ABSENCE OF RESONANCES NEAR CRITICAL LINE FOR CC MANIFOLDS

LEONARDO MARAZZI

ABSTRACT. We find a resonance free region polynomially close to the critical line on Conformally compact manifolds with polyhomogeneous metric.

1. Introduction

In this note we prove that there is a resonance free region polynomially close to the critical line for Conformally compact (CC) manifolds with polyhomogeneous metric near the boundary.

CC manifolds appear in both physical and mathematical settings. From the physical perspective CC Einstein manifolds, which have a polyhomogeneous metric when their dimension is odd, are related to string theory via the AdS/CFT correspondence. In most of the cases in the literature the sectional curvatures of a CC manifold are constant, at least at the boundary of the manifold, this is seen in the generic examples: Asymptotically hyperbolic (AH), Schwarzchild, de Sitter-Schwarzchild, etc. In general the sectional curvatures of a CC manifold are not constant neither near the boundary nor at the boundary. From a mathematical perspective this new parameter that appears, related to the curvature at the boundary, changes the geometry of the CC manifolds at the boundary. The question is how much it affects the resolvent? In this note we show that the resonances free regions close to the critical line still appear. It is still an open question whether there is a resonance free strip near the critical line.

Scattering theory on AH manifolds has been studied by many authors originating with the paper of Mazzeo-Melrose [MaMe]. One of the main philosophies of scattering theory is to study the distribution of resonances (poles of the resolvent). In AH manifolds (c.f. [Guil1]) there is a region free of resonances close to the critical line (c.f. [Burq]). Guillarmou (c.f. [Guil1]) proved there is a resonance free region exponentially close to the critical line on AH manifolds and for AH manifolds with constant curvature near infinity he proves that there is a strip free of resonances close to the critical line.

Other important questions on scattering theory refer to upper and lower bounds on the number of resonances on balls (c.f. [GuZw1, GuZw2]). As far as the author knows resonances have not been studied on CC manifolds. Questions regarding upper and lower bounds on the number of resonances on balls are unknown for a general AH manifold let alone CC manifolds.

The method we use is more or less standard, the main point is to carry out the parametrix taking into account the parameter $\alpha(y)$ that appears in the Laplace-Beltrami operator and which corresponds to the sectional curvatures at the boundary. The parametrix consists of an approximation near and away from the boundary. Near the boundary we use a CC metric with polyhomogeneus metric together with the uniform estimates on the resolvent up to the critical line $\Re\xi = n/2$ given in Proposition 2.1, and then take a suitable polyhomogeneous metric. Away from the boundary we use the hyperbolic model, for which we prove the necessary energy estimates in Proposition 1.1.

Partially supported by CNPq-Brasil.

Scattering theory on CC manifolds was first studied in this generality by Borthwick [Borth]. In [Mar09] the author proves inverse theorems on CC manifolds. In this setting we have from [Mar11] combined with Theorem 1.1 of [Borth] the following

Theorem 1.1. *The essential spectrum of Δ_g is $[\alpha_0^2\frac{n^2}{4},\infty)$ and there are no embedded eigenvalues except possibly at $\alpha_0^2\frac{n^2}{4}$.*

We can meromorphically extend the resolvent $R(\xi) := (\Delta_g - \alpha_0^2\xi(n-\xi))^{-1}$ to the whole complex plane without some intervals (c.f. [Borth]). A vertical interval near $\xi = n/2$, and some horizontal intervals to the left of $\Re\xi = n/2$. (See Figure 1)

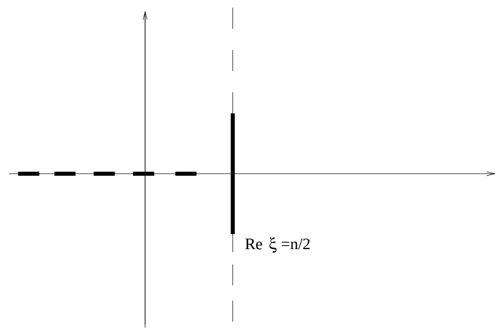

FIGURE 1. Bold lines are not covered by the meromorphic extention

Let X be a compact C^∞ manifold with boundary ∂X which is equipped with a Riemannian metric g such that for any defining function "x" of ∂X $x^2 g$ is a C^∞ non-degenerate Riemannian metric up to ∂X. It can be shown (c.f. [Grah1, Mar09]) that there exists a diffeomorphism $\Psi : V \longrightarrow [0,\epsilon)\times\mathbb{R}^n$ such than g satisfies

$$\Psi^* g = \frac{dx^2}{\alpha^2(y)x^2} + \frac{h(x,y,dy)}{x^2}, \tag{1.1}$$

for V a collar neighborhood of the boundary ∂X. Here $h(x)$, $x\in[0,\epsilon)$, is a family of metrics on ∂X, and $\alpha \in C^\infty(\partial X)$. To simplify notation we drop the pull-back coordinate function Ψ and call the compact manifold (X,g) with boundary ∂X and metric g a CC manifold. We let $\alpha_0 = \min_{\partial X}\alpha$ and $\alpha_1 = \max_{\partial X}\alpha$. Without loss of generality we could assume that $\alpha_0 = 1$, however we keep α_0 for clarity purposes. We assume that $h(x)$, $x\in[0,\epsilon)$, is a family of metrics on ∂X which has a polyhomogeneous expansion near the boundary ∂X of the form

$$h(x,y,dy) \sim h_0(y,dy) + \sum_{0<i\in\mathbb{N}} x^i \sum_{0\le j\le U_i} (\ln x)^j h_{ij}(y,dy), \tag{1.2}$$

where $U_i \in \mathbb{N}_0$ and h_{ij} are symmetric 2-tensors at ∂X.

We say that the metric g is non-trapping if every geodesic approaches the boundary ∂X. Under this assumption using Propositions 1.1 and 2.1 we prove in Section 4 our main theorem

Theorem 1.2. *Let (X,g) be a CC manifold with metric g as in (1.1), and x a boundary defining function. If g is non-trapping, there exists $C_1, C_2 > 0$ such that the weighted resolvent $x^{1/2}R(\xi)x^{1/2}$ extends analytically across $\{\xi\in\mathbb{C} : |\Im\xi| > C_2, \Re\xi > \frac{n}{2}\}$ to*

$$\{\xi\in\mathbb{C} : |\Im\xi| > C_2, \Re\xi > \frac{n}{2} - \frac{C_1}{\Im\xi}\},$$

as a bounded operator in $L^2(X)$.

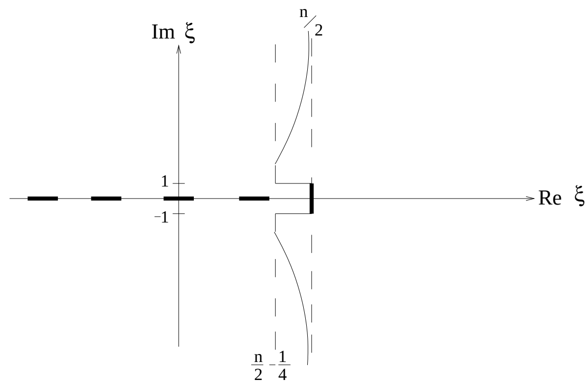

FIGURE 2. Region with no resonances

Let (M, H_0) be a compact Riemannian manifold, for the parametrix near ∂X we use (X_0, g_0) the manifold

$$X_0 := (0,\infty)_x \times M, \qquad g_0 := \frac{dx^2 + H_0}{x^2} \tag{1.3}$$

and them take $H_0 = \alpha^2 h_0$. We are interested in the behavior of the resolvent near $x = 0$, thus we consider functions supported in $(0,1)_x \times M$ that could be written in local coordinates as

$$\sum_{i+|\beta|\leq k} a_{i,k}(x,y)(x\partial_x)^i x^{|\beta|}(\partial_y)^\beta,$$

with $a_{i,k}(x,y)$ polyhomogneous in x.

Modulo the inclusions necessary to be in the right manifold, we construct a parametrix which satisfies

$$\begin{aligned}\Delta_g - \alpha^2\Delta_{g_0}\psi_3 &= xD_R + (x^2\Delta_h - \alpha^2 x^2\Delta_{H_0}),\\ \Delta_g - \alpha^2\psi_3\Delta_{g_0} &= xD_L + (x^2\Delta_h - \alpha^2 x^2\Delta_{H_0}),\end{aligned}$$

and by taking $H_0 = \alpha^2 h_0$ we obtain

$$\begin{aligned}\Delta_g - \alpha^2\Delta_{g_0}\psi_3 &= xD_R,\\ \Delta_g - \alpha^2\psi_3\Delta_{g_0} &= xD_L.\end{aligned} \tag{1.4}$$

After obtaining such a parametrix the method of [Guil1] is applied using the stronger estimates of the following proposition we prove in section 3:

Proposition 1.1. *Let (X_0, g_0) be as before, and x a boundary defining function. Then there exists $C > 0$ such that the weighted resolvent $x^{1/2}R_0(\xi)x^{1/2} = x^{1/2}(\Delta_{g_0} - \alpha_0^2\xi(n-\xi))^{-1}x^{1/2}$ extends continuously from $\{\Re\xi > n/2, |\Im\xi| \geq 1\}$ to $\{\Re\xi > n/2 - 1/4, |\Im\xi| \geq 1\}$, as a map in $\mathcal{L}(L^2(X_0,g_0)), H^p(X_0,g_0))$ and the extension satisfies*

$$||\partial_\xi^q x^{1/2}R(\xi)x^{1/2}||_{\mathcal{L}(L^2,H^p)} \leq C|\xi - \frac{n}{2}|^{-2+p}, \tag{1.5}$$

for $p = 0,1,2$, $q = 0,1$, and $\xi \neq n/2$.

Throughout this note C is an arbitrary constant that can change every time it is written.

In the final section we include an application to the wave equation. To state the corollary let $f_1, f_2 \in C_0^\infty(\overset{\circ}{X})$, and $u(t,z) \in C^\infty(\mathbb{R}_+ \times \overset{\circ}{X})$ satisfy:

$$
\begin{gathered}
\Box u = \left(D_t^2 - \Delta_g\right) u(t,z) = 0, \quad \text{on} \quad \mathbb{R}_+ \times \overset{\circ}{X}, \\
u(0,z) = f_1(z), \quad D_t u(0,z) = f_2(z).
\end{gathered} \tag{1.6}
$$

Here $D_t = -\alpha_0 \frac{\partial_t}{i}$. Our result will only hold for high energies so we need to take $v = \chi(t)u$ and $0 < \epsilon << 1$ so that $\chi(t)$ is a smooth function so that

$$
\chi(t) = \begin{cases} 0, & t < \epsilon, \\ 1, & t > 1. \end{cases}
$$

We prove the following corollary

Corollary 1.1. *Let u be a solution to* (1.6) *and let v be as above. Then*

$$
||v|| \leq C_N t^{-N}, \qquad \forall N.
$$

Acknowledgements The author acknowledges support from CNPq and thank F. Cardoso for encouragement while writing this note. The author also thanks A. Sá Barreto for suggesting the problem and P. Hislop for many helpful discussions.

2. Uniform estimates up to the critical line

In [CardVod] Cardoso-Vodev obtained uniform estimates for the resolvent of the Lapacian for a metric of the form

$$
g_r = dr^2 + \sigma(r), \tag{2.1}
$$

where $\sigma(r)$ is a one-parameter family of Riemannian metrics. The transformation $x = e^{-r}$ puts the AH metric (1.1) into the form of the metric studied in [CardVod] with $\sigma(r) = e^{2r} h(e^{-r})$. In that same spirit we consider the metric

$$
g_r = (\alpha(y))^{-2} dr^2 + \sigma(r), \tag{2.2}
$$

where the metric σ no longer has a polyhomogeneous expansion, it is actually in the class studied in [CardVod]. Combining [Mar09] and [Mar11] we can prove that the resolvent $\mathcal{R}(\xi)$ is analytic for $\Re\xi > n/2$.

We denote by $|\sigma|$ the determinant of σ. The corresponding Laplace-Beltrami operator is

$$
\Delta_g = -\alpha^2(y)\partial_r^2 - \frac{\alpha^2(y)\partial_r(|\sigma|^{1/2})}{|\sigma|^{1/2}}\partial_r - \Delta_\sigma + \frac{\partial_{y_i}\alpha(y)}{\alpha(y)}\sigma^{ij}\partial_{y_j}. \tag{2.3}
$$

We follow the notation of [CardVod] and denote $|\sigma|^{1/2}$ by p, we also write α instead of $\alpha(y)$ keeping in mind that α is a function on the boundary. If we conjugate Δ_g by $p^{1/2}$, we obtain

$$
p^{1/2}\Delta_g p^{-1/2} = -\alpha^2(y)\partial_r^2 + \Lambda + q, \tag{2.4}
$$

with $\Lambda = -\partial_{y_i}\sigma^{ij}\partial_{y_j}$, and

$$
q = -\frac{(\partial_r p_\sigma)^2}{4p_\sigma^2} - \frac{\partial_{y_i}p_\sigma\partial_{y_j}p_\sigma}{4p_\sigma^2}\sigma^{ij} + \frac{p_\sigma\Delta_g p_\sigma^{-1}}{2} - \frac{\partial_{y_i}p_\sigma}{4}\sigma^{ij}\partial_{y_j}\alpha - (\partial_{y_i}\alpha)\sigma^{ij}\frac{\partial_{y_i}p}{2p_\sigma} + \frac{\partial_{y_i}\alpha}{\alpha}\sigma^{ij}\partial_{y_j}. \tag{2.5}
$$

The method of [CardVod] under the following assumptions

$$
|q| \leq C, \qquad \partial_r q \leq C r^{-1-\delta} \tag{2.6}
$$

and

$$
-\partial_r(\sigma^{-1})(r,y,\xi) \geq \frac{C}{r}\sigma^{-1}(r,y,\xi) \qquad \forall (y,\xi) \in T^*S_r. \tag{2.7}
$$

gives the following:

Proposition 2.1. *Let (X,g) be a conformally compact manifold with metric g as in* (1.1)*, and x a boundary defining function. Then there exists $c > 0$ such that the weighted resolvent $x^{1/2}R(\xi)x^{1/2} = x^{1/2}(\Delta_g - \alpha_0^2\xi(n-\xi))^{-1}x^{1/2}$ extends continuously from $\{\Re\xi > n/2, |\Im\xi| \geq 1\}$ to $\{\Re\xi \geq n/2, |\Im\xi| \geq 1\}$ on $L^2(X)$ and the extension satisfies*

$$||x^{1/2}R(\xi)x^{1/2}||_{\mathcal{L}(L^2,H^p)} \leq Ce^{C|\xi|}, \qquad C > 0, \tag{2.8}$$

for $p = 0, 1$, $|\Im\xi| \geq 1$, and $0 \leq \Re\xi - n/2 \leq 1$; where the Sobolev norm is with respect to the metric g. Moreover if g is non-trapping we have

$$||x^{1/2}R(\xi)x^{1/2}||_{\mathcal{L}(L^2,H^p)} \leq C|\Im\xi|^{-1+p}, \qquad C > 0. \tag{2.9}$$

for $p = 0, 1$, $|\Im\xi| \geq 1$, and $0 \leq \Re\xi - n/2 \leq 1$.

Proof. The technique is essentially found in [CardVod] however our case is a bit simpler in the sense that we are only concerned with what happens near the boundary ∂X of the CC manifold X which corresponds to the elliptic ends of the manifolds considered in [CardVod], the manifolds they considered include also manifolds with cusp ends. Away from the boundary the operator Δ_g is elliptic and the result follows from the existing results for asymptotically hyperbolic manifolds (e.g. [CardVod, Guil1]).

We prove the theorem using the coordinates (r, y) with corresponding Laplace-Beltrami operator

$$\Delta_g = -\alpha^2(y)\partial_r^2 - \frac{\alpha^2(y)\partial_r(|\sigma|^{1/2})}{|\sigma|^{1/2}}\partial_r - \Delta_\sigma + \frac{\partial_{y_i}\alpha(y)}{\alpha(y)}\sigma^{ij}\partial_{y_j}. \tag{2.10}$$

We denote by

$$P = p^{1/2}((\alpha_0\lambda)^{-2}\Delta_g - 1 + i(\alpha_0\lambda)^{-2}\epsilon)p^{-1/2} = -\alpha^2(y)(\alpha_0\lambda)^{-2}\partial_r^2 + (\alpha_0\lambda)^{-2}\Lambda + (\alpha_0\lambda)^{-2}q + i(\alpha_0\lambda)^{-2}\epsilon, \tag{2.11}$$

with Λ and q as before. We rename $L_r = (\alpha_0\lambda)^{-2}\Lambda$, $V = (\alpha_0\lambda)^{-2}q$ and $D_r = (i\alpha_0\lambda)^{-1}\alpha\partial_r$ to simplify notation.

The theorem follows from the following

Proposition 2.2. *Let $u \in H^2(X,g)$ be such that $r^sPu \in L^2(X,g)$ for $1/2 < s \leq 1/2 + \delta_0$. Then for all γ such that $0 < \gamma << 1$ there exist constants C_1, C_2, $\lambda_0 > 0$ independent of λ and ϵ, such that for $\lambda \geq \lambda_0$ we have*

$$||r^{-s}u||^2_{H^1(V,g)} \leq C_1\lambda^2||r^sPu||^2_{L^2(V,g)} - C_2\lambda^{-1}\Im\langle\alpha_0^{-2}\alpha\partial_r u, u\rangle_{L^2(\partial X,g)}. \tag{2.12}$$

Proof. Integration by part gives

$$\langle r^{-2s}(L_r - 1 + V)u, u\rangle_{L^2(V,g)} + ||r^{-s}\alpha D_r u||^2_{L^2(V,g)} = \Re\langle r^{-2s}Pu, u\rangle_{L^2(V,g)} + 2s(\alpha_0\lambda)^{-2}\Re\langle r^{-2s-1}\alpha^2 u', u\rangle_{L^2(V,g)}.$$

Thus by Cauchy-Schwarz we have

$$\begin{aligned} &|\langle r^{-2s}(L_r - 1 + V)u, u\rangle_{L^2(V,g)} + ||r^{-s}\alpha D_r u||^2_{L^2(V,g)}| \\ &\qquad \leq O(\lambda)||r^{-s}Pu||^2_{L^2(V,g)} + O(\lambda^{-1})\left(||r^{-s}\alpha D_r u||^2_{L^2(V,g)} + ||r^{-s}u||^2_{L^2(V,g)}\right). \end{aligned} \tag{2.13}$$

Since $\Im\langle\gamma^{-1}\lambda Pu, \gamma\lambda^{-1}u\rangle_{L^2(V,g)} = \Im\langle Pu, u\rangle_{L^2(V,g)} = (\lambda)^{-2}\Im\langle\alpha_0^{-2}\alpha\partial_r u, u\rangle_{L^2(\partial V)} + \epsilon||u||_{L^2(V,g)}$, we have that

$$\epsilon||u||_{L^2(V,g)} \leq \gamma^{-1}\lambda||r^sPu||^2_{L^2(V,g)} + \gamma\lambda^{-1}||r^{-s}u||^2_{L^2(V,g)} - (\lambda)^{-2}\Im\langle\alpha_0^{-2}\alpha\partial_r u, u\rangle_{L^2(\partial V)} \tag{2.14}$$

for all positive γ. Again taking imaginary parts since ϵ is small and α bounded we have

$$||D_r u||^2_{L^2(V,g)} \leq C||u||^2_{L^2(V,g)} + ||Pu||^2_{L^2(V,g)}. \tag{2.15}$$

The previous two equations give
(2.16)
$$\epsilon\lambda\left(||u||^2_{L^2(V,g)}+||D_ru||^2_{L^2(V,g)}\right)\le O(\lambda^2)||r^sPu||^2_{L^2(V,g)}+\gamma||r^{-s}u||^2_{H^1(V,g)}-C\lambda^{-1}\Im\langle\alpha_0^{-2}\alpha\partial_r u,u\rangle_{L^2(\partial V)}.$$
for all $\gamma>0$. Let
$$E(r)=-\langle(L_r-1+V)u(r,\cdot),u(r,\cdot)\rangle+||D_ru(r,\cdot)||^2,$$
where $||u(r,\cdot)||$ means the L^2 norm in the rest of the variables that do not include r. Taking the derivative with respect to r we have

$$\begin{aligned}E'(r)=&-\langle[\partial_r,L_r]u(r,\cdot),u(r,\cdot)\rangle-\langle V'u(r,\cdot),u(r,\cdot)\rangle\\&-2\epsilon\Im\langle u(r,\cdot),u'(r,\cdot)\rangle-2\lambda\alpha_0\Im\langle Pu(r,\cdot),\frac{1}{\alpha(\cdot)}D_ru(r,\cdot)\rangle.\end{aligned}\tag{2.17}$$

Writing $\langle Pu(r,\cdot),\frac{1}{\alpha(\cdot)}D_ru(r,\cdot)\rangle=\langle r^{-2s}\lambda Pu(r,\cdot),\frac{1}{\alpha(\cdot)\lambda}r^{2s}D_ru(r,\cdot)\rangle$, and using our assumptions (2.6), (2.7) and Cauchy-Schwarz we have

$$\begin{aligned}E'(r)\ge&\frac{C}{r}\langle L_ru(r,\cdot),u(r,\cdot)\rangle-\epsilon\lambda(||u(r,\cdot)||^2+||\frac{\alpha_0}{\alpha(\cdot)}D_r(r,\cdot)||^2)\\&-r^{-2s}(||u(r,\cdot)||^2+||D_r(r,\cdot)||^2)-\lambda^2r^{2s}||Pu(r,\cdot)||^2.\end{aligned}\tag{2.18}$$

Integrating and using that $L_r\ge0$ we get

$$\begin{aligned}E(r)=-\int_r^\infty E'(t)dt\le&\int_r^\infty\left[\epsilon\lambda(||u(t,\cdot)||^2+||\frac{\alpha_0}{\alpha(\cdot)}D_t(t,\cdot)||^2)\right]dt+||r^{-s}u(r,\cdot)||^2_{H^1(V,g)}\\&+\lambda^2||r^sPu(r,\cdot)||^2_{L^2(V,g)},\end{aligned}\tag{2.19}$$

and using (2.16) we obtain

$$E(r)\le\gamma||r^{-s}u(r,\cdot)||^2_{H^1(V,g)}+C\lambda^2||r^sPu(r,\cdot)||^2_{L^2(V,g)}-C\lambda^{-1}\Im\langle\alpha_0^{-2}\alpha\partial_ru,u\rangle_{L^2(\partial V)}.\tag{2.20}$$

On the other hand

$$\begin{aligned}r^{1-2s}E'(r)\ge&Cr^{-2s}\langle L_ru(r,\cdot),u(r,\cdot)\rangle-r^{1-2s}\epsilon\lambda(||u(r,\cdot)||^2+||\frac{\alpha_0}{\alpha(\cdot)}D_r(r,\cdot)||^2)\\&-r^{1-4s}(||u(r,\cdot)||^2+||D_r(r,\cdot)||^2)-r^{1-2s}\lambda^2r^{2s}||Pu(r,\cdot)||^2\end{aligned}\tag{2.21}$$
,

and $\int_a^\infty r^{1-2s}E'(r)dr=(2s-1)\int_a^\infty r^{-2s}E(r)dr$, thus integrating (2.21) from a to ∞ we get

$$\begin{aligned}&C||r^{-s}(L_r)^{1/2}||^2_{L^2(V,g)}-\epsilon\lambda(||r^{1-2s}u(r,\cdot)||^2_{L^2(V,g)}+||r^{1-2s}\frac{\alpha_0}{\alpha(\cdot)}D_r(r,\cdot)||^2_{L^2(V,g)})\\&-(||r^{1-4s}u(r,\cdot)||^2_{L^2(V,g)}+||r^{1-4s}D_r(r,\cdot)||^2_{L^2(V,g)})-\lambda^2||rPu(r,\cdot)||^2_{L^2(V,g)}\\&\le\gamma||r^{-s}u(r,\cdot)||^2_{H^1(V,g)}+C\lambda^2||r^sPu(r,\cdot)||^2_{L^2(V,g)}-C\lambda^{-1}\Im\langle\alpha_0^{-2}\alpha\partial_ru,u\rangle_{L^2(\partial V)}.\end{aligned}\tag{2.22}$$

Since $1/2<s\le1/2+\delta_0$ and $\alpha_0\le\alpha\le\alpha_1$ the terms with negative sign in left hand side of the previous equation can be absorbed by the terms in the right hand side to get

(2.23)
$$||r^{-s}(L_r)^{1/2}||^2_{L^2(V,g)}\le\gamma||r^{-s}u(r,\cdot)||^2_{H^1(V,g)}+C\lambda^2||r^sPu(r,\cdot)||^2_{L^2(V,g)}-C\lambda^{-1}\Im\langle\alpha_0^{-2}\alpha\partial_ru,u\rangle_{L^2(\partial V)}.$$

The proposition now follows from the previous inequality, (2.20) and (2.13).

□

The theorem follows from the previous proposition by noticing that

$$
\begin{aligned}
(2.24)\quad -\Im\langle\alpha_0^{-2}\alpha\partial_r u,u\rangle_{L^2(\partial V)} &= -\Im\langle\alpha_0^{-2}(\Delta_g-\alpha_0^2\lambda^2-i\epsilon)u,u\rangle_{L^2(V,g)}-\epsilon||\alpha_0^{-2}u||^2_{L^2(V,g)}\\
&\leq C||(\Delta_g-\alpha_0^2\lambda^2-i\epsilon)u||^2_{L^2(V,g)}+C\epsilon||u||^2_{L^2(V,g)},
\end{aligned}
$$

this inequality into (2.12) gives

$$
||r^{-s}u||^2_{H^1(V,g)}\leq C_1\lambda^2||r^sPu||^2_{L^2(V,g)}+C||(\Delta_g-\alpha_0^2\lambda^2-i\epsilon)u||^2_{L^2(V,g)}+C\epsilon||u||^2_{L^2(V,g)}, \tag{2.25}
$$

letting $\epsilon\to 0$ since $|\lambda|>K$ we obtain

$$
||r^{-s}u||^2_{H^1(V,g)}\leq C_2\lambda||r^sPu||_{L^2(V,g)}. \tag{2.26}
$$

The theorem, with $r^{\frac{1+\delta}{2}}=(\ln x)^{\frac{1+\delta}{2}}$ instead of x, follows from the last inequality by factoring out λ^{-2} from P. Lastly, noticing that $x^{1/2}<|\ln x|^{\frac{1+\delta}{2}}$ for $x>0$ sufficiently small we obtain the theorem. □

3. Model

Writing the Laplace-Beltrami operator Δ_g in local coordinates $z=(x,y)$, logaritmic terms appear. It is a differential operator of order two in $(x\partial_x,x\partial_y)$ with polyhomogeneous coefficients. Hence we denote by ${}^{pol}\mathrm{Diff}_0^k(\bar X)$ the space of polyhomogeneous differential operators of order k that could be written in local coordinates as

$$
\sum_{i+|\beta|\leq k}a_{i,k}(x,y)(x\partial_x)^i x^{|\beta|}(\partial_y)^\beta,
$$

with $a_{i,k}(x,y)$ polyhomogneous in x.

The variable sectional curvature gives a resolvent which lives on spaces of polyhomogeneous operators $\mathcal{A}(X)$, however the definition of such spaces is well known and we do not discuss these spaces in more detail here since we are just going to look at the resolvent as a linear map. We refer the interested reader to [Borth, Mar09, Mar11].

The argument of [FrHis] can be extended to show ${}^{pol}\mathrm{Diff}_0^k(\bar X)\subset\mathcal{L}(H^s(X),H^{s-k}(X))$, where as usual $H^k(X):=\mathrm{Dom}(1+\Delta_g)^{k/2}$. Also $x^{-\beta}D^kx^\beta\in{}^{pol}\mathrm{Diff}_0^k(X)$ for $D^k\in{}^{pol}\mathrm{Diff}_0^k(X)$.

The part of the parametrix near the boundary will be given by (X_0,g_0) we define next: let (M,H_0) be a compact Riemannian manifold then (X_0,g_0) is the manifold

$$
X_0:=(0,\infty)_x\times M,\qquad g_0:=\frac{dx^2+H_0}{x^2}. \tag{3.1}
$$

Note that

$$
\bar X_0:=[0,\infty)\times M. \tag{3.2}
$$

The elements of ${}^{pol}\mathrm{Diff}_0^k(\bar X_0)$ with support in $[0,1]\times M$ could be written in local coordinates as

$$
\sum_{i+|\beta|\leq k}a_{i,k}(x,y)(x\partial_x)^i x^{|\beta|}(\partial_y)^\beta,
$$

with $a_{i,k}(x,y)$ polyhomogneous in x. Via the change of variables $r=\ln x$, Δ_{g_0} is unitarily equivalent to

$$
P_0=-\partial_r^2+e^{2r}\Delta_{H_0}+\frac{n^2}{4}.
$$

We now prove Proposition 1.1

Proof. We prove the proposition for $p = 0$, the other cases follow from ${}^{pol}\mathrm{Diff}_0^k(\bar{X}_0)$ being contained in $\mathcal{L}(H^s(X_0), H^{s-k}(X_0))$ ([FrHis]). By the spectral theorem we can decompose

$$P_0 = \bigoplus_j P_0^{(j)}, \qquad P_0^{(j)} = -\partial_r^2 + e^{2r}\mu_j^2 + \frac{n^2}{4}.$$

with $\{\mu_j\}_{j\in\mathbb{N}_0}$ the eigenvalues of Δ_{H_0} associated to an orthonormal basis $\{\psi_j\}_{j\in\mathbb{N}_0}$ of $L^2(M)$ eigenvectors and counted with multiplicities.

We have the decomposition

$$\rho(P_0 - \xi(n-\xi))^{-1}\rho f = \sum_{j\in\mathbb{N}_0} \rho(P_0^{(j)} - \xi(n-\xi))^{-1} < f, \psi_j > \psi_j \rho.$$

We let $U_J : L^2(\mathbb{R}, dr) \longrightarrow L^2(\mathbb{R}, dr)$ be the isometric translation

$$U_J = f(\cdot) \mapsto f(\ln(\mu_j) + \cdot),$$

for $\mu_j \neq 0$, we have $U_j P_0^{(j)} U_j = Q$, with $Q = -\partial_r^2 + e^{2r} + \frac{n^2}{4}$. We set $k = \xi - n/2$ to simplify notation.

It is well known that we can decompose the Green kernel

$$R_Q(\xi; r, t) = (Q - \xi(n-\xi))^{-1}(r,t) = K_{-k}(e^r)I_k(e^t)H(r-t) - I_k(e^r)K_{-k}(e^t)H(t-r),$$

where H is the Heaviside function, and K_{-k}, I_k are given by

$$I_k(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos(u)}\cos(ku)du - \frac{\sin(k\pi)}{\pi}\int_0^\infty e^{-z\cosh(u) - ku}du, \tag{3.3}$$

$$K_{-k}(z) = \int_0^\infty \cosh(ku)e^{-z\cosh(u)}du. \tag{3.4}$$

If $|\Re(k)| \leq 1/4$ and $|\Im(k)| \geq 1$, we have, for $t > 0$, that

$$\begin{aligned} |I_k(e^t)| = \left|\frac{1}{\pi}\int_0^\pi e^{e^t\cos(u)}\cos(ku)du - \frac{\sin(k\pi)}{\pi}\int_0^\infty e^{-e^t\cosh(u)-ku}du\right| \leq \\ C\left|\int_0^\pi e^{e^t e^{|k|u}}e^{|k|u}du\right| + Ce^{-e^t}\left|\int_0^\infty e^{-ku}du\right| \leq C\frac{e^{e^t}}{e^t}|k|^{-1}, \end{aligned} \tag{3.5}$$

$$\left|K_{-k}(e^t)\right| = \left|\int_0^\infty \cosh(ku)e^{-e^t\cosh(u)}du\right| \leq C\left|\int_0^\infty \sinh(ku)e^{-e^t\cosh(ku)}du\right| \leq C\frac{e^{-e^t}}{e^t}|k|^{-1} \tag{3.6}$$

the proof of the first inequality of (3.6) is included in the appendix. For $t \leq 0$ we have

$$\begin{aligned} |I_k(e^t)| = \left|\frac{1}{\pi}\int_0^\pi e^{e^t\cos(u)}\cos(ku)du - \frac{\sin(k\pi)}{\pi}\int_0^\infty e^{-e^t\cosh(u)-ku}du\right| \leq \\ C\left|\int_0^\pi \cos(ku)du\right| + C\left|\int_0^\infty e^{-ku}du\right| \leq C|k|^{-1} \end{aligned} \tag{3.7}$$

and

$$\left|K_{-k}(e^t)\right| = \left|\int_0^\infty \cosh(ku)e^{-e^t\cosh(u)}du\right| \leq C|k|^{-1}. \tag{3.8}$$

We suppose, without loss of generality, that $\rho(e^r) = e^{r/2}\chi(r)$ with χ a smooth function so that $\chi(r) = 1$ for $r \leq -1$ and $\chi(r) = 0$ for $r \geq 1$. Thus the last inequalities give

$$|K_{-k}(e^r)\rho(e^{r-\ln\mu_j})| \leq \begin{cases} C|k|^{-1}e^{-e^r} & r > 0 \\ C|k|^{-1}e^{r/2} & r \leq 0 \end{cases} \tag{3.9}$$

$$|I_k(e^t)\rho(e^{t-\ln\mu_j})| \leq \left\{ \begin{array}{ll} C|k|^{-1}e^{e^t} & t>0 \\ C|k|^{-1}e^{t/2} & t\leq 0 \end{array}\right. \tag{3.10}$$

Thus we obtain

$$\begin{aligned} |I_k(e^t)\rho(e^{t-\ln\mu_j})| \int_r^\infty |K_{-k}(e^r)\rho(e^{r-\ln\mu_j})| \leq \frac{C}{|k|^2} \\ |K_{-k}(e^r)\rho(e^{r-\ln\mu_j})| \int_{-\infty}^r |I_k(e^t)\rho(e^{t-\ln\mu_j})| \leq \frac{C}{|k|^2} \end{aligned} \tag{3.11}$$

When $\mu_j = 0$, for $\xi \notin \mathbb{C}\backslash 0$, the euclidean resolvent

$$R_0^{(j)}(\xi; r, t) = |2k|^{-1}e^{-k|r-t|}.$$

Thus for $\Re\xi > \frac{n}{2} - \frac{1}{4}$, $|\Im\xi| \geq 1$, and $p = 0$:

$$||\partial_r^p \rho(e^\cdot)R_0^{(j)}(\xi)\rho(e^\cdot)||_{L\{L^2(\mathbb{R})\}} \leq |k|^{-2+p}. \tag{3.12}$$

For $(p,q) = (0,0)$ the Lemma follows from (3.11) and (3.12), and the cases $(p,q) = (1,0)$ and $(p,q) = (2,0)$ follow from ${}^{pol}\mathrm{Diff}_0^k(X_0) \subset \mathcal{L}(H^s(X_0), H^{s-k}(X_0))$. The case $q = 1$ follows from the Cauchy formula. □

4. Absence of resonances near critical line

In this section we prove Theorem 1.2.

Proof. We define the resolvent $R_0(\xi) := (P_0 - \alpha_0^2\xi(n-\xi))^{-1}$ and let $R(\xi)$ be the resolvent as defined before, which extends to the physical sheet $\{\Re(\xi) > n/2\}$. We work in V be a collar neigborhood of ∂X in the conformally compact manifold (X, g) isometric to $U := (0, \delta)_x \times \partial X$ equipped with the metric $(\alpha x)^{-2}dx^2 + x^{-2}h(x)$, via the isometry $i : V \to U$. We assume $\delta = 1$ without loss of generality.

Let $\mathcal{I}_U : L^2(X_0, dvol_{g_0}) \to L^2(U, dvol_g)$ and $\mathcal{R}_U : L^2(U, dvol_g) \to L^2(X_0, dvol_{g_0})$ be the bounded operators given by

$$\begin{aligned} \mathcal{R}_U &: f \mapsto f(\iota_U(\cdot)), \\ \mathcal{I}_U &: f \mapsto 1_U f; \end{aligned}$$

where ι_U is the inclusion $U \hookrightarrow X_0$, and 1_U is the characteristic function of U. Since the function $\alpha = \alpha(y)$ satisfies $0 < \alpha_0 < \alpha < \alpha_1$, i^*g_0 and g are quasi-isometric. The pullback and push-forward of i map $i^* : L^2(U, dvol_{g_0}) \to L^2(V, dvol_g)$ and $i_* : L^2(V, dvol_g) \to L^2(U, dvol_{g_0})$ respectively as bounded operators. We also have $I^* := \mathcal{I}_V i^* \mathcal{R}_U$, and $I_* := \mathcal{I}_U i_* \mathcal{R}_V$ as linear operator from $L^2(X_0, dvol_{g_0})$ to $L^2(X, dvol_g)$ and from $L^2(X, dvol_g)$ to $L^2(X_0, dvol_{g_0})$ respectively.We let for $j = 1, ..., 4$; $\psi_j(x)$ be defined by

$$\psi_j(x) := \left\{ \begin{array}{ll} 1, & x \in [0, j/5] \\ 0, & x \in [(j+1)/5, \infty) \end{array}\right. .$$

We have $I_*I^*\psi_j = \psi_j$ and $I_*I^*i^*\psi_j = i^*\psi_j$.

For the first step of the parametrix we note the there exist operators D_R and D_L in ${}^{pol}\mathrm{Diff}_0^2(X)$ such that

$$\begin{aligned} \Delta_g - \alpha^2 I^* \Delta_{g_0}\psi_3 I_* = xD_R + (x^2\Delta_h - \alpha^2x^2\Delta_{H_0}), \\ \Delta_g - \alpha^2 I^* \psi_3\Delta_{g_0} I_* = xD_L + (x^2\Delta_h - \alpha^2x^2\Delta_{H_0}), \end{aligned}$$

since

$$\Delta_g = \alpha^2[-(x\partial_x)^2 + nx\partial_x - x^2(\partial_x \ln\sqrt{h})\partial_x] + x^2\Delta_h - x^2(\partial_{y_i}\ln\alpha)h^{ij}\partial_{y_j}.$$

Notice that

$$x^2\Delta_h = x^2\frac{1}{|h|^{1/2}}\sum \partial_i h^{ij}|h|^{1/2}\partial_j = x^2\sum h^{ij}\partial_i\partial_j + xD_1, \tag{4.1}$$

with D_1 in ${}^{pol}\mathrm{Diff}^1_0(X)$. We assume we can write

$$h(x,y,dy) \sim h_0(y,dy) + \sum_{i\in\mathbb{N}_0} x^i \sum_{0\le j\le U_i} (\ln x)^j h_{ij}(y,dy). \tag{4.2}$$

thus

$$x^2\Delta_h = x^2\sum h_0^{ij}\partial_i\partial_j + xD_2, \tag{4.3}$$

with D_2 in ${}^{pol}\mathrm{Diff}^2_0(X)$. Thus taking $H_0 = \alpha^2 h_0$ we get that $\alpha^2 x^2\Delta_{H_0} - x^2\Delta_h = xD_3$, with D_3 in ${}^{pol}\mathrm{Diff}^2_0(X)$. Thus we obtain

$$\begin{aligned} \Delta_g - \alpha^2 I^*\Delta_{g_0}\psi_3 I_* &= xD_R,\\ \Delta_g - \alpha^2 I^*\psi_3\Delta_{g_0} I_* &= xD_L. \end{aligned} \tag{4.4}$$

where D_R and D_L are not necessarily the same as before.

By (4.4) we can now apply the same parametrix as the one used in [Guil1] to prove the absence of resonances exponentially close to the critical line. For $\Re(\xi) > n/2$ we have

$$(\Delta_{g_0}\psi_3 - \xi(n-\xi))\psi_2 R_0(\xi)\psi_1 = \psi_1 + [\Delta_{g_0},\psi_2]R_0(\xi)\psi_1.$$

We let $\chi_1 := 1 - i^*\psi_1$, and χ_0 a smooth function with compact support on X which is equal to 1 on the support of χ_1. We denote by D^p (resp. D_0^p) all differential operator in $\mathrm{Diff}^p(X)$ with support in $\mathrm{Supp}(i^*\psi_3\ )$ (resp. $\mathrm{Diff}^p_0(\bar{X}_0)$ with support in $\mathrm{Supp}(\psi_3\ )$). We have $D^p x = xD^p$, $D_0^p I_* = I_* D^p$, and $D^p I^* = I^* D_0^p$.

Let $\Xi := \xi(n-\xi)$, and $\Xi_0 := \alpha_0^2\xi_0(n-\xi_0)$, for ξ_0 fixed such that $\Re(\xi_0) > n/2$, we have

$$(\Delta_g - \Xi)E_R(\xi) = 1 + L_R(\xi), \tag{4.5}$$

with

$$E_R(\xi) = R_{0R}(\xi) + \chi_0 R(\xi_0)\chi_1, \qquad R_{0R}(\xi) := I^*\psi_2 R_0(\xi)\frac{\psi_1}{\alpha^2}I_*,$$

and

$$L_R(\xi) = [\Delta_g,\chi_0]R(\xi_0)\chi_1 + (\Xi-\Xi_0)\chi_0 R(\xi_0)\chi_1 + I^*\alpha^2[\Delta_{g_0},\psi_2]R_0(\xi)\frac{\psi_1}{\alpha^2}I_* + xD_R R_{0R}(\xi). \tag{4.6}$$

Thus we obtain

$$x^{-1/2}L_R = (D^1 + (\Xi-\Xi_0)x^{-1}\chi_0)R(\xi_0)\chi_1 + x^{1/2}I^*\alpha^2 D_0^2 R(\xi)\frac{\psi_1}{\alpha^2}I_*. \tag{4.7}$$

On the other hand

$$E_L(\xi)(\Delta_g - \Xi) = 1 + L_L(\xi), \tag{4.8}$$

with

$$E_L(\xi) = R_{0L}(\xi) + \chi_0 R(\xi_0)\chi_1, \qquad R_{0R}(\xi) := I^*\frac{\psi_1}{\alpha^2}R_0(\xi)\psi_2 I_*, \tag{4.9}$$

and

$$L_L(\xi) = \chi_1 R(\xi_0)[\Delta_g,\chi_0] + (\Xi-\Xi_0)\chi_1 R(\xi_0)\chi_0 + I^*\frac{\psi_1}{\alpha^2}R_0(\xi)\alpha^2[\Delta_{g_0},\psi_2]I_* + R_{0L}(\xi)xD_R. \tag{4.10}$$

From equations (4.5) and (4.8) we get

$$R(\xi) = E_R(\xi) - R(\xi)L_R(\xi), \qquad R(z) = E_L(z) - L_L(z)R(z). \tag{4.11}$$

Substituting the first equation in (4.11) into the resolvent identity $R(\xi) - R(z) = (\Xi - Z)R(\xi)R(z)$, we get $x^{1/2}R(\xi)x^{1/2}(1 + K(\xi, z)) = K_1(\xi, z)$, where

$$K(\xi, z) = (\Xi - Z)x^{-1/2}L_R(\xi)R(z)x^{1/2} \quad \text{and} \quad K_1(\xi, z) = x^{1/2}R(z)x^{1/2} + (\Xi - Z)x^{1/2}E_R(\xi)R(z)x^{1/2}.$$

From (4.9) and (4.10) we have

$$\begin{aligned} R(z)x^{1/2} = E_L(z)x^{1/2} - L_L(z)R(z)x^{1/2} = I^* \frac{\psi_1}{\alpha^2} R_0(z)(\psi_2 I^* x^{1/2} + I^*\alpha^2 x^{1/2}D^2x^{1/2}R(z)x^{1/2}) \\ + \chi_1 R(\xi_0)x^{1/2}[\chi_0 + D^1x^{1/2}R(z)x^{1/2} + (\Xi_0 - Z)x^{-1/2}\chi_0 R(z)x^{1/2}]. \end{aligned} \tag{4.12}$$

Putting the last equation together with $x^{-1/2}L_R(\xi)$ into

$$\frac{K(\xi, z)}{\Xi - Z} = x^{-1/2}L_R(\xi)R(z)x^{1/2}$$

we get

$$\begin{aligned} \frac{K(\xi, z)}{\Xi - Z} &= (D^1 + (\Xi_0 - \Xi)x^{-1}\chi_0)x^{1/2}R(\xi_0)\chi_1 R(z)x^{1/2} \\ &+ x^{1/2}I^*\alpha^2 D_0^2\psi_4 R_0(\xi)\frac{\psi_1}{\alpha^2}I_* x^{1/2}x^{-1/2}\chi_1 R(\xi_0)x^{1/2}\left(\chi_0 + D^1x^{1/2}R(z)x^{1/2} + (\Xi_0 - Z)x^{-1/2}\chi_0 R(z)x^{1/2}\right) \\ &+ x^{1/2}I^*\alpha^2 D_0^2\psi_4 R_0(\xi)\frac{\psi_1^2}{\alpha^4}R_0(z)I_* x^{1/2}(\alpha^2 x^{1/2}D^2x^{1/2}R(z)x^{1/2} + \tilde{\psi_2}). \end{aligned} \tag{4.13}$$

The first line in (4.13) extends to $\{\Re(\xi) > n/2 - 1/4\} \cap \{|\Im(\xi)| \geq 1\}$ and $\{\Re(z) \geq n/2\} \cap \{|\Im(z)| \geq 1\}$ as an operator with L^2 norm bounded by

$$C\left(\frac{|\xi|^2}{|z|} + 1\right) \leq C\frac{|\xi|^2}{|z|}.$$

The second line can be extended to $\{\Re(\xi) > n/2 - 1/4\} \cap \{|\Im(\xi)| \geq 1\}$ and $\{\Re(z) \geq n/2\} \cap \{|\Im(z)| \geq 1\}$ as an operator with L^2 norm bounded by

$$C(|z| + C_1 + 1/|z|) \leq C|z|$$

Finally we analyze the third line, by Proposition 2.1 $D^2x^{1/2}R(z)x^{1/2}$ can be extended to $\{\Re(z) \geq n/2\} \cap \{|\Im(z)| \geq 1\}$ as an operator with L^2 norm bounded by $C|z|$. Using the trick of [Guil1] i.e. writing

$$\begin{aligned} x^{1/2}I^*D_0^2\psi_4 R_0(\xi)\frac{\psi_1^2}{\alpha^4}R_0(z)\psi_4 I_* x^{1/2} = \\ (\frac{\psi_1}{\alpha^2} - i_*(x^{1/2})\psi_4[\frac{\psi_1}{\alpha^2}, \Delta_{g_0}]x^{-1/2})\frac{i_* x^{1/2}\psi_4 R_0(\xi)i_* x^{1/2}\psi_4 - i_*(x^{1/2}\psi_4)R_0(z)i_* x^{1/2}\psi_4}{\Xi - Z}\times \\ \times(\frac{\psi_1}{\alpha^2} - x^{-1/2}[\frac{\psi_1}{\alpha^2}, \Delta_{g_0}]i_* x^{1/2}\psi_4), \end{aligned} \tag{4.14}$$

and using Proposition 1.1 for $q = 1$ we see that we can extend (4.14) to $\{\Re(\xi) > n/2-1/4\} \cap \{|\Im(\xi)| \geq 1\}$ and $\{\Re(z) \geq n/2\} \cap \{|\Im(z)| \geq 1\}$ as an operator with L^2 norm bounded by $C\frac{1}{|\xi|^2}$. Combining the last two estimates we get that the third line of (4.13) can be extended to $\{\Re(\xi) > n/2 - 1/4\} \cap \{|\Im(\xi)| \geq 1\}$ and $\{\Re(z) \geq n/2\} \cap \{|\Im(z)| \geq 1\}$ as an operator with L^2 norm bounded by $C\frac{|z|}{|\xi|^2}$.

These three bounds together give that

$$||K(\xi, z)||_{\mathcal{L}(L^2)} \leq C|\xi - z|\left(\frac{|z|}{|\xi|^2} + \frac{|\xi|^2}{|z|} + |z|\right). \tag{4.15}$$

Fixing $z = n/2 + is$, with $|s| >> 0$ and $\Im(\xi) = s$, $\Re(\xi) > n/2 - 1/4$, the last inequality becomes

$$\begin{aligned}\|K(\xi,z)\|_{\mathcal{L}(L^2)} &\le C|\Re(\xi) - \frac{n}{2}|\left(\frac{\sqrt{n^2/4+s^2}}{(\Re\xi)^2+s^2} + \frac{(\Re\xi)^2+s^2}{\sqrt{n^2/4+s^2}} + \sqrt{n^2/4+s^2}\right) \\ &\le C|\Re(\xi) - \frac{n}{2}|(\sqrt{n^2/4+s^2} + \frac{(\Re\xi)^2+s^2}{\sqrt{n^2/4+s^2}}).\end{aligned} \tag{4.16}$$

Thus taking

$$|\Re(\xi) - \frac{n}{2}| < C(\Im(\xi))^{-1} \tag{4.17}$$

$K(\xi,z)$ is holomorphic in ξ for $\{\Re(\xi) > n/2 - C(\Im(\xi))^{-1}\} \cap \{|\Im(\xi)| \ge 1\}$ and we can invert $1 + K(\xi,z)$ holomorphically. The term $K_1(\xi,z)$ can be handled using the estimates above. □

5. Asymptotics of the wave equation

In this section we give a second application of the resolvent estimate given in Theorem (2.1) to the wave equation on a CC manifold. We prove Corollary 1.1.

Let (X,g) be a CC manifold and $f_1, f_2 \in C_0^\infty(\mathring{X})$, and $u(t,z) \in C^\infty(\mathbb{R}_+ \times \mathring{X})$ satisfy:

$$\begin{gathered}\Box u = \left(D_t^2 - \Delta_g\right) u(t,z) = 0, \quad \text{on} \quad \mathbb{R}_+ \times \mathring{X}, \\ u(0,z) = f_1(z), \;\; D_t u(0,z) = f_2(z).\end{gathered} \tag{5.1}$$

Here $D_t = -\alpha_0 \frac{\partial_t}{i}$. Our result will only hold for high energies so we need to take $v = \chi(t)u$ and $0 < \epsilon << 1$ so that $\chi(t)$ is a smooth function so that

$$\chi(t) = \begin{cases} 0, & t < \epsilon, \\ 1, & t > 1. \end{cases}$$

Then we have that v satisfies

$$\begin{gathered}\Box u = \left(D_t^2 - \Delta_g\right) v(t,z) = F := [\Box, \chi]u, \quad \text{on} \quad \mathbb{R}_+ \times \mathring{X}, \\ v(0,z) = 0, \;\; D_t v(0,z) = 0.\end{gathered} \tag{5.2}$$

Thus taking the Fourier transform in t we get that $\mathcal{F}v$ satisfies

$$\left(\alpha_0 \lambda^2 - \Delta_g\right) (\mathcal{F}v)(\lambda, z) = \mathcal{F}F. \tag{5.3}$$

Now we can use Theorem (2.1), since $(\mathcal{F}v)(\lambda,z) = R(\lambda)(\mathcal{F}F)(\lambda,z)$. Thus taking the inverse Fourier transform we have

$$v(t,z) = \int e^{it\lambda} R(\lambda, z, z') \hat{F}(\lambda, z') dz' d\lambda.$$

The corollary now follows since we obtained polynomial bounds on the resolvent $R(\lambda)$ and F is Schwartz.

Appendix: Proof of inequality in (3.6)

We prove that if $|\Re(k)| \le 1/4$ and $|\Im(k)| \ge 1$,

$$\left|\int_0^\infty \cosh(ku) e^{-e^t \cosh(u)} du\right| \le C \left|\int_0^\infty \sinh(ku) e^{-e^t \cosh(ku)} du\right|. \tag{.4}$$

The left hand side of (.4) is

$$\left|\int_0^\infty \cosh(ku)e^{-e^t\cosh(u)}du\right|^2 = \tag{.5}$$

$$\left|\int_0^\infty \cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}e^{-e^t\cosh(u)}du\right|^2 + \left|\int_0^\infty \sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}e^{-e^t\cosh(u)}du\right|^2.$$

The right hand side of (.4) is

$$\left|\int_0^\infty \sinh(ku)e^{-e^t\cosh(ku)}du\right|^2 = \tag{.6}$$

$$\Bigg|\int_0^\infty \left(\cos((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2} + i\sin((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}\right)e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}}\times$$

$$\left[\cos[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}] - i\sin[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\right]du\Bigg|^2 =$$

$$\Bigg|\int_0^\infty du\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\cos((\Im k)u)\cos[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]$$

$$+\int_0^\infty du\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\sin((\Im k)u)\sin[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\Bigg|^2$$

$$+\Bigg|\int_0^\infty du\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\sin((\Im k)u)\cos[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]$$

$$-\int_0^\infty du\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\cos((\Im k)u)\sin[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\Bigg|^2$$

$$=$$

$$\left[\int_0^\infty du\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\cos((\Im k)u)\cos[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\right]^2$$

$$+\left[\int_0^\infty du\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\sin((\Im k)u)\sin[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\right]^2$$

$$+\left[\int_0^\infty du\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\cos((\Im k)u)\sin[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\right]^2$$

$$+\left[\int_0^\infty du\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\sin((\Im k)u)\cos[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\right]^2$$

$$+2\int_0^\infty du\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\cos((\Im k)u)\cos[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\times$$

$$\int_0^\infty du\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\sin((\Im k)u)\sin[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]$$

$$-2\int_0^\infty du\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\cos((\Im k)u)\sin[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\times$$

$$\int_0^\infty du\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\sin((\Im k)u)\cos[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}].$$

The inequality follows by noticing that the difference of the last two terms is bounded by the sum of the first four and that

$$(.7)\quad \left|\int_0^\infty \cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}e^{-e^t\cosh(u)}du\right|^2\leq$$

$$C\left[\left[\int_0^\infty du\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\cos((\Im k)u)\cos[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\right]^2\right.$$

$$\left.+\left[\int_0^\infty du\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\cos((\Im k)u)\sin[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\right]^2\right]$$

and that

$$\left|\int_0^\infty \sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}e^{-e^t\cosh(u)}du\right|^2\leq$$

$$C\left[\left[\int_0^\infty du\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\sin((\Im k)u)\sin[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\right]^2\right.$$

$$\left.+\left[\int_0^\infty du\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}(e^{-e^t\cos((\Im k)u)\frac{e^{(\Re k)u}+e^{-(\Re k)u}}{2}})\sin((\Im k)u)\cos[e^t\sin((\Im k)u)\frac{e^{(\Re k)u}-e^{-(\Re k)u}}{2}]\right]^2\right].$$

References

[Borth] D. Borthwick. *Scattering Theory for Conformally Compact Metrics with Variable Curvature at Infinity* J. of Func. Anal. **184**, 313-376, (2001).

[Burq] N. Burq. *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et adsence de résonances au voisinage du réet,* Acta. Math. **180**, 1-29, (1998).

[CardVod] F. Cardoso, G.Vodev. *Uniform estimates of the Laplace-Beltrami operator on infinite volume Riemannian manifolds II,* Ann. H. Poincaré **3** (2002), 673-691.

[FrHis] R. Froese, P. Hislop. *Spectral analysis of second order elliptic operators on noncompact manifolds.* Duke Math. J. **58**, 103–129.

[Grah1] C.R. Graham. *Volume and area renormalizations for conformally compact Einstein Metrics.* Rend. Circ. mat. Palermo **2** Suppl. **63**, 31-42, (2000).

[Guil1] C. Guillarmou. *Absence of resonance near the critical line on asymptotically hyperbolic spaces,* Asymptot. Anal. **42** (2005), no. 1-2, 105–121.

[GuZw1] L.Guillopé and M. Zworski. *Polynomial bounds on the number of resonanses for some complete spaces of constant negative curvature near infinity,* Asymptot. Anal. **11** (1995), 1-22.

[GuZw2] L.Guillopé and M. Zworski. *Upper bounds on the number of resonances for non-compact Riemannian surfaces.* J. Funct. Anal. **129** (1995), 364–389.

[JoSB] M.S. Joshi and A. Sá Barreto. *Inverse scattering on asymptotically hyperbolic manifolds.* Acta Math. **184**, 41-86, (2000)

[Mar09] L.Marazzi. *Inverse scattering on conformally compact manifolds.* Inverse problems and Imaging **3**, No. 3, 537–550 (2009)

[Mar11] L.Marazzi. *Asymptotically hyperbolic manifolds with polyhomogeneous metric.* To appear in Differential and integral equations, arXiv:0811.4473.

[MaMe] Mazzeo, R.R.; Melrose, R. B. *Meromorphic extention of the resolvent on complete spaces with asymptotically constant negative curvature.* Journal of Functional analysis **75**, 260-310 (1987)

[Vod02] G. Vodev. *Uniform estimates of the Laplace-Beltrami operator on infinite volume Riemannian manifolds with cusps,* Comm. Partial Diff. Equations **27** (2002), 1437-1465.

[Vod04] G. Vodev, *Local energy decay of solutions to the wave equation for non trapping metrics,* Ark. Mat., **42** (2004), 379-397.

Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA

E-mail address: `leonardo.marazzi@uky.edu`