Inverse Systems of Zero-dimensional Schemes in $\mathbb{P}^n$

Young Hyun Cho

Department of Mathematics, Seoul National University, Seoul 151-742, Korea.

Anthony Iarrobino

Department of Mathematics, Northeastern University, Boston, MA 02115, USA.

July 7, 2000

Abstract

The authors construct the global Macaulay inverse system $L_{\mathfrak{Z}}$ for a zero-dimensional subscheme $\mathfrak{Z}$ of projective n-space $\mathbb{P}^n$ over an algebraically closed field K, from the local inverse systems of the irreducible components of $\mathfrak{Z}$. In Section 2, they show how to find "generators" of $L_{\mathfrak{Z}}$ from generators of the local inverse systems (Lemma 2.9, Theorems 2.24, 2.29). They also give examples showing a somewhat surprising behavior of this globalization with respect to the regularity degree $\sigma(\mathfrak{Z})$: in particular, although $L_{\mathfrak{Z}}$ is determined by $(L_{\mathfrak{Z}})_{\sigma(\mathfrak{Z})}$, the degree i component $(L_{\mathfrak{Z}})_i$ may not be simply a homogenization of $(L_{\mathfrak{Z}})_{\sigma(\mathfrak{Z})}$ to degree i (Examples 2.13 to 2.17). This concrete globalization, building the Macaulay dual $L_{\mathfrak{Z}}$ of the one-dimensional coordinate ring $\mathcal{O}_{\mathfrak{Z}}$, directly from the Artinian local inverse systems is the main result.

As application they show in Theorem 3.3 that when $\mathfrak{Z}$ is a (locally) Gorenstein zero-dimensional subscheme of $\mathbb{P}^n$, and the positive integer j is sufficiently large, then a general enough graded Artinian Gorenstein quotient A of $\mathcal{O}_{\mathfrak{Z}}$ of socle degree j, has the maximum possible Hilbert function $H(A) = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$, given the Hilbert function $H_{\mathfrak{Z}}$ of $\mathcal{O}_{\mathfrak{Z}}$. Also, the algebra A, or, equivalently, a general enough degree-j form F annihilated by the defining ideal $I_{\mathfrak{Z}}$ in the Macaulay duality, determines $\mathfrak{Z}$.

The main tools are elementary, but delicate: a careful study of how to homogenize a local inverse system (Definition 2.4, Proposition 2.11), and of the behavior of the homogenization under a change of coordinates (Comparison Theorem 2.24).

In a sequel paper the authors determine the global Hilbert functions $H_{\mathfrak{Z}}$ for compressed Gorenstein subschemes $\mathfrak{Z} \subset \mathbb{P}^n$. Then, using Theorem 3.3, they exhibit families $\mathrm{PGOR}(T)$ of graded Gorenstein Artin algebras of embedding dimension r and certain Hilbert functions $T = H(s, j, r), r \geq 5, s$ large enough, that contain several irreducible components [ChoI1]. In a second related paper, they show that Theorem 3.3 cannot be simply extended to schemes $\mathfrak{Z}$ locally of type two, and type two level Artinian quotients — having two-dimensional socle in a single degree — by showing that there are no level Artinian algebras of Hilbert function $H = (1, 3, 5, 6, \ldots, 2)$ [ChoI2].

1 Introduction

We study Macaulay's inverse systems for the defining ideals of punctual subschemes $\mathfrak{Z}$ of the projective space $\mathbb{P}^n$ over an algebraically closed field K. Of course, we may suppose that such schemes are contained in an affine subspace $\mathbb{A}^n$ of $\mathbb{P}^n$. For any graded ideal I in the coordinate ring $R = K[x_1, \dots, x_{n+1}]$ of $\mathbb{P}^n$, Macaulay's inverse system I^{-1} is an R-submodule of the dual ring, the divided power series ring $\Gamma = K_{DP}[X_1, \dots, X_{n+1}]$; and I^{-1} contains the same information as is in the original ideal. Thus, it is not hard to determine which inverse systems arise from punctual schemes (Proposition 1.13), or which arise from a punctual scheme concentrated at a single point (Lemma 2.1).

Our main work here begins with an Artinian quotient $A = R'/J$ of the coordinate ring $R' = K[y_1, \dots, y_n]$ of affine n-space $\mathbb{A}^n \subset \mathbb{P}^n : x_{n+1} = 1$ that defines a punctual subscheme $\mathfrak{Z} \subset \mathbb{A}^n$, concentrated at a finite set of points. Its "local", or affine inverse system $L'(J)$ is an R'-submodule of the completion $\widehat{\Gamma'}$ of the divided power ring $\Gamma' = K_{DP}[Y_1, \dots, Y_n]$ dual to R' — this completion is the R'-injective envelope of K. We then determine from $L'(J)$ the "global" inverse system $L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1} \subset \Gamma$ over $\mathbb{P}^n$, of the defining ideal $I_{\mathfrak{Z}} \subset R$ for $\mathfrak{Z}$. Our goal is to write "generators" (in a suitable sense) of the global inverse system $L_{\mathfrak{Z}}$, in terms of generators of the local inverse systems of the irreducible components of $\mathfrak{Z}$.

Suppose that $\mathfrak{Z}$ has degree s. Then $A = R'/J$ has dimension s as K-vector space. The local, or affine inverse system $L'(J)$ also has dimension $\dim_K L'(J) = s$. Since $J = \cap_k J(k)$, the intersection of its primary components, the inverse system $L'(J)$ is a direct sum of the local inverse systems $L'(J(k)) = L'(J\mathcal{O}_{p(k)}) \subset \widehat{\Gamma'}$ at the points $p(k)$ of support of $\mathfrak{Z}$. The scheme $\mathfrak{Z}$ has a unique saturated global defining ideal $I_{\mathfrak{Z}}$, and its coordinate ring $\mathcal{O}_{\mathfrak{Z}} = R/I_{\mathfrak{Z}}$ has Krull dimension one. The global Hilbert function $H_{\mathfrak{Z}} = H(\mathcal{O}_{\mathfrak{Z}})$, satisfies $H_{\mathfrak{Z}} = (1, \dots, s, s, \dots)$, the first difference $\Delta H_{\mathfrak{Z}}$ being an O-sequence of total length s (see Theorem 1.12). The global inverse system $L_{\mathfrak{Z}}$ is a non-finitely generated, graded R-submodule of Γ, whose global Hilbert function $H(L_{\mathfrak{Z}})$ satisfies $H(L_{\mathfrak{Z}}) = H_{\mathfrak{Z}}$. Suppose now that $\mathfrak{Z}$ is concentrated at a single point p. Since $(H_{\mathfrak{Z}})_i = s$ for $s \geq \tau(\mathfrak{Z})$, an invariant of $\mathfrak{Z}$, it is natural to believe that $(L_{\mathfrak{Z}})_i$ should be a homogenization of $(L'(J))_{\leq i}$ where $L'(J) = (L_{\mathfrak{Z}})_{x_{n+1}=1}$, at least for $i \geq \alpha(\mathfrak{Z})$, the socle degree of A (Proposition 2.11, Theorem 2.24).

It is well known that the global Hilbert function $H_{\mathfrak{Z}}$ is not determined by the local Hilbert functions of $\mathcal{O}_p/J\mathcal{O}_p$ at the points of its support — even when the support of $\mathfrak{Z}$ is a single point. An exception is when $J\mathcal{O}_p$ is "conic", itself a graded ideal in the local ring $\mathcal{O}_p$ (see Example 1.11). In this "conic" local case, we have $\Delta H_{\mathfrak{Z}} = H(\mathcal{O}_p/J\mathcal{O}_p)$, and the ideal $I_{\mathfrak{Z}}$ and its global inverse system $L_{\mathfrak{Z}}$ is easily read from I', L' ([IK, Lemma 6.1], Proposition 2.18 below). In general, how do we determine the global inverse system $L_{\mathfrak{Z}}$ from the local inverse systems $L'(J(k))$? This is the main question that we answer explicitly below, through suitably homogenizing the local inverse systems (Definition 2.4, Lemma 2.9), and through the Comparison Theorem 2.24, and Decomposition Theorem 2.29.

In Section 2.1 we consider the case $\mathfrak{Z}$ has support a coordinate point $p = p_0 = (0 : \ldots : 0 : 1) \in \mathbb{P}^n$. Since $\mathcal{I}_p = J\mathcal{O}_p$ defines an Artinian quotient, we have $J \supset {M'}^{j+1}, M' = (y_1, \dots, y_n)$ for some integer $j > 0$, and we may replace $\mathcal{O}_p$ by R'. Thus J has an inverse system $L'(J) \subset \Gamma'$: there is no need to complete to $\widehat{\Gamma'}$ for $p = p_0$, however $L'(J)$ will usually not be graded. We define the homogenization of $L'(J)$ in Definition 2.4, then show it is the same as $L_{\mathfrak{Z}}$ and find suitable "generators" of $L_{\mathfrak{Z}}$ in Lemmas 2.7, and 2.9. We give examples showing that the regularity degree component $(L_{\mathfrak{Z}})_{\sigma(\mathfrak{Z})}$ of the global inverse system, although it determines all other components, need not do so simply by a homogenization-related process (Examples 2.13 to 2.17); however the socle-degree component $L_{\alpha(\mathfrak{Z})}$ does so determine $L_{\mathfrak{Z}}$ (Proposition 2.11, (ii)).

In Section 2.2 we similarly determine the global inverse system $L_{\mathfrak{Z}}$ for a scheme concentrated at an arbitrary point $p \in \mathbb{A}^n \subset \mathbb{P}^n$. We then prove a projective space "Comparison Theorem" (Theorem 2.24) relating the inverse system at p to one concentrated at the origin. This result is different from our version of Macaulay's Comparison Lemma, which describes local inverse systems at points $p \in \mathbb{A}^n$, as the product of a local inverse system at the origin and an "exponential" power series f_p (Lemma 2.22). Rather, the "Comparison Theorem" shows that $L_{\mathfrak{Z}}$ and its "generators" can be obtained from $L_{\mathfrak{Z}'}$, the corresponding inverse system L_o at the origin, by suitably substitut-

[0]MSC Primary: 14N05; Secondary: 13N10,13D40,13H10, 14C05.

ing divided powers of a linear form $L_p = \sum_k a_k X_k$ determined by the coordinates $(a_1, \ldots, a_n, 1)$ of p, for powers of $Z = X_{n+1}$ in L_o.

We complete our study of globalization in Section 2.3. The Decomposition Theorem 2.29 handles the transition to arbitrary punctual schemes. We briefly discuss regularity degree, giving an upper bound in terms of the invariant $\alpha(\mathfrak{Z})$ when the number of components is less or equal $n+2$ (Proposition 2.34). Our concrete globalization $L_{\mathfrak{Z}}$ of the local Macaulay inverse system is new, and is a main contribution of this article.

In Section 3 we give the application that motivated our globalization of inverse systems. When $\mathfrak{Z}$ is a (locally) Gorenstein punctual subscheme of $\mathbb{P}^n$, there is an obvious upper bound $\mathrm{Sym}(H_{\mathfrak{Z}}, j)$ for the Hilbert function $H(A)$ of an Artinian Gorenstein (GA) quotient A of $\mathcal{O}_{\mathfrak{Z}}$, having socle degree j. As a consequence of our construction of "generators" for the global inverse system, we show that this upper bound is always achieved by some GA quotient of $\mathcal{O}_{\mathfrak{Z}}$, hence for almost all GA quotients of socle degree j, provided that j is sufficiently large (Theorem 3.3).

1.1 Inverse systems and Gorenstein subschemes of $\mathbb{P}^n$

Macaulay used his inverse systems, a version of the classical notion of apolarity, to develop a theory of primary decomposition of ideals [Mac1]. Consider a subscheme $\mathfrak{Z} = \mathrm{Spec}(\mathcal{O}_p/\mathcal{I}_p)$ of affine n-space concentrated at the point p of $\mathbb{A}^n$, whose maximal ideal is $m_p \subset R' = k[y_1, \ldots, y_n]$, and local ring $\mathcal{O}_p$. We may write also $\mathfrak{Z} = \mathrm{Spec}(A)$, $A = R'/I'$, where A has finite length, and where I' satisfies $m_p \supset I' \supset m_p^{\alpha+1}$ for some α. The affine inverse system $L(I') \subset \widehat{\Gamma'}$ is a finite R'-module, isomorphic to the dualizing module $\Omega(A)$. Macaulay's "Comparison Lemma" relates the quotient $R' \to A$ and inverse system $L(I')$ to its translation, an isomorphic Artin quotient $R' \to A_o$ and inverse system L_o concentrated at the origin p_0 of affine space: we have $L(I') = L_o \cdot f_p$, for a certain rational power series f_p. We give a second version of the Comparison Lemma when char $K = 0$, using the partial differentiation action of R on a dual polynomial ring $\mathcal{R}$ — then $L(I') = L_o \cdot F_p \subset \widehat{\mathcal{R}}$ where F_p an exponential power series; and we compare with Macaulay's original version (Lemmas 2.21, 2.22, and Remark 2.23). The number of generators of the submodule $L(I') \subset \widehat{\Gamma'}$ is the "type" of A, the vector space dimension of the *socle* $\mathrm{SOC}(A) = (0 : m)$ (Definition 1.7). In particular when A is a Gorenstein Artin algebra — one whose socle is a vector space of dimension 1 — the local inverse system has a single generator, and was termed by Macaulay a "principal system" [Mac1, §60].

Inverse systems under the name "apolarity" were known to the early Italian algebraic geometers: they were used classically by A. Terracini, and others more recently to translate questions about the Hilbert function of ideals of functions vanishing to specified order at a set of general enough points in $\mathbb{P}^n$ (the "interpolation problem"), to questions concerning the Hilbert functions of ideals generated by powers of the corresponding linear forms. This translation has led to new insights — if still conjectural — when $n \geq 3$ [Ter1, Ter2, EhR, I4], and also contributed to the solution of a "Waring problem" for forms, via J. Alexander and A. Hirschowitz's solution of the order-two interpolation problem (see [Ter2, AlH, Cha2, I3],[IK, §2.1]). Principal inverse systems generated by certain forms associated to partitions, occur as spaces of "harmonics" in a recent "n-factorial conjecture" in combinatorics and geometry [Ha]; they are also related to constant-coefficient partial differential equations [Rez]. Inverse systems have been studied further, sometimes as Matlis duality/injective envelope (see [No, NR],[BS, Chap. 10]). Related to the simpler Matlis duality are the deeper topics of dualizing modules, residues, and local cohomology (see [L-J, BS, Schz]).

Macaulay introduced his inverse systems in the context of affine space, although he also studied their homogenization in the case of ideals concentrated at the origin of affine space (see §59, §75 of [Mac1]). However, to our knowledge, there has hitherto been no systematic study of inverse systems in the context of projective spaces, beyond the case of fat points considered by A. Terracini and others ([Ter1, Ter2, EhR, EmI2, Ge, I3], see [Tes] for an exception). We here carry out this study of inverse systems for arbitrary punctual subschemes $\mathfrak{Z}$ of projective space $\mathbb{P}^n$. In particular we determine how to suitably "homogenize" the local or affine inverse systems studied by Macaulay, to obtain the global inverse system to the global ideal $I_{\mathfrak{Z}}$ defining $\mathfrak{Z}$ (Lemma 2.7, Theorem 2.24).

Since a zero-dimensional scheme $\mathfrak{Z} \subset \mathbb{P}^n$ is the union of a finite number of schemes $\mathfrak{Z}(i)$, each supported at a single point $p(i) \in \mathbb{P}^n$, we may define dualizing modules $D(V)$ for B-modules V, where $B = R/I_{\mathfrak{Z}}$, as direct sums of the dualizing modules at the finite number of points: thus $D(V) = \mathrm{Hom}(V, \oplus E(i))$, where $E(i) = E(R/M(i))$ is the injective hull of the residue field K of B at the maximal ideal $M(i)$ at the i-th point $p(i)$ of the support. This viewpoint is adopted by Curtis-Reiner [CR, p.37], and was used by R. Michler in [Mi]. However, our task is in one sense easier, and in another different. Easier, since our ideal $I_{\mathfrak{Z}}(i)$ includes a power of the maximal ideal $m_{p(i)}$ at $p(i)$ so we may avoid the full injective hull and deal locally with "dual polynomials", or dual polynomials times an exponential (see [L-J, Mac1] and Lemma 2.22, Remark 2.23 below). Different, since we wish here to consider a global inverse system for $R/I_{\mathfrak{Z}}$ that is embedded in Γ, rather than simply being an R-module. We pass from the local inverse systems for the ideals $I_{\mathfrak{Z}(i)}$ at each point, finite submodules of $\widehat{\Gamma}'$, to the global inverse system for the ideal $I_{\mathfrak{Z}} \subset R$, which is not finitely generated, but, rather, is determined by a finite number of its elements, as we shall see. Thus, we regard the inverse system $L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1}$ as a subspace of Γ, the R-injective envelope of K, and keep track of the transition from local to global. It may be that a different approach would be more general; but we have chosen to be quite concrete. One reason for our choice to consider the inverse system $L_{\mathfrak{Z}}$ of $I_{\mathfrak{Z}}$ inside Γ, is that our main application concerns Artinian Gorenstein algebras determined by a general element F of the degree-j component $(L_{\mathfrak{Z}})_j$ of the inverse system.

Artinian Gorenstein algebras (which we will henceforth usually call "Gorenstein Artin" (GA), the more common term among specialists) are minimal reductions of Gorenstein algebras. Gorenstein algebras are a natural generalization of complete intersections. Artin algebras, and in particular GA algebras occur in the study of mapping germs of differentiable maps; GA algebras that are in general non-standard — have generators of different degrees occur as the homology rings of manifolds. Recently a category of commutative Frobenius algebras, that are products of fields, and usually non-standard, non-graded Gorenstein Artin algebras — have been identified with the category of two-dimensional topological quantum field theories [Ab]. However, J. Watanabe showed that the family ZGOR(T) of all – not necessarily graded – standard GA algebras of a symmetric Hilbert function T is fibered by the map $A \to Gr_m(A)$ to the associated graded algebra, over the family PGOR(T) parametrizing graded GA algebras of Hilbert function T ([Wa],[I2, Prop. 1.7]. Our work here relates to the knowledge of PGOR(T), particularly of its component structure and we hope there could be application to these other fields. Certain standard graded GA algebras that are "generic" — have no deformations to GA algebras of different Hilbert function — have been already used by V. Puppe to construct manifolds having no circle action [Pup].

The inverse system viewpoint can be used to parametrize Gorenstein Artin algebra quotients of R' having a given Hilbert function (see, for example [I2]). Several authors have studied from this or related viewpoints "compressed algebras" — those having a maximum possible Hilbert function, given the socle degree and embedding dimension (see [I1, FL, Bo2]). One application of our work will be to construct irreducible components of PGOR(T) for certain T, in embedding dimension at least five, using as an ingredient the local punctual schemes corresponding to "compressed algebras" [ChoI1].

We first translate into the language of global inverse systems, some basic algebraic properties of the coordinate ring $R/I_{\mathfrak{Z}}$, where $I_{\mathfrak{Z}}$ is the defining ideal of a zero-dimensional subscheme of $\mathbb{P}^n$: we consider such properties as "there is a linear non zero-divisor ℓ on $R/I_{\mathfrak{Z}}$", and the "type" of $R/I_{\mathfrak{Z}}$. Then we use the inverse sytems to study such questions as "When is $\mathfrak{Z}$ arithmetically Gorenstein (aG)?", and "When can $I_{\mathfrak{Z}}$ be recovered from a general form F annihilated by $I_{\mathfrak{Z}}$?" (F must have sufficiently high degree). We discuss the former question, "when is $\mathfrak{Z}$ aG?", in Example 2.14, Proposition 2.18, Corollary 2.20, Remark 2.31, and in Examples 2.32, 3.14, 3.15. As to the latter question, it is not hard to see that if $\mathfrak{Z}$ is Gorenstein and is also either smooth, or concentrated at a single point and "conic" — defined by a homogeneous ideal $\mathcal{I}_p$ of the local ring $\mathcal{O}_p$ — then we can recover $I_{\mathfrak{Z}}$ from F (see [Bo2],[IK, Lemma 6.1]). However, it is also easy to see that the second order neighborhood of a point $p \in \mathbb{P}^n, n \geq 2$, a non-Gorenstein scheme defined by ${m_p}^2$, cannot be recovered in this manner (Example 3.1). What is the context in which we might recover $\mathfrak{Z}$ from an Artinian Gorenstein quotient?

We answer this question in Theorem 3.3, our main application to Artinian Gorenstein quotients of the coordinate rings of locally Gorenstein schemes $\mathfrak{Z}$. Given a positive integer j and a sequence $H_{\mathfrak{Z}}$, we let $\mathrm{Sym}(H_{\mathfrak{Z}}, j)$ be the sequence

$$\mathrm{Sym}(H_{\mathfrak{Z}}, j)_i = \begin{cases} (H_{\mathfrak{Z}})_i \ , & \text{if } i \leq j/2; \\ (H_{\mathfrak{Z}})_{j-i} \ , & \text{if } i \geq j/2. \end{cases} \tag{1.1}$$

We denote by $\sigma(\mathfrak{Z})$ the Castelnuovo-Mumford regularity of $\mathfrak{Z}$, we set $\tau(\mathfrak{Z}) = \sigma(\mathfrak{Z}) - 1$, and let $\alpha(\mathfrak{Z})$ be the maximum socle degree of the local coordinate ring of any irreducible component $\mathfrak{Z}(i)$ (see Definition 2.3). We let $\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \max\{\tau(\mathfrak{Z}), \alpha(\mathfrak{Z})\}$, $I_{\mathfrak{Z}}$ be the defining ideal, $L_{\mathfrak{Z}}$ its inverse system, and now state Theorem 3.3.

Theorem. Recovering the scheme $\mathfrak{Z}$ from a Gorenstein Artin quotient. *Let $\mathfrak{Z}$ be a (locally) Gorenstein zero-dimensional subscheme of $\mathbb{P}^n$ over an algebraically closed field K,* char $K = 0$ *or* char $K > j$, *and let $L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1}$. Then we have*

1. *If $j \geq \beta(\mathfrak{Z})$, and F is a general enough element of $(L_{\mathfrak{Z}})_j$, then $H(R/\mathrm{Ann}\ (F)) = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$.*
2. *If $j \geq \beta(\mathfrak{Z})$, and F is a general enough element of $(L_{\mathfrak{Z}})_j$, then for i satisfying $\tau(\mathfrak{Z}) \leq i \leq j - \alpha(\mathfrak{Z})$ we have $\mathrm{Ann}\ (F)_i = (I_{\mathfrak{Z}})_i$. Equivalently, we have $R_{j-i} \circ F = (L_{\mathfrak{Z}})_i$.*
3. *If $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z}) + 1\}$, and $F \in (L_{\mathfrak{Z}})_j$ is general enough, then $\mathrm{Ann}\ (F)$ determines $\mathfrak{Z}$ uniquely. If $I_{\mathfrak{Z}}$ is generated in degree $\tau(\mathfrak{Z})$, then $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z})\}$ suffices.*

Thus, we may recover $\mathfrak{Z}$ from a general dual form F when $\mathfrak{Z}$ is locally Gorenstein and j is large enough. The authors show elsewhere that Theorem 3.3 does not extend simply to subschemes $\mathfrak{Z}$ that are not Gorenstein, by showing that the sequence $H = (1, 3, 4, 5, ..., 6, 2)$ cannot occur as the Hilbert function of a level algebra — one having socle in a single degree ([ChoI2]).

The question of which symmetric sequences T of integers are Gorenstein sequences — can occur as Hilbert functions of a Gorenstein Artin algebra — is open in embedding dimension $r \geq 4$. Note that we do not here find any new Gorenstein sequences of the form $T = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$, since each sequence $H_{\mathfrak{Z}}$ already occurs for a smooth scheme $\mathfrak{Z}$ by [Mar], (see Theorem 1.12 below), and the above Theorem was already known for smooth schemes [Bo2],[IK, Theorem 5.3E, Lemma 6.1]. Rather, the importance of this result is that it allows us to relate the postulation punctual Hilbert scheme $\mathrm{Hilb}^s_{\mathrm{Gor,H}}(\mathbb{P}^n)$ parametrizing degree-s Gorenstein subschemes $\mathfrak{Z} \subset \mathbb{P}^n$, satisfying $H_{\mathfrak{Z}} = H$, with the scheme $\mathrm{PGOR}(T)$ parametrizing graded Gorenstein Artin algebras of Hilbert function $T = \mathrm{Sym}(H, j)$.

In a sequel paper [ChoI1] we determine the global Hilbert functions $H_{\mathfrak{Z}}$ for compressed Gorenstein subschemes $\mathfrak{Z} \subset \mathbb{P}^n$. Let $H_s(r), r = n+1$, satisfy $H_s(r)_i = \min\{\dim_K R_i, s\}$; then $H_s(r)$ is the global Hilbert function of a generic degree-s smooth scheme. We show that if $\mathfrak{Z}$ is a general enough compressed local Gorenstein scheme of degree s, then $H_{\mathfrak{Z}} = H_s(r)$. Using Theorem 3.3, we will exhibit families $\mathrm{PGOR}(T)$ of graded Gorenstein Artin algebras of embedding dimension r and certain Hilbert functions $T = H(s, j, r) = \mathrm{Sym}(H_s(r), j), r \geq 5, s$ large enough given r, that contain several irreducible components. Each component is fibred over a family of Gorenstein zero-dimensional schemes, with fibre an open in a projective space $\mathbb{P}^{s-1}$. One component is fibred over general enough smooth schemes $\mathfrak{Z} \subset \mathbb{P}^n, n = r - 1$ of degree s. The other component is fibred over a family of compressed Gorenstein subschemes. Here

$T = H(s,j,r) = \mathrm{Sym}(H_s(r), j) : H(s,j,r)_i = \min\{r_i, r_{j-i}, s\}$, is the Hilbert function of GA algebras $R/\mathrm{Ann}\ (F), F = L_1^s + \ldots + L_s^j \in \Gamma$, determined by a dual generator F that is a sum of s general enough (divided) powers of linear forms. Some of these results were reported in [IK, §6.4].

In Section 3.3 we explore a second viewpoint on our construction of a global inverse system from the local inverse system related to "generalized additive decomposition" of a form F when $r > 2$. A binary form F of degree j, always has a length-s generalized additive decomposition (GAD), with $s \leq (j+2)/2$: this is either a sum of j-th powers of s distinct linear forms, or a sum

$$F = \sum_i B_i L_i^{j+1-s_i}, \deg B_i = s_i - 1, \deg L_i = 1, s = \sum s_i. \tag{1.2}$$

The existence of such an additive decomposition when $r = 2$ is equivalent to there being a form $h \in \mathrm{Ann}\ (F)$ that can be written $h = \prod_i \ell_i^{s_i}$, where $\ell_i \circ L_i = 0$. Thus, the additive decomposition of equation (1.2) corresponds to a punctual scheme $\mathfrak{Z} : h = 0 \subset \mathbb{P}^1$, whose irreducible components $\mathfrak{Z}_i : \ell_i^{s_i} = 0$ have specified multiplicities s_i. If $2s \leq j+1$ then it is classical that the GAD as in equation (1.2) is unique: for an exposition see §1.3 of [IK], especially Prop. 1.36, Theorem 1.43.

For any embedding dimension, we say that a punctual scheme $\mathfrak{Z} \subset \mathbb{P}^n$ is an *annihilating scheme* of the form $F \in R$, if $I_{\mathfrak{Z}} \circ F = 0$. Since for a punctual scheme, $I_{\mathfrak{Z}} = \cap I_{\mathfrak{Z}_i}$, where $\mathfrak{Z}_i$ are the irreducible components of $\mathfrak{Z}$, we have ${I_{\mathfrak{Z}}}^{\perp} = \sum_i {I_{\mathfrak{Z}_i}}^{\perp}$, it follows that any form F annihilated by $\mathfrak{Z}$ can be written as a "generalized sum", a sum of forms annihilated by the components $\mathfrak{Z}_i$. In determining very concretely the inverse systems of $I_{\mathfrak{Z}_i}$, we are partially answering the question, what is a generalized additive decomposition? In particular, when $r = 3$, many forms F have a "tight" annihilating scheme $\mathfrak{Z}_F \subset \mathbb{P}^2$ that is unique; as well, there is often a unique "generalized additive decomposition", up to trivial multiplications. This occurs when the Hilbert function $H(R/\mathrm{Ann}\ (F))$ contains as a subsequence (s,s,s). Then, there is a unique degree-s annihilating scheme, according to [IK, Theorem 5.31], and, as we shall see, a corresponding unique "generalized additive decomposition" for F (Theorems 3.19 and 3.20).

1.2 Notation and Basic Facts

We now introduce notation, following [IK, Appendix A]. We will assume throughout, unless specifically stated otherwise, that the base field K satisfies char $K = 0$, or char $K = p > j$, where j is the maximum degree of any form considered (see Example 2.2 for the necessity of this assumption). We will also assume either that K is algebraically closed, or that all punctual schemes considered have as support K-rational points. Let C denote the K-vector space $\langle x_1, \ldots, x_{n+1}\rangle$, and $C^* = \langle X_1, \ldots, X_{n+1}\rangle$ denote its dual; recall that the divided power ring $\Gamma = \Gamma(C^*) = K_{DP}[X_1, \ldots, X_{n+1}]$ satisfies,

$$\Gamma = \oplus\Gamma_j = \oplus\mathrm{Hom}(R_j, K), \text{ with } \Gamma_j = \langle\{X^{[U]} | |U| = j\}\rangle, \text{ the span of the dual generators to } x^U \in R,$$

where here U denotes the multiindex $U = (u_0, \ldots u_n)$, of length $|U| = \sum u_i$. For convenience we set $X^{[U]} = 0$ if any component of U is negative. The multiplication in Γ is defined by

$$X^{[U]} \cdot X^{[V]} = \binom{U+V}{U} X^{[U+V]}. \tag{1.3}$$

We denote by R', Γ', respectively, the corresponding rings $R' = K[y_1, \ldots, y_n]$ and $\Gamma' = K_{DP}[Y_1, \ldots, Y_n]$, respectively. We have $\Gamma' \cong E' = \mathrm{Hom}_{R'}(R', R'/M'), M' = (y_1, \ldots, y_n)$. We denote by $\widehat{\Gamma'}$ the completion of Γ' with respect to M'; thus, $\widehat{\Gamma'}$ is a divided power series ring. The rings R', Γ' correspond to the point $p_0 = (0 : \cdots : 0 : 1)$ in $\mathbb{P}^n$, whose maximal ideal is $m_{p_0} = (x_1, \ldots, x_n) \subset R$. We recall below the contraction action of $R = K[x_1, \ldots, x_{n+1}]$ on the divided power ring. Note that, given our assumption excluding low characteristics, each theorem about inverse systems stated in the context of the contraction action of R on Γ, has an analogue for the partial differential operator (PDO) action of R on $\mathcal{R} = K[X_1, \ldots, X_{n+1}]$, a second copy of the polynomial ring. When char $K = 0$, there is a natural Gl-invariant homomorphism $\phi : \mathcal{R}$ to Γ, $\phi(X^U) = U!X^{[U]} \in \Gamma$ (for a discussion see [IK, Appendix A]). To keep the exposition simple, we will in general restrict ourselves to the contraction action. Note that we use here a different notation than Macaulay's $K[x_1^{-1}, ..., x_{n+1}^{-1}]$ for the injective envelope $E = E(K)$ ([Mac1],[Ei, Theorem 21.6]). The claims implicit in (d),(e) of the following Definition are shown in Lemmas 1.4 and 1.6 below.

Definition 1.1. Inverse Systems

(a) (contraction action) If $h = \sum a_K x^K \in R, F = \sum b_U X^{[U]} \in \Gamma$, *then*

$$h \circ F = \sum_{K,U} a_K b_U X^{[U-K]}$$

(b) (partial differentiation action — PDO) If $h \in R, F \in \mathcal{R} = K[X_1, \ldots, X_{n+1}]$, *then*

$$h \circ F = h(\partial\ /\partial X_1, \ldots, \partial\ /\partial X_{n+1})(F) \in \mathcal{R}.$$

(c) A homogeneous inverse system $W \subset \Gamma$ *is a graded* R*-submodule of* Γ *under the contraction action. Thus* $W = W_0 \oplus \cdots \oplus W_j \oplus \cdots \subset \Gamma$ *is an inverse system iff* $\forall i \leq j, R_i \circ W_j \subset W_{j-i}$.

(d) (inverse system of a graded ideal) If I *is a graded ideal of* R*, we will denote by* I^{-1} *or by* $I^{\perp}$ *the homogeneous inverse system of* I*, namely the* R*-submodule of* Γ *given by* $I^{\perp} = \oplus {I_j}^{\perp}$*, where*

$${I_j}^{\perp} = \{F \in \Gamma_j | \, h \circ F = 0 \quad \forall h \in I_j\}.$$

(e) (ideal of an inverse system) If $W \subset \Gamma$ *is an inverse system, then we denote by* I_W *the ideal* $I_W = \text{Ann } (W)$ *where* $(I_W)_j = \{h \in R_j \mid h \circ w = 0, \ \ \forall w \in W\}$.

(f) (local inverse system) An inverse system in $\widehat{\Gamma'}$ *is an* R'*-submodule of* $\widehat{\Gamma'}$ *under the contraction action. If* J *is any ideal of* $R' = K[y_1, \ldots, y_n]$*, then we denote by* $J^{\perp} = J^{-1} \in \widehat{\Gamma'}$*, the inverse system of all elements of* $\widehat{\Gamma'}$ *annihilated by* J*, in the contraction action. The ideal* $I_W \subset R'$ *of an inverse system* $W \subset \widehat{\Gamma'}$ *is the annihilator of* W *under the contraction action. [Warning: in general neither* J *nor* $J^{\perp}$ *is homogeneous].*

Henceforth in this paper, inverse systems in Γ (but not in $\Gamma', \widehat{\Gamma'}$) are assumed to be homogeneous.

We will later need that the elements of R_1 act as differentials on Γ (see Lemma 2.22).

Lemma 1.2. *If* ℓ *is an element of* R_1*, and* $F, G \in \Gamma_u, \Gamma_v$*, respectively, then*

$$\ell \circ (F \cdot G) = (\ell \circ F) \cdot G + F \cdot (\ell \circ G). \tag{1.4}$$

Proof. By bilinearity, it suffices to show (1.4) when ℓ is a variable, and F, G are monomials, whence it suffices to show it when $R = K[x], \Gamma = K[X]$ in a single variable, and for $\ell = x, F = X^{[a]}, G = X^{[b]}$. There, it results from the definition of the multiplication in the divided power ring Γ, and the usual Pascal triangle binomial identity. □

We need a simple result relating inverse systems and ideals. First we recall

Definition 1.3. *(a) If* $V \subset R_j$*, and* $i \geq 0$*, we have* $R_i \cdot V = \langle hv \mid h \in R_i, v \in V\rangle$*; if also* $i \leq j$ *we have* $V : R_i = \langle h \in R_{j-i} \mid R_i h \subset V\rangle$.

(b) *If $W \subset \Gamma_j$ and $i \geq 0$, we have $R_{-i} \circ W =_{def} W : R_i = \langle \{F \in \Gamma_{j+i} \mid R_i \circ F \subset W\}\rangle$. If $W \subset \Gamma_j$ and $0 \leq i \leq j$ we have $R_i \circ W = \langle h \circ w \mid h \in R_i, w \in W\rangle$.*

Lemma 1.4. Inverse system and Matlis dual. *Assume that (V, W) is a pair of vector spaces satisfying $V \subset R_j$, $W \subset \Gamma_j$ and $V^\perp \cap \Gamma_j = W$. Then*

(i.) If $0 \leq i$, $\quad (R_i \cdot V)^\perp \cap \Gamma_{j+i} = W : R_i$.

(ii.) If $0 \leq i \leq j$, $\quad (V : R_i)^\perp \cap \Gamma_{j-i} = R_i \circ W$.

(iii.) If $L \subset \Gamma$ is a homogeneous inverse system, then Ann $(L) \subset R$ *is a graded ideal of R; if I is a graded ideal of R, then $I^{-1} \subset \Gamma$ is a homogeneous inverse system. Furthermore,* Ann $(L)^{-1} = L$; *and* Ann $(I^{-1}) = I$. *Also $I^{-1} \cong \mathrm{Hom}_K(R/I, K)$, the Matlis dual of R/I.*

(iv.) If the inverse system $L' \subset \Gamma'$ (not necessarily graded) has finite dimension as K-vector space, then $I' =$ Ann (L') *is an M'-primary ideal of R', where $M' = (y_1, \ldots, y_n)$. Conversely, an M'-primary ideal I' of R' determines a finite-dimensional inverse system of $L(I') \subset \Gamma'$.*

(v.) If $I' \subset R'$ is an ideal of finite colength c, defining an Artin quotient R'/I' with s distinct maximal ideals, then ${I'}^{-1} \subset \widehat{\Gamma'}$ is a dimension-c inverse system of the form ${I'}^{-1} = \oplus_1^s L'(i), L'(i) = V'(i)f_{p(i)}$, where $V'(i) \subset \Gamma'$ is a finite inverse system, and $f_{p(i)}$ is a specific power series (see (2.17)*).*

Proof. For (i), note that $(R_i \cdot V) \circ F = 0 \iff V \circ (R_i \circ F) = 0 \iff R_i \circ F \subset W$.
For (ii), note that if $h \in R_{j-i}$ then $h \circ (R_i \circ W) = 0 \iff R_i \circ (h \circ W) = 0 \iff R_i h \subset V$.
For (iii), note that I is a graded ideal of R if for each pair of non-negative integers (i, j), $\quad R_i \cdot I_j \subset I_{i+j}$, or, equivalently, if $I_{i+j} : R_i \supset I_j$. By (ii) the latter is equivalent to $R_i \circ \langle I_{i+j}^{-1}\rangle \subset I_j^{-1}$, implying that I^{-1} is a homogeneous inverse system. One shows similarly that the annihilator Ann $(L) \subset R$ of a homogeneous inverse system L is an ideal, using (i). That the double duals are the identities in this case follows from the exactness of the pairing $R_i \circ \Gamma_i \to k$. For (iv), note that if $L' \subset \Gamma'$ is finite dimensional then $L' \subset \Gamma'_{\leq j}$ for some integer j, hence Ann $(L') \supset {M'}^{j+1}$, and conversely. For (v), note that since $I' = \cap I'(i)$, the inverse system ${I'}^{-1}$ is the direct sum of the inverse systems $L'(i)$ of the components $I(i)$ at the points $p(i)$ of support. Then use Lemmas 2.21 and 2.22 below. □

Usually, a homogeneous inverse system $W \subset \Gamma$ is not finitely generated: if W is finitely generated, then $\dim_k W$ is finite, and by Lemma 1.4(iv) W determines an Artin algebra $A_W = R/I, I =$ Ann (W) with I an $M = (x_1, \ldots, x_{n+1})$-primary ideal. Recall

Definition 1.5. *A graded ideal $I \subset R$ is* saturated *if it has no irreducible component primary to the irrelevent ideal M, equivalently, if $I = I : M^\infty = \{f \mid \exists k \geq 0, M^k \cdot f \subset I\}$. This is equivalent to,*

$$\forall a, b \in \mathbb{N}, a \leq b \quad I_a = I_b : R_{b-a} = \{f \in R_a \mid R_{b-a} \cdot f \subset I_b\}. \tag{1.5}$$

If $\dim(R/I) = 1$, *I is saturated iff there is a (linear) non-zero divisor for R/I in R.*

Note that the condition of equation (1.5) results from the more usual saturation condition,

$$\exists N \in \mathbb{N} \mid \forall a, \forall b \geq \max(N, a),\ I_a = I_b : R_{b-a}. \tag{1.6}$$

Lemma 1.6. Macaulay's correspondence. *There is a one-to-one correspondence between homogeneous inverse systems $W \subset \Gamma$ and graded ideals I of R, given by $I \to I^{-1} \subset \Gamma$, and $W \to I_W =$* Ann $(W) \subset R$. *The ideal I_W is saturated iff the inverse system W satisfies*

$$\forall a, b \in \mathbb{N}, a \leq b \quad W_a = R_{b-a} \circ W_b. \tag{1.7}$$

Furthermore, the element $\ell \in R_i$ is a non-zero divisor for R/I iff $W = I^{-1}$ satisfies

$$\forall b \in \mathbb{N}, b \geq i, \text{ we have } \ell \circ W_b = W_{b-i}. \tag{1.8}$$

Proof. The 1-1 correspondence has been shown in Lemma 1.4. The relation (1.7) follows from (1.5), using Lemma 1.4ii. That ℓ is a non zero-divisor for R/I is equivalent to

$$\text{for each integer } b \geq i, \text{and } \forall h \in R_{b-i}, \text{ we have } \ell \cdot h \in I_b \Rightarrow h \in I_{b-i}. \tag{1.9}$$

Letting $W = I^{-1}$, we may translate the implication in (1.9) as

$$\begin{aligned}(\ell \cdot h) \circ W_b = 0 &\Longrightarrow h \circ W_{b-i} = 0, \text{ or, equivalently,}\\ h \circ (\ell \circ W_b) = 0 &\Longrightarrow h \circ W_{b-i} = 0, \text{ or}\\ (\ell \circ W_b)^{\perp} \cap R_{b-i} &\subset (W_{b-i})^{\perp} \cap R_{b-i} \text{ or}\\ \ell \circ W_b &\supset W_{b-i}.\end{aligned}$$

Since by definition $R_i \circ W_b \subset W_{b-i}$, this shows the criterion (1.8). □

We will term an inverse system W of Γ *saturated* if W satisfies (1.7) — that is, W arises from a saturated ideal. We now recall the definitions of socle, and type.

Definition 1.7. *The* socle $\mathrm{SOC}(A)$ *of an Artin algebra A with a single maximal ideal m is* $(0 : m) \subset A$. *The* type *of A is the dimension* $\dim_K \mathrm{SOC}(A)$, *and the socle degree is the maximum degree i in which* $\mathrm{SOC}(A)_i$ *is nonzero. Suppose the ring $B = R/I$ is Cohen-Macaulay, of Krull dimension one, and let $\ell \in R$ be a linear non-zero divisor for B. Then the* $\mathrm{type}(B) = \dim_K \mathrm{SOC}(A), A = B/(\ell \cdot B) = R/(I, \ell)$*; here the maximal ideal m is the image in A of $M = (x_1, \ldots, x_{n+1}) \subset R$. If $\mathfrak{Z} \subset \mathbb{P}^n$ is a zero-dimensional scheme, then its type is that of $\mathcal{O}_{\mathfrak{Z}} = R/I_{\mathfrak{Z}}$.*

It is well known that this notion of type does not depend on the non-zero divisor ℓ used: the type may also be defined as the rank of the last module in a free R-resolution of A, and these ranks remain the same when we divide by any non-zero divisor. See also [BH, Lemma 1.2.19] for the analogue in the case B is local of arbitrary dimension.

Corollary 1.8. *Suppose $I \subset R$ has inverse system $W \subset \Gamma$. The vector space $I_j/\langle R_1 \cdot I_{j-1}\rangle$ of degree-j generators of I is dual to the vector space $\langle W_{j-1} : R_1\rangle/W_j$. The vector space $(I_{j+1} : R_1)/I_j$ of degree-j socle elements of $A = R/I$ is dual to the vector space W_j/R_1W_{j+1} of degree-j generators of W.*

Proof. This is immediate from Lemma 1.6 and Lemma 1.4 (i),(ii). □

We now show how to recognize the type of I from the inverse system; as well, we wish to describe the inverse system of the projective closure of a scheme. Finally, we complete our listing of basic facts by characterizing the ideals defining zero-dimensional schemes (Theorem 1.12), and their inverse systems (Proposition 1.13).

Lemma 1.9. *Let $I = I_{\mathfrak{Z}}$ be the homogeneous saturated ideal defining a zero-dimensional subscheme $\mathfrak{Z} \subset \mathbb{P}^n$, and let $W = I^{-1} \subset \Gamma$ be the inverse system of I. Let $\ell \in R_1$ be a non-zero divisor for $B = R/I$, let $A = B/\ell B$, with maximal ideal m, denote by $\Gamma_\ell = \ell^{\perp} \subset \Gamma$ the R-submodule of Γ perpendicular to ℓ, and let $W_\ell = W \cap \Gamma_\ell$. Then*

i. W_ℓ is the dual module of A.

ii. $W_\ell/\langle M \circ W_\ell\rangle \cong (\mathrm{SOC}(A))^{\vee}$, the dual space to $\mathrm{SOC}(A)$.

Proof. Since $A = B/\ell B$ is isomorphic to $R/(I,\ell)$, its dual module is the inverse system of (I,ℓ), so $A^\vee \cong I^{-1}\cap\ell^\perp = W_\ell$: this shows (i). Also, $(I,\ell) : M$ is perpendicular to $M\circ W_\ell$. Thus, we have $A = R/(I,\ell)$ and $\mathrm{SOC}(A) = (0 : m) = ((I,\ell) : M)/(I,\ell)$, hence $(\mathrm{SOC}(A))^\vee = (R/(I,\ell))^\vee/(R/((I,\ell) : M))^\vee = (I,\ell)^\perp/((I,\ell) : M)^\perp \cong W_\ell/\langle M \circ W_\ell\rangle$. This is (ii), so completes the proof. □

When $\mathfrak{Z}$ is a punctual subscheme of $\mathbb{A}^n \subset \mathbb{P}^n$, its projective closure has an empty intersection with the hyperplane at infinity, $z = 0$, since $\mathfrak{Z}$ is already closed. However, in fact there is a graded Artinian algebra $R'/(I_\mathfrak{Z})_{z=0}$ lying on the hyperplane at infinity, uniquely determined by $\mathfrak{Z}$, and whose Hilbert function determines $H(R/I_\mathfrak{Z})$. We also show the connection with the global inverse system. Recall that the Hilbert function $H(B)$ for an R-module B is the sequence $H(B)_i = \dim_k B_i$, with B_i the degree-i component of the associated graded module $Gr_M(B)$. We will write Hilbert functions of submodules of Γ in the order of increasing degrees, so that $H(\Gamma) = H(R)$. We define the sequence ΔH by $\Delta H_i = H_i - H_{i-1}$.

Lemma 1.10. Projective Closure. *When $R = K[x_1,\ldots,x_n,z]$ and $\ell = z$ then $\Gamma_z = z^\perp = K_{DP}[X_1,\ldots,X_n]$. Suppose that $\mathfrak{Z}$ is a punctual subscheme of $\mathbb{A}^n : z = 1$, with global inverse system $W = L_\mathfrak{Z}$. Then z is a non-zerodivisor for $R/I_\mathfrak{Z}$, and $W_z = W \cap \Gamma_z$ satisfies*

a. $\Delta H(R/I_\mathfrak{Z}) = H(R/(I_\mathfrak{Z},z)) = H(W_z)$.

b. There is an exact sequence, $0 \to W_z(i) \to W(i) \xrightarrow{z\circ} W(i-1) \to 0$, *where the homomorphism* $z\circ :\ W(i) \to W(i-1)$ *is the contraction action of $z \in R$ on Γ (see Definition 1.1(a)).*

c. The above sequence is dual to $0 \to (R/I_\mathfrak{Z})(i-1) \xrightarrow{m_z\cdot} (R/I_\mathfrak{Z})(i) \to K[x_1,\ldots,x_n]/(I_\mathfrak{Z})_{z=0} \to 0$ *where the homomorphism* $m_z\cdot$ *is multiplication by z.*

In (a),(b) *above, z, W_z may be replaced by ℓ, W_ℓ, when $\mathfrak{Z}$ is an arbitrary punctual subscheme of $\mathbb{P}^n$, provided ℓ is a non-zerodivisor for $R/I_\mathfrak{Z}$.*

Proof. If z were a zerodivisor for $R/I_\mathfrak{Z}$, then z would be contained in an associated prime of $I_\mathfrak{Z}$, contradicting the assumption $\mathfrak{Z} \subset \mathbb{A}^n$. Then (c) is immediate, since z is a non-zerodivisor. The statement (b), follows from (c) by dualizing, and (a) follows from these exact sequences by taking vector space dimensions. □

Note: henceforth we will in examples sometimes use $Z^u = u!Z^{[u]}$ in writing monomials, in place of $Z^{[u]}$.

Example 1.11. *Let $I_\mathfrak{Z} = (xy, x^2z - y^3, x^3) \subset R = K[x,y,z]$; then $\mathfrak{Z}$ is a degree-5 scheme concentrated at the point $p_0 = (0 : 0 : 1)$ of $\mathbb{P}^2$ (the origin of $\mathbb{A}^2$), having global Hilbert function $H_\mathfrak{Z} = H(R/I_\mathfrak{Z}) = (1,3,5,5,\ldots)$. The Artin algebra $A = R/(I_\mathfrak{Z},z) \cong K[x,y]/(xy,x^3,y^3)$ has Hilbert function $H(A) = \Delta H_\mathfrak{Z} = (1,2,2,0)$, and is what we might regard as the "boundary" of $\mathfrak{Z}$ on the line at infinity, $z = 0$. The inverse system $W = (I_\mathfrak{Z})^{-1} \subset \Gamma = K_{DP}[X,Y,Z]$ satisfies*

$$W_3 = \langle X^{[2]}Z + Y^{[3]}, Y^2Z, YZ^2, XZ^2, Z^3\rangle$$
$$W_2 = \langle X^{[2]}, Y^2, YZ, XZ, X^2\rangle$$
$$W_1 = \langle X, Y, Z\rangle; \quad W_0 = \langle 1\rangle,$$

and $W_z = \langle 1, X, Y, X^2, Y^2\rangle = W \cap \Gamma_z = W \cap K_{DP}[X,Y] \subset \Gamma$ is the dual module to A.

When we consider $\mathfrak{Z} \subset \mathbb{A}^2$, by setting $z = 1$ in $I_\mathfrak{Z}$, we find $I' = (xy, x^2 - y^3)$, which defines a scheme concentrated at p_0 of local Hilbert function $H' = (1,2,1,1)$, different from $\Delta H_\mathfrak{Z}$.

If we consider instead $\mathfrak{Z}'$, defined by (x^2, xy, y^4), we would find the same local Hilbert function H' for $\mathfrak{Z}'$, but now $H_{\mathfrak{Z}'} = (1,3,4,5,\ldots)$, the sum function, since $\mathfrak{Z}'$ is "conic". This example shows that the local Hilbert function H' does not determine the global Hilbert function $H_\mathfrak{Z}$.

We recall next a well known result, see for example [GeM, Mar, Or]. We quote most of it from [IK, Theorem 1.69]. A scheme $\mathfrak{Z} \subset \mathbb{P}^n$ is *arithmetically Cohen-Macaulay* if $R/I_{\mathfrak{Z}}$ is Cohen-Macaulay: if $\dim \mathfrak{Z} = 0$, this is equivalent to there being a non-zero divisor in R for $R/I_{\mathfrak{Z}}$. Recall that $\tau(\mathfrak{Z}) = \min\{i \mid \dim_K((R/I_{\mathfrak{Z}})_i) = s\}$.

Theorem 1.12. *[Mar, Or, GeM]* Zero-dimensional schemes. *Let $\mathfrak{Z}$ be a degree-s zero-dimensional subscheme of $\mathbb{P}^n$, and let $I = I_{\mathfrak{Z}}$ be its saturated defining ideal. Then $\mathfrak{Z}$ is arithmetically Cohen-Macaulay, and*

i. The Hilbert function $H(R/I)$ is nondecreasing in i, and stabilizes at the value s for $i \geq \tau(\mathfrak{Z})$. We have $\tau(\mathfrak{Z}) \leq s-1$, with equality iff $\mathfrak{Z}$ is contained in a line.

ii. The Castelnuovo-Mumford regularity $\sigma = \sigma(\mathfrak{Z})$ satisfies $\sigma = \tau(I) + 1$. In particular, if $i \geq \sigma$, then $I_i = R_{i-\sigma} \cdot I_\sigma$. Thus, I is generated by degree σ.

iii. The first difference $\Delta(H(R/I)) = C = (1, c_1, \ldots c_\tau, 0)$ is an O-sequence (the Hilbert function of some Artin quotient of R'), with $s = \sum c_i$.

iv. [Mar] Every O-sequence $C = (1, c_1, \ldots, c_\tau, 0)$, $c_1 \leq n$, $c_\tau \neq 0, \sum c_i = s$, occurs as $\Delta H(R/I)$ for some degree-s punctual scheme $\mathfrak{Z}$ with $\tau(\mathfrak{Z}) = \tau$, consisting of smooth points.

Conversely, any saturated ideal $I \subset R$ satisfying the Hilbert function conditions i,iii above for $H(R/I)$ is the defining ideal of such a zero-dimensional subscheme, namely $\mathfrak{Z} = \mathrm{Proj}\ (R/I) \subset \mathbb{P}^n$.

Proof outline. There are direct proofs of (i) −(iii) in [Or, GeM]; see also [IK, Theorem 1.69]. Let I be an ideal of R, such that R/I has dimension one. One can show cohomologically that I saturated is equivalent to R/I being Cohen-Macaulay (see, for example, [IK, Lemma 1.67]), and this is equivalent to a general element ℓ of R_1 being a non-zerodivisor for R/I. Then ΔH is the Hilbert function of $R/(I_{\mathfrak{Z}}, \ell)$, so is an O-sequence. That $\sigma = \tau + 1$ is the Castelnuovo-Mumford regularity is shown cohomologically. That $\tau = s-1$ iff $\mathfrak{Z}$ is on a line is a consequence of ΔH being an O-sequence summing to s: so $\tau = s-1$ iff $\Delta H = (1, 1, \ldots, 1)$, which is equivalent to $(H_{\mathfrak{Z}})_1 = 2$. P. Maroscia's result (iv) is shown by deforming monomial ideals defining Artin quotients of R' having Hilbert function C (see [Mar, GeM]). The last statement concerning a converse follows from the 1-1 correspondence between saturated ideals of R and subschemes of $\mathbb{P}^n$. □

The first difference $\Delta H = (1, c_1, \ldots, c_{\sigma-1}, 0, \ldots)$ is sometimes termed the *h-vector* of $\mathfrak{Z}$ (see, for example, [Mig, §1.4]).

Proposition 1.13. Inverse system of a punctual scheme. *The inverse system W is the inverse system of a saturated ideal $I_{\mathfrak{Z}}$, $\mathfrak{Z}$ a degree-s zero-dimensional scheme of $\mathbb{P}^n$, regular in degree σ iff*

a. $\dim_K W_j = s \ \forall j \geq \sigma - 1$, *and*

b. $\exists N \in \mathbb{N} \mid \forall a, \forall b \geq \max(N, a),\ W_a = R_{b-a} \circ W_b$.

The condition (b) *implies the apparently stronger* (1.7). *Furthermore, if $\mathfrak{Z}$ is such a degree s scheme regular in degree σ, then for all $b \geq \sigma$,*

$$W_b = W_\sigma : R_{b-\sigma} = \{f \in \Gamma_b \mid R_{b-\sigma} \circ f \subset W_\sigma\}. \tag{1.10}$$

Proof. That an inverse system W arising from such a scheme $\mathfrak{Z}$ must satisfy (a),(b), is immediate from Lemma 1.6, and Theorem 1.12. Suppose conversely that W satisfies (a),(b). The condition (b) implies that $I = \mathrm{Ann}\ (W)$ is a saturated ideal, by Lemma 1.4ii applied to (1.6). By (a), its Hilbert polynomial is s, so I defines a zero-dimensional scheme of degree-s; and having regularity degree no greater than σ; condition (a) implies that $H(R/I) = (1, \ldots, s, s, \ldots)$, with the first s

occurring before degree $\sigma - 1$. The two imply that R/I is Cohen-Macaulay of dimension 1, and regularity degree no greater than σ (see Theorem 1.12). By Theorem 1.12 (ii) if $\mathfrak{Z}$ is such a scheme, the ideal $I = I_{\mathfrak{Z}}$ is generated by degree σ; the last equation (1.10) is a translation of this generation fact into the inverse system language, using Lemma 1.4 (i). $\square$

2 Inverse system of a punctual scheme

In Section 2.1 we consider schemes $\mathfrak{Z} \subset \mathbb{P}^n$ concentrated at a single point p_0 that is a coordinate point; these are simpler since the local inverse system lies in the ring Γ'. In Section 2.2 we study a scheme $\mathfrak{Z}$ concentrated at an arbitary point p, for which the local inverse system lies in the completion $\widehat{\Gamma'}$; finally we consider schemes for a general zero-dimensional scheme $\mathfrak{Z}$ with finite support in $\mathbb{A}^n \subset \mathbb{P}^n$, in Section 2.3. In each case we show how to directly homogenize the local inverse system for $\mathfrak{Z}$, or its components, to obtain the global inverse system $L_{\mathfrak{Z}} \subset \Gamma$ of the global defining ideal $I_{\mathfrak{Z}} \subset R$. Recall that we denote by $m_p \subset R$ the homogeneous ideal of the point p: if $p = (a_1 : \ldots : a_n : 1)$, then $m_p = (a_1 z - x_1, \ldots, a_n z - x_n)$. Recall also that the homogeneous ideal $I \subset R$ is concentrated at the point $p \in \mathbb{P}^n$ iff there exists an integer $u > 0$ such that

$$m_p \supset I \supset {m_p}^u. \tag{2.1}$$

According to both [Ter1] and [EmI2, Theorem I] (see also [EhR], which treats the special case $u = 2$), we have — under the restriction char $K = 0$ or char $K > j$

$$({m_p}^u)^\perp \cap \Gamma_j = \Gamma_{u-1} \cdot L_p^{[j+1-u]}. \tag{2.2}$$

Here, the right hand side is interpreted as Γ_j if $j < u$. Thus, the condition (2.1) corresponds to the following condition on the inverse system

$$L_p^{[j]} \subset [I^{-1}]_j \subset \Gamma_{u-1} \cdot L_p^{[j+1-u]}, \tag{2.3}$$

where if $p = (a_1 : \cdots : a_n : 1)$ then $L_p = a_1 X_1 + \cdots + a_n X_n + X_{n+1}$, and $L_p^{[j]}$ denotes the form $L_p^{[j]} = {L_p}^j / j! = \sum_{J || J| = j} a^J \cdot X^J$, proportional to the divided power L_p^j. We have shown

Lemma 2.1. *The homogeneous ideal I of R defines a zero-dimensional scheme concentrated at the point p of $\mathbb{P}^n$ iff equivalently (TFAE)*

(i) There exists an integer u such that $m_p \supset I \supset {m_p}^u = (a_1 z - x_1, \ldots, a_n z - x_n)^u$.

(ii) There exists an integer $\alpha = u - 1$ such that the inverse system $I^\perp$ satisfies

$$K_{DP}[L_p] \subset I^\perp \subset ({m_p}^u)^\perp = \Gamma_{\leq \alpha} \cdot K_{DP}[L_p]. \tag{2.4}$$

In particular, if the homogeneous ideal J of R defines a zero-dimensional scheme concentrated at the point $p_0 = (0 : \ldots : 0 : 1) \in \mathbb{P}^n$, then $K[Z] \subset J^\perp \subset \Gamma_{\leq a} \cdot K[Z], Z = X_{n+1}$ for some $a \geq 0$.

The following example shows the need for our limitation on the characteristic of K (§1.2).

Example 2.2. *Let $n = 1, R = K[x, y], \Gamma = K[X, Y]$; choose the point $p = (a_1 : 1) \in \mathbb{P}^1$, and $I = {m_p}^2 = (x - a_1 y)^2$, then we have that $[I^\perp]_2$ satisfies*

$$\begin{aligned}(a_1 X + Y)^{[2]} \subset [I^\perp]_2 \subset \Gamma_1 \cdot L_p &= \langle X, Y \rangle \cdot (a_1 X + Y) \\ &= \langle 2a_1 X^{[2]} + XY, a_1 XY + 2Y^{[2]} \rangle,\end{aligned} \tag{2.5}$$

provided char $K \neq 2$. *When* char $K = 2$ *and $a_1 = 0$ the space on the right is just $\langle XY \rangle$, so is one-dimensional, and is not all of $({m_p}^2)^\perp{}_2$, which also includes ${L_p}^{[2]} = a_1^2 X^{[2]} + a_1 XY + Y^{[2]}$. Thus, equation (2.2) and the equality on the right of Lemma 2.1 (2.4) do not extend to characteristic $p \neq 0$, when p is less than or equal to the degree j (here $j = 2$) of the forms being considered.*

Recall that the *socle degree* α of a local Artin algebra A of maximal ideal m is the highest integer such that $m^\alpha A \neq 0$, but $m^{\alpha+1}A = 0$, and that the point $p_0 = (0 : \ldots : 0 : 1)$. For a punctual scheme $\mathfrak{Z}$, we now define $\alpha(\mathfrak{Z})$ to be the maximum local socle degree of $\mathfrak{Z}$. More precisely,

Definition 2.3. *If $\mathfrak{Z}$ is a scheme concentrated at p_0, we let $\alpha(\mathfrak{Z})$ denote the highest socle degree of (R'/J), where $J \subset R'$ defines $\mathfrak{Z}$. Equivalently, $\alpha(\mathfrak{Z})$ is the highest degree of an element of $J^{-1} \in \Gamma'$. If $\mathfrak{Z}$ is concentrated at a point p, then $\alpha(\mathfrak{Z})$ is defined similarly using the local ring at p (see Section 2.2). More generally, if the punctual scheme $\mathfrak{Z}$ has decomposition $\mathfrak{Z} = \mathfrak{Z}(1) \cup \cdots \cup \mathfrak{Z}(k)$ as the union of irreducible components $\mathfrak{Z}(1), \ldots, \mathfrak{Z}(k)$, each concentrated at (distinct) points $p(1), \ldots, p(k)$, then $\alpha(\mathfrak{Z}) = \max\{\alpha(\mathfrak{Z}(1)), \ldots, \alpha(\mathfrak{Z}(k))\}$ of the local socle degrees.*

2.1 Schemes concentrated at a coordinate point

We will fix the coordinate point as $p = p_0 = (0 : \ldots : 0 : 1)$, and will for short use z, Z to denote x_{n+1}, X_{n+1}, respectively. We let $R' = K[y_1, \ldots, y_n]$ be the coordinate ring of affine space $\mathbb{A}^n$, the locus on $\mathbb{P}^n$ where $X_{n+1} \neq 0$; and we let $\Gamma' = K_{DP}[Y_1, \ldots, Y_n]$, the divided power ring. Note that if $\mathcal{I}_p \subset \mathcal{O}_p$ is an ideal defining a punctual scheme $\mathfrak{Z}$ concentrated at p, then $\mathcal{I}_p \supset m_p^{\alpha(\mathfrak{Z})+1}$, and each element of $\mathcal{I}_p$ may be written mod $m_p^{\alpha(\mathfrak{Z})+1}$ as a polynomial h in $R' = K[y_1, \ldots, y_n]$ of some degree t no greater than α: then its homogenization to degree u is

$$\mathrm{Homog}(h, z, u) = z^u \cdot h(x_1/z, \ldots, x_n/z), \tag{2.6}$$

if $u \geq t$, and 0 otherwise. The homogenization $I_{\mathfrak{Z}}$ of $\mathcal{I}_p$ is spanned by ${m_p}^{\alpha(\mathfrak{Z})+1}$, and by all the homogenizations of such elements $h \in \mathcal{I}_p$:

$$I_{\mathfrak{Z}} = \big(\mathrm{Homog}(h, z, u) \mid u \in \mathbb{Z}^+, h \in \mathcal{I}_p \text{ degree } h \leq \alpha(\mathfrak{Z})\big) + {m_p}^{\alpha(\mathfrak{Z})+1} \tag{2.7}$$

Recall that the inverse system $L_{\mathfrak{Z}}$ in Γ of $I_{\mathfrak{Z}}$ consists of all elements of Γ, annihilated by $I_{\mathfrak{Z}}$. Given a point $p = (a_1 : \ldots : a_n : 1)$ of $\mathbb{P}^n$, we let $L_p = a_1X_1 + \cdots + a_nX_n + Z \in \Gamma$.

Definition 2.4. Homogenization of an inverse system at a point.

1. *Suppose $F \in \Gamma[1/Z, 1/Z^{[2]}, \ldots]$. We denote by $F \cdot_{rp} Z^{[u]}$ the result of raising the Z-degree of the Z-factor in each term by u, without changing the coefficients that appear. For example, if $F = X_1X_2/Z^{[2]} + X_2^{[4]}/Z^{[4]} \in K_{DP}[Y_1, Y_2]$, then $F \cdot_{rp} Z^{[4]} = X_1X_2Z^{[2]} + X_2^{[4]}$. We may also write $Z^{[u]} \cdot_{rp} F$ for $F \cdot_{rp} Z^{[u]}$. If $w \in \Gamma$ has the form $w = \sum w_i \cdot L_p^{[k-i]}, w_i \in \Gamma'$, then we denote by $w \cdot_{rp} L_p^{[u]}$ the product*

$$w \cdot_{rp} L_p^{[u]} = \sum w_i \cdot L_p^{[k+u-i]}. \tag{2.8}$$

2. *Suppose that $f \in \Gamma' = K_{DP}[Y_1, \ldots, Y_n]$ satisfies $f = \oplus f_i, f_i \in \Gamma'_i$, and let $L_p = a_1X_1 + \cdots + a_nX_n + Z$. Then for any integer $u \geq 0$ we define the inverse-system homogenization*

$$\mathrm{Homog}(f, L_p, u) = \sum_{0 \leq i \leq u} f_i(X_1, \ldots, X_n) \cdot L_p^{[u-i]}. \tag{2.9}$$

For example, if $f = Y_1Y_2 + Y_2^{[4]}$, then $f(X_1/Z, X_2/Z) = F$ above, and $\mathrm{Homog}(f, Z, 4) = X_1X_2Z^{[2]} + X_2^{[4]}$, *while* $\mathrm{Homog}(f, Z, 3) = X_1X_2Z$.

3. *Let $L' \subset \Gamma'$ be an inverse system (so L' is an R'-submodule of Γ'), and suppose p fixed. Then we define*

$$L'[u] = \langle \{\mathrm{Homog}(f, L_p, u)\ \forall f \in L'\} \rangle$$

and we define the homogenization of the inverse system L',

$$\text{Homog}(L', L_p) = \oplus_{u \geq 0} L'[u] = \langle \text{Homog}(f, L_p, u) \mid f \in L', u \geq 0 \rangle. \tag{2.10}$$

If we leave out the homogenizing form or do not specify p, then we assume $L_p = Z$, $p = p_0$.

Note that this definition allows $\text{Homog}(f, Z, u)$ to be nonzero even if u is smaller than the degree of f; this is natural here, since the global inverse system is closed under the contraction action of R. Thus, for example

$$z \circ \left(X_1 X_2 Z^2 + X_2^4 \right) = X_1 X_2 Z.$$

We of course wish to show that if $L' \subset \Gamma'$ is the inverse system of $\mathcal{I}_p \subset \mathcal{O}_p$, then $L = \text{Homog}(L', Z) \subset \Gamma$ is the inverse system of $I_{\mathfrak{Z}}$ (Lemma 2.7). We also wish to show how to obtain from L' the key "generators" of L — which is infinitely generated since $I_{\mathfrak{Z}}$ is zero-dimensional, not Artin. To this end, we need a basic result.

Lemma 2.5. Homogenization and duality. *Suppose that $h' \in R'$ has degree a, that $f' \in \Gamma'$, that $i \geq a$, and that $w \in \mathbb{Z}$. Let $h = h'[i] = \text{Homog}(h', z, i)$ and $f = f'[i+w] = \text{Homog}(f', Z, i+w)$. Then*

$$h \circ f = h'[i] \circ f'[i + w] = (h' \circ f')[w]. \tag{2.11}$$

In particular,

$$h' \circ f' = 0 \Rightarrow (h' \circ f')_{\leq w} = 0 \Leftrightarrow (h' \circ f')[w] = 0 \Leftrightarrow h'[i] \circ f'[i + w] = 0; \tag{2.12}$$

and if f' has degree b, then

$$h' \circ f' = 0 \Leftrightarrow (h' \circ f')_{\leq b} = 0 \Leftrightarrow (h' \circ f')[b] = 0 \Leftrightarrow h'[i] \circ f'[i + b] = 0. \tag{2.13}$$

Proof. Let $h' = \sum_{u=0}^{a} h_u$ and $f' = \sum_{v=0}^{b} f_v$. Then $h = h'[i] = \sum_{u=0}^{a} h_u z^{i-u}$ and $f'[i + w] = \sum_{v=0}^{\min\{b,i+w\}} f_v Z^{[i+w-v]}$. Now, we have formally (below, $Z^{[c]} = 0$ if $c < 0$),

$$\begin{aligned} h'[i] \circ f'[i + w] &= \sum_{u=0}^{a} \left(\sum_{v=u}^{\min\{b,u+w\}} h_u(x_1, \ldots, x_n) \circ f_v(X_1, \ldots, X_n) \cdot Z^{[w+u-v]} \right) \\ &= \sum_{u=0}^{a} \left(\sum_{v=u}^{b} h_u \circ f_v \cdot Z^{[w+u-v]} \right) = h' \circ f'[w]. \end{aligned}$$

The second equation is immediate from the first, and the fact, homogenization to degree w in Γ' annihilates terms in $h \circ f$ having degree greater than w. The third is immediate from the second. □

Corollary 2.6. *Suppose $h' \in R'$ has degree no greater than a, and $f' \in \Gamma'_{\leq b}$, and let $h = \text{Homog}(h', z, a) \in R$, $f = \text{Homog}(f', Z, b) \in \Gamma$. Then*

$$h' \circ f' = 0 \Leftrightarrow h \circ (f \cdot_{rp} Z^{[a]}) = 0. \tag{2.14}$$

Proof. In (2.13), take $i = a$, and note that $\text{Homog}(f', Z, a + b) = \text{Homog}(f', Z, b) \cdot_{rp} Z^{[a]}$. □

Lemma 2.7. local to global inverse systems. *Suppose that $p = p_0 = (0 : \ldots : 0 : 1)$ in $\mathbb{P}^n$, and that $L' \subset \Gamma'$ is the inverse system of $\mathcal{I}_p \subset \mathcal{O}_p$, where $\mathcal{I}_p$ defines a degree-s 0-dimensional scheme $\mathfrak{Z}$ concentrated at p. Then $\text{Homog}(L', Z) \subset \Gamma$ is the inverse system $L_{\mathfrak{Z}}$ of $I_{\mathfrak{Z}} \subset R$.*

Proof. Let $S = \mathrm{Homog}(L', Z)$. It is immediate from (2.12) in Lemma 2.5 that $I_{\mathfrak{Z}} \circ S = 0$, so $L_{\mathfrak{Z}} \supset S$. Also, note that S is an R-module: $R \circ S \subset S$. To show this, it suffices to check that if $f \in S_u, h \in R_1$, then $h \circ f \in S_{u-1}$. Let $f = \mathrm{Homog}(f', Z, u), f' \in L'$. Note that $z \circ f = \mathrm{Homog}(f', Z, u-1)$, so is in S. Also, if $1 \le i \le n$ then considering each term, it is easy to see that $x_i \circ f = \mathrm{Homog}(y_i \circ f', Z, u-1)$, so is in S. This shows $R_1 \circ S \subset S$, and by induction that S is an R-module.

For $i \ge \tau(\mathfrak{Z})$, $\dim_K (L_{\mathfrak{Z}})_i = s$. For $i \ge \alpha(\mathfrak{Z})$, the socle degree, $\dim_K S_i = s$, since the homomorphism $f \in L' \to f[i]$ is an isomorphism of $\Gamma'_{\le i}$ into Γ_i, and $\dim_K L' = s$. Since $S \subset L_{\mathfrak{Z}}$, we have $S_i = (L_{\mathfrak{Z}})_i$ for $i \ge \max\{\alpha(\mathfrak{Z}), \tau(\mathfrak{Z})\}$. Since $I_{\mathfrak{Z}}$ is saturated, by Lemma 1.6 we have that there is an integer N such that $(L_{\mathfrak{Z}})_u = R_{i-u} \circ (L_{\mathfrak{Z}})_i$ for all $i \ge N$ and $u \le i$. We conclude that $L_{\mathfrak{Z}} \subset S$, completing the proof of the Lemma. □

We give a direct proof of the following result, that is also a consequence of Lemma 2.7.

Lemma 2.8. *When $j \ge \alpha(\mathfrak{Z})$, $(L_{\mathfrak{Z}})_{\ge j}$ is closed under the raised power action of Z: $f \to f \cdot_{rp} Z^{[u]}$.*

Proof. Let $f \in (L_{\mathfrak{Z}})_j$, set $f_1 = f \cdot_{rp} Z, I = I_{\mathfrak{Z}}$, and suppose by way of contradiction that $h \in I_{j+1}$ satisfies $h \circ f_1 \ne 0$. Then $h = zh_1 + h'(x), h_1 \in R_j, h' \in R'_{j+1}$. Since $j \ge \alpha(\mathfrak{Z})$, $h' \in J_{j+1}$, where J is the ideal defining $\mathfrak{Z} \subset \mathbb{A}^n$; hence $zh_1 \in I_{j+1}$, implying $h_1 \in I_j$, since the homogenizing variable is a non-zero-divisor of $R/I_{\mathfrak{Z}}$. But we have $h'(x) \circ f_1 = 0$ (as each term of f_1 has a Z-factor), hence $zh_1 \circ f_1 = (h \circ f_1 - h'(x) \circ f_1) \ne 0$. Then $h_1 \circ f = zh_1 \circ f_1 \ne 0$, a contradiction since $h_1 \in I_j$. □

The assumption $j \ge \alpha(\mathfrak{Z})$ in the above Lemma is necessary (see Example 2.17). We now state a key result concerning the generation of the homogenized inverse system.

Lemma 2.9. "Generators" for the global inverse system. *Suppose that $V' \subset \Gamma'_{\le \alpha}$ generates the inverse system L' of $\mathcal{I}_p$, and denote by $I_{\mathfrak{Z}}$ the homogenization of $\mathcal{I}_p$, and by V the subspace* $\mathrm{Homog}(V', Z, \alpha)$ *of Γ_α. Then the inverse system $L_{\mathfrak{Z}} = {I_{\mathfrak{Z}}}^{-1} \subset \Gamma$ satisfies*

$$\begin{aligned}(L_{\mathfrak{Z}})_j &= \mathrm{Homog}(L'_{\le \alpha}, Z, j) \\ &= R_\alpha \circ (V \cdot_{rp} Z^{[j]}).\end{aligned} \tag{2.15}$$

Proof. Since $L' = L'_{\le \alpha}$, the first equality follows from Lemma 2.7. That V' generates L' is equivalent to $L' = R'_{\le \alpha} \circ V'$. If $h' \in R'_{\le \alpha}$ and $v' \in V'$, let $v = v'[\alpha]$; then by Lemma 2.5 $h'[\alpha] \circ v \cdot_{rp} Z^{[j]} = h'[\alpha] \circ v'[j + \alpha] = (h' \circ v')[j] \in \mathrm{Homog}(L', Z, j)$. This shows $\mathrm{Homog}(L', Z, j) \subset R_\alpha \circ (V \cdot_{rp} Z^{[j]})$. Lemmas 2.8 and 2.7 show that $V \cdot_{rp} Z^{[j]} \subset \mathrm{Homog}(L', Z, j + \alpha) = (L_{\mathfrak{Z}})_{j+\alpha}$, implying the opposite inclusion. This completes the proof of (2.15). □

Example 2.10. *The above Lemma 2.9 can be used to calculate the homogenization of an ideal, given generators of the local inverse system. Suppose we begin with the local ideal $I' \subset R' = K[y_1, y_2]$, $I' = \mathrm{Ann}\ (f'), f' = Y_1^{[8]} + Y_2^{[8]} + Y_1^{[3]}Y_2^{[3]} + (Y_1 + Y_2)^{[6]}$. Then*

$$I' = (3y_1^6 - 4y_1y_2^5 + y_2^6 + y_1^2y_2^2 - 2y_1y_2^3, y_1^6 - y_2^6 + y_1^3y_2 - y_1y_2^3, m_{p_0}^9),$$

of local Hilbert function $H' = H(R'/I') = (1, 2, 3, 4, 3, 2, 2, 2, 1)$, and I' defines a degree-20 punctual scheme $\mathfrak{Z} = \mathrm{Spec}(R'/I')$ concentrated at $p_0 = (0 : 0 : 1) \in \mathbb{P}^2$, with $\alpha(\mathfrak{Z}) = 8$. The homogenized ideal $I = I_{\mathfrak{Z}} \subset R$ has more than two generators, and is tricky to find directly — we may homogenize a standard basis, but some computer algebra programs do not provide a standard basis for a non-graded ideal. However, by homogenizing f', forming $f = \mathrm{Homog}(f', Z, 8) = X_1^{[8]} + X_2^{[8]} + X_1^{[3]}X_2^{[3]}Z^{[2]} + (Y_1 + Y_2)^{[6]}Z^{[2]}$, we may calculate $W_8 = R_8 \circ f$, and we can find $J = \mathrm{Ann}\ W_8$: in the Macaulay *algebra program [BSE] we find the contraction of R_8 with f, then* *use the script "<l_from_dual" to find J). Then the homogenized ideal* $I = J_{\leq 8} + M_p^9$. *In this case* $J_{\leq 8}$ *already generates* I, *since* $\Delta(H(R/J_{\leq 8}))$ *has the correct degree, 20. We found in this way that*

$$I = (x_1^3x_2^2 + x_1^2x_2^3 - 3x_1x_2^4,\ x_1^4x_2 - x_1x_2^4,\ x_1x_2^5 - x_2^6 + (3/4)x_1^3x_2z^2 - (1/4)x_1^2x_2^2z^2 - (1/4)x_1x_2^3z^2,\ x_1^6 - x_2^6 + x_1^3x_2z^2 - x_1x_2^3z^2),$$

of Hilbert function $H_{\mathfrak{Z}} = H(R/I)$ *satisfying* $\Delta H_{\mathfrak{Z}} = (1,2,3,4,5,4,1)$, *and* $\sigma(\mathfrak{Z}) = 7$.

The following Proposition summarizes and extends some of the above results. Recall that we use the notation z for x_{n+1} and Z for X_{n+1}; and we denote by m_z or m_{z^a} multiplication by x_{n+1} or by x_{n+1}^a in R or in $A = R/I$. We denote by $L = \mathrm{Homog}(J^{-1}, Z) \subset \Gamma$, the homogenization of the inverse system $J^{-1} \subset \Gamma' = K_{DP}[Y_1, \ldots, Y_n]$; we let $L_i[j] = \mathrm{Homog}(L_{i\{X_{n+1}=1\}}, X_{n+1}, j)$. This is just $z^{i-j} \circ L_i \in \Gamma_j$ if $j \leq i$, and $Z^{[j-i]} \cdot_{rp} L_i$ if $j \geq i$; and is obtained in any case by changing each $X_{n+1}^{[u]}$ factor appearing in a monomial term of an element $F \in L_i$ to $X_{n+1}^{[u+j-i]}$, forming an element $F[j] \in \Gamma_j$. Note that if $j > i$, and $F \in L_i$, then $F[j]$ is not necessarily in L_j (see Remark 2.12 and Example 2.15 below). Recall that M' is the maximal ideal of $R' = K[y_1, \ldots, y_n]$ at the origin, and $M_0 = m_p = (x_1, \ldots, x_n) \subset R$ is the homogeneous maximal ideal of R at the corresponding point $p = p_0 \in \mathbb{P}^n$.

Proposition 2.11. Homogenization for schemes with support p_0. *Suppose that* $J \subset R'$ *defines an zero-dimensional scheme* $\mathfrak{Z} = \mathrm{Spec}(R'/J)$ *concentrated at the origin, and let* $\alpha = \alpha(\mathfrak{Z})$ *be the socle degree of* R'/J *(Definition 1.7) and let* $L' = J^{-1} \subset \Gamma'$ *be the affine inverse system of* $\mathfrak{Z}$. *Let* $I = \mathrm{Homog}(J, z) \subset R$ *and let* $L = \mathrm{Homog}(L', Z) = \oplus_i L'[i]$. *Then we have*

(i) $I = I_{\mathfrak{Z}}$, *and is a saturated ideal primary to the maximal ideal* $m_p, p = (0 : \ldots : 0 : 1) \in \mathbb{P}^n$, *and it satisfies* $m_p \supset I \supset {m_p}^{\alpha+1}$. *In particular,* $I_a = I_b : R_{b-a}$ *for* $a \leq b$, *and* $I_b = R_{b-a}I_a$ *for* $b \geq a \geq \sigma(\mathfrak{Z})$. *Furthermore, if* $a \leq b$, *then* $I_a = I_b : z^{b-a}$, *and if* $b \geq \alpha$ *we have* $I_b = z^{b-\alpha} \cdot I_\alpha + ({m_p}^{\alpha+1})_b$.

(ii) $L = L_{\mathfrak{Z}}$, *and satisfies,* $L_a = z^{b-a} \circ L_b$ *for* $a \leq b$, *and* $K_{DP}[Z] \subset L \subset \Gamma_{\leq \alpha} \cdot K_{DP}[Z]$. *Furthermore, for* $\alpha \leq a \leq b$ *the map* $F \to F[b]$ *taking* L_a *to* L_b, *and the map* $z^a \circ : L_b \to L_a$ *are inverse isomorphisms. Also,* L *satisfies* $L_b = L_a[b]$ *for any pair* (a, b) *satisfying* $a \geq \alpha$.

(iii) A *satisfies* $m_{z^{b-a}} : A_a \to A_b$ *is injective for* $a \leq b$, *and furthermore* $m_{z^{b-a}}$ *defines an isomorphism* $A_a \cong A_b$ *for* $\alpha \leq a \leq b$. *In particular, for* $k > 0$, $m_{z^k} : A_{\alpha-k} \to A_\alpha$ *is an injection, and* $m_{z^k} : A_\alpha \to A_{\alpha+k}$ *is an isomorphism onto.*

(iv) Let $\dim_K R'/J = s$. *The subscheme* $\mathfrak{Z} = \mathrm{Proj}\ (A)$ *of* $\mathbb{P}^n$ *has degree* s, *and* $J \supset {M'}^s$. *Furthermore, the regularity* $\sigma(\mathfrak{Z})$ *satisfies* $\sigma(\mathfrak{Z}) = \tau(\mathfrak{Z}) + 1 \leq \alpha + 1 \leq s$.

(v) Let $\mathfrak{L} \subset \Gamma$ *be an inverse system satisfying* $K_{DP}[Z] \subset \mathfrak{L} \subset \Gamma_{\leq \alpha} \cdot K_{DP}[Z]$, *and let* $\mathfrak{I} = \mathrm{Ann}\ (\mathfrak{L})$. *Then, letting* $\mathfrak{J} = (\mathfrak{I})_{(z=1)} \subset R'$, *and* $\mathfrak{L}' = (\mathfrak{L}_b)_{Z=1}, b \geq \alpha$ *we have* $\mathfrak{L}' = (\mathfrak{J})^{-1}$, $\mathfrak{J}$ *defines a scheme* $\mathfrak{Z}$ *concentrated at the origin with* $\mathfrak{I} = I_{\mathfrak{Z}}, \mathfrak{L} = L_{\mathfrak{Z}}$, $\alpha(\mathfrak{Z}) \leq \alpha$, *and* $\mathfrak{L} = \mathrm{Homog}(\mathfrak{L}', Z)$.

Proof. That $I = I_{\mathfrak{Z}}$ and is saturated is well-known, since the primary decomposition of an ideal carries over to its homogenization (see §VII.5 Theorem 17 of [ZarS]). The next statements are standard, since z is a non-zero divisor in $R/I_{\mathfrak{Z}}$. That $J = (J_{\leq \alpha}) + (M')^{\alpha+1}$ implies the last statement of (i).

That $L = L_{\mathfrak{Z}}$, and $K_{DP}[Z] \subset L \subset \Gamma_{\leq \alpha} \cdot K_{DP}[Z]$ in (ii) follow from Lemma 2.9 and (i). That $L_a = z^{b-a} \circ L_b$ follows from z being a non-zero divisor in $R/I_{\mathfrak{Z}}$, and Lemma 1.6. Lemma 2.8 and an easy verification implies that the two maps given are inverse isomorphisms when $a, b \geq \alpha$. For any $f \in J^{-1}$, deg $f \leq \alpha$, so if $a \geq \alpha$ we have $f[b] = (f[a])[b]$: this implies the last statement. The statements of (iii) follow from and are weaker than those of (i) or (ii): they are about the quotient algebra A, rather than the ideal I or inverse system L. Now (ii) and (iii) imply the key inequality $\tau(\mathfrak{Z}) \leq \alpha(\mathfrak{Z})$ for (iv) since $\tau(\mathfrak{Z}) = \min\{i \mid \dim_K (R/I_{\mathfrak{Z}})_i = s\}$. That $J \supset M'^s$ is well known: any monomial of M'^s has a length-$s+1$ chain of monomials that divide it, so some linear combination of elements of the chain must be in J — since the degree of $\mathfrak{Z}$ is only s — implying the monomial itself is in J.

Note that the main condition of (v) is that of (2.3) with $L_p = Z$: this is the condition for $\mathfrak{I}$ to define a zero-dimensional scheme $\mathfrak{Z}$ concentrated at p_0, so (v) would follow from the standard fact, $J = (I_{\mathfrak{Z}})_{z=1}$ defines the portion of $\mathfrak{Z}$ in $\mathbb{A}^n : z = 1$, and (ii), provided we show that $\mathfrak{L}' = J^{-1}$. Directly, we have $1 \subset \mathfrak{L}' \subset \Gamma'_{\leq \alpha}$; thus, identifying x and y variables (since we have taken $z = 1$) and letting $\mathfrak{J}' = \text{Ann}\ (\mathfrak{L}') \subset R'$, we have $(x_1, \ldots, x_n) \supset \mathfrak{J}' \supset (x_1, \ldots, x_n)^{\alpha+1}$. Also, we have $\dim_K \mathfrak{L}' = \dim_K \mathfrak{L}_b = H(R/I_{\mathfrak{Z}})_b = s$, with $s = \deg(\mathfrak{Z})$, as there is no kernel in dehomogenizing from a vector subspace of Γ_b. Clearly $\mathfrak{L}'$ is independent of the choice of $b \geq \alpha$ by (ii). Taking $b = 2\alpha$ and using equation (2.13) of Lemma 2.5 we can see that $\mathfrak{J} \subset \mathfrak{J}'(Y)$, but we have $\dim_K(K[x_1, \ldots, x_n]/\mathfrak{J}') = \dim_K(R/\mathfrak{J}) = s$, implying $\mathfrak{J} = \mathfrak{J}'$. It likewise follows from the equality of dimensions that $\mathfrak{L}' = \mathfrak{J}^{-1}$. This completes the proof of (v), and of Proposition 2.11. □

Remark 2.12. Homogenized component L_i is not determined by L'_i. *We note here a perhaps surprising property of the homogenized inverse system $L = \text{Homog}(J^{-1}, Z)$, where $J^{-1} \subset \Gamma'$ is the inverse system of an ideal $J \subset R'$ defining a zero-dimensional scheme $\mathfrak{Z}$. Namely, $F \in L_i, i < \alpha(\mathfrak{Z})$, does not imply that there is a (possibly nonhomogenous) element $f \in J^{-1}$ of degree i such that $F = f[i]$. There are also elements of L_i arising from homogenizing to degree i those elements of J^{-1} having higher degree. See Example 2.15 below, where $X_1^{[2]} \in L_2, X_1^{[2]} = z \circ (X_1^{[2]}Z - X_1X^{[2]})$, but is not a homogenization of an element of $J^{-1}_{\leq 2}$. Likewise, as mentioned earlier, $X_1^{[2]} \in L_2$ does* not *imply* $\text{Homog}(X_1^{[2]}, Z, 3) = X_1^{[2]}Z \in L_3$*: rather the corresponding element of L_3 is $X_1^{[2]}Z - X_1X_2^{[2]}$. However, if $i \geq \alpha(\mathfrak{Z})$, then $F \in L_i$ and $j \geq i \Rightarrow F[j] \in L_j$ by Proposition 2.11ii. For similar reasons, the condition $b \geq \alpha$ in Proposition 2.11v cannot be removed, and we may have $((L_{\mathfrak{Z}})_a)_{Z=1} \not\subset ((L_{\mathfrak{Z}})_b)_{Z=1}$ when $a < b$. (See Example 2.15 below).*

Note that $\sigma(\mathfrak{Z})$ may be rather less than $\alpha+1$, the upper bound of (iv), and is almost always less than $\alpha+1$ when the defining ideal of $\mathfrak{Z}$ in R' is non-homogeneous. (See Examples 2.10,2.13,2.17).

If we write $\alpha(\mathfrak{Z}) = \sigma(\mathfrak{Z}) + k(\mathfrak{Z})$, it is not clear how to bound $k(\mathfrak{Z})$ above. The examples where $\mathfrak{Z}$ is defined locally by a general enough compressed Gorenstein ideal of $\mathcal{O}_p$, in the sequel article [ChoI1] show that there is no constant upper bound. On the other hand, these examples satisfy $k(\mathfrak{Z}) \leq \sigma(\mathfrak{Z})$, suggesting that the latter bound might be valid for $\mathfrak{Z}$ supported at a single point.

For any zero-dimensional scheme $\mathfrak{Z}$, Lemma 1.10 shows that the inverse system L of $I_{\mathfrak{Z}}$ is determined by L_σ: thus $L_i = L_\sigma : R_{i-\sigma}$ if $i \geq \sigma$, and $L_i = R_{\sigma-i} \circ L_\sigma$ if $i \leq \sigma$. However, when both $\alpha, i > \sigma$, L_i may not obtained by simply raising the Z-power of elements of L_σ — even when $\mathfrak{Z}$ is concentrated at p_0 (see Example 2.17).

In the following examples we sometimes use $Z^u = u!Z^{[u]}$ instead of $Z^{[u]}$ in a monomial $g \in \mathcal{D}$, as a constant coefficient will not affect the vector space span of g. Below we set $Z^{[u]} = 0$ if $u < 0$.

Example 2.13. L_τ may not determine L. *Let $R = K[x_1, x_2, x_3], p = p_0 = (0:0:1), \mathcal{I}_p = (y_1y_2, y_1^2 - y_2^3), f' = (Y_1^{[2]} + Y_2^{[3]})$; then $\mathcal{I}_p \supset (y_1y_2, y_1^3, y_2^4)$ and $H(R_p/\mathcal{I}_p) = (1, 2, 1, 1)$, and $\alpha(\mathfrak{Z}) = 3$. The homogenization $I_{\mathfrak{Z}} = (x_1x_2, x_1^2z - x_2^3, x_1^3, x_2^4)$, and $H(R/I_{\mathfrak{Z}}) = (1, 3, 5, 5, \ldots)$, so $\tau(\mathfrak{Z}) = 2, \sigma(\mathfrak{Z}) = 3$. The inverse system $L = {I_{\mathfrak{Z}}}^{-1}$ satisfies, by Lemma 2.9*

$$\begin{aligned} L_j =& \langle X_1^{[2]}Z^{[j-2]} + X_2^{[3]}Z^{[j-3]}, X_2^2Z^{j-2}, X_2Z^{j-1}, X_1Z^{j-1}, Z^j \rangle \\ =& R_3 \circ (\text{Homog}(f', Z, j+3)) = R_3 \circ \left(X_1^{[2]}Z^{[j+1]} + X_2^{[3]}Z^{[j]} \right) \\ =& \text{Homog}(V', Z, j), \text{ where } V' = R' \circ f' = \langle f', Y_2^2, Y_2, Y_1, 1 \rangle. \end{aligned}$$

Note that $L_\sigma = L_3$ determines L, but the space $L_\tau = L_2$ does not. This corresponds to $(I_{\mathfrak{Z}})_\sigma$ determining $I_{\mathfrak{Z}}$ (see Theorem 1.12ii). Also, since $\Delta H(R/I_{\mathfrak{Z}}) = (1,2,2,0)$, which is not symmetric, $\mathfrak{Z}$ is not arithmetically Gorenstein; however, $\mathfrak{Z}$ is locally Gorenstein and has a single point of support.

Example 2.14. *More generally, with R, p as above, let $f' = (Y_1^{[2]} + Y_2^{[j]}), j \geq 3$; then $\mathcal{I}_p = (y_1y_2, y_1^2 - y_2^j)$ and $H(R_p/\mathcal{I}_p) = (1,2,1,\ldots,1_j)$, with $j-1$ ones at the end, determining a punctual scheme $\mathfrak{Z}$ at p of degree $j+2$, for which $\alpha(\mathfrak{Z}) = j$. The homogenization $I_{\mathfrak{Z}} = (x_1x_2, x_1^2z^{j-2} - x_2^j, x_1^3)$, so $H = (1,3,5,6,\ldots,j+2,j+2,\ldots)$, and $\Delta H(R/I_{\mathfrak{Z}}) = (1,2,2,1,\ldots,1,0)$, with $j-3$ ones at the end, so $\tau(\mathfrak{Z}) = j-1, \sigma(\mathfrak{Z}) = j$; the inverse system $L = I^{-1}$ is determined by*

$$L_\sigma = \langle X_1^{[2]} \cdot Z^{[j-2]} + X_2^{[j]}, X_2^{j-1}Z, \ldots, X_2Z^{j-1}, X_1Z^{j-1}, Z^j \rangle,$$

in the same sense as Example 2.13, but not in that sense by L_τ.

When $j = 3$ or $j \geq 5$ then $\mathfrak{Z}$ is not arithmetically Gorenstein, since $\Delta H(R/I_{\mathfrak{Z}})$ is not symmetric. When $j = 4$ then $H_{\mathfrak{Z}} = (1,3,5,6,6,\ldots), \Delta H = (1,2,2,1)$, and $\mathfrak{Z}$ is arithmetically Gorenstein iff it satisfies the Cayley-Bacharach property that $H(R/I_{\mathfrak{Z}'})_{\tau-1} = H(R/I_{\mathfrak{Z}})_{\tau-1}$ for any subscheme $\mathfrak{Z}' \subset \mathfrak{Z}$ of degree $s-1$ (see [Kr2],[Mig, Theorem 4.1.10]). Here $\tau(\mathfrak{Z}) = 3, \sigma(\mathfrak{Z}) = 4$, and the local inverse system is $L' = \langle 1, Y_1, Y_2, Y_2^2, Y_2^3, Y_1^2 + Y_2^4 \rangle$. The only R'-closed system L'' of codimension one is $L'' = \langle 1, Y_1, Y_2, Y_2^2, Y_2^3 \rangle$. Since L'', and thus $J'' = \mathrm{Ann}\ (L'') \subset R''$ defining the subscheme $\mathfrak{Z}''$ is graded, Proposition 2.18 implies that $H(R/I_{\mathfrak{Z}''}) = (1,3,4,5,5,\ldots)$, the sum function of $H(R'/J'') = (1,2,1,1)$. Thus $\mathfrak{Z}$ does not satisfy the Cayley-Bacharach condition, and is not arithmetically Gorenstein.

Example 2.15. Component L_2 not the homogenization of L_2'. *If r=3, $R = K[x_1,x_2,x_3]$, $\Gamma' = K[Y_1,Y_2], f = Y_1^{[2]} - Y_1Y_2^{[2]} \in \Gamma'$, then $I' = (y_1^2 + y_1y_2^2, y_2^3)$, of Hilbert function $H(R'/I') = (1,2,2,1)$, and $\alpha(\mathfrak{Z}) = 3$. The related homogeneous ideal I in R determining the degree-6 scheme $\mathfrak{Z}$ concentrated at $p_0 = (0:0:1)$ in $\mathbb{P}^2$ is*

$$I = (x_1^2z + x_1x_2^2, x_2^3, x_1^3, x_1^2x_2),$$

of Hilbert function $H_{\mathfrak{Z}} = (1,3,6,6,\ldots)$, so $\tau(\mathfrak{Z}) = 2, \sigma(\mathfrak{Z}) = 3$. Here the homogenization of f to degree α is $G = f[3] = X_1^{[2]}Z - X_1X_2^{[2]}$. By Lemma 2.9, letting $L = L_{\mathfrak{Z}} = {I_{\mathfrak{Z}}}^{-1}$, we have that L is simply determined by the actions of the pair $(z,Z) = (x_3,X_3)$ on L_α, which satisfies

$$\begin{aligned} L_3 = R_3 \circ F[6] &= R_3 \circ (X_1^{[2]}Z^{[4]} - X_1X_2^{[2]}Z^{[3]}) \\ &= \langle G, Z^3, X_1Z^2, X_1X_2Z, X_2^2Z, X_2Z^2 \rangle. \end{aligned}$$

Likewise, $L_2 = R_1 \circ L_3 = \langle X_1^2, X_1Z, X_1X_2, X_2^2, X_2Z, Z^2 \rangle \subset \Gamma_2$. Note that L_2 contains X_1^2, which is the partial of $G = f[3]$ with respect to z, but is not the homogenization of an element of $L'_{\leq 2} = {I'^{-1}}_{\leq 2}$, as $L' = \langle f, Y_2^2, Y_1Y_2, Y_2, Y_1, 1 \rangle$.

Example 2.16. *When R', p are as above, and $I' = (y_1^2, y_2^3), f' = Y_1Y_2^2$, the local Hilbert function is $H(R'/I') = (1,2,2,1)$, $\alpha(\mathfrak{Z}) = 3$, then $I = I_{\mathfrak{Z}} = (x_1^2, x_2^3)$, $H(R/I_{\mathfrak{Z}}) = (1,3,5,6,6,\ldots)$, so $\sigma(\mathfrak{Z}) = 4$, $\tau(\mathfrak{Z}) = 3 = \alpha(\mathfrak{Z})$, and $L = I^{-1}$ is determined by $L_\tau = \langle X_1X_2^2, ZX_2^2, Z^2X_2, ZX_1X_2, Z^2X_1, Z^3 \rangle$, even in the stronger sense that $L_j = R_\tau \circ (L_\tau \cdot_{rp} Z^j)$ if $j \geq \tau$, and $L_j = R_{\tau-j} \circ L_\tau$ when $j \leq \tau$. This example and Example 2.13 above illustrate that L must be determined by L_σ, but L is also determined by L_τ if I is generated in degrees less or equal to τ. However, the next example shows that this determination by L_τ (or by L_σ) is usually in a "weaker" sense than here.*

Example 2.17. How does L_σ determine L? *We choose a curvilinear ideal (one not contained in ${m_p}^2$), $\mathcal{I}_p = (y_1 + y_2^2 + y_2^3 + y_2^4, y_2^5) \subset \mathcal{O}_p$, of local Hilbert function $H(\mathcal{O}_p/\mathcal{I}_p) = (1,1,1,1,1)$. Using the computer algebra program* MACAULAY *[BSE] we calculated its homogenization as $I_{\mathfrak{Z}} =$*

$(x_1z+x_2^2-x_1x_2, x_1z^2+x_2^2z+x_1^2z+x_2^3, x_1^3)$, of Hilbert function $H_\mathfrak{Z}=(1,3,5,5,\ldots)$, so $\sigma(\mathfrak{Z}) = 3 < \alpha(\mathfrak{Z}) = 4$. A local dual generator $f' \in L' = (I')^{-1}$ is $f' = Y_2^{[4]} - Y_1Y_2^{[2]} + Y_1^{[2]} - Y_1Y_2 - Y_2^{[2]}$. Note that since $\mathcal{I}_p$ is not homogeneous, its dual generator is not unique, up to multiple by a nonzero constant in K; rather, f' is unique up to the action $\lambda \circ f'$ by a unit λ of R'. We have $L' = \langle f', Y_2^{[3]} - Y_1Y_2 - Y_1, Y_2^{[2]} - Y_1, Y_2, 1\rangle$, and, letting $F = \operatorname{Homog}(f', Z, 4)$, we have

$$L_4 = \langle F, X_2^{[3]}Z - X_1X_2Z^2 - X_1Z^{[3]}, X_2^{[2]}Z^{[2]} - X_1Z^{[3]}, X_2Z^3, Z^4\rangle.$$

Here L_3 contains $f'[3] = \operatorname{Homog}(f', Z, 3) = -X_1X_2^{[2]} + X_1^{[2]}Z - X_1X_2Z - X_2^{[21}Z$. Note that $Z \cdot_{rp} f'[3] \notin L_4$. Thus, while $L_4 = L_3 : R_1$, so L_3 determines L_4 (Proposition 1.13 Equation (1.10)), $L_4 \neq L_3 \cdot_{rp} Z$ — unlike the simple relation $L_{i+1} = L_i \cdot_{rp} Z$ when $i \geq \alpha(\mathfrak{Z})$. Also $L_4 \neq R_3 \circ (f'[3] \cdot_{rp} Z^{[4]})$, rather, by Lemma 2.9 we need to use $f'[\alpha]$: so $L_j = R_4 \circ (f'[4] \cdot_{rp} Z^{[j]})$.

We now return to one of our themes, deciding when a (locally) Gorenstein 0-scheme is arithmetically Gorenstein, with the aid of the inverse system.

Proposition 2.18. Cones that are aG. *Suppose that $\mathfrak{Z} \subset \mathbb{P}^n$ is a a degree-s zero-dimensional locally Gorenstein subscheme, concentrated at a single point $p \in \mathbb{P}^n$. Suppose further that $\mathfrak{Z}$ is defined by a homogeneous ideal $\mathcal{I}_p$ of the local ring $\mathcal{O}_p$ at p (we say that $\mathfrak{Z}$ is "conic", see [IK, Lemma 6.1]). Then $\mathfrak{Z}$ is arithmetically Gorenstein.*

Proof. We may suppose that the point is $p = (0 : \ldots : 0 : 1)$, and that $\mathcal{I}_p$ is defined by $I' \subset R' = k[y_1, \ldots, y_n]$. Then, letting $z = x_{n+1}$ we have $(I_\mathfrak{Z})_i = \oplus_0^i z^{i-a} \cdot I'(X)$, whence it follows that $R/(I_\mathfrak{Z}, z) \cong R'/I'$, implying that $R/I_\mathfrak{Z}$ is Gorenstein, and that $\Delta H_\mathfrak{Z} = H(R'/I')$. □

Remark 2.19. *The converse of Proposition 2.18 is false in $\mathbb{P}^2$: the ideal $\mathcal{I}_p = (y_1^2, y_1y_2 - y_2^3)$ in $\mathcal{O}_p, p = (0 : 0 : 1)$ has local Hilbert function $H(\mathcal{O}_p/\mathcal{I}_p) = (1,2,1,1,1)$, and is not homogeneous. The homogenized ideal $I_\mathfrak{Z} \subset R = K[x_1, x_2, x_3]$ is $I_\mathfrak{Z} = (x_1^2, x_1x_2z - x_2^3, x_2^5)$, of Hilbert function $H_\mathfrak{Z} = (1,3,5,6,6,\ldots)$; here z is a nonzero divisor for $R/I_\mathfrak{Z}$, and the quotient $R/(z, I_\mathfrak{Z}) \cong K[y_1, y_2]/(y_1^2, y_2^3)$, so $\mathfrak{Z}$ is arithmetically Gorenstein.*

The examples of D. Bernstein and the second author [BeI], and of M. Boij and D. Laksov [BoL], of Gorenstein Artin algebras having non-unimodal Hilbert functions, and, later of M. Boij of such algebras whose Hilbert functions have arbitrarily many maxima [Bo1], are all graded. It follows from Proposition 2.18 that these examples lead to "thick points" that are arithmetically Gorenstein schemes $\mathfrak{Z}$ in $\mathbb{P}^n$, with non-unimodal first difference $\Delta H_\mathfrak{Z}$ Hilbert functions. We give the first such example, constructed in this manner below.

Corollary 2.20. *There is an arithmetically Gorenstein, "conic", zero-dimensional scheme $\mathfrak{Z}$ concentrated at a single point $p \in \mathbb{P}^5$ with $\Delta H_\mathfrak{Z}$ non-unimodal, and satisfying*

$$\Delta H_\mathfrak{Z} = (1,5,12,22,35,51,70,91,90,91,\ldots,5,1). \tag{2.16}$$

The homogeneous form $F' \in \Gamma' = K_{DP}[U,V,W,X,Y]$ defining $\mathcal{I}_p = \operatorname{Ann}(F')$ is $F' = Uf + Vg$ where f, g are general enough degree-15 forms in W, X, Y.

Remark. *When $\mathfrak{Z}$ is concentrated at a single point, defined by $\mathcal{I}_p \subset \mathcal{O}_p$, and the local Hilbert function $H(\mathcal{O}_p/\mathcal{I}_p)$ is symmetric, then it is known that $\mathcal{I}_p$ is Gorenstein iff the associated graded ideal $\mathcal{I}_p^*$ is also Gorenstein ([Wa, Proposition 1.9],[I2, Proposition 1.7]). It is not hard to show that when $H(\mathcal{O}_p/\mathcal{I}_p)$ is symmetric, $\mathfrak{Z}$ is arithmetically Gorenstein iff $\mathfrak{Z}$ is Gorenstein and $\mathcal{I}_p = \mathcal{I}_p^*$.*

2.2 Schemes concentrated at an arbitrary point of $\mathbb{P}^n$

We now extend the results of the previous subsection to any point $p \in \mathbb{A}^n \subset \mathbb{P}^n$. We translate the point to the origin using the linear group action, and use the adjoint representation on Γ, to translate the inverse system. Following F. H. S. Macaulay, we take $\widehat{\Gamma'} = K_{DP}\{\{Y_1, \dots, Y_n\}\}$, the divided power analog of the power series ring, upon which the polynomial ring $R' = K[y_1, \dots, y_n]$ acts by contraction, as before. The rings R, Γ, remain the same, but a finite inverse system will be an R'-submodule of $\widehat{\Gamma'}$ having finite dimension as K-vector space. When $p = (a_1 : \dots : a_n : 1) \in \mathbb{P}^n$, we will sometimes use $q = (a_1, \dots, a_n)$ to specify the point $q = (a_1, \dots, a_n)$ of $\mathbb{A}^n$ without regard to $\mathbb{P}^n$. We let

$$f_q = (1 - \sum a_i Y_i)^{-1} = 1 + \sum_{k \geq 1} (\sum a_i Y_i)^{[k]} = \sum_k \sum_{U | |U| = k} a^U Y^{[U]}. \tag{2.17}$$

Here f_q is the divided power analog of the exponential series $F_q = \exp(\sum a_i Y_i)$ in the usual power series ring $\widehat{\mathcal{R}'}$. We will sometimes use f_p, F_p to denote the corresponding f_q, F_q.

Lemma 2.21. *[Mac1, §64, p. 73]* Inverse systems for ideals with support an arbitrary point. *The finite inverse system $J \subset \widehat{\Gamma'}$ (respectively, $J' \subset \widehat{\mathcal{R}'}$ in the differentiation action of R' on $\widehat{\mathcal{R}'}$) is the inverse system of an ideal of R' with support the point $p = (a_1, \dots, a_n) \in \mathbb{A}^n$ iff there exists an integer N such that*

$$f_q \subset J \subset \Gamma'_{\leq N} \cdot f_q \subset \widehat{\Gamma'} \tag{2.18}$$

or, respectively,

$$\exp(\sum a_i Y_i) \subset J' \subset \mathcal{R}'_{\leq N} \cdot \exp(\sum a_i Y_i) \subset \widehat{\mathcal{R}'}. \tag{2.19}$$

Proof Outline. Here (2.18) is the divided power analog of (2.19). To show (2.19), note first that $m_q = (y_1 - a_1, \dots, y_n - a_n) \subset R'$ annihilates the one-dimensional vector space $\exp(\sum a_i Y_i) \in \widehat{\mathcal{R}'}$, since y_i acting by differentiation on this series is the same as multiplication by a_i. Likewise, $(m_q^{N+1})^\perp \subset \mathcal{R}'_{\leq N} \cdot \exp(\sum a_i Y_i)$ is immediate, and a dimension check shows (2.19). □

The following lemma is a consequence of [Mac1, §64,66] (see Remark 2.23 below). Given $q = (a_1, \dots, a_n) \in \mathbb{A}^n$ and an ideal J of R' concentrated at the origin, we denote by $T_q(J)$ the translated ideal $T_q(J) = (h(y_1 - a_1, \dots, y_n - a_n) \mid h \in J)$. Clearly, $T_q(J)$ is concentrated at q.

Lemma 2.22. Macaulay's Comparison Lemma: change of "origin" in $\mathbb{A}^n$.

- *(i.) If $h' \in R'$, f' in $\widehat{\Gamma'}$, then $h'(y_1 - a_1, \dots, y_n - a_n) \circ (f' \cdot f_q) = (h'(y_1, \dots, y_n) \circ f') \cdot f_q$.*
- *(ii.) If $L'' \subset \widehat{\Gamma'}$ is the inverse system of an ideal J of R' that is concentrated at the origin, then $L'' \cdot f_q$ is the inverse system of $T_q(J)$.*
- *(iii.) Let $\mathcal{I}_q \subset \mathcal{O}_q$ be the ideal $J' \cdot \mathcal{O}_q$, where $J' \subset R'$ has support q. Then the inverse system $L' = (J')^{-1} \subset \widehat{\Gamma'}$ has the form $L' = L'' \cdot f_q$, where $L'' \subset \Gamma'$ is the inverse system of the ideal $J = (T_q)^{-1}(J')$, concentrated at the origin.*
- *(iv.) The R' submodules of $\widehat{\Gamma'}$ generated by L' and by L'' in* (iii) *are isomorphic.*
- *(v.) The analogous statements to* (i)-(iv) *are true for the partial differentiation action of R' on the power series ring $\widehat{\mathcal{R}'}$, with f_q replaced by $F_q = \exp(\sum a_i Y_i)$.*

Proof. The part (ii) is implied by (i). It suffices to show part (i) for monomials h'; by induction on degree, we may suppose that $h' = y_i$ (since the statement is obvious for $h' =$ constant). Then we have by additivity of contraction, and Lemma 1.2,

$$\begin{aligned}(y_i - a_i)\circ (f'\cdot f_q) &= y_i \circ (f'\cdot f_q) - a_i\circ (f'\cdot f_q)\\ &= (y_i\circ f'\cdot f_q + f'\cdot y_i\circ f_q) - a_i f'\cdot f_q\\ &= y_i\circ f'\cdot f_q + f'\cdot a_i f_q - a_i f'\cdot f_q\\ &= (y_i\circ f')\cdot f_q,\end{aligned}$$

as claimed. This completes the proof of (ii). Any ideal J' of R' concentrated at p satisfies, $J' = T_q(J), J = (T_q)^{-1}(J')$, so (ii) implies (iii). Also, (iv) is immediate. □

Remark 2.23. Macaulay's notation, *Macaulay [Mac1, §64, p.72] describes the same transform as in Lemma 2.22, as follows. If $F = \sum a_{p_1,\dots,p_n} y_1^{p_1}\cdots y_n^{p_n}$ is a polynomial, and $E = \sum c_1^{p_1}\cdots c_n^{p_n}\,(y_1^{p_1}\cdots y_n^{p_n})^{-1}$ is a modular equation, and the new origin is $(-a_1,-a_2,\dots,-a_n)$, then the transformed polynomial $F' = \sum a_{p_1,\dots,p_n}(y_1-a_1)^{p_1}\cdots(y_n-a_n)^{p_n}$, and the transformed modular equation is $E' = \sum (c_1+a_1)^{p_1}\cdots(c_n+a_n)^{p_n}\,(y_1^{p_1}\cdots y_n^{p_n})^{-1}$. Here the coefficients c are in symbolic notation: that is, after expanding the expressions, $c_1^{p_1}\cdots c_n^{p_n}$ is to be put equal to the coefficient $c_{p_1,\dots,p_n}$. In particular if $E = 1$, then $E' = \sum a_1^{p_1}\cdots a_n^{p_n}\,(y_1^{p_1}\cdots y_n^{p_n})^{-1}$, the inverse function of $(x_1 - a_1,\dots,x_n-a_n)$.*

Macaulay is here translating a mutually perpendicular polynomial/inverse system pair at the origin, to one concentrated at the point $q = (a_1,\dots,a_n)$. We may rewrite Macaulay's formula for E',using the multiindex $U = (u_1,\dots,u_n)$ where $U \le P$ means, $u_i \le p_i$ for each i, as follows:

$$\begin{aligned}E' &= \sum_{U,P|0\le U\le P} c_{u_1,\dots,u_n}\binom{p_1}{u_1}\cdots\binom{p_n}{u_n} a_1^{p_1-u_1}\cdots a_n^{p_n-u_n}\cdot (y_1^{p_1}\cdots y_n^{p_n})^{-1} \quad \textit{(Macaulay's notation)}\\ &= \sum_{U,P|0\le U\le P} c_{u_1,\dots,u_n}\binom{p_1}{u_1}\cdots\binom{p_n}{u_n} a_1^{p_1-u_1}\cdots a_n^{p_n-u_n}\cdot Y_1^{[p_1]}\cdots Y_n^{[p_n]} \quad \textit{(our notation)}\\ &= \sum_{U,P-U|0\le U,0\le P-U} c_{u_1,\dots,u_n}\left(Y_1^{[u_1]}\cdots Y_n^{[u_n]}\right) a_1^{p_1-u_1}\cdots a_n^{p_n-u_n}\cdot Y_1^{[p_1-u_1]}\cdots Y_n^{[p_n-u_n]}\\ &= \quad E\cdot f_q \ \textit{(here and in the previous step we use the product in the divided power ring } \widehat{\Gamma'}).\end{aligned}$$

Note that our Lemma 2.22, Equation (i) *when $h'\circ f' = 0$, is equivalent to Macaulay's formula, so Lemma 2.22* (ii) *is a consequence of Macaulay's formula for changing the point of origin.*

Fix a point $q = (a_1,\dots,a_n) \in \mathbb{A}^n \subset \mathbb{P}^n$ with projective coordinates $p = (a_1 : \dots : a_n : 1)$. Let $J' \subset R'$ be an ideal supported at q, so that $(R'/J') \cong \mathcal{O}_q/\mathcal{I}_q, \mathcal{I}_q = J'\cdot\mathcal{O}_q$ defines an Artin quotient. By Lemma 2.22 its inverse system $L' = (J')^{-1} \subset \widehat{\Gamma'}$ satisfies $L' = L''\cdot f_q$, where $L'' \subset \Gamma'$ is the inverse system of $J = (T_q)^{-1}(J')$. Recall that $L_p = a_1X_1 + \cdots + a_nX_n + Z$. We have defined homogenization $\mathrm{Homog}(L'', L_p, u)$ for inverse systems $L'' \subset \Gamma'$ in Definition 2.4.

Theorem 2.24. Comparison Theorem. *Let $I_{\mathfrak{Z}} \subset R$ be the saturated ideal defining the scheme $\mathfrak{Z}$ concentrated at the point $p \in \mathbb{A}^n \subset \mathbb{P}^n$, and $L_{\mathfrak{Z}} = I_{\mathfrak{Z}}^{-1} \subset \Gamma$ its global inverse system. Let $J' \subset R'$ be the ideal defining $\mathfrak{Z} \subset \mathbb{A}^n$ and $L' = (J')^{-1} \subset \widehat{\Gamma'}$ its affine inverse system. Let $J = T_q^{-1}(J')$, and $L'' = J^{-1} \subset \Gamma'$ its inverse system. Let $\alpha = \alpha(\mathfrak{Z})$ and suppose that $V'' \subset \Gamma'_{\le\alpha}$ generates L'' (so $L'' = R'\circ V''$), and set $V = \mathrm{Homog}(V'', L_p, \alpha)$. Then the global inverse system $L_{\mathfrak{Z}}$ satisfies*

$$\begin{aligned}(L_{\mathfrak{Z}})_i &= \mathrm{Homog}(L''_{\le\alpha}, L_p, i)\\ &= R_\alpha \circ (V\cdot_{rp} L_p^{[i]}). \qquad (2.20)\end{aligned}$$

Furthermore, let g denote the linear transformation of R taking p to the origin, and g^* the contragradient transform on Γ, and set $\mathfrak{Z}_o = \text{Proj }(R/g(I_{\mathfrak{Z}}))$, $L_o = (I_{\mathfrak{Z}_o})^{-1}$. Then we have

$$L = g^* \circ L_o. \tag{2.21}$$

The R-module $L_{\mathfrak{Z}}$ is isomorphic to L_o. Also, if $\mathfrak{Z}$ is any punctual scheme concentrated at p, then $L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1}$ satisfies the first part of (2.20)*, for a suitable $L'' \subset \Gamma'_{\leq \alpha}$, where $\alpha = \alpha(\mathfrak{Z})$; conversely, if an inverse system L satisfies $L_i = \text{Homog}(L''_{\leq\alpha}, L_p, i)$, then $L = L_{\mathfrak{Z}}$ for a punctual scheme $\mathfrak{Z}$ concentrated at p.*

Proof. The linear transformation of R taking $p_0 = (0 : \ldots : 0 : 1)$ to $p = (a_1 : \ldots : a_n : 1)$ is $g(x_1) = x'_1 = x_1 - a_1 z, \ldots, g(x_n) = x'_n = x_n - a_n z, g(z) = z' = z$. The contragradient transform of $\Gamma = R^\vee$ satisfies $g^*(v^*)(v) = v^*(g^{-1}v)$, and is readily seen to be $g^*(X_i) = X_i, 1 \leq i \leq n$; and $g^*(Z) = L_p$. The contraction map is equivariant (see, for example [Mac1], or [IK, Prop. A3]), so for $h \in R, F \in \Gamma$, $g^*(h \circ F) = g(h) \circ (g^* F)$. Thus, (2.20) follows from Lemma 2.9 and in particular Equation (2.15). The last statement follows from Proposition 2.11 (v), similarly by translation to p. □

Remark. *We believe that equation* (2.20) *could also be approached directly from Lemma 2.22, using the fact, homogenizing $v \cdot f_q, v \in \Gamma'$ to a given degree, with respect to Z, is the same as homogenizing v with respect to L_p, since $Z^{[j]} \cdot_{rp} \left(1 + \sum_{U | 1 \leq |U| \leq j} a^U (X/Z)^{[U]}\right) = L_p^{[j]}$. Note that if V'' in* (2.20) *has the minimum possible dimension, then by Corollary 1.8, $\dim_K V'' = \text{type}\mathcal{O}_{\mathfrak{Z}}$.*

Example 2.25. *Let $\mathfrak{Z}$ denote the degree-4 scheme concentrated at $p_1 = (1 : 0 : 1)$, determined by $f' = (Y_1^{[2]} + Y_2^{[2]}) \cdot f_{p_1}$. Then $I_{\mathfrak{Z}}$ is the translation to p_1 of $(x_1x_2, x_1^2 - x_2^2)$, so $I_{\mathfrak{Z}} = (x_1x_2 - zx_2, x_1^2 - 2zx_2 + z^2 - x_2^2)$, of Hilbert function $H_{\mathfrak{Z}} = (1, 3, 4, 4, \ldots)$, with $\tau(\mathfrak{Z}) = \alpha(\mathfrak{Z}) = 2$. Here $L_{\mathfrak{Z}}$ determines $L = (I_{\mathfrak{Z}})^{-1}$, and satisfies, by Theorem 2.24,*

$$\begin{aligned} L_{\mathfrak{Z}} =& \text{Homog}(V, X_1 + Z, 3), \ \textit{where } V = \langle X_1^{[2]} + X_2^{[2]}, X_1, X_2, 1\rangle \\ =& \langle 3X_1^{[3]} + X_1^{[2]} Z + X_1 X_2^{[2]} + X_2^{[2]} Z, \langle X_1, X_2\rangle \cdot (X_1 + Z)^{[2]}, (X_1 + Z)^{[3]}\rangle \end{aligned}$$

Example 2.26. *Again, consider the point $p = (1 : 0 : 1) \in \mathbb{P}^2$, but let the ideal $\mathcal{I}_p$ in the local ring $\mathcal{O}_p$ be defined by $\mathcal{I}_p = \text{Ann }(f' \cdot f_p)$ where $f' = Y_1^{[2]} + Y_2^{[3]}$; this ideal is the translation to p of the ideal found in Example 2.13, concentrated at $p_0 = (0 : 0 : 1)$. We have $\mathcal{I}_p = \big((y_1 - 1)y_2, (y_1 - 1)^2 - y_2^3, (y_1 - 1)^3, y_2^4\big)$, and its homogenization in $R = K[x_1, x_2, z]$ is $I = \big((x_1 - z)x_2, (x_1 - z)^2 z - x_2^3, (x_1 - z)^3, x_2^4\big)$, of Hilbert function $H_{\mathfrak{Z}} = (1, 3, 5, 5, 5, \ldots)$, defining a scheme $\mathfrak{Z}$ of regularity $\sigma(\mathfrak{Z}) = 3$. By Theorem 2.24 the inverse system $L = L_{\mathfrak{Z}}$ is determined by the "generator" element $F = \text{Homog}(f', L_p, 3) = X_1^{[2]} \cdot L_p + X_2^{[3]} \in L$, $L_p = X_1 + Z$: so $L_i = R_3 \circ G_{i+3}, G_{i+3} = F \cdot_{rp} L_p^{[i]}$. Thus we have for L_3, which determines L,*

$$\begin{aligned} L_3 &= R_3 \circ F \cdot_{rp} (X + Z)^{[3]} = R_3 \circ \left(X_1^{[2]} \cdot (X_1 + Z)^{[4]} + X_2^{[3]} \cdot (X_1 + Z)^{[3]}\right) \\ &= R_3 \circ \left[\left(15X_1^{[6]} + 10X_1^{[5]} Z + 6X_1^{[4]} Z^{[2]} + 3X_1^{[3]} Z^{[3]} + X_1^{[2]} Z^{[4]}\right) + \right. \\ &\qquad \left. X_2^{[3]} \cdot \left(X_1^{[3]} + X_1^{[2]} Z + X_1 Z^{[2]} + Z^{[3]}\right)\right]. \end{aligned}$$

Also, this is, by the first part of (2.20) *in Theorem 2.24, and Example 2.13*

$$L_3 = \text{Homog}(V'', L_p, 3), \ \textit{where } V'' = R' \circ f' = \langle f', Y_2^2, Y_2, Y_1, 1\rangle.$$

So $z^3 \circ G_6 = 3X_1^{[3]} + X_1^{[2]}Z + X_2^{[3]} \in L_3$. *Note the coefficient* 3 *on the first term, and that since* $I \circ L = 0$, *we have* $I \circ (z^3 \circ G_6) = 0$. *Thus, for example, we have*

$$\begin{aligned}(x_1 - z)^3 \circ (z^3 \circ G_6) &= \left(x_1^3 - 3x_1^2 z + 3x_1 z^2 - z^3\right) \circ (3X_1^{[3]} + X_1^{[2]}Z + X_2^{[3]}) \\ &= x_1^3 \circ (3X_1^{[3]}) - 3x_1^2 z \circ (X_1^{[2]}Z) + 0 - 0 \\ &= 0\end{aligned}$$

Proposition 2.27. Inverse system of a scheme concentrated at a single point. *Assume that* char $K = 0$, *or* char $K > j$ *below. The inverse system* $W \subset \Gamma$ *is the inverse system of a degree-s punctual subscheme* $\mathfrak{Z} \subset \mathbb{P}^n$ *concentrated at the point* $p = (a_1 : \ldots : a_n : 1)$, *and regular in degree* σ *with* $\alpha(\mathfrak{Z}) = \alpha$ *iff (TFAE)*

i. $\exists \alpha \in \mathbb{N} \mid K_{DP}[L_p] \subset W \subset \Gamma_{\leq \alpha} \cdot K_{DP}[L_p]$.

ii. *a.* $\dim_K W_j = s \;\; \forall j \geq \tau =_{def} \sigma - 1$ *and*

b. $\forall (i, n) \mid n \geq \max\{i, \sigma\}$, $R_{n-i} \circ W_n = W_i$.

c. $W \subset \Gamma_{\leq \alpha} \cdot K_{DP}[L_p]$.

iii. *a. ii.a and ii.b above, and* $\forall j$, $L_p^{[j]} \in W_j$, *and*

b. $\forall j$, $W_j = R_\alpha \circ (W_\alpha \cdot_{rp} L_p^{[j]})$.

Proof. The condition (iib) above implies the corresponding condition of Proposition (1.13), so that (iia),(iib) are equivalent to $I = \text{Ann }(W)$ being the saturated ideal defining a degree-s zero-dimensional scheme $\mathfrak{Z} \subset \mathbb{P}^n$. The third condition (iic) is that of Lemma 2.1, and assures that $\mathfrak{Z}$ has support the point p; the specific bound α arises from the change of coordinates of Lemma 2.22 applied to the formulas $L_j = L_\alpha[j]$ and $L \subset \Gamma_{\leq \alpha} \cdot K[Z]$ (note that $K[Z] = K_{DP}[Z]$) of Proposition 2.11 (ii). Thus, the hypotheses on $\mathfrak{Z}$ implies the first condition (ii) and conversely. That the hypotheses imply (iiib) follows from Theorem 2.24; also, (iiib) evidently implies (iic). □

2.3 Schemes with finite support

We now combine the results of previous sections, to determine the inverse system of schemes concentrated at several points. We will assume that coordinates are chosen so that any punctual scheme $\mathfrak{Z}$ considered lies entirely within the affine chart $\mathbb{A}^n$ where $x_{n+1} \neq 0$. Recall that for punctual subschemes $\mathfrak{Z} \subset \mathbb{P}^n$, $\tau(\mathfrak{Z}) = \sigma(\mathfrak{Z}) - 1$. If $\mathfrak{Z} = \cup \mathfrak{Z}(i)$ with $\mathfrak{Z}(i)$ concentrated at p_i, we let $\alpha(i) = \alpha(\mathfrak{Z}(i))$, the local socle degree (Definition 2.3).

Proposition 2.28. *The saturated ideal* $I_{\mathfrak{Z}}$ *is that of a scheme with support* $p_1, \ldots, p_k$ *iff*

$$M(p_1) \cap \cdots \cap M(p_k) \supset I_{\mathfrak{Z}} \supset M(p_1)^{\alpha(1)+1} \cap \cdots \cap M(p_k)^{\alpha(k)+1}. \tag{2.22}$$

The inverse system L *is that of such a scheme iff it is saturated (see Lemma 1.6, equation* (1.7)*), and*

$$\langle L_{p_1}^i, \ldots, L_{p_k}^i \rangle \subset L_i \subset \langle \Gamma_{\alpha(1)} L_{p_1}^{i-\alpha(1)}, \ldots, \Gamma_{\alpha(k)} L_{p_k}^{i-\alpha(k)} \rangle \tag{2.23}$$

Proof. The condition (2.22) is the condition for the primary decomposition of $I_{\mathfrak{Z}}$ to have $p_1, \ldots, p_k$ as the associated points; the condition (2.23) is its translation by (2.2) (see also Lemma 2.1). □

Theorem 2.29. Decomposition of the inverse system of a punctual scheme. *Let $I = \mathcal{I}_\mathfrak{Z}$ be the (saturated) defining ideal of a zero-dimensional, degree-s scheme $\mathfrak{Z} = \mathfrak{Z}(1) \cup \ldots \cup \mathfrak{Z}(k)$ in $\mathbb{P}^n$ over the field K, whose irreducible components $\mathfrak{Z}(1) = \mathfrak{Z}_{p_1}, \ldots, \mathfrak{Z}(k) = \mathfrak{Z}_{p_k}$, have degrees $s_1, \ldots, s_k$, are concentrated at the distinct K-rational points $p_1, \ldots, p_k$, respectively, and whose (saturated) defining ideals are $I(1), \ldots, I(k) \in R$. Let $I, I(1), \ldots, I(k)$ have (global) inverse systems $L = I^{-1}, L(1), \ldots L(k) \subset \Gamma$, respectively. We denote the regularity degree of $\mathfrak{Z}$ by σ, and that of each $\mathfrak{Z}(u)$ by $\sigma(u)$, and let $\alpha(u) = \alpha(\mathfrak{Z}(u))$, and set $\mathfrak{Z}'(u) = \text{Proj }(R/(M(p_u)^{\alpha(u)+1} \cap (I(1) \cap \cdots \widehat{I(u)} \cdots \cap I(k)))$. Then we have,*

(i) $L = L(1) + \cdots + L(k)$,

(ii) When $i \geq \sigma - 1$, then $L_i = L(1)_i \oplus \cdots \oplus L(k)_i$, and $I(1)_i, \ldots, I(k)_i$ intersect properly in R_i.

(iii) $L(u)_i \subset L_i \cap (\Gamma_{\alpha(u)} \cdot L_{p_u}^{(i-\alpha(u))})$, with equality for $i \geq \min\{i \mid \dim(L_i \cap (\Gamma_{\alpha(u)} \cdot L_{p_u}^{(i-\alpha(u))})) = s_u\}$. Certainly there is equality for $i \geq \tau(\mathfrak{Z}'(u))$. Also $L(u)_i = R_{j-i} \circ L(u)_j$ if $i \leq j$ and $j \geq \sigma(u)$.

Proof. First, (i) follows from the exactness of the action of R_i on Γ_i: the perpendicular space in Γ_i to an intersecton $I(1)_i \cap \cdots \cap I(k)_i$ is the sum $L(1)_i + \cdots + L(k)_i$. That the sum is direct when $i \geq \sigma - 1$ arises from $H(R/I_\mathfrak{Z})_i = s = \sum_u s_u = \sum_u H(R/I_{\mathfrak{Z}_u})_i$ when $i \geq \sigma - 1$: this shows the first statement of (ii), which is equivalent by duality to the second. The inclusion of (iii) arises from the inclusion $I(u) \supset I + M(p_u)^{\alpha(u)+1}$ by duality, using Equation (2.2). When $i \geq \tau(\mathfrak{Z}'(u))$ we have that $(M(p_u)^{\alpha(u)+1})_i$ and $\left(I(1) \cap \cdots \widehat{I(u)} \cdots \cap I(k)\right)_i$ intersect properly in R_i by (ii), whence it is not hard to show $I(u)_i = (I + M(p_u)^{\alpha(u)+1})_i$. Here is a proof: let $L'(u) = L(1) \oplus \cdots \oplus \widehat{L(u)} \oplus \cdots \oplus L(k)$. Then

$$L_i \cap (\Gamma_{\alpha(u)} \cdot L_{p_u}^{(i-\alpha(u))}) = (L'(u) + L(u))_i \cap (\Gamma_{\alpha(u)} \cdot L_{p_u}^{(i-\alpha(u))}) = (L(u)_i + K_i), \tag{2.24}$$

where, when $i \geq \tau$ we may assume $K_i \subset L'(u)_i$, since the sum $L'(u)_i \oplus L(u)_i$ is then direct; but when $i \geq \tau(\mathfrak{Z}'(u))$ we must have $K = 0$. The last statement follows from Lemma 1.6. □

Algorithm 2.30. *Suppose that the degree s of a punctual scheme $\mathfrak{Z} \subset \mathbb{P}^n$ is given, also an upper bound $N \geq \sigma(\mathfrak{Z})$ on the regularity degree; and suppose the inverse system $(L_\mathfrak{Z})_i$ in any degree can be calculated. We may find the primary decomposition of $I_\mathfrak{Z}$ as follows: first, determine the points p_u of support by testing which powers $L_{p_u}^{[N]} \in (L_\mathfrak{Z})_N$. Following Theorem 2.29* (iii)*, then choose $i \geq s + \dim_K(R/m^{s+1})$ and form the intersection $L(u)_i = (L_\mathfrak{Z})_i \cap (\Gamma_s \cdot L_{p_u}^{(i-s)})$, from which $I(u)$ can be determined (see Example 3.15). However, this may require working in a high degree. Can we obtain $L(u)_N$ directly from $(L_\mathfrak{Z})_N$, which contains $L(u)_N$ as a direct summend by Theorem 2.29* (ii)*?*

Remark 2.31. Determining when $\mathfrak{Z}$ is arithmetically Gorenstein. *An Artin Gorenstein local algebra has a unique minimum length ideal, its socle, of dimension one as K-vector space. Thus if $\mathfrak{Z}$ is a zero-dimensional locally Gorenstein punctual scheme in $\mathbb{P}^n$, each irreducible component $\mathfrak{Z}_i$ has a unique proper subscheme of degree one less than $\mathfrak{Z}_i$: we denote by $\mathfrak{Z}'_i$ its union with the remaining components. To use the Cayley-Bacharach (*CB*) criterion (see Example 2.14) for a Gorenstein punctual scheme with k irreducible components, one needs to check the Hilbert function for the k different subschemes $\mathfrak{Z}'_1, \ldots, \mathfrak{Z}'_k$: the* CB *criterion is that (with $\tau = \tau(\mathfrak{Z})$)*

$$H(R/I_{\mathfrak{Z}'_k})_{\tau-1} = H(R/I_\mathfrak{Z})_{\tau-1} \text{ for each } \mathfrak{Z}'_i, i = 1, \ldots, k. \tag{2.25}$$

We have seen in Example 2.14 that when $\mathfrak{Z}$ is local, not "conic", but $\mathfrak{Z}'$ is conic, then $\mathfrak{Z}$ fails the CB *criterion. Since being arithmetically Gorenstein is a global property, there are no local criterion for it. Nevertheless, the above equation* (2.25)*, or even the inverse system can be used to check the* CB *criterion, as we illustrate in the next example.*

Example 2.32. NON AG SCHEME. *Suppose* $R = K[x_1, x_2, x_3, z]$, *and* $\Gamma = K_{DP}[X_1, X_2, X_3, Z]$, *let* $I_{\mathfrak{Z}} = m_p \cap I(2)$, *where* $M(p) = (x_1 - z, x_2 - z, x_3 - z)$, *the maximal ideal at* $p = (1, 1, 1, 1)$, *and* $I(2) = (x_1, x_2^2, x_3^2)$, *a complete intersection concentrated at* $p_0 = (0 : 0 : 0 : 1)$. *Then* $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2) \subset \mathbb{P}^3$ *with* $\mathfrak{Z}(1) = p$, $\mathfrak{Z}(2) = \text{Proj}\ (R/I(2))$, *and*

$$I = I_{\mathfrak{Z}} = (x_1^2 - x_1x_3, x_1x_2 - x_1x_3, x_1x_3 - x_1z, x_2^2 - x_1z, x_3^2 - x_1z)$$

$H_{\mathfrak{Z}} = (1, 4, 5, 5, \ldots)$, *with* $\tau(\mathfrak{Z}) = 2$. *A calculation shows* $\mathfrak{Z}'(2) = \text{Proj}\ (R/I'(2))$, *where* $I'(2) = M(p) \cap (I(2), x_1x_2)$, *has Hilbert function* $H'(2) = (1, 4, 4, \ldots)$, *satisfying the criterion, but* $\mathfrak{Z}'(1) = \mathfrak{Z}(2)$, *of Hilbert function* $H'(1) = H(2) = (1, 3, 4, \ldots)$, *so* $\mathfrak{Z}$ *is not arithmetically Gorenstein.*

This can be seen using the inverse systems as follows: taking $W = L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1}, W(1) = L_{\mathfrak{Z}(1)}, W(2) = L_{\mathfrak{Z}(2)}, L_p = X_1 + X_2 + X_3 + Z$, *we have*

$$W_j = W(1)_j + W(2)_j = L_p^{[j]} + \langle X_2Z^{j-1}, X_3Z^{j-1}, X_2X_3Z^{j-2}, Z^j \rangle.$$

The inverse system $W'(2)$ *to* $I_{\mathfrak{Z}'(2)}$ *is obtained by removing from* W_j *the "generator" element* $X_2X_3Z^{j-2}$ *of* $W(2)$, *not affecting* $\dim_K W'(2)_1 = 4$, *the dual module* $W'(1)$ *to* $I'_{\mathfrak{Z}'(1)}$ *is obtained by removing from* W_j *the "generator"* $L_p^{[j]}$, *of* $W(1)$ *which gives* $\dim_K W'(1)_1 = 3$, *not 4, as required by the Cayley-Bacharach criterion* (2.25).

Remark 2.33. REGULARITY DEGREE. *When* $\mathfrak{Z}$ *is concentrated at a single point we showed that the regularity and local socle degree are related by* $\sigma(\mathfrak{Z}) \leq \alpha(\mathfrak{Z}) + 1$ *(see Proposition 2.11 (iv)). This result cannot extend to arbitrary punctual schemes. When the degree-s scheme* $\mathfrak{Z}$ *is smooth, we have* $\alpha(\mathfrak{Z}) = 0$, *but* $H_{\mathfrak{Z}}$ *can be any sequence such that* $\Delta H_{\mathfrak{Z}}$ *is an O-sequence of length-s, by Theorem 1.12 (iv). Since for a punctual scheme* $\mathfrak{Z}$, $\sigma(\mathfrak{Z}) = 1 + \tau(\mathfrak{Z})$, *with* $\tau(\mathfrak{Z}) = \max\{i \mid (\Delta H_{\mathfrak{Z}})_i \neq 0\}$, *the maximum regularity degree is s, when* $\Delta H_{\mathfrak{Z}} = (1, 1, \ldots, 1)$. *Even the degree* τ *component of the ideal* $I_{\mathfrak{Z}}$ *or of the inverse system* $I_{\mathfrak{Z}}^{-1}$, *may be far from determining the support of* $\mathfrak{Z}$. *For a simplest example, if* $s = 2$, *the smooth scheme* $\mathfrak{Z} = (1 : 0 : 1) \cup (0 : 0 : 1) \subset \mathbb{P}^2$ *has inverse system* I^{-1} *satisfying* $(I^{-1})_i = (I_{\mathfrak{Z}}^{-1})_i = \langle Z^i, (X + Z)^{[i]} \rangle$, $\Delta H_{\mathfrak{Z}} = (1, 1)$, *so* $\tau(\mathfrak{Z}) = 1$. *But the degree-*τ *component of the inverse system,* $(I^{-1})_1 = \langle Z, X + Z \rangle$, *only restricts the two points of* $\mathfrak{Z}$ *to lie on the line* $y = 0$.

There has been much study of regularity questions for zero-dimensional schemes. For example M. Chardin and P. Philippon show that if there are forms $f_1, \ldots, f_n$ *of degrees* $d_1, \ldots, d_n$ *in* $\mathbb{P}^n$, *such that* $f_1 = \cdots = f_n = 0$ *contains* $\mathfrak{Z}$, *and they form a local complete intersection (LCI) at each support point of* $\mathfrak{Z}$, *then the regularity degree of* $\mathfrak{Z}$ *is at most* $d_1 + \cdots + d_n - n$ *[CharP, Theorem A]. LCI schemes* $\mathfrak{Z}$ *occur naturally in both singularity theory (see [Mi]) and also in the study of certain hyperplane arrangements (see [Schk]). It could be of interest to explore such punctual schemes from an inverse system point of view, however, to detect CI or LCI from the inverse system is not so easy: it is simpler to detect if* $\mathfrak{Z}$ *is Gorenstein.*

We give the following basic result bounding the regularity degree, in terms of the socle degrees of the irreducible components of $\mathfrak{Z}$, when the number of components is small. We say that k points in $\mathbb{P}^n$ are in (linearly) general position if each subset of s points spans a $\mathbb{P}^{s-1}$, for $s \leq n + 1$.

Proposition 2.34. *Let* $\mathfrak{Z}$ *be a zero-dimensional scheme, supported at* $p(1), \ldots, p(k) \subset \mathbb{P}^n$, *and suppose the socle degrees of the irreducible components* $\mathfrak{Z}(1), \ldots, \mathfrak{Z}(k)$ *are* $\alpha(1) \leq \cdots \leq \alpha(k)$. *If* $k \leq n + 2$ *and the k points are in linearly general position, then the regularity degree* $\sigma(\mathfrak{Z})$ *satisfies*

$$\sigma(\mathfrak{Z}) \leq \alpha(k) + \alpha(k-1) + 2. \tag{2.26}$$

Proof. The inverse system $W \subset \mathcal{R}$ for $m_{p(1)}^{\alpha(1)+1} \cap \cdots \cap m_{p(k)}^{\alpha(k)+1}$ satisfies, by the analogue for the partial derivative action of R on $\mathcal{R}$ of (2.2), $W_i = (L(1)^{i-\alpha(1)}, \ldots, L(k)^{i-\alpha(k)})_i$. The hypothesis that the points are in linearly general position, implies that the ideal $(L(1)^{i-\alpha(1)}, \ldots, L(k)^{i-\alpha(k)})$

is a complete intersection when $k \le n+1$, and an almost complete intersection when $k = n+2$. Using the Hilbert function of CI's, or a result of R. Stanley (see [I4, Lemma C]) when $k = n+2$, we have $\dim_K W_i = \sum_u \dim_K R_{\alpha(u)}$ iff $i < i - \alpha(k) + i - \alpha(k-1)$, or $i \ge \alpha(k) + \alpha(k-1) + 1$; for such i the sum $(L_{\mathfrak{Z}(1)})_i + \cdots + (L_{\mathfrak{Z}(k)})_i$ is direct: since each $L_{\mathfrak{Z}}(u)$ has $\tau(\mathfrak{Z}(u)) \le \alpha(u)$, we have $\tau(\mathfrak{Z}) \le \alpha(k) + \alpha(k-1) + 1$, implying (2.26). □

Analogous inequalities when $k \ge n+3$ can be shown in some special cases, with the hypothesis that the points of support are "generic": however, the general problem of bounding $\sigma(\mathfrak{Z})$ in terms of the $\alpha(u)$ is equivalent to the interpolation problem, of determining the Hilbert function of higher order vanishing ideals at the k points: this problem is open in general, unless $\alpha(u) \le 2$, or $k \le n+2$ (see [AlH, Cha1, Cha2, I3]). When $k = 6$ points on $\mathbb{P}^3$, there is exceptional behavior: calculation for $\alpha = 3, 4, \ldots$ shows that if $\mathfrak{Z}(u) = \text{Proj } (R/m_{p(u)}^{\alpha+1}), u = 1, \ldots, 6$, then $\sigma(\mathfrak{Z}) = 2\alpha + 3$.

3 When can we recover the scheme $\mathfrak{Z}$ from a dual form F?

Here we study when a scheme $\mathfrak{Z} \subset \mathbb{P}^n$ can be recovered from a general element F in $(I_{\mathfrak{Z}})_j^{-1}$, the degree-$j$ component of its inverse system. Is $\mathfrak{Z}$ determined by a general dual form $F \in \Gamma_j$ — a form annihilated by $I_{\mathfrak{Z}}$? We begin with two examples, first of the scheme Proj $(R/{m_p}^2)$, which cannot be so recovered, and, second, of a non-CM scheme $\mathfrak{Z}$— having components of different dimension — that can be recovered. We then restate and prove our main result, giving a sufficient condition when $\dim \mathfrak{Z} = 0$ (Theorem 3.3). We give several Corollaries related to improvements in special cases, and Corollary 3.10, a consequence concerning subfamilies of the parameter space PGOR(T). In Section 3.2 we briefly describe linkage as viewed through the lens of inverse systems, and in Section 3.3 we interpret our results in terms of generalized additive decompositions of forms (Theorems 3.19 and 3.20). The following example is similar to [IK, Example 5.10].

Example 3.1. Non-recoverable scheme. *On $\mathbb{P}^2$ with coordinate ring $R = K[x, y, z]$, consider the non-Gorenstein ideal ${m_p}^2, p = (0 : 0 : 1)$ which defines a degree-3 subscheme $\mathfrak{Z} \subset \mathbb{P}^2$. Thus $I = I_{\mathfrak{Z}} = (x^2, xy, y^2) \subset R = K[x, y, z]$, of Hilbert function $H(R/I_{\mathfrak{Z}}) = (1, 3, 3, \ldots,)$, and local Hilbert function $H(R'/I') = (1, 2)$. Thus $\tau(\mathfrak{Z}) = 1$, $\sigma(\mathfrak{Z}) = 2$, and we have*

$$(I_{\mathfrak{Z}})_i^{-1} \cap \Gamma_i = \{Z^i, Z^{i-1}X, Z^{i-1}Y\}. \tag{3.1}$$

Taking a general element $F = \alpha Z^j + \beta XZ^{j-1} + \kappa YZ^{j-1}$, we find that Ann (F) *contains $\kappa x - \beta y$, so we cannot recover the ideal $I_{\mathfrak{Z}}$ from a single form F. However, we can recover $I_{\mathfrak{Z}}$ using two forms F, G, thus from a level algebra of type 2.*

Example 3.2. Line with embedded point. *Let $R = K[x, y, z], \Gamma = K_{DP}[X, Y, Z]$. Consider $F = XZ^{[3]} + Y^{[3]}Z \in \Gamma_4$. Then* Ann $(F) = (x^2, xy, xz^2 - y^3, z^4)$ *defines an Artin algebra $R/$*Ann (F) *of Hilbert function $T = (1, 3, 4, 3, 1)$. However,* Ann $(F)_{\le 2} = (x^2, xy)$, *defines a scheme $\mathfrak{Z} \subset \mathbb{P}^2$ consisting of a line with an embedded point, whose Hilbert function satisfies $H_{\mathfrak{Z}} = (1, 3, 4, 5, 6, \ldots)$.*

Taking instead $F_1 = XYZ^2$, we find Ann $(F_1) = (x^2, y^2, z^3)$, *also of Hilbert function T, and* Ann $(F)_{\le 2}$ *defines a degree-4 scheme $x^2 = y^2 = 0$. More generally let $\mathfrak{Z}_1 =$* Proj $(R/(g, h))$ *be any complete intersection scheme concentrated at $p_0 \in \mathbb{P}^2$, of local Hilbert function $H(R'/(g, h)) = (1, 2, 1)$, and let $f_1 \in \Gamma'$ be a generator of the local inverse system $F = f_1 Z^2$. Then it is easy to see directly (or by Corollary 3.4) that* Ann $(F)_{\le 2} = (g, h)$, *so determines $\mathfrak{Z}_1$.*

Remark. Nonexistence of a morphism from Gor(T) to the Hilbert scheme of points. *Example 3.2 shows that when $T = (1, 3, 4, 3, 1)$, it is not possible to define a morphism from all of* PGOR(T) *(the family of Gorenstein ideals of Hilbert function T, see Definition 3.9 below) to the punctual Hilbert scheme* $\text{Hilb}^4(\mathbb{P}^2)$ *parametrizing degree-4 zero-dimensional subschemes of $\mathbb{P}^2$. The above example also answers negatively a question asked in [IK, p. 142], whether $\mathfrak{Z}$ locally Gorenstein might be a necessary condition for $I_{\mathfrak{Z}}$ to occur as the ideal generated by the lower degree* *generators of a Gorenstein Artin quotient of $R/I_{\mathfrak{Z}}$ — as here $\mathfrak{Z}$ is not even Cohen-Macaulay. The question of which $\mathfrak{Z}$ occur is open, even when $\mathfrak{Z}$ is restricted to be pure zero-dimensional. See [IK, Remark 5.73 and Chapter 6] for further discussion.*

3.1 Recovering $\mathfrak{Z}$: main results

We now show our main result about recovering the scheme $\mathfrak{Z}$ from a general element $F \in L_{\mathfrak{Z}}$. Recall that for a zero-dimensional degree- s scheme $\mathfrak{Z} \subset \mathbb{P}^n$ we denote by $\tau(\mathfrak{Z}) = \sigma(\mathfrak{Z}) - 1 = \min\{i \mid (H_{\mathfrak{Z}})_i = s\}$. We denote by $\alpha(\mathfrak{Z})$ the maximum local socle degree of a component of $\mathfrak{Z}$ (see Definition 2.3). We let $\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \max\{\tau(\mathfrak{Z}), \alpha(\mathfrak{Z})\}$, and $L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1}$. It is evident that for any $F \in (L_{\mathfrak{Z}})_j$, we have $I_{\mathfrak{Z}} \subset \text{Ann}\ (F)$. We assumed throughout the paper that char $K = 0$, or char $K = p > j$, where j is the maximum degree of any form considered, here the degree of F (see Example 2.2 for the necessity of this assumption). We assumed that K is algebraically closed in order for the support of $\mathfrak{Z}$ to consist of K-rational points. The sequence $\text{Sym}(H_{\mathfrak{Z}}, j)$ is defined in equation (1.1).

Theorem 3.3. Recovering the scheme $\mathfrak{Z}$ from a Gorenstein Artin quotient. *Let $\mathfrak{Z}$ be a (locally) Gorenstein zero-dimensional subscheme of $\mathbb{P}^n$ over an algebraically closed field K,* char $K = 0$ *or* char $K > j$, *and let $L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1}$. Then we have*

1. *If $j \geq \beta(\mathfrak{Z})$, and F is a general enough element of $(L_{\mathfrak{Z}})_j$, then $H(R/\text{Ann}\ (F)) = \text{Sym}(H_{\mathfrak{Z}}, j)$.*
2. *If $j \geq \beta(\mathfrak{Z})$, and F is a general enough element of $(L_{\mathfrak{Z}})_j$, then for i satisfying $\tau(\mathfrak{Z}) \leq i \leq j - \alpha(\mathfrak{Z})$ we have $\text{Ann}\ (F)_i = (I_{\mathfrak{Z}})_i$. Equivalently, we have $R_{j-i} \circ F = (L_{\mathfrak{Z}})_i$.*
3. *If $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z}) + 1\}$, and $F \in (L_{\mathfrak{Z}})_j$ is general enough, then* Ann (F) *determines $\mathfrak{Z}$ uniquely. If $I_{\mathfrak{Z}}$ is generated in degree $\tau(\mathfrak{Z})$, then $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z})\}$ suffices.*

Proof. Since $H(R/\text{Ann}\ (F))$ is symmetric about $j/2$, part (1) follows immediately from (2). We now show (2). Suppose first that $\mathfrak{Z}$ has support the single point $p_0 = (0 : \cdots : 0 : 1)$. Let f' of degree $\alpha = \alpha(\mathfrak{Z})$ generate the local inverse system at p_0 of $\mathfrak{Z}$, let $f = \text{Homog}(f', Z, \alpha)$, and let $L = L_{\mathfrak{Z}}$. Lemma 2.9 shows that $\forall i, L_i = R_\alpha \circ (f \cdot_{rp} Z^{[i]})$. Taking $G = f \cdot_{rp} Z^{[j-\alpha]}$, we have $G \in L_j$, and for $i' \geq \alpha$, we have by Proposition 2.11 ii

$$R_{i'} \circ G = R_{i'-\alpha} \circ (R_\alpha \circ G) = R_{i'-\alpha} \circ L_{j-\alpha} = L_{j-i'}. \tag{3.2}$$

Taking $F = G$, this proves (2) in this case. Next, if $\mathfrak{Z}$ has support an arbitrary single point $p \in \mathbb{P}^n$, the proof of (2) is made similar, using Theorem 2.24 and (2.20).

Next, suppose that $\mathfrak{Z}$ has degree s, and support $p(1), ..., p(k)$; thus $I_{\mathfrak{Z}} = I(1) \cap \cdots \cap I(k)$ with $I(u)$ being the ideal of R defining a scheme $\mathfrak{Z}(u)$ having degree s_u, and concentrated at the point $p(u)$, with $\sum s_u = s$. Suppose that $\mathfrak{Z}(u) \subset \mathbb{A}^n$ is defined by $I'(u) \subset R'$ whose inverse system has generator $f'(u)$ (since $I'(u)$ is Gorenstein) in the sense $I'(u)^{-1} = (R' \circ f'(u)) \cdot f_{q(u)}$ where if $p(u) = (a_1(u) : \ldots : a_n(u) : 1)$, we denote by $q(u) = (a_1(u), \ldots, a_n(u))$ the coordinates of $p(u)$ in $\mathbb{A}^n$. Let $G(1), ..., G(k)$ in Γ_j be the homogenizations $G(u) = \text{Homog}(f'(u), L_{p(u)}, j)$ (see Definition 2.4). Suppose that $i \geq \tau(\mathfrak{Z})$. Denote by $\overline{h}$ the class of $h \mod I_{\mathfrak{Z}}$, and similarly for ideals, and let $V(u) = I(1) \cap \cdots \cap \widehat{I(u)} \cap \cdots \cap I(k)$. We will show first

Claim. *For each $u, 1 \leq u \leq k$ we have*

$$(\overline{I(u)}_i) \oplus (\overline{I(1)} \cap \cdots \cap \widehat{I(u)} \cap \cdots \cap \overline{I(k)})_i = R_i/(I_{\mathfrak{Z}})_i. \tag{3.3}$$

Furthermore, if $i \geq \tau(\mathfrak{Z})$, then cod $I(u)_i = s_u$ *in R_i, and* $\dim_K \overline{V(u)}_i = s_u$, *and also the codimension of $V(u)_i$ in R_i satisfies* cod $V(u)_i = s - s_u$.

Proof of Claim. That the sum in (3.3) is direct is immediate, since the intersection of the two summends is $(I_{\mathfrak{Z}})_i$. Since $\mathfrak{Z}(u)$ has degree s_u, $\overline{I(u)_i}$ has codimension no greater than s_u in $R_i/(I_{\mathfrak{Z}})_i$; likewise the vector space $V(u)_i$ has codimension in R_i at most $\left(\sum_{v\neq u} s_v\right) = s - s_u$; likewise, $\overline{V(u)_i}$ has codimension at most $s - s_u$ in $R_i/(I_{\mathfrak{Z}})_i$. Since $i \geq \tau(\mathfrak{Z})$ we have $\dim_K(R_i/(I_{\mathfrak{Z}})_i) = s$; thus we have likewise $\dim_K(R_i/(I(u))_i = s_u$, $\dim_K R_i/V(u)_i = s - s_u$; this shows the equality of the Claim. □

Now let $F = \lambda_1 \cdot G'(1) + \cdots + \lambda_k \cdot G'(k)$, where $G'(u) \in W(u)_j, W(u) = I(u)^{-1}$ satisfies (3.2), with G, W there replaced by $G'(u), W(u)$, where $\lambda_u \in K$ and each $\lambda_u \neq 0$. Consider $w = h \circ G'(u), h \in R_{i'}$; by applying (3.3), we conclude that $h = h' + h'', h' \circ G'(u) = 0, h'' \in V(u)$, thus $h \circ G'(u) = h'' \circ G'(u) = h'' \circ F$. Thus, we have $i' \geq \tau \Rightarrow R_{i'} \circ F \supset R_{i'} \circ G'(u)$. Since evidently $R_{i'} \circ F \subset (R_{i'} \circ G'(1) + \cdots + R_{i'} \circ G'(k))$ there is for $i' \geq \tau$ an equality of vector spaces

$$R_{i'} \circ F = (R_{i'} \circ G'(1) + \cdots + R_{i'} \circ G'(k)). \tag{3.4}$$

If we take $i' \geq \max\{\tau(\mathfrak{Z}), \alpha(\mathfrak{Z})\}$, we may take $G'(u) = G(u)$ and apply (3.2) to each term $G'(u)$ of (3.4), and conclude, letting $W(u) = (I_{\mathfrak{Z}(u)})^{-1}$, and taking F as above, $i = j - i'$

$$R_{i'} \circ F = W(1)_i + \cdots + W(k)_i \subset W_i \tag{3.5}$$

When $i \geq \tau = \tau(\mathfrak{Z})$, the sum in (3.5) is direct, and the inclusion on the right is an equality. That a particular $F \in W_j$ satisfies $\dim_K R_{j-i} \circ F = s$, the maximum value possible (so there is equality on the right of (3.5)) implies a fortiori that a general element $F \in W_j$ will have the same property. This completes the proof of (2). If $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z}) + 1\}$, we have that Ann $(F)_{\sigma(\mathfrak{Z})} = (I_{\mathfrak{Z}})_{\sigma(\mathfrak{Z})}$, so by (3.5), and Theorem 1.12 (ii), F determines $\mathfrak{Z}$, showing (3). This completes the proof of Theorem 3.3 □.

Corollary 3.4. *A necessary and sufficient condition for $F = \lambda_1 \cdot G(1) + \cdots + \lambda_k \cdot G(k)$ in Theorem 3.3 to be general enough to satisfy the conclusion, is for each $\lambda_1, \ldots, \lambda_k$ to be nonzero.*

Proof. The sufficiency was just shown, see especially (3.4). For the necessity, note that if we form F' by omitting the term G_i from F then $I(F')_{\leq\tau} = I(\mathfrak{Z}')_{\leq\tau}$ where $\mathfrak{Z}' = \mathfrak{Z} - \mathfrak{Z}_i$. □

Remark. *We have found no counterexample to show that we could not replace β in Theorem 3.3 by some smaller value, $\beta' \geq 2\tau(\mathfrak{Z})$. What is needed is to establish (3.2) for $i' \geq \alpha' = \beta' - \tau$ — for example (3.2) for $i' \geq \tau(\mathfrak{Z})$ would allow us to replace $j \geq \beta(\mathfrak{Z})$ in Theorem 3.3 (2) by $j \geq 2\tau(\mathfrak{Z})$, and to simply omit $j \geq \beta(\mathfrak{Z})$ from the statement of Theorem 3.3 (3) (See Corollary 3.6 below). A measure of the specialness of our result, and a hope for improvement, is given by the rather special form of F in (3.4), far from a generic element of $(L_{\mathfrak{Z}})_j$. The special case $\mathfrak{Z}$ smooth of Theorem 3.3 was shown by M. Boij [Bo2], and the cases $\mathfrak{Z}$ smooth or local "conic" by the second author and V. Kanev [IK, Theorem 5.3E, Lemma 6.1].*

The following Corollary, which determines $\beta(\mathfrak{Z})$ in special cases, shows that we indeed recover the previous results of M. Boij and V. Kanev and the second author, when $\mathfrak{Z}$ is smooth, or conic.

Corollary 3.5. *If $\mathfrak{Z}$ is supported at a single point p, then $\tau(\mathfrak{Z}) \leq \alpha(\mathfrak{Z})$, and $\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z})$. Then a general $F \in W_j$ determines $\mathfrak{Z}$ if $j \geq \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z}) + 1$, or if $j = \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z})$ and $I_{\mathfrak{Z}}$ is generated in degrees less or equal $\tau(\mathfrak{Z})$. If also $\mathfrak{Z}$ is conic, then $\tau(\mathfrak{Z}) = \alpha(\mathfrak{Z})$ and $\beta(\mathfrak{Z}) = 2\tau(\mathfrak{Z})$. If instead $\mathfrak{Z}$ is smooth, then $\alpha(\mathfrak{Z}) = 0$, and also $\beta(\mathfrak{Z}) = 2\tau(\mathfrak{Z})$. In either the conic or smooth case, a general $F \in W_j$ determines $\mathfrak{Z}$ if either $j \geq 2\tau(\mathfrak{Z}) + 1$, or if both $j \geq 2\tau(\mathfrak{Z})$ and $I_{\mathfrak{Z}}$ is generated in degrees less or equal $\tau(\mathfrak{Z})$.*

We now state the Corollary mentioned in the Remark above: we show that if the statements of Theorem 3.3 are true for each component $\mathfrak{Z}(u)$ of $\mathfrak{Z}$, but with β replaced by $\beta' = \tau(\mathfrak{Z})+\alpha'$, then they are true for $\mathfrak{Z}$ with β replaced by β'. We let $L(u) = L_{\mathfrak{Z}(u)}$.

Corollary 3.6. *Suppose that $\mathfrak{Z} = \mathfrak{Z}(1) \cup \ldots \cup \mathfrak{Z}(k)$, and that there is an integer $\alpha' \geq \tau(\mathfrak{Z})$ for which (3.2) holds for each $\mathfrak{Z}(u), u = 1,\ldots,k$, with G, L there replaced by a suitable choice of general enough $G'(u) \in L(u)_j$, with $j = \tau(\mathfrak{Z}) + \alpha'$ and $i' = \alpha'$ Then the conclusions of Theorem 3.3 hold with $\beta(\mathfrak{Z})$ replaced by $\beta' = \tau(\mathfrak{Z}) + \alpha'$.*

Proof. Taking $F = \sum \lambda(u)G'(u)$ after (3.4), the proof is essentially the same (except we no longer take $G'(u) = G(u)$). Since $G'(u)$ is assumed to satisfy (3.2) for $j = \tau(\mathfrak{Z}) + \alpha'$, with $i' = \alpha'$ in place of $i = \alpha$, we obtain the conclusion of Theorem 3.3 (2) but with $j = \tau(\mathfrak{Z}) + \alpha'$. For larger $j' = j + c, c \geq 0$, we note that (3.2) is still satisfied, replacing $G'(u)$ by $G'(u) \cdot_{rp} L_p^{[c]} \in \Gamma_{j'}$, and $i' = \alpha'$ by $i' = \alpha' + c$. This implies Theorem 3.3 (2),(3), but with β replaced by β'. This complete the proof of Corollary 3.6. $\square$

Example 3.7. *Let $R = K[X_1, X_2, Z]$ and $f = \mathrm{Homog}(f', Z, 4)$ from Example 2.17 where $f' = Y_2^{[4]} - Y_1Y_2^{[2]} + Y_1^{[2]} - Y_1Y_2 - Y_2^{[21}$. Here $\mathfrak{Z}$ is concentrated at a single point $p_0 = (0:0:1) \in \mathbb{P}^2$, the Hilbert function $H(R/I_{\mathfrak{Z}}) = (1,3,5,5,\ldots)$, so $\tau(\mathfrak{Z}) = 2, \alpha(\mathfrak{Z}) = 4$, and $\beta(\mathfrak{Z}) = 2+4 = 6$. The Corollary 3.5 implies that for $j \geq 6$, a general $F \in L_j, L = (I_{\mathfrak{Z}})^{-1}$ has $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$. However, a calculation shows that this occurs for a general $F \in L_4$ (see Example 2.17 for L_4), hence for $j \geq 4$. In particular, if F is a general element of L_5, $H(R/\mathrm{Ann}\ F) = (1,3,5,5,3,1)$, and $\mathrm{Ann}\ (F)_{\leq 3} = \left(x_1x_2 - x_2^2 - x_1z, x_2^3 + x_1^2z + x_2^2z + x_1z^2, x_1^3\right) = (I_{\mathfrak{Z}})_{\leq 3}$; thus F determines $\mathfrak{Z}$ since $\sigma(\mathfrak{Z}) = 3$.*

Example 3.8. *Consider the subscheme $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2)$ of $\mathbb{P}^2$, with $\mathfrak{Z}(1)$ the scheme of Example 3.7 concentrated at $p_1 = (0:0:1)$ and and $\mathfrak{Z}(2)$ the degree-4 scheme concentrated at $p_2 = (1:0:1)$, determined by $f' = (Y_1^{[2]} + Y_2^{[2]}) \cdot f_{p_2}$, of Example 2.25, where $\tau(\mathfrak{Z}(2)) = \alpha(\mathfrak{Z}(2)) = 2$. The intersection $I_{\mathfrak{Z}} = I_{\mathfrak{Z}(1)} \cap I_{\mathfrak{Z}(2)}$ satisfies (calculated in* MACAULAY*)*

$$I_{\mathfrak{Z}} = \left(x_1^3 + x_1^2x_2 - 2x_1x_2^2 - 2x_1^2z - x_1x_2z + x_2^2z + x_1z^2,\right. \\ \left. x_1^2x_2^2 - 4x_1x_2^3/3 - x_1^2x_2z - x_1x_2^2z + x_2^3z + x_1x_2z^2, x_1x_2^3, x_2^4 - x_1^2x_2z + x_2^3z + x_1x_2z^2\right),$$

of Hilbert function $H_{\mathfrak{Z}} = (1,3,6,9,9,\ldots)$, $\tau(\mathfrak{Z}) = 3, \alpha(\mathfrak{Z}) = 4$. Corollary 3.6, and the calculation of Example 3.7 for $\mathfrak{Z}(1)$, as well as Corollary 3.5 applied to $\mathfrak{Z}(2)$, show that we may replace $\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z}) = 3+4$ in Theorem 3.3 for $\mathfrak{Z}$ by $\beta' = 3+3 = 6$. Thus, a general $F \in (L_{\mathfrak{Z}})_6$ satisfies $H(R/\mathrm{Ann}\ (F)) = \mathrm{Sym}(H_{\mathfrak{Z}}, 6) = (1,3,6,9,6,3,1)$.

We now derive some further consequence of our main theorem, along the lines of Lemma 6.1 of [IK], shown there in the special case of $\mathfrak{Z}$ conic or smooth. We introduce first some definitions from [IK]. For $F \in \Gamma_j$ we let $H_F = H(R/\mathrm{Ann}\ (F))$.

Definition 3.9. *A punctual scheme $\mathfrak{Z}$ is an* annihilating scheme *for $F \in \Gamma$ if $I_{\mathfrak{Z}} \subset I_F = \mathrm{Ann}\ (F)$. An annihilating scheme is* tight *if also $\deg \mathfrak{Z} = \max_i\{(H_F)_i\}$. If $T = (1,\ldots,1)$ is a sequence of integers symmetric about $j/2$ we denote by $\mathrm{PGOR}(T)$ the (locally closed) subvariety of $\mathbb{P}(\Gamma_j)$ parametrizing forms $F \in \Gamma_j$ — up to constant multiple — such that $H_F = T$. We denote by* $\mathbf{PGOR}(T)$ *(in boldface) the corresponding scheme, whose scheme structure is defined by determinantal ideals of certain catalecticant matrices, corresponding to the conditions $(H_F)_u = T_u$ (see [IK]).*

The tangent space $\mathcal{T}_F$ to the affine cone over $\mathbf{PGOR}(T)$ at F is isomorphic to $R_j/((\mathrm{Ann}\ F)^2)_j$ [IK, Theorem 3.9]. We denote by $\nu = \nu(\mathfrak{Z})$ the order $\nu(\mathfrak{Z}) = \min\{i | (H_{\mathfrak{Z}})_i \neq r_i\}$ of $I_{\mathfrak{Z}}$. We denote by $\mathrm{U}_{\mathfrak{Z}} \subset \mathrm{PGOR}(T), T = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$ or more precisely by $\mathrm{U}_{\mathfrak{Z}}(j)$ the family of $F \in \Gamma_j$, up to constant multiple, such that $F \in {(I_{\mathfrak{Z}})_j}^{\perp}$ and $H_F = T$. Evidently $F \in \Gamma_j$ satisfies $F \in \mathrm{U}_{\mathfrak{Z}}(j)$ iff Ann $(F)_i = (I_{\mathfrak{Z}})_i$ for $i \le j/2$ (since $I_{\mathfrak{Z}} \subset$ Ann (F) when $F \in {(I_{\mathfrak{Z}})_j}^{\perp}$). Below, we will usually omit to include the phrase "up to constant mutiple" when this is clear from the context, or unimportant. The punctual Hilbert scheme $\mathrm{Hilb}^s(\mathbb{P}^n)$ parametrizes degree-s subschemes of $\mathbb{P}^n$ (see [IKl]).

Corollary 3.10. *Let $\mathfrak{Z}$ be a zero-dimensional degree s locally Gorenstein scheme of $\mathbb{P}^n$ having regularity degree $\sigma(\mathfrak{Z})$, let $j \ge 2\tau(\mathfrak{Z})$, and let $F \in {(I_{\mathfrak{Z}})_j}^{\perp}$.*

(i) If $j \ge \beta(\mathfrak{Z})$ (or if $\mathfrak{Z}$ satisfies the hypothesis of Corollary 3.6 and $j \ge \beta'(\mathfrak{Z})$), there is an open dense family $F \in {(I_{\mathfrak{Z}})_j}^{\perp}$ such that $F \in \mathrm{U}_{\mathfrak{Z}}(j)$. For such F, we have $(\mathrm{Ann}\ (F))_i = (I_{\mathfrak{Z}})_i$ *for $i \le j - \tau(\mathfrak{Z})$, and $\mathfrak{Z}$ is a tight annihilating scheme of F.*

(ii) If $j \ge 2\tau(\mathfrak{Z})$, and F satisfies $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$, and if $Y \subset \mathbb{P}^n$ is any zero-dimensional subscheme satisfying $\deg(Y) \le s$ and $I_Y \subset \mathrm{Ann}(f)$, then $\deg(Y) = s$ and $(\mathcal{I}_Y)_i = (\mathcal{I}_{\mathfrak{Z}})_i$ for $i \le j - \tau(\mathfrak{Z})$.

(iii) If F satisfies $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$, and if also either

(a) $j \ge 2\tau(\mathfrak{Z}) + 1$, or

(b) $j \ge 2\tau(\mathfrak{Z})$,and $((\mathcal{I}_{\mathfrak{Z}})_{\le \tau}) = \mathcal{I}_{\mathfrak{Z}}$,

then $\mathfrak{Z}$ is the unique tight annihilating scheme of F.

(iv) If F satisfies $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$, then Ann $(F)^2{}_i = (I^2_{\mathfrak{Z}})_i$ *for $i \le j - (\tau - \nu)$.*

If also $\tau \le \nu$ and $\mathfrak{Z}$ is a tight annihilating scheme of F, then the tangent space $\mathcal{T}_F$ to the affine cone over **PGOR**$(T), T = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$ *at F satisfies*

$$\dim_K \mathcal{T}_F = s + dim_K((\mathcal{I}_{\mathfrak{Z}}/(\mathcal{I}_{\mathfrak{Z}})^2)_j).$$

(v) If $Y \subset Hilb^s(\mathbb{P}^n)$ is locally closed, and $\mathfrak{Z}_y, y \in Y$ is the corresponding family of degree s zero-dimensional subschemes of P^n, if $H(R/\mathcal{I}_{\mathfrak{Z}_y}) = H$ for all $y \in Y$, if $\sigma = \tau + 1$ is the generic regularity degree of $\mathfrak{Z}_y, y \in Y$ (attained for an open subset of Y), and if $j, \mathcal{I}_{\mathfrak{Z}}$ satisfy (iii.a) or (iii.b) above, and $T = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$, then there exists a subfamily $U_Y \subset \mathrm{PGOR}(T)$ satisfying

(c) $F \in U_y \Leftrightarrow H_F = T$ and $\mathfrak{Z}_y$ is a tight annihilating scheme of F,

(d) $dim(U_Y) = dim(Y) + s - 1$.

Proof. Here the main assertion (i) follows directly from Theorem 3.3 and the proof of [IK, Lemma 6.1]: we need in (i) the hypothesis $j \ge \beta(\mathfrak{Z})$ in order to use Theorem 3.3. For any $j \ge 2\tau(\mathfrak{Z})$, the assumption $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$, and that $I_{\mathfrak{Z}} \subset$ Ann (F) entail most of (ii)-(iv). □

Example 3.11. *Consider the subscheme $\mathfrak{Z}$ of Example 3.8, for which Corollary 3.6 applies for $\beta'(\mathfrak{Z}) = 6$, and choose a general $F \in (I_{\mathfrak{Z}})_6^{-1}$; then $T = H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, 6) = (1, 3, 6, 9, 6, 3, 1)$. A calculation shows that $\dim_K R/((I_{\mathfrak{Z}})^2)_6 = 27$. Since $\nu = \tau$ for $\mathfrak{Z}$, Theorem 3.10 (iv) implies that $\dim_K \mathcal{T}_F = 27$; this is easy to check directly since* Ann $(F)_{\le 3} = \langle h_3 \rangle$, *so* Ann $(F)^2_6 = \langle h_3^2 \rangle$ *of codimension 1 in R_6. Since $r = 3$,* PGOR(T) *is smooth: this here corresponds to the smoothability of degree-9 schemes in $\mathfrak{Z}$: the dimension of* PGOR(T) *is 27, since* $\dim(Hilb^9(\mathbb{P}^2)) = 18$*, and the dimension of the fiber of* PGOR(T) *over $Hilb^9(\mathbb{P}^2)$ is 9.*

Strikingly, if $j = 7$, so $T' = (1, 3, 6, 9, 9, 6, 3, 1)$, the analogous dimension is $\dim_K \mathcal{T}_F = 30$ (since $(\mathrm{Ann}\ (F)^2)_7 = (I_{\mathfrak{Z}})^2_7 = h_3 \cdot (I_{\mathfrak{Z}})_4$*, of dimension 6); when $j \ge 8$ the dimension is again 27, as can be checked by caculating $H(R/(I_{\mathfrak{Z}})^2)$.*

3.2 Dualizing module as ideal, and Linkage

We first recall a result of M. Boij, stating when the dualizing module of $\mathfrak{Z}$ is an ideal of $R/I_{\mathfrak{Z}}$. A consequence of his criterion and Theorem 3.3 is that the dualizing module can always be so realized when $\mathfrak{Z}$ has dimension zero, and is (locally) Gorenstein (Corollary 3.13). We then give an example to illustrate how the inverse systems behave in linkage.

M. Boij's theorem pertains to $d-$dimensional Cohen Macaulay rings $B = R/I$, and $d-1$ dimensional Gorenstein quotients. Let $\kappa(B)$ denote the degree of the polynomial $(1-z)^d \cdot \mathrm{Hilb}_X(z)$: here $\mathrm{Hilb}_X(z)$ is the Hilbert series $\sum H_{\mathfrak{Z}}(i)\cdot z^i$, so $\kappa(B)$ is the highest socle degree of a minimal reduction of B.

Theorem. *[Bo2, Theorem 3.3] Let $B = R/I$ be a Cohen-Macaulay algebra of dimension d, and let $J \subset B$ be an ideal of initial degree at least $\kappa(B)+2$ such that B/J is Gorenstein of dimension $d-1$.*

Then there is an isomorphism $J \to \mathrm{Ext}_R^{r-d}(B,R) = \omega_B$, which is homogeneous of degree $-\kappa(B/J) - r + d - 1$.

We consider the special case $d = 1$, and $I = I_{\mathfrak{Z}}$, the homogeneous defining ideal of a zero-dimensional scheme $\mathfrak{Z}$. Then Boij's theorem becomes,

Corollary 3.12. *Let $\mathfrak{Z}$ be a zero-dimensional subscheme of $\mathbb{P}^n$, and let J be an ideal of $B = \mathcal{O}_{\mathfrak{Z}} = R/I_{\mathfrak{Z}}$ having initial degree at least $\tau(\mathfrak{Z})+2$, such that B/J is Gorenstein of dimension zero and socle degree j. Then there is an isomorphism $J \to \mathrm{Ext}_R^{r-1}(B,R) = \omega_B$, which is homogeneous of degree $-j-r$.*

Our Main Theorem 3.3 and M. Boij's theorem imply

Corollary 3.13. *If $\mathfrak{Z}$ is a (locally) Gorenstein zero-dimensional scheme of $\mathbb{P}^n$, then there are ideals J of $\mathcal{O}_{\mathfrak{Z}} = R/I_{\mathfrak{Z}}$ satisfying the conclusions of Corollary 3.12, with $\mathcal{O}_{\mathfrak{Z}}/J$ of socle degree j, provided $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z})+1\}$. Any such ideal has the form $J = \mathrm{Ann}\ (F)/I_{\mathfrak{Z}}, F \in (I_{\mathfrak{Z}}^{-1})_j \subset \Gamma_j$. Also, if $j \geq 2\tau(\mathfrak{Z})+1$ and F is any element of $(I_{\mathfrak{Z}}^{-1})_j$, such that $H_F = H(R/\mathrm{Ann}\ (F)) = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$, then $J = \mathrm{Ann}\ (F)/I_{\mathfrak{Z}}$ is isomorphic to the dualizing module of $\mathfrak{Z}$.*

Proof. The second statement follows from Boij's theorem for $B = \mathcal{O}_{\mathfrak{Z}}$, and Macaulay's result connecting the socle of R/J' for an Artinian quotient, and generators of the inverse system of J' (see Corollary 1.8): J' is Gorenstein iff ${J'}^{-1}$ is principal. The third statement follows from Corollary 3.12 and the definition of $\mathrm{Sym}(H_{\mathfrak{Z}}, j)$ (see Equation (1.1)): the restriction $j \geq 2\tau(\mathfrak{Z})+1$ and $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$ implies that the order of $\mathrm{Ann}\ (F)/I_{\mathfrak{Z}}$ is at least $\tau(\mathfrak{Z})+2$, satisfying the hypotheses of M. Boij's theorem, and $\mathcal{O}_{\mathfrak{Z}}/J \cong R/\mathrm{Ann}\ (F)$, so is Gorenstein. By Theorem 3.3 and Corollary 3.10, such F exist with $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$ if $j \geq \beta(\mathfrak{Z})$. □

M. Boij showed that when $\mathfrak{Z}$ is smooth, then the conclusions of Corollary 3.13 hold also for $j \geq 2\tau(\mathfrak{Z})-1$. His work is related to that of M. Kreuzer in [Kr1, Kr2]. Corollary 3.13 can be used as a test of whether a Gorenstein scheme is arithmetically Gorenstein, since $\mathfrak{Z}$ is aG iff the dualizing module is principal.

Example 3.14. *Let $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2) \subset \mathbb{P}^3$ be the scheme of Example 2.32, where $\mathfrak{Z}(1) = p, p = (1,1,1,1)$ is a smooth point, and $\mathfrak{Z}(2) = \mathrm{Proj}\ (R/I(2))$ where $I(2) = (x_1, x_2^2, x_3^2)$, is a CI at $p_0 = (0:0:0:1)$. We found there that $\mathfrak{Z}$ was not arithmetically Gorenstein, although $\Delta H_{\mathfrak{Z}} = (1,3,1)$ is the h-vector of a Gorenstein ideal (it is a "Gorenstein sequence"). Since $\alpha(\mathfrak{Z}) = \tau(\mathfrak{Z}) = 2$, we have $\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z}) = 4$. By Corollary 3.13, it suffices to take a general element $F \in (L_{\mathfrak{Z}})_5$, to see the dualizing module as the ideal $J = \mathrm{Ann}\ (F)/I_{\mathfrak{Z}}$. We have, taking $L_p = X_1 + X_2 + X_3 + Z$*

$$(L_{\mathfrak{Z}})_5 = \langle X_2Z^4, X_3Z^4, X_2X_3Z^3, Z^5, L_p^{[5]}\rangle.$$

A calculation shows that with $F = X_2Z^4 + X_3Z^4 + X_2X_3Z^3 + Z^5 + L_p^{[5]}$, *we have* $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, 5) = (1,4,5,5,4,1)$, *and that* $\mathrm{Ann}\,(F) = (I_{\mathfrak{Z}}, x_2x_3z^2 - x_2z^3 - x_3z^3 + z^4, x_1z^4 - z^5/2)$. *The dualizing module* $\mathrm{Ann}\,(F)/I_{\mathfrak{Z}}$ *is not principal, confirming that* $\mathfrak{Z}$ *is not arithmetically Gorenstein.*

We now give an example showing how inverse systems behave in linkage: here $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2)$ is AG (even CI).

Example 3.15. INVERSE SYSTEMS OF LINKED LOCAL CI'S. *We consider inverse systems of three ideals in* $R = K[x_1, x_2, z]$ *defining punctual subschemes* $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2)$ *of* $\mathbb{P}^2$. *The ideal* $I(1) = (x_1 - z, x_2) = M(p_1)$ *defines the simple point* $\mathfrak{Z}(1) = p_1 = (1, 0, 1)$. *The ideal* $I(2)$, *concentrated at* $p = (0:0:1)$ *defines a degree 5 scheme* $\mathfrak{Z}(2)$, *that of Example 2.13, (there termed* $\mathfrak{Z}$*), which is a local complete intersection:*

$$I(2) = I_{\mathfrak{Z}(2)} = (x_1x_2, x_1^2z - x_2^3, x_1^3),$$

of Hilbert function $H_{\mathfrak{Z}(2)} = (1,3,5,5,\ldots)$. *Their intersection is the ideal* $I = (x_1 - z, x_2) \cap I_{\mathfrak{Z}(2)} = (x_1x_2, x_1^3 + x_2^3 - x_1^2z)$, *a complete intersection defining the degree 6 punctual scheme* $\mathfrak{Z}$, *of Hilbert function* $H_{\mathfrak{Z}} = (1,3,5,6,6,\ldots)$. *Thus* $I(1)$ *and* $I(2)$ *determine the two irreducible components of* $\mathfrak{Z}$, *which are linked through* $\mathfrak{Z}$. *Letting* $W = I^{-1}, W(1) = I(1)^{-1}$, *and* $W(2) = I(2)^{-1}$ *denote the corresponding inverse systems, we have* $W = W(1) + W(2)$, *where the sum must be direct in degrees at least* $\tau(\mathfrak{Z}) = 3$ *by Theorem 2.29 ii. The inverse system* $W(1)$ *satisfies* $W(1)_i = \langle (X_1 - Z)^{[i]} \rangle$, *while* $W(2)$ *satisfies, from Example 2.13*

$$W(2)_i = \langle X_1^{[2]}Z^{[i-2]} + X_2^{[3]}Z^{[i-3]}, X_2^2Z^{i-2}, X_2Z^{i-1}, X_1Z^{i-1}, Z^i \rangle$$

By the Decomposition Theorem 2.29i., we have

$$\begin{aligned} W_i &= W(1)_i + W(2)_i \\ &= \langle (X_1 - Z)^{[i]}, X_1^{[2]}Z^{[i-2]} + X_2^{[3]}Z^{[i-3]}, X_2^2Z^{i-2}, X_2Z^{i-1}, X_1Z^{i-1}, Z^i \rangle. \end{aligned}$$

Note that the above sum is direct in degrees at least three, but not direct in degrees less or equal two, as is evident by regarding $H_{\mathfrak{Z}(1)} = (1,1,\ldots)$ *and* $H_{\mathfrak{Z}(2)}, H_{\mathfrak{Z}}$. *Furthermore, by the Decomposition Theorem 2.29iii. we have*

$$W(2)_i = W_i \cap \langle K[X_1, X_2]_{\leq 3} \cdot_{rp} K[Z] \rangle_i$$

the intersection of W *and the inverse system of* ${m_p}^4$ *(here* $4 = \alpha(\mathfrak{Z}(2)) + 1$*), whenever the dimension of the right side is 5, which occurs for* $i \geq 4$.

That $\mathfrak{Z}$ *is AG can be seen from the inverse system, following Lemma 1.9, by showing that* $L_{\mathfrak{Z}} \cap \Gamma_z = W \cap K_{DP}[X_1, X_2]$ *is a principal* $R' = k[x_1, x_2]$*-module: in fact,* $G = X_1^3 - X_2^3$ *generates this intersection.*

Finally, from the properties of linkage, $I(1)/I$ *has dualizing module isomorphic to* $R/(I(2))$, *and conversely* $I(2)/I$ *has dualizing module* $R/I(1)$; *in particular the number of generators of* $I(1)/I$ *(here two) is the same as the dimension of* $\mathrm{SOC}(R/I(2))$; *and the number of generators of* $I(2)/I$ *(here one) is* $\dim_K \mathrm{SOC}(R/I(1))$. *In addition, since* R/I *is locally Gorenstein, similar properties hold for the localizations at* p, p_1: *here at* p_1, $(R/I(1))_{p_1} \cong R'/m_{p_1}$, *has one-dimensional socle, and* $m_{p(1)} \cong I_{p_1}$, *the localization, so there are zero generators of the quotient; also,* $(R'/I(2))_{p_1} = 0$, *so has zero socle, and* $I(2)_{p_1} = R'_{p_1}$ *has one generator.*

3.3 Generalized Additive Decompositions

We recall the GAD given in (1.2) for a degree-j form of $\Gamma = K[X,Y]$, namely

$$F = \sum_i B_i L_i^{j+1-s_i}, \deg B_i = s_i - 1, \deg L_i = 1, s = \sum s_i.$$

Each term $B_i L_i^{j+1-s_i}$ corresponds to a single support point $p_i : l_i = 0$ of $\mathbb{P}^1$, occuring with multiplicity s_i. Our aim is to model this kind of decomposition in $r \geq 3$ variables. The following definition is more general than that of [I3, Def. 4A], but is related to the concept of annihilating scheme introduced there [I3, Def. 4D] (see Definition 3.9 above).

Definition 3.16. *For $F \in \Gamma_j$, we say that $F = F_1 + \ldots + F_k$ is a* generalized additive decomposition *(GAD) of F, having (total) length $s = \sum s_i$, of partition $\pi = (s_1, \ldots, s_k)$, with k parts, associated to the scheme $\mathfrak{Z}$, if $\mathfrak{Z}$ is a degree-s punctual scheme $\mathfrak{Z}$ whose decomposition into irreducible schemes is $\mathfrak{Z} = \cup \mathfrak{Z}_i$, where $\deg \mathfrak{Z}_i = s_i$, and each $F_i \in {I_{\mathfrak{Z}_i}}^\perp$ for $i = 1, \ldots, k$. We say that a GAD of F is "tight" if $\mathfrak{Z}$ is a tight annihilating scheme of F: namely, if $s = \deg \mathfrak{Z} = \max_i\{(H_F)_i\}$ (Definition 3.9). We say that a GAD is* unique *if the k summends $F_1, \ldots, F_k$ are unique.*

The form of each term F_i — corresponding to $\mathfrak{Z}_i$ — can be read from Theorem 2.24 or Proposition 2.27: F_i is an element of the degree-j homogenization of the local inverse system of $\mathfrak{Z}_i$.

Lemma 3.17. *If F has a length-s GAD, then $\forall i \geq 0$, we have $(H_F)_i \leq s$.*

Proof. We have Ann $(F) \supset \mathcal{I}_{\mathfrak{Z}}$, hence $(H_F)_i \leq (H_{\mathfrak{Z}})_i$, but $(H_{\mathfrak{Z}})_i$ is bounded above by $\deg \mathfrak{Z}$. □

Which forms F have a length-s GAD? When is the GAD for F unique? Recall that we denote by $\sigma(\mathfrak{Z})$ the regularity degree of $\mathfrak{Z}$ (see Theorem 1.12), and by $\tau(\mathfrak{Z}) = \sigma(\mathfrak{Z}) - 1$. Evidently we have

Lemma 3.18. *If F is annihilated by a punctual scheme $\mathfrak{Z}, \mathfrak{Z} = \mathfrak{Z}_1 \cup \ldots \cup \mathfrak{Z}_k$ as in Definition 3.9, then F has a GAD of length $\leq s$ associated to $\mathfrak{Z}$; if also $\deg F \geq \tau(\mathfrak{Z})$, then the GAD has length s, is of partition $(s_1, \ldots, s_k), s_i = \deg \mathfrak{Z}_i$, and this GAD is the unique GAD of F that is associated to $\mathfrak{Z}$.*

Proof. For $j \geq \tau(\mathfrak{Z})$ we have $(H_{\mathfrak{Z}})_j = s$, hence $(I_{\mathfrak{Z}})_j^\perp = (I_{\mathfrak{Z}_1})_j^\perp \oplus \cdots \oplus (I_{\mathfrak{Z}_k})_j^\perp$, and the GAD is unique. □

The following result is an immediate consequence of [IK, Theorem 5.31], and Definition 3.16. It does not extend simply to $r > 3$ (see [Bo3, Theorem 6.42], [ChoI1], and the discussion in [IK, §6.4]).

Theorem 3.19. Uniqueness of GAD when $r = 3$. *If $r = 3$ and $H_F \supset (s, s, s)$ then F has a unique tight GAD of length s, up to permutation and change of scale, and no GAD's of smaller length than s.*

Proof. By Theorem 5.31 of [IK], F has a unique tight annihilating scheme $\mathfrak{Z}$; this determines a unique GAD by Lemma 3.18, since $j = \deg F > \sigma(\mathfrak{Z})$ (as here we have $j \geq 2\sigma(\mathfrak{Z})$). □

Recall from Definition 2.3 that $\alpha(\mathfrak{Z})$ is the highest socle degree of a component of $\mathfrak{Z}$. Finally we have,

Theorem 3.20. *Suppose that $\mathfrak{Z}$ is a Gorenstein punctual subscheme of $\mathbb{P}^n$, and that $j \geq \max\{\tau(\mathfrak{Z}) + \alpha(\mathfrak{Z}), 2\tau(\mathfrak{Z}) + 1\}$. If F is a general enough element of ${(I_{\mathfrak{Z}})_j}^\perp$, then F has a unique GAD of length s associated to $\mathfrak{Z}$, and no GAD's of length less than s.*

Proof. Let $t = \lfloor j/2 \rfloor$. By Theorem 3.3 the hypotheses on F and $\mathfrak{Z}$ imply that $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$; furthermore, the assumption on j implies that $(H_F)_t = (H_F)_{t+1} = s$, so $H_F \supset (s, s)$. It follows that any scheme $\mathfrak{Z}'$ of degree at most s, such that $I_{\mathfrak{Z}'} \subset I_F$, satisfies $(H_{\mathfrak{Z}'})_t = (H_{\mathfrak{Z}'})_{t+1} = s$, hence $(I_{\mathfrak{Z}'})_t = (I_F)_t$ and $(I_{\mathfrak{Z}'})_{t+1} = (I_F)_{t+1}$. By Theorem 1.12 this equality implies that $\mathfrak{Z}'$ is regular in degree $t+1$ (so $\sigma(\mathfrak{Z}') \leq t+1$), and is determined by F, so we must have $\mathfrak{Z} = \mathfrak{Z}'$. Uniqueness of the GAD now follows from Lemma 3.18. □

Example 3.21. *Let $R = K[x, y, z]$ and denote by Υ the degree 3 scheme $\Upsilon = \mathrm{Proj}\ (R/(x, y^3))$ concentrated at the origin $p_0 = (0 : 0 : 1)$ of $\mathbb{P}^2$; and denote by $\mathfrak{Z} = \mathfrak{Z}_1 \cup \ldots \cup \mathfrak{Z}_k$ the union of k distinct subschemes, where $\mathfrak{Z}_i$ denotes a translation of Υ to a point $p_i = (a_{i0} : a_{i1} : 1) \in \mathbb{P}^2$ (by T_{p_i} as in Lemma 2.22). By Theorem 2.24, we have that $\mathfrak{Z}_i = \mathrm{Proj}\ (R/(x - a_{i0}z, (y - a_{i1}z)^3))$; and since the inverse system $L(\Upsilon) \subset \Gamma = K_{DP}[X, Y, Z]$ satisfies $L(\Upsilon)_u = (I_\Upsilon)^\perp = R_u \circ (Y^{[2]} \cdot Z^u) = \langle Y^{[2]}Z^{u-2}, YZ^{u-1}, Z^u\rangle$, we have*

$$\begin{aligned} L(\mathfrak{Z}_i)_u &= R \circ (Y^{[2]} \cdot (a_{i0}X + a_{i1}Y + Z)^{[u]}) \\ &= \langle Y^{[2]} \cdot (a_{i0}X + a_{i1}Y + Z)^{[u-2]}, Y \cdot (a_{i0}X + a_{i1}Y + Z)^{[u-1]}, (a_{i0}X + a_{i1}Y + Z)^{[u]}\rangle. \end{aligned}$$

Taking $k = 2$, letting $p_1 = p_0, p_2 = (1 : 1 : 1)$, we have

$$\mathcal{I}_{\mathfrak{Z}} = (x, y^3) \cap (x - z, (y - z)^3) = (x^2 - xz, 3xy^2 - y^3 - 3xyz + xz^2),$$

and $\Delta H_{\mathfrak{Z}} = (1, 2, 2, 1), H_{\mathfrak{Z}} = (1, 3, 5, 6, 6, \ldots)$. By Lemma 3.18, a form $F \in (I_{\mathfrak{Z}})_j^\perp$ for $j \geq 3$ has a unique decomposition associated to $\mathfrak{Z}$, into $k = 2$ parts, each of length 3,

$$\begin{aligned} F = F_1 + F_2\ \ |F_1 &\in \langle Y^{[2]}Z^{j-2}, YZ^{j-1}, Z^j\rangle, \\ F_2 &\in \langle Y^{[2]} \cdot (X + Y + Z)^{[j-2]}, Y(X + Y + Z)^{[j-1]}, (X + Y + Z)^{[j]}\rangle. \end{aligned}$$

When $j = 3$, the form $F = 3Y^{[3]} + XY^{[2]} \in L_{\mathfrak{Z}}$, as it is evidently annihilated by $\mathcal{I}_{\mathfrak{Z}}$ acting as contraction. Thus F has a GAD into 2 parts, each of length 3,

$$F = Y^{[2]} \cdot (X + Y + Z) - Y^{[2]}Z. \tag{3.6}$$

By Theorem 3.20, since when $k = 2$, $\mathfrak{Z}$ is Gorenstein with $\tau(\mathfrak{Z}) = 3$ and $\alpha(\mathfrak{Z}) = 2$, we have for $j \geq 7$ that a general $F \in (L_{\mathfrak{Z}})_j$ has tight annihilating scheme $\mathfrak{Z}$, so a unique GAD of length 6. However, if $j \geq 6$, and$F$ includes $Y^{[2]}(X + Y + Z)^{[j-2]}$ and $Y^{[2]}(Z^{[j-2]}$ terms, then it is easily seen that F determines $\mathfrak{Z}$, as $I_{\mathfrak{Z}}$ is generated in degree 3, and $H_F = \mathrm{Sym}(H_{\mathfrak{Z}}, j)$ by calculation.

Taking $k = 3$, using translates of Υ at the three points p_1, p_2, and $p_3 = (2, 3, 1)$ we find $\Delta H_{\mathfrak{Z}} = (1, 2, 3, 3)$; taking $k = 4$ and points p_1, p_2, p_3, and $p_4 = (7, 11, 1)$ we find $\Delta H_{\mathfrak{Z}} = (1, 2, 3, 4, 2)$. (However, if we take instead $p_4' = (2, 5, 1)$ we find $\Delta H_{\mathfrak{Z}'} = (1, 2, 3, 3, 2, 1)$.) We might ask, for a generic choice of k points $\{p_i\}$, do we obtain a degree $3k$ scheme $\mathfrak{Z}$ in "general position" - having the same Hilbert function as $3k$ generic smooth points? This is not the case for $k = 2$ here, but is for $k = 3, 4$, and presumably for higher k.

Also, we may ask, what is the dimension of the family $\mathcal{F}(\Upsilon, k, \mathbf{P}^2)$ of all degree $3k$ punctual subschemes of $\mathbb{P}^2$ having the form $\mathfrak{Z} = \mathfrak{Z}_1 \cup \cdots \cup \mathfrak{Z}_k$, with $\mathfrak{Z}_i$ a translate of Υ? In this direction, the tangent space to such families have been studied classically for power sum representations $F = \sum L_{p_i}^j$ (see [Ter2, Bro, AlH, I3], [IK, §2.1,2.2], and for GAD's see [Eh, Tes], also [Cha2]).

Acknowledgements

We gratefully acknowledge suggestions of Tony Geramita and Ruth Michler, of V. Kanev concerning the GAD viewpoint, a discussion with Yuri Berest on projective closure, and with Peter Schenzel concerning the connections with dualizing modules; and as well the comments of Bae-Eun Jung. In addition, the second author gratefully acknowledges the influence of many

discussions begun some years ago with Jacques Emsalem as he wrote [Em], concerning inverse systems, in particular our discussions of the exponentials introduced in Section 3.

The first author was supported in part by Grant No. 98-0701-01-5-L from the Korea Science and Engineering Foundation and 1998 Basic Science Research Institute Program, BSRI-98-1414 from Ministry of Education.

References

[Ab] Abrams L.: *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications 5 (1996), no. 5, 569 – 587.

[AlH] Alexander J., Hirschowitz A.: *Polynomial interpolation in several variables*, J. Algebraic Geometry 4 (1995), 201–222.

[BSE] Bayer D., Stillman M., Eisenbud, D.: *Macaulay, A system for computation in algebraic geometry and commutative algebra*, 1982-1990, Source and object code available for Unix and Macintosh computers, from http://www.math.columbia.edu/ bayer/Macaulay.html.

[BeI] Bernstein D., Iarrobino A.: *A nonunimodal graded Gorenstein Artin algebra in codimension five*, Comm. in Algebra **20** # 8 (1992), 2323–2336.

[Bo1] Boij M.: *Graded Gorenstein Artin algebras whose Hilbert functions have a large number of valleys*, Comm. Algebra 23(1), 97–103 (1995).

[Bo2] ______: *Gorenstein Artin algebras and points in projective space*, Bull. London Math. Soc. **31** (1999), no. 1, 11–16.

[Bo3] ______: *Components of the space parametrizing graded Gorenstein Artin algebras with a given Hilbert function*, Pacific J. Math. **187** (1999), No. 1, 1–11.

[BoL] ______,Laksov, D.: *Nonunimodality of graded Gorenstein Artin algebras*, Proc. A.M.S. **120** (1994), 1083–1092.

[BS] Brodman M., Sharp R.: *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics,**60**, Cambridge University Press, Cambridge, 1998. xvi+416 pp.

[Bro] Bronowski J.: *The sums of powers as canonical expression*, Proc. Cambridge Philos. Soc. **29** (1933), 69–82.

[BH] Bruns W. , Herzog J.: *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics # 39, Cambridge University Press, Cambridge, U.K., 1993; revised paperback edition, 1998.

[Cha1] Chandler K.: *Higher infinitesimal neighborhoods*, J. Algebra **205** (1998), 460–479.

[Cha2] ______: *A brief proof of the maximal rank theorem for generic double points in projective space*, preprint, 15p. 1998.

[CharP] Chardin M., Philippon P.: *Regularité et interpolation*, J. Alg. Geometry **8** (1999), 471–481.

[ChoI1] Cho Y., Iarrobino A.: *Gorenstein Artin algebras arising from punctual schemes in projective space*, preprint (2000).

[ChoI2] ______,______: *Hilbert functions and level algebras*, preprint (2000).

[CR] Curtis C., Reiner I.: *Representation theory of finite groups and associative algebras*, J. Wiley, London, 1962.

[Eh] Ehrenborg R.:*On apolarity and generic canonical forms*, J. Algebra 213 (1999), no. 1, 167–194.

[EhR] Ehrenborg R., Rota G.-C.: *Apolarity and canonical forms for homogeneous polynomials*, European J. of Combinatorics **14** (1993), 157–182.

[Ei] Eisenbud D.: *Commutative algebra with a view toward algebraic geometry.* Graduate Texts in Math. # 150, Springer-Verlag, Berlin and New York, 1994.

[Em] Emsalem J.: *Géomètrie des Points Epais*, Bull. Soc. Math. France **106** (1978), 399–416.

[EmI1] ______, Iarrobino A.: *Some zero-dimensional generic singularities; finite algebras having small tangent space*, Compositio Math. **36** (1978), 145–188.

[EmI2] ______,______: *Inverse system of a symbolic power I*, J. Algebra. **174** (1995), 1080–1090.

[FL] ______, Laksov D.: *Compressed algebras*, Conf. on Complete Intersections in Acireale, (S.Greco and R. Strano, eds), Lecture Notes in Math. # 1092, Springer-Verlag, Berlin and New York, 1984, pp. 121–151.

[Ge] Geramita A. V.: *Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals*, The Curves Seminar at Queen's, Vol X, (Kingston, 1995), Queen's Papers in Pure and Appl. Math., **102** (1996), 1–114.

[GeM] ______, Maroscia P.: *The ideal of forms vanishing at a finite set of points in* $\mathbb{P}^n$, J. Algebra, **90** (1984), 528–555.

[Ha] Haiman M.: *Conjectures on the quotient ring by diagonal invariants*, J. Algebraic Combin. 3 (1994), no. 1, 17–76.

[I1] ______: *Compressed algebras: Artin algebras having given socle degrees and maximal length*, Trans. AMS **285** (1984), 337–378.

[I2] ______: *Associated graded algebra of a Gorenstein Artin algebra*, Mem. Amer. Math. Soc. **107** (1994) # 514, viii+115p.

[I3] ______: *Inverse system of a symbolic power II: The Waring problem for forms*, J. Algebra **174** (1995), 1091–1110.

[I4] ______: *Inverse system of a symbolic power III: thin algebras and fat points*, Compositio Math. **108** (1997), 319–356.

[IK] ______, Kanev V.: *Power Sums, Gorenstein Algebras, and Determinantal Loci*, 345+xxvii p. (1999) Springer Lecture Notes in Mathematics # 1721, Springer, Heidelberg.

[IKl] ______, Kleiman S.: *The Gotzmann theorems and the Hilbert scheme*, Appendix C., p. 289–312, in A. Iarrobino and V. Kanev, Power Sums, Gorenstein Algebras, and Determinantal Loci, (1999), Springer Lecture Notes in Mathematics # 1721, Springer, Heidelberg.

[Kr1] Kreuzer M.: *On 0-dimensional complete intersections*, Mathematische Annalen **292** (1992), 43–58.

[Kr2] ______: *Some applications of the canonical module of a zero- dimensional scheme*, Zero-dimensional Schemes, F. Orecchia and L. Chiantini, 1994, Walter de Gruyter, Berlin, pp. 243–252.

[L-J] Lejeune-Jalabert M.: *Liaison et résidu*, in Algebraic geometry (La Rábida, 1981), 233–240, Lecture Notes in Math., 961, Springer, Berlin-New York, 1982.

[Mac1] Macaulay F.H.S.: *The Algebraic Theory of Modular Systems*, Cambridge Univ. Press, Cambridge, U. K., 1916; reprinted with a foreword by P. Roberts, Cambridge Univ. Press, London and New York, 1994.

[Mar] Maroscia P.: *Some problems and results on finite sets of points in* $\mathbb{P}^n$, Open Problems in Algebraic Geometry, VIII, Proc. Conf. at Ravello, (C. Cilberto, F. Ghione, and F. Orecchia, eds.) Lecture Notes in Math. # 997, Springer-Verlag, Berlin and New York, 1983, pp. 290–314.

[Mi] Michler R.: *The dual of the torsion module of differentials*, preprint, 1999.

[Mig] Migliore J.: *Introduction to Liaison Theory and Deficiency Modules*, Progress in Math. # 165, Birkhaäuser, Boston, 1998, 224 p.

[No] Northcott D. G.: *Injective envelopes and inverse polynomials.*, J. London Math. Soc. (2), **8** (1974), 290–296.

[NR] ______, Rees D.: *Principal systems*, Quart. J. Math. Oxford (2), **8** (1957), 119–127.

[Or] Orecchia F.: *One-dimensional local rings with reduced associated graded rings, and their Hilbert functions*, Manuscripta Math. **32** (1980), no. 3-4, 391–405.

[Pup] Puppe V.: *Simply connected manifolds without* S^1 *symmetry*, Algebraic Topology and Transformation Groups (Gottingen, 1987), Lecture Notes in Math. # 1361, Springer-Verlag, Berlin and New York (1988) pp. 261–268.

[Rez] Reznick B.: *Homogeneous polynomial solutions to constant coefficient PDE's*, Adv. Math. **117** (1996), no. 2, 179–192.

[Schk] Schenck H.: *Stability and Jump Loci for a family of bundles on* $\mathbb{P}^2$, preprint (1999), 9p.

[Schz] Schenzel P.: *Local cohomology and duality*, Six lectures on commutative algebra (Bellaterra, 1996), 241–292, Progr. Math., 166, Birkhauser, Basel, 1998

[Ter1] Terracini A.: *Sulle* V_k *per cui la variet'a degli* $S_h(h+1)$*-seganti ha dimensione minore dell'ordinario*, Rend. Circ. Mat. Palermo **31** (1911), 527–530.

[Ter2] ______: *Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari*, Annali di Mat. Pura Appl. Serie III, **24** (1915), 1–10.

[Tes] Teschke O.: *Moduli of rank two bundles with Chern classes* $(0, 2)$ *on quartic double solids*, preprint, 22 p. (1999).

[Wa] Watanabe J.: *The Dilworth number of Artin Gorenstein rings*, Advances in Math. **76** (1989), 194–199.

[ZarS] Zariski O. and Samuel P.: *Commutative Algebra, Vol II.*, D. Van Nostrand Co, Princeton (1960).