimensions of algebraic varieties, and various other algebraic constructions that require the computation of certain integers may often be translated to the computation of the Schur coefficients of a given function. Thus a quintessential problem in symmetric functions is to prove that a given function has nonnegative integer coefficients when expressed as a sum of Schur functions. In combinatorics, we hope to achieve this by finding a collection of combinatorial objects enumerated by the coefficients.

The purpose of dual equivalence graphs, introduced in [Ass], is to provide a universal tool to establish the symmetry and Schur positivity of functions expressed in terms of fundamental quasisymmetric functions. In the present paper, we reformulate this machinery in terms of involutions on a set and give a more explicit characterization of the Schur coefficients. The general setup is as follows. Begin with a set $\mathcal{A}$ of combinatorial objects together with a notion of a descent set Des sending an object to a subset of positive integers. Optionally, we may also have a nonnegative, possibly multivariate, integer statistic stat associated to each object. Define the quasi-symmetric generating function by

$$f(X;q) \;=\; \sum_{T\in\mathcal{A}} q^{\mathrm{stat}(T)} Q_{\mathrm{Des}(T)}(X),$$

where Q denotes the fundamental basis for quasi-symmetric functions [Ges84].

A *dual equivalence* for $(\mathcal{A},\mathrm{Des})$ is a family of involutions on $\mathcal{A}$ with prescribed fixed point sets depending on Des that satisfies certain commutativity relations. A dual equivalence is compatible with stat when the involutions preserve the statistic.

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From this framework, we obtain an explicit set $\mathrm{Dom} \subset \mathcal{A}$ such that

$$f(X;q) \;=\; \sum_{\lambda} \left(\sum_{\substack{S \in \mathrm{Dom}(\mathcal{A}) \\ \alpha(S)=\lambda}} q^{\mathrm{stat}(S)} \right) s_{\lambda}(X),$$

where α is a map derived from Des that associates to each element of Dom a partition. In particular, giving a dual equivalence for the data $(\mathcal{A}, \mathrm{Des})$ that is compatible with stat proves that the generating function $f(X;q)$ is symmetric and Schur positive and provides an explicit combinatorial formula for the Schur coefficients.

In practice, $\mathcal{A}$ often comes with a notion of a reading word, and the descent set of an object is usually defined to be the descent set or inverse descent set of the reading word of that object. We present simplifications of this machinery specific to this ubiquitous case.

2. Combinatorics of Young tableaux

A *partition* λ is a decreasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0.$$

A basis for the space of symmetric functions homogeneous of degree n is naturally indexed by partitions λ whose parts λ_i sum to n, denoted $|\lambda| = n$.

Similarly, a *composition* α is an ordered sequence of positive integers

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell), \quad \alpha_1, \alpha_2, \cdots, \alpha_\ell > 0.$$

A basis for the space of quasi-symmetric functions homogeneous of degree n is naturally indexed by compositions α whose parts α_i sum to n, denoted $|\alpha| = n$. Compositions of n are in bijection with subsets of $[n-1] = \{1, 2, \ldots, n-1\}$ via

$$\begin{array}{rcl} (\alpha_1, \alpha_2, \ldots, \alpha_\ell) & \longmapsto & \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1}\}, \\ (D_1, D_2 - D_1, \ldots, n - D_{\ell-1}) & \longleftarrow\!\shortmid & \{D_1 < D_2 < \ldots < D_{\ell-1}\}. \end{array}$$

Therefore bases of quasi-symmetric functions homogeneous of degree n are also indexed by subsets of $[n-1]$.

We identify a partition λ with its Young diagram, the collection of left-justified cells with λ_i cells in row i. A *standard Young tableau*, or simply *tableau*, of shape λ is a bijective filling of the cells of λ with entries from $[n]$, where $n = |\lambda|$, such that entries increase along rows and up columns. For example, see Figure 1.

$$\begin{array}{|c|c|cc} \hline 7 & 9 & \multicolumn{1}{|c}{} & \\ \hline 3 & 4 & \multicolumn{1}{c|}{6} & \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

Figure 1. A standard Young tableau of shape $(4,3,2)$.

To each tableau T, we associate the following combinatorial data. The *reading word of* T, denoted $w(T)$, is the permutation obtained by reading the rows of T from top to bottom. For example, for T the tableau in Figure 1, we have

$$w(T) = (7, 9, 3, 4, 6, 1, 2, 5, 8).$$

The *descent set of* T, denoted $\mathrm{Des}(T)$, is the subset of $[n-1]$ consisting of all entries i for which $i+1$ lies in a higher row than i. Equivalently, $\mathrm{Des}(T)$ is the inverse descent set of the permutation $w(T)$. For the example in Figure 1, we have

$$\mathrm{Des}(T) = \{2, 5, 6, 8\}.$$

To make the connection with symmetric and quasi-symmetric functions, let X denote the variables $x_1, x_2, \ldots$. Recall Gessel's fundamental basis for quasi-symmetric functions homogeneous of degree n [Ges84] given by

$$Q_D(X) \;=\; \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j = i_{j+1} \Rightarrow j \notin D}} x_{i_1} \cdots x_{i_n}, \tag{2.1}$$

where the indexing set is a subset $D \subseteq [n-1]$.

The most fundamental basis for symmetric functions homogeneous of degree n is the Schur function basis, which Gessel [Ges84] showed may be defined by

$$s_\lambda(X) \;=\; \sum_{T \in \mathrm{SYT}(\lambda)} Q_{\mathrm{Des}(T)}(X). \tag{2.2}$$

where $\mathrm{SYT}(\lambda)$ denotes the set of all standard Young tableau of shape λ.

Define Haiman's *elementary dual equivalence involutions* [Hai92], denoted d_i with $1 < i < n$, on permutations as follows. For w a permutation, if i lies between $i-1$ and $i+1$ in w, then $d_i(w) = w$. Otherwise, d_i interchanges i and whichever of $i \pm 1$ is further away. Two permutations w and u are *dual equivalent* if there exists a sequence $i_1, \ldots, i_k$ such that $u = d_{i_k} \cdots d_{i_1}(w)$. For examples, see Figure 2.

$$\begin{array}{lll}
\{2314 \xleftrightarrow{d_2} 1324 \xleftrightarrow{d_3} 1423\} & \{2143 \underset{d_3}{\overset{d_2}{\leftrightarrows}} 3142\} & \{1432 \xleftrightarrow{d_2} 2431 \xleftrightarrow{d_3} 3421\} \\
\{2341 \xleftrightarrow{d_2} 1342 \xleftrightarrow{d_3} 1243\} & & \{4312 \xleftrightarrow{d_2} 4213 \xleftrightarrow{d_3} 3214\} \\
\{2134 \xleftrightarrow{d_2} 3124 \xleftrightarrow{d_3} 4123\} & \{2413 \underset{d_3}{\overset{d_2}{\leftrightarrows}} 3412\} & \{4132 \xleftrightarrow{d_2} 4231 \xleftrightarrow{d_3} 3241\}
\end{array}$$

FIGURE 2. The nontrivial dual equivalence classes of $\mathfrak{S}_4$.

Haiman [Hai92] showed that the dual equivalence involutions on permutations extend to tableaux via their reading words and that, under this action, the dual equivalence classes correspond precisely to all tableaux of the same shape. For examples, see Figure 3. Note that taking the reading words of these tableaux gives the leftmost two classes in the lower row of Figure 2.

$$\begin{array}{|c|cc} \hline 2 & & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \xleftrightarrow{d_2} \begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \xleftrightarrow{d_3} \begin{array}{|c|cc} \hline 4 & & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \underset{d_3}{\overset{d_2}{\leftrightarrows}} \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$$

FIGURE 3. Two dual equivalence classes of SYT of size 4.

Given this, we may rewrite (2.2) in terms of dual equivalence classes as

$$s_\lambda(X) \;=\; \sum_{T \in [T_\lambda]} Q_{\mathrm{Des}(T)}(X), \tag{2.3}$$

where $[T_\lambda]$ denotes the dual equivalence class of some fixed $T_\lambda \in \mathrm{SYT}(\lambda)$.

This paradigm shift to summing over objects in a dual equivalence class is generalized in the following section to give a universal method for proving that a quasi-symmetric generating function is symmetric and Schur positive.

3. Characterization of dual equivalence

Suppose we are given a function $f(X;q)$ of the form

$$f(X;q) \;=\; \sum_{T\in\mathcal{A}} q^{\mathrm{stat}(T)} Q_{\mathrm{Des}(T)}(X), \tag{3.1}$$

where $\mathcal{A}$ is some finite set of combinatorial objects, Des is a notion of descent set for objects in $\mathcal{A}$, and stat is some nonnegative integer statistic on $\mathcal{A}$. Motivated by (2.3), we will define an equivalence relation on objects in $\mathcal{A}$ so that the sum over objects in any single equivalence class is a single Schur function.

Definition 3.1. Let $\mathcal{A}$ be a finite set of combinatorial objects, and let Des be a descent set map on $\mathcal{A}$ such that $\mathrm{Des}(T) \subseteq [n-1]$ for all $T \in \mathcal{A}$. A *dual equivalence for* $(\mathcal{A}, \mathrm{Des})$ is a family of involutions $\{\varphi_i\}_{1<i<n}$ on $\mathcal{A}$ satisfying the following conditions:

(i) (*fixed points*) The fixed points of φ_i are given by
$$\mathcal{A}^{\varphi_i} \;=\; \{T \in \mathcal{A} \mid i-1 \in \mathrm{Des}(T) \Leftrightarrow i \in \mathrm{Des}(T)\}\,.$$

(ii) (*descent set*) For $T \in \mathcal{A} \setminus \mathcal{A}^{\varphi_i}$, we have
$$\begin{aligned} \{j\} \cap \mathrm{Des}(T) &\neq \{j\} \cap \mathrm{Des}(\varphi_i(T)) && \text{for } j = i-1 \text{ or } j = i, \\ \{j\} \cap \mathrm{Des}(T) &= \{j\} \cap \mathrm{Des}(\varphi_i(T)) && \text{for } j < i-2 \text{ or } i+1 < j. \end{aligned}$$

(iii) (*equality*) For $T \in \mathcal{A} \setminus \mathcal{A}^{\varphi_i}$, we have
$$\begin{aligned} \{i-2\} \cap \mathrm{Des}(T) \neq \{i-2\} \cap \mathrm{Des}(\varphi_i(T)) &\;\Leftrightarrow\; \varphi_i(T) = \varphi_{i-1}(T), \\ \{i+1\} \cap \mathrm{Des}(T) \neq \{i+1\} \cap \mathrm{Des}(\varphi_i(T)) &\;\Leftrightarrow\; \varphi_i(T) = \varphi_{i+1}(T). \end{aligned}$$

(iv) (*commutativity*) For $T \in \mathcal{A}$ and $|i-j| \geq 3$, we have
$$\varphi_j \circ \varphi_i(T) = \varphi_i \circ \varphi_j(T).$$

(v) (*minimality*) For any $2 \leq h \leq i < n$, if $S = \varphi_{i_\ell} \circ \cdots \circ \varphi_{i_1}(T)$ for indices $h \leq i_1, \ldots, i_\ell \leq i$, then there exist indices $h \leq j_1, \ldots, j_m \leq i$ with at most one $j_k = i$ such that $S = \varphi_{j_m} \circ \cdots \circ \varphi_{j_1}(T)$.

The first step towards justifying this definition is to verify that Haiman's dual equivalence involutions indeed satisfy these conditions.

Proposition 3.2. *The involutions d_i give a dual equivalence for* $(\mathrm{SYT}(n), \mathrm{Des})$.

Proof. The fixed points of d_i are tableaux where $i-1, i, i+1$ appear in increasing or decreasing order in the reading word which precisely corresponds to condition (i). When a tableau is not a fixed point for d_i, the potential inverse descents between $i-1, i$ and $i, i+1$ are switched while all other letters remain fixed, thus establishing condition (ii). For condition (iii), note that the potential inverse descent between $i-2$ and $i-1$ differs between T and $d_i(T)$ if and only if $i-1$ and i are exchanged with $i-2$ lying between them. In this case, d_{i-1} also exchanges $i-1$ and i, so $d_{i-1}(T) = d_i(T)$. The analogous argument holds for the potential inverse descent between i and $i+1$, thereby establishing condition (iii). Condition (iv) follows from the fact that for $|i-j| \geq 3$, the sets $\{i-1, i, i+1\}$ and $\{j-1, j, j+1\}$ are disjoint, and so d_i and d_j commute.

For condition (v), we may use Haiman's result that two skew tableaux are dual equivalent if and only if they *jeu de taquin* to the same shape [Hai92] to reduce to the case $h = 2$. As the condition is vacuously true for $n = 1$, we may use induction on n and the fact that the restriction of a tableau to cells containing entries less than $i+1$ is again a tableau to reduce to the case $i = n-1$.

Let $T, S \in \mathrm{SYT}(\lambda)$ with $|\lambda| = n$, and suppose $S = d_{i_\ell} \circ \cdots \circ d_{i_1}(T)$ with $i_1, \ldots, i_\ell \leq n-1$. If the positions of the cells containing n are the same for both T and S, then removing the cells containing n results in two tableaux, say T' and S', of the same shape of size $n-1$. By Haiman's result, these must be dual equivalent, i.e. $S' = d_{j_m} \circ \cdots \circ d_{j_1}(T')$ with $j_1, \ldots, j_m \leq n-2$. Otherwise, choose $U \in \mathrm{SYT}(\lambda)$ so that n lies in the same position in U and T, the cell containing $n-1$ in U lies in the same position as the cell containing n in S, and $n-2$ lies between n and $n-1$ in the reading word for U. Since the cell containing n must be an outer corner for both T and S, there always exists such a U. Then, since U and T have the same shape with n removed, $U = d_{j_{k-1}} \circ \cdots \circ d_{j_1}(T)$ for some $j_1, \ldots, j_{k-1} \leq n-2$. Since $d_{n-1}(U)$ interchanges n and $n-1$ in U, $d_{n-1}(U)$ and S have the same shape with n removed, and so $S = d_{j_m} \circ \cdots \circ d_{j_{k+1}}(d_{n-1}(U))$ for some $j_{k+1}, \ldots, j_m \leq n-2$. Taking $j_k = n-1$ and substituting establishes condition (v). □

By (2.3), dual equivalence classes of tableaux precisely correspond to Schur functions. Definition 3.1 was formulated so that the same property holds true for dual equivalence classes for any pair $(\mathcal{A}, \mathrm{Des})$.

Theorem 3.3. *If $\{\varphi_i\}_{1<i<n}$ is a dual equivalence for $(\mathcal{A}, \mathrm{Des})$, then for any dual equivalence class $\mathcal{C}$, we have*

$$\sum_{T \in \mathcal{C}} Q_{\mathrm{Des}(T)}(X) \;=\; s_\lambda(X) \tag{3.2}$$

for some partition λ of n. In particular, if $\mathrm{stat}(\varphi_i(T)) = \mathrm{stat}(T)$ for all $1 < i < n$ and all $T \in \mathcal{A}$, then $f(X;q)$ is symmetric and Schur positive.

In order to prove Theorem 3.3, we show that a dual equivalence for $\mathcal{A}$ is equivalent to a dual equivalence graph with vertex set $\mathcal{A}$.

Recall that a signed, colored graph $(\mathcal{A}, \sigma, \Phi)$ consists of the following data: a finite vertex set $\mathcal{A}$, a signature function $\sigma : \mathcal{A} \to \{\pm 1\}^{n-1}$, and for each $1 < i < n$, a collection Φ_i of pairs of distinct elements of $\mathcal{A}$.

To make the connection with a dual equivalence for $(\mathcal{A}, \mathrm{Des})$, regard σ as the indicator function for Des. That is,

$$\sigma(T)_i = +1 \;\Leftrightarrow\; i \in \mathrm{Des}(T). \tag{3.3}$$

Recall the axiomatic definition of a dual equivalence graph [Ass](Definition 3.4).

Definition 3.4. A signed, colored graph $(\mathcal{A}, \sigma, \Phi)$ is a *dual equivalence graph* if the following hold:

(ax1) For $T \in \mathcal{A}$ and $1 < i < n$, $\sigma(T)_{i-1} = -\sigma(T)_i$ if and only if there exists $S \in \mathcal{A}$ such that $\{T, S\} \in \Phi_i$. Moreover, S is unique when it exists.

(ax2) For $\{T, S\} \in \Phi_i$, $\sigma(T)_j = -\sigma(S)_j$ for $j = i-1, i$, and $\sigma(T)_h = \sigma(S)_h$ for $h < i-2$ and $h > i+1$.

(ax3) For $\{T, S\} \in \Phi_i$, if $\sigma(T)_{i-2} = -\sigma(S)_{i-2}$, then $\sigma(T)_{i-2} = -\sigma(T)_{i-1}$, and if $\sigma(T)_{i+1} = -\sigma(S)_{i+1}$, then $\sigma(T)_{i+1} = -\sigma(T)_i$.

(ax4) For every $i < n$, every nontrivial connected component of $\Phi_{i-1} \cup \Phi_i$ and $\Phi_{i-2} \cup \Phi_{i-1} \cup \Phi_i$ appears in Figure 4 and Figure 5, respectively.

(ax5) If $\{T, S\} \in \Phi_i$ and $\{S, R\} \in \Phi_j$ for $|i - j| \geq 3$, then $\{T, U\} \in \Phi_j$ and $\{U, R\} \in \Phi_i$ for some $U \in \mathcal{A}$.

(ax6) Any two vertices of a connected component of $(\mathcal{A}, \sigma, \Phi_2 \cup \cdots \cup \Phi_i)$ may be connected by a path crossing at most one Φ_i edge.

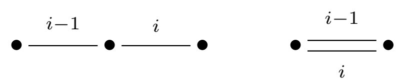

FIGURE 4. Allowed components for $\Phi_{i-1} \cup \Phi_i$.

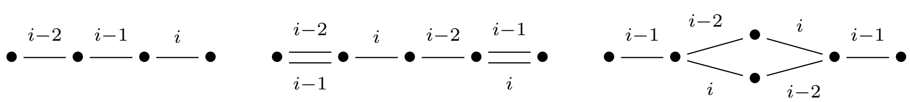

FIGURE 5. Allowed components for $\Phi_{i-2} \cup \Phi_{i-1} \cup \Phi_i$.

We begin the correspondence by showing that every dual equivalence graph $(\mathcal{A}, \sigma, \Phi)$ gives rise to a dual equivalence for $(\mathcal{A}, \mathrm{Des})$.

Theorem 3.5. *Let* $(\mathcal{A}, \sigma, \Phi)$ *be a dual equivalence graph, and let* Des *be the descent function on* $\mathcal{A}$ *satisfying* (3.3). *Then the maps* $\varphi_i : \mathcal{A} \to \mathcal{A}$, $1 < i < n$, *defined by*

$$\varphi_i(T) \;=\; \begin{cases} S & \text{if } \{T, S\} \in \Phi_i \\ T & \text{if } \{T, S\} \notin \Phi_i \text{ for all } S \in \mathcal{A} \end{cases} \tag{3.4}$$

give a dual equivalence for $(\mathcal{A}, \mathrm{Des})$.

Proof. Axiom 1 shows that each φ_i is an involution and that condition (i) holds. Similarly, axioms 2, 5, and 6 directly translate to conditions (ii), (iv), and (v), respectively. Therefore we need only establish condition (iii).

To that end, let $T \in \mathcal{A} \setminus \mathcal{A}^{\varphi_i}$. If $\varphi_i(T) = \varphi_{i-1}(T)$, then, by condition (ii),

$$\{i-2\} \cap \mathrm{Des}(T) \neq \{i-2\} \cap \mathrm{Des}(\varphi_{i-1}(T)) = \{i-2\} \cap \mathrm{Des}(\varphi_i(T)),$$

and similarly for $\varphi_i(T) = \varphi_{i+1}(T)$. Conversely, we have

$$\{i-2\} \cap \mathrm{Des}(T) \neq \{i-2\} \cap \mathrm{Des}(\varphi_i(T)) \Leftrightarrow \sigma(T)_{i-2} = -\sigma(\varphi_i(T))_{i-2}$$

By axiom 2, $\sigma(T)_{i-3} = \sigma(\varphi_i(T))_{i-3}$, and so by axiom 1 exactly one of T and $\varphi_i(T)$ will have an $i-2$ edge. Inspecting Figure 5, axiom 4 implies that we must have $\varphi_i(T) = \varphi_{i-1}(T)$. Again, the analogous argument holds for $\{i+1\} \cap \mathrm{Des}(T) \neq \{i+1\} \cap \mathrm{Des}(\varphi_i(T))$, and so $\{\varphi_i\}$ is a dual equivalence for $(\mathcal{A}, \mathrm{Des})$. □

The converse of Theorem 3.5 requires a more elaborate diagram chase, though the proof is equally straightforward.

Theorem 3.6. *Let $\{\varphi_i\}$ be a dual equivalence for $(\mathcal{A}, \mathrm{Des})$, and let σ be the signature function satisfying* (3.3)*. Then the graph $(\mathcal{A}, \sigma, \Phi)$, where Φ_i is given by*

$$\Phi_i \;=\; \{\; \{T, \varphi_i(T)\} \mid T \notin \mathcal{A}^{\varphi_i} \,\}\,, \tag{3.5}$$

is a dual equivalence graph.

Proof. Condition (i) and the fact that φ_i is an involution shows that axiom 1 holds. Similarly, axioms 2, 5, and 6 are direct translations of conditions (ii), (iv), and (v), respectively. Therefore we need only establish axioms 3 and 4.

To show axiom 3 holds, let $\{T, S\} \in \Phi_i$. Then since $S = \varphi_i(T)$, we have

$$\begin{array}{rcl}
\sigma(T)_{i-2} = -\sigma(S)_{i-2} & \Leftrightarrow & \{i-2\} \cap \mathrm{Des}(T) \neq \{i-2\} \cap \mathrm{Des}(\varphi_i(T)) \\
& \stackrel{\text{(iii)}}{\Rightarrow} & T \neq \varphi_i(T) = \varphi_{i-1}(T) \\
& \stackrel{\text{(i)}}{\Rightarrow} & (i-2 \in \mathrm{Des}(T) \Leftrightarrow i-1 \notin \mathrm{Des}(T)) \\
& \Leftrightarrow & \sigma(T)_{i-2} = -\sigma(T)_{i-1}.
\end{array}$$

The analogous argument holds when $\sigma(T)_{i+1} = -\sigma(S)_{i+1}$, and so axiom 3 follows.

To show that nontrivial connected components of $\Phi_{i-1} \cup \Phi_i$ appear in Figure 4 requires more work. First, suppose $T \in \mathcal{A}^{\varphi_i}$. By (i),

$$\{i-2, i-1, i\} \cap \mathrm{Des}(T) = \{i-2\} \text{ or } \{i-1, i\}.$$

Without loss of generality, we assume the former is the case. Then

$$\begin{array}{rl}
\text{(ii)} \;\Rightarrow & \{i-2, i-1\} \cap \mathrm{Des}(\varphi_{i-1}(T)) = \{i-2, i-1\} \setminus \mathrm{Des}(T) = \{i-1\}, \\
\text{(iii)} \;\Rightarrow & \{i\} \cap \mathrm{Des}(\varphi_{i-1}(T)) = \{i\} \cap \mathrm{Des}(T) = \varnothing.
\end{array}$$

Therefore, by (i), we have $\varphi_{i-1}(T) \notin \mathcal{A}^{\varphi_i}$. Thus

$$\begin{array}{rl}
\text{(ii)} \;\Rightarrow & \{i-1, i\} \cap \mathrm{Des}(\varphi_i(\varphi_{i-1}(T))) = \{i-1, i\} \setminus \mathrm{Des}(\varphi_{i-1}(T)) = \{i\}, \\
\text{(iii)} \;\Rightarrow & \{i-2\} \cap \mathrm{Des}(\varphi_i(\varphi_{i-1}(T))) = \{i-2\} \cap \mathrm{Des}(\varphi_{i-1}(T)) = \varnothing.
\end{array}$$

Therefore, by (i), we have $\varphi_i(\varphi_{i-1}(T)) \in \mathcal{A}^{\varphi_{i-1}}$. The analogous arguments hold if the component contains an element $T \in \mathcal{A}^{\varphi_{i-1}}$. In either case, the connected component has the structure of the left graph of Figure 4.

Now suppose the component contains only elements of $\mathcal{A} \setminus (\mathcal{A}^{\varphi_{i-1}} \cup \mathcal{A}^{\varphi_i})$. Then, condition (i) forces

$$\{i-2, i-1, i\} \cap \mathrm{Des}(T) = \{i-1\} \text{ or } \{i-2, i\}$$

for every T in the component. Since $T \neq \varphi_i(T)$, by condition (ii) we have

$$\{i-2, i-1, i\} \cap \mathrm{Des}(T) \neq \{i-2, i-1, i\} \cap \mathrm{Des}(\varphi_i(T)).$$

In particular, $\{i-2\} \cap \mathrm{Des}(T) \neq \{i-2\} \cap \mathrm{Des}(\varphi_i(T))$. Condition (iii) therefore ensures that $\varphi_i(T) = \varphi_{i-1}(T)$. Thus the connected component has the structure of the right graph of Figure 4.

Finally, we must show that nontrivial connected components of $\Phi_{i-1} \cup \Phi_i \cup \Phi_{i+1}$ appear in Figure 5. First note that conditions (i), (ii) and (iii) ensure that every component will contain a vertex $T \in \mathcal{A}^{\varphi_i}$.

If $T \in \mathcal{A}^{\varphi_i} \cap \mathcal{A}^{\varphi_{i\pm 1}}$, then the same argument and descent set analysis used for the case $T \in \mathcal{A}^{\varphi_i}$ for connected components of $\Phi_{i-1} \cup \Phi_i$ shows that the component must have the form of the far left graph in Figure 5. Therefore we may assume that $T \in \mathcal{A}^{\varphi_i}$ and $T \notin \mathcal{A}^{\varphi_{i\pm 1}}$. Consider the sequence

$$\cdots\ T_{-2} \overset{\varphi_{i+1}}{\longleftrightarrow} T_{-1} \overset{\varphi_{i-1}}{\longleftrightarrow} T_0 \overset{\varphi_{i+1}}{\longleftrightarrow} T_1 \overset{\varphi_{i-1}}{\longleftrightarrow} T_2 \overset{\varphi_{i+1}}{\longleftrightarrow} T_3\ \cdots$$

defined by $T_0 = T$ and for $k \in \mathbb{Z}$,

$$T_k = \begin{cases} \varphi_{i-1}(T_{k-1}) & k \text{ even} \\ \varphi_{i+1}(T_{k-1}) & k \text{ odd.} \end{cases}$$

Since $\mathcal{A}$ is finite, the set of vertices that appear in this sequence must also be finite. This can happen in one of two ways. Either the sequence loops: $T_{-h} = T_j$ for some $h, j > 0$; or both directions hit fixed points: $T_{-(h+1)} = T_{-h}$ and $T_j = T_{j+1}$ for some $h, j > 0$. We will use condition (v) to show that, in fact, either $T_{-2} = T_2$ or $T_{-3} = T_{-2}$ and $T_3 = T_2$.

Assuming the sequence loops, by tracking $i - 2$ and $i + 1$ in $\mathrm{Des}(T_k)$ using conditions (ii) and (iii), we see that $T_{-h} = T_j$ only if $j + h \equiv 0 \mod 4$. By symmetry, we may always take $j = h$. In this case, tracking descents reveals that $T_{2k} \in \mathcal{A}^{\varphi_i}$ and $\varphi_i(T_{2k+1}) \in \mathcal{A}^{\varphi_{i-1}} \cap \mathcal{A}^{\varphi_{i+1}}$. Therefore the only sequences of dual equivalences taking T_k to $T_{k'}$ are alternating between φ_{i-1} and φ_{i+1}. If $j > 2$, then T_{-2} and T_1 violate condition (v). Therefore $j = 2$, resulting in the far right graph of Figure 5.

Now assume both directions reach fixed points. By (iii), $T_{\pm k} = T_{\pm(k+1)}$ with $|k|$ minimal if and only if $T_{\pm k} = \varphi_i(T_{\pm(k-1)})$. Chasing descent sets reveals that the first point where this happens in either direction must be for k even. If $T_{-3} \neq T_{-2}$, then T_{-2} and T_1 violate condition (v), and if $T_3 \neq T_2$, then T_3 and T_0 violate condition (v). Therefore $\varphi_i(T_{-2}) = \varphi_{i+1}(T_{-2})$ and $\varphi_i(T_2) = \varphi_{i-1}(T_2)$ resulting in the middle graph of Figure 5. □

By construction, dual equivalence classes precisely contain the vertices of a connected component of the corresponding dual equivalence graph. Therefore Theorem 3.3 now follows from Theorem 3.6 and [Ass](Corollary 3.11).

Remark 3.7. In the proof of Theorem 3.6, condition (v) was only used when $h = 2$ and $h = i - 2$. Therefore the statement in Definition 3.1 could be reduced to this smaller case.

4. Schur expansions

Theorem 3.3 shows that dual equivalence may be used to prove that a function is symmetric and Schur positive, and it gives a combinatorial interpretation of the Schur coefficients as the number of equivalence classes of a certain type. In this section, we give a more direct interpretation of the Schur coefficients in terms of distinguished elements of $\mathcal{A}$.

For $T \in \mathcal{A}$, let $\alpha(T)$ be the composition of n corresponding to $\mathrm{Des}(T)$. Recall the dominance order on partitions of n, which we extend to compositions of n by

$$\alpha \geq \beta \ \Leftrightarrow\ \alpha_1 + \cdots + \alpha_k \geq \beta_1 + \cdots + \beta_k \ \forall k. \tag{4.1}$$

We can now define the set of distinguished elements.

Definition 4.1. Let $\{\varphi_i\}_{1<i<n}$ be a dual equivalence for $(\mathcal{A}, \mathrm{Des})$. Then $T \in \mathcal{A}$ is called *dominant* if $\alpha(T) \geq \alpha(S)$ for every S in the dual equivalence class of T.

Since dominance order is a partial order, it is not immediately obvious that dominant vertices exist. In fact, not only do they exist, but each dual equivalence class contains a unique dominant element, and $\alpha(T)$ is a partition for T dominant.

Theorem 4.2. *Let* $\{\varphi_i\}$ *be a dual equivalence for* $(\mathcal{A}, \mathrm{Des})$ *preserving* stat. *Then*

$$f(X;q) = \sum_{T \in \mathrm{Dom}(\mathcal{A})} q^{\mathrm{stat}(T)} s_{\alpha(T)}(X) = \sum_{\lambda} \left(\sum_{\substack{T \in \mathrm{Dom}(\mathcal{A}) \\ \alpha(T) = \lambda}} q^{\mathrm{stat}(T)} \right) s_{\lambda}(X), \tag{4.2}$$

where $\mathrm{Dom}(\mathcal{A})$ *is the set of dominant objects of* $\mathcal{A}$ *with respect to* $\{\varphi_i\}$.

Proof. Given λ, let $T_\lambda \in \mathrm{SYT}(\lambda)$ denote the *superstandard tableau of shape* λ obtained by filling the first row with $1, 2, \ldots, \lambda_1$, the second row with $\lambda_1+1, \ldots, \lambda_1+\lambda_2$, and so on. For example, see Figure 6. For any $T \in \mathrm{SYT}(\lambda)$, we have $\alpha(T) \leq \lambda$ with equality if and only if $T = T_\lambda$. Since the dual equivalence classes on tableaux include all tableaux of a given shape, each dual equivalence class contains a unique dominant element, and the map alpha gives the corresponding Schur function for the class. The result for arbitrary $\mathcal{A}$ now follows from Theorem 3.3 since the descent sets must be the same for the elements of a dual equivalence class and the set of tableaux of shape λ for some partition λ. $\square$

$$\begin{array}{|c|c|c|c|} \hline 8 & 9 \\ \cline{1-2} 5 & 6 & 7 \\ \cline{1-3} 1 & 2 & 3 & 4 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c|c|} \hline 3 & 6 \\ \cline{1-2} 2 & 5 & 8 \\ \cline{1-3} 1 & 4 & 7 & 9 \\ \hline \end{array}$$

FIGURE 6. The superstandard (dominant) and substandard (subordinate) tableaux of shape $(4, 3, 2)$.

Remark 4.3. Theorem 4.2 makes use of the implicit bijection between $\mathcal{A}$ and tableaux that exists whenever there is a dual equivalence for $\mathcal{A}$. This bijection can be realized by identifying each $T \in \mathrm{Dom}(\mathcal{A})$ with the superstandard tableau $T_{\alpha(T)}$ and then applying the same sequence of dual equivalence involutions to both.

Remark 4.4. There is another distinguished element that can be chosen from each equivalence class which is almost as natural as the dominant element. Say that $T \in \mathcal{A}$ is *subordinate* if $\alpha(T) \leq \alpha(S)$ for every S in the dual equivalence class of T. For example, the right tableaux in Figure 6 is the subordinate tableaux of shape $(4, 3, 2)$. Then each dual equivalence class contains a unique subordinate vertex. Define a map β on subordinate vertices by sending T to $\alpha([n-1] \setminus \mathrm{Des}(T))'$. That is, complement the set and conjugate the shape. Then in Theorem 4.2, the set Dom of dominant vertices may be replaced with the set Sub of subordinate vertices when α is replaced with β.

In order to find the dominant vertices for a dual equivalence $\{\varphi_i\}$ for $(\mathcal{A}, \mathrm{Des})$, one can immediately eliminate all $T \in \mathcal{A}$ where $\alpha(T)$ is not a partition. However, since the definition of dominant depends on the entire equivalence class, this is not sufficient. For example, neither of the tableaux in Figure 6 is dominant, though both have α equal to a partition. Still one need not compare T with every element of the equivalence class in order to decide whether or not T is dominant.

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} 8 \\ \cline{1-3} 5 & 6 & 7 \\ \hline 1 & 2 & 3 & 4 & 9 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|}\cline{1-1} 8 \\ \cline{1-3} 5 & 6 & 9 \\ \hline 1 & 2 & 3 & 4 & 7 \\ \hline\end{array}$$

FIGURE 7. Two tableaux of shape $(5, 3, 1)$ with $\alpha = (4, 3, 2)$.

It may happen that the dual equivalence is natural enough that the set of dominant elements can be described explicitly. However, if that is not the case and one is interested in making fast computations of the Schur expansion, then the methods of Egge, Loehr and Warrington [ELW10] using a modified inverse Kostka matrix to give a non-positive integral formula may be of use.

5. Reading words

In practice, the descent set associated to the set of objects $\mathcal{A}$ is almost always defined via a *reading word* for elements of $\mathcal{A}$. That is, there is a map $w : \mathcal{A} \to \mathfrak{S}_n$ such that for $T \in \mathcal{A}$, either $\mathrm{Des}(T) = \mathrm{Des}(w(T))$ or $\mathrm{Des}(T) = \mathrm{Des}(w(T)^{-1})$. Though the latter case, i.e. inverse descent set of $w(T)$, is more commonly used, we still refer to this as the descent set of T. Given the prevalence of this model, we consider how the definition of dual equivalence may be simplified given a sufficiently nice reading word.

For $\mathfrak{S}_3$, there are only two possible dual equivalences when $\mathrm{Des}(w)$ is taken to be the inverse descent set of w. One is given by Haiman's dual equivalence involutions, and the other is given by the *twisted dual equivalence involutions*, denoted $\widetilde{d}_i$ with $1 < i < n$, defined as follows. For w a permutation, if i lies between $i - 1$ and $i + 1$ in w, then $\widetilde{d}_i(w) = w$. Otherwise, $\widetilde{d}_i$ cyclically rotates $i - 1, i, i + 1$ so that i lies on the other side of $i - 1$ and $i + 1$. For example, see Figure 8.

$$d_2 \circlearrowright 1\ 2\ 3 \circlearrowleft \widetilde{d}_2 \qquad \begin{tikzcd} 2\ 1\ 3 \arrow[rr, leftrightarrow, "d_2"] \arrow[drr, leftrightarrow, "\widetilde{d}_2"'] & & 3\ 1\ 2 \arrow[dll, leftrightarrow, "\widetilde{d}_2"] \\ 2\ 3\ 1 \arrow[rr, leftrightarrow, "d_2"'] & & 1\ 3\ 2 \end{tikzcd} \qquad d_2 \circlearrowright 3\ 2\ 1 \circlearrowleft \widetilde{d}_2$$

FIGURE 8. The two dual equivalences for $\mathfrak{S}_3$.

It is easy to see that any family of involutions such that $\varphi_i(w) = d_i(w)$ or $\widetilde{d}_i(w)$ must satisfy the fixed points (i) and descent set (ii) conditions. Moreover, if the decision of which involution to apply is independent of the positions of $[n] \setminus \{i - 1, i, i + 1\}$, then the commutativity (iv) condition also holds. Unfortunately, the equality (iii) condition and minimality (v) condition will often fail, even for a rule as simple as $\varphi_i(w) = \widetilde{d}_i(w)$ for all w. However, under most circumstances, the dual equivalence classes will still be Schur positive, though not necessarily equal to a single Schur function. Therefore we introduce a more general notion to capture the Schur positivity for equivalence classes.

For $J \subseteq [n]$ and $w \in \mathfrak{S}_n$, let w_J denote the subword of w using letters from J, that is $w_{j_1} \cdots w_{j_m}$ where $J = \{w_{j_1}, \ldots, w_{j_m}\}$ and $j_1 < \cdots < j_m$. For any subword u of a permutation, define the *standardization of* u, denoted $\mathrm{st}(u)$, to be the word obtained by replacing u_i with i, where $u_1 < u_2 < \cdots < u_k$ are the letters of u taken in increasing order.

Definition 5.1. Let $\mathcal{A}$ be a finite set of combinatorial objects of fixed degree n, and let $w : \mathcal{A} \to \mathfrak{S}_n$ be a reading word on $\mathcal{A}$. A *D equivalence for* $(\mathcal{A}, w)$ is a family of involutions $\{\varphi_i\}_{1<i<n}$ on $\mathcal{A}$ satisfying the following conditions:

(a) (*restriction*) For $T \in \mathcal{A}$ and $J = \{i-1, i, i+1\}$, we have

$$\begin{array}{rcl} w\left(\varphi_i(T)\right)_J & = & d_i(w(T))_J \ \text{ or } \ \widetilde{d}_i(w(T))_J \\ w\left(\varphi_i(T)\right)_{[n]\setminus J} & = & w(T)_{[n]\setminus J} \end{array}$$

(b) (*equality*) For $T \in \mathcal{A}$, if $\varphi_{i-1}(T) \in \mathcal{A}^{\varphi_{i+1}}$ and $\varphi_{i+1}(T) \in \mathcal{A}^{\varphi_{i-1}}$, then either $\varphi_{i-1}(T) = T = \varphi_{i+1}(T)$ or $\varphi_i(T) = \varphi_{i-1}(T)$ or $\varphi_i(T) = \varphi_{i+1}(T)$.

(c) (*commutativity*) For $T \in \mathcal{A}$ and $|i-j| \geq 3$, we have

$$\varphi_j \circ \varphi_i(T) = \varphi_i \circ \varphi_j(T).$$

(d) (*local Schur positivity*) For $J = \{i-1, i\}$ or $\{i-1, i, i+1\}$ and $\mathcal{C}$ an equivalence class under $\{\varphi_j\}_{j \in J}$, the restricted generating function

$$\sum_{T \in \mathcal{C}} Q_{\mathrm{Des}(\mathrm{st}(w(T)_J))}(X),$$

where st is the standardization map defined above, is Schur positive.

Unlike the definition of dual equivalence, the defining conditions for D equivalence are completely local. Under this relaxation, a D equivalence class will not, in general, correspond to a single Schur function, but it will still be Schur positive.

Theorem 5.2. *If $\{\varphi_i\}_{1<i<n}$ is a D equivalence for $(\mathcal{A}, w)$, then for any D equivalence class $\mathcal{C}$, we have*

$$\sum_{T \in \mathcal{C}} Q_{\mathrm{Des}(T)}(X) \tag{5.1}$$

is Schur positive. In particular, if $\mathrm{stat}(\varphi_i(T)) = \mathrm{stat}(T)$ *for all* $1 < i < n$ *and all* $T \in \mathcal{A}$*, then* $f(X; q)$ *is symmetric and Schur positive.*

Similar to the proof of Theorem 3.3, we will prove Theorem 5.2 by showing that a D equivalence for $\mathcal{A}$ gives rise to a generalized dual equivalence graph, called a *D graph*, with vertex set $\mathcal{A}$. The following definition is based on [Ass](Definitions 4.5 and 5.1), which are the defining characteristics of the graphs studied there.

Definition 5.3. A *D graph* is a locally Schur positive graph satisfying dual equivalence graph axioms $1, 2, 3$ and 5 such that for every connected component of $\Phi_{i-1} \cup \Phi_{i+1}$ with exactly two edges, say with vertices $\Phi_{i-1}(w), w, \Phi_{i+1}(w)$, either $\Phi_i(w) = \Phi_{i-1}(w)$ or $\Phi_i(w) = \Phi_{i+1}(w)$.

Theorem 5.4. *Let $\{\varphi_i\}$ be a D equivalence for $(\mathcal{A}, w)$, and define σ and Φ by*

$$\sigma(T)_i = +1 \ \Leftrightarrow \ i \in \mathrm{Des}(w(T)) \quad \textit{and} \quad \Phi_i \ = \ \left\{ \, \{T, \varphi_i(T)\} \mid T \notin \mathcal{A}^{\varphi_i} \right\},$$

respectively. Then the graph $(\mathcal{A}, \sigma, \Phi)$ is a D graph.

Proof. The outline of the proof is to show that condition (a) implies axioms 1 and 2, condition (d) implies axiom 3 and local Schur positivity, condition (b) implies the restriction on connected components of $\Phi_{i-1} \cup \Phi_{i+1}$, and condition (c) implies axiom 5. Of these assertions, the only one that is not immediately obvious is that condition (d) implies axiom 3.

Suppose, for contradiction, that axiom 3 fails for some $\{T, S\} \in \Phi_i$. That is to say, $\sigma(T)_{i-2} = -\sigma(S)_{i-2}$ and $\sigma(T)_{i-2} = \sigma(T)_{i-1}$. We may assume

$$\{i-2, i-1, i\} \cap \mathrm{Des}(T) \; = \; \{i-2, i-1\}.$$

Since $S = \varphi_i(T)$, these assumptions and condition (a) imply that

$$\{i-2, i-1, i\} \cap \mathrm{Des}(S) \; = \; \{i\}.$$

By condition (a) again, this means that $T, S \in \mathcal{A}^{\varphi_{i-1}}$, and so the equivalence class of T under the actions of φ_{i-1} and φ_i is simply $\{T, S\}$. Therefore the generating function for this class is $Q_{\{i-2,i-1\}} + Q_{\{i\}}$, which is not symmetric, much less Schur positive, violating condition (d). $\square$

By construction, D equivalence classes precisely contain the vertices of a connected component of the corresponding D graph. Therefore Theorem 5.2 now follows from Theorem 5.4 and [Ass](Theorem 5.9).

6. Examples

As a first example, define a family of involutions $\{\varphi_i\}_{1<i<n}$ on $\mathfrak{S}_n$ by

$$\varphi_i(w) = \begin{cases} \widetilde{d}_i(w) & \text{if } i-1, i, i+1 \text{ are consecutive in } w, \\ d_i(w) & \text{otherwise.} \end{cases} \tag{6.1}$$

In [Ass08], we construct a simple bijection f on $\mathfrak{S}_n$ with the property that $f(d_i(w)) = \varphi_i(f(w))$ for all $w \in \mathfrak{S}_n$. This proves that $\{\varphi_i\}$ gives a dual equivalence for $\mathfrak{S}_n$ by Proposition 3.2. Moreover, the set of dominant vertices coincide for φ_i and d_i. For example, compare Figure 2 with Figure 9.

$$\{2314 \overset{\varphi_2}{\longleftrightarrow} 3124 \overset{\varphi_3}{\longleftrightarrow} 4123\} \quad \{2143 \underset{\varphi_3}{\overset{\varphi_2}{\Longleftrightarrow}} 3142\} \quad \{1432 \overset{\varphi_2}{\longleftrightarrow} 2431 \overset{\varphi_3}{\longleftrightarrow} 3241\}$$

$$\{2341 \overset{\varphi_2}{\longleftrightarrow} 1342 \overset{\varphi_3}{\longleftrightarrow} 1423\} \qquad \{4312 \overset{\varphi_2}{\longleftrightarrow} 4231 \overset{\varphi_3}{\longleftrightarrow} 3421\}$$

$$\{2134 \overset{\varphi_2}{\longleftrightarrow} 1324 \overset{\varphi_3}{\longleftrightarrow} 1243\} \quad \{2413 \underset{\varphi_3}{\overset{\varphi_2}{\Longleftrightarrow}} 3412\} \quad \{4132 \overset{\varphi_2}{\longleftrightarrow} 4213 \overset{\varphi_3}{\longleftrightarrow} 3214\}$$

Figure 9. The nontrivial dual equivalence classes of $\mathfrak{S}_4$.

For the next example, consider the involutions $\{\widetilde{d}_i\}_{1<i<n}$ on $\mathfrak{S}_n$. Once again, it is not difficult to show that $\{\widetilde{d}_i\}$ gives a D equivalence for $\mathfrak{S}_n$, and an elementary bijective proof is given in [Ass08]. However, this is not a dual equivalence for $n > 3$. For example, see Figure 10.

It is shown in [Ass](Theorem 4.10) that the Schur expansions for the D equivalence classes in this case have the following simple expansion. Recall that a *ribbon* is a skew diagram containing no 2×2 block of cells. Let ν be a ribbon of size n. Label the cells of ν from 1 to n from northwest to southeast. Define the *descent*

$$\{2314 \overset{\widetilde{d}_2}{\longleftrightarrow} 3124 \overset{\widetilde{d}_3}{\longleftrightarrow} 2143 \overset{\widetilde{d}_2}{\longleftrightarrow} 1342 \overset{\widetilde{d}_3}{\longleftrightarrow} 1423\} \qquad \{1432 \overset{\widetilde{d}_2}{\longleftrightarrow} 2413 \overset{\widetilde{d}_3}{\longleftrightarrow} 3214\}$$
$$\{2341 \overset{\widetilde{d}_2}{\longleftrightarrow} 3142 \overset{\widetilde{d}_3}{\longleftrightarrow} 4123\} \qquad \{4312 \overset{\widetilde{d}_2}{\longleftrightarrow} 4231 \overset{\widetilde{d}_3}{\longleftrightarrow} 3421\}$$
$$\{2134 \overset{\widetilde{d}_2}{\longleftrightarrow} 1324 \overset{\widetilde{d}_3}{\longleftrightarrow} 1243\} \qquad \{4132 \overset{\widetilde{d}_2}{\longleftrightarrow} 4213 \overset{\widetilde{d}_3}{\longleftrightarrow} 3412 \overset{\widetilde{d}_2}{\longleftrightarrow} 2431 \overset{\widetilde{d}_3}{\longleftrightarrow} 3241\}$$

FIGURE 10. The nontrivial D equivalence classes of $\mathfrak{S}_4$.

set of ν, denoted $\mathrm{Des}(\nu)$, to be the set of indices i such that the cell labelled $i+1$ lies immediately south of the cell labelled i. Define the *major index of a ribbon* by

$$\mathrm{maj}(\nu) = \sum_{i \in \mathrm{Des}(\nu)} i. \tag{6.2}$$

Then for any D equivalence class $\mathcal{C}$ of $\mathfrak{S}_n$ under the action of $\widetilde{d}_2, \ldots, \widetilde{d}_{n-1}$, we have

$$\sum_{w \in \mathcal{C}} Q_{\mathrm{Des}(w)}(X) = \sum_{\nu \in \mathrm{Rib}(\mathcal{C})} s_\nu(X), \tag{6.3}$$

where for some (equivalently, every) $w \in \mathcal{C}$, $\mathrm{Rib}(\mathcal{C})$ is the set of ribbons of length n with major index equal to the inversion number of w such that $n-1$ is a descent of the ribbon if and only if $w_1 > w_n$.

Despite the simple Schur expansion for the D equivalence classes in this case, there is no known analog for dominant elements under a general D equivalence. This is the main drawback with using the weaker conditions of D equivalence over the stronger conditions of dual equivalence.

Both of these examples are special cases of Macdonald polynomials, or more generally LLT polynomials, where the reading word is an injection from the combinatorial objects to permutations [Ass]. In a recent preprint [AB], this machinery is applied to k-Schur functions. For this application, the reading word is not injective since there may be multiple starred strong tableaux in a given D equivalence class with the same reading word. For these applications, the involutions defined form a D equivalence, and so the corresponding functions are proved to be Schur positive. Except for certain special cases, there is no known dual equivalence.

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