CALABI-YAU PROBLEM FOR LEGENDRIAN CURVES IN $\mathbb{C}^3$ AND APPLICATIONS

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Abstract. We construct a complete, bounded Legendrian immersion in $\mathbb{C}^3$. As direct applications of it, we show the first examples of a weakly complete bounded flat front in hyperbolic 3-space, a weakly complete bounded flat front in de Sitter 3-space, and a weakly complete bounded improper affine front in $\mathbb{R}^3$.

1. Introduction

In a series of previous papers, the authors have constructed the first examples of complete bounded null holomorphic immersion

$$\nu : \mathbb{D}_1 \longrightarrow \mathbb{C}^3$$

of the unit disc $\mathbb{D}_1 \subset \mathbb{C}$, where *null* means that $\nu_z \cdot \nu_z$ vanishes identically, here $\nu_z := d\nu/dz$ is the derivative of ν with respect to the complex coordinate z of $\mathbb{D}_1$ and the dot denotes the canonical complex bilinear form. The existence of such an immersion has important consequences. Actually, as a short and direct application of the main result in [14], by using different kinds of transformations, the following objects were constructed:

(1) complete bounded minimal surfaces in the Euclidean 3-space $\mathbb{R}^3$ ([14, Theorem A]),
(2) complete bounded holomorphic curves in $\mathbb{C}^2$ ([14, Corollary B]).
(3) weakly complete bounded maximal surfaces in the Lorentz-Minkowski 3-space $\mathbb{R}^3_1$ ([14, Corollary D]),
(4) complete bounded constant mean curvature one surfaces in the hyperbolic 3-space H^3 ([14, Theorem C]).

Moreover, we constructed higher genus examples of the first three objects in [15]. Recently, Alarcón and López [1] have constructed a complete bounded null proper holomorphic immersion of a given Riemann surface of an arbitrary topology into a convex domain in $\mathbb{C}^3$ (see also [2]). Their method is different from ours.

It is known that null curves in $\mathbb{C}^3$ are closely related to Legendrian curves in $\mathbb{C}^3$ (cf. Bryant [4] and also Ejiri-Takahashi [5] for the corresponding $\mathrm{SL}(2,\mathbb{C})$-case). In this paper, we use the techniques develop by the authors in [14] to produce a

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complete bounded Legendrian holomorphic immersion

$$F : \mathbb{D}_1 \longrightarrow \mathbb{C}^3.$$

Recall that F is called *Legendrian* if the pull-back of the canonical contact form

$$\Omega_{\mathbb{C}} := dx_3 + x_2 dx_1 \tag{1.1}$$

by F vanishes, where (x_1, x_2, x_3) is the canonical complex coordinate system of $\mathbb{C}^3$. The existence of such an F is non-trivial, since the correspondences between null curves and Legendrian curves given in [4] and [5] seem not to preserve neither boundedness nor completeness. Also, the authors do not know whether the method in [1] can be applied for Legendrian holomorphic immersions by using a suitable modification or not.

As applications, we are able to construct the following new examples:

(1) a weakly complete bounded flat front in H^3 (Theorem 4.1),
(2) a weakly complete bounded flat front in the de Sitter 3-space S^3_1 (Theorem 4.2),
(3) a weakly complete bounded improper affine front in $\mathbb{R}^3$ (Theorem 4.3).

It should be remarked that there are no compact flat fronts in H^3 and S^3_1 (resp. improper affine fronts in $\mathbb{R}^3$). See Remark 4.4. A holomorphic map $E\colon \mathbb{D}_1 \to \mathrm{SL}(2,\mathbb{C})$ is called *Legendrian* if the pull-back $E^*\Omega_{\mathrm{SL}}$ vanishes on $\mathbb{D}_1$, where Ω_{SL} is the complex contact form on $\mathrm{SL}(2,\mathbb{C})$ defined as

$$\Omega_{\mathrm{SL}} := x_{11} dx_{22} - x_{12} dx_{21}. \tag{1.2}$$

Here, elements in $\mathrm{SL}(2,\mathbb{C})$ are represented by matrices $(x_{ij})_{i,j=1,2}$. A holomorphic immersion $E\colon \mathbb{D}_1 \to \mathrm{SL}(2,\mathbb{C})$ is said to be *complete* if the pull-back metric $E^* g_{\mathrm{SL}}$ of the canonical Hermitian metric g_{SL} on $\mathrm{SL}(2,\mathbb{C})$ is complete, see (2.2).

To construct the bounded holomorphic immersion $F : \mathbb{D}_1 \to \mathbb{C}^3$, we show the following

Main Theorem. *There exists a complete holomorphic Legendrian immersion of the unit disk* $\mathbb{D}_1 \subset \mathbb{C}$ *into* $\mathrm{SL}(2,\mathbb{C})$ *such that its image is contained an arbitrary bounded domain in* $\mathrm{SL}(2,\mathbb{C})$.

By Darboux's theorem, the contact structure of $\mathrm{SL}(2,\mathbb{C})$ is locally Legendrian equivalent to that of $\mathbb{C}^3$. Moreover, the following explicit transformation

$$F\colon \mathbb{C}^3 \ni (x,y,z) \longmapsto \begin{pmatrix} e^{-z} & xe^{-z} \\ ye^{z} & e^{z}(1+xy) \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

maps holomorphic contact curves in $\mathbb{C}^3$ to those in $\mathrm{SL}(2,\mathbb{C})$. Then if one take a complete Legendrian immersion of $\mathbb{D}_1$ into $\mathrm{SL}(2,\mathbb{C})$ with sufficiently small image in $\mathrm{SL}(2,\mathbb{C})$, a Legendrian immersion into $\mathbb{C}^3$ is obtained. Completeness follows from the same argument as [14, Lemma 3.1]. In fact, since the image is bounded, the metrics induced from $\mathrm{SL}(2,\mathbb{C})$ and $\mathbb{C}^3$ are equivalent.

The paper is organized as follows: In Section 2, we establish our formulations and state the key-lemma to prove the main theorem, which is proved in Section 3. In Section 4, we give the applications as above. In the appendix, we prepare a Runge-type theorem for Legendrian curves in $\mathrm{SL}(2,\mathbb{C})$ which is needed in Section 3.

Finally, we mention the corresponding real problem, that is, the existence of complete bounded Legendrian submanifolds immersed in $\mathbb{R}^{2n+1}$. When $n = 1$, there exists a closed Legendrian curve immersed in an arbitrarily given open subset in $\mathbb{R}^3$: In fact, in [9, Section 2], it is shown the existence of a Legendrian curve contained in an arbitrary given open ball of $P^3 = T_1S^2$ (i.e. the unit cotangent bundle of 2-sphere) as a lift of an eye-figure curve. Since any contact structure is locally rigid, it gives an existence of a closed Legendrian curve immersed in any ball of $\mathbb{R}^3$. Also, as an application of our construction, we can construct a complete bounded Legendrian immersion $L : \mathbb{D}_1 \to B(\subset \mathbb{R}^5)$: There exists a canonical projection (cf. [11, Page 159])

$$\pi : \mathrm{SL}(2, \mathbb{C}) \longrightarrow T_1^* H^3,$$

where $T_1^*H^3$ is a unit cotangent bundle of the hyperbolic 3-space H^3. Then the projection of our complete bounded holomorphic Legendrian curve gives a complete bounded Legendrian submanifold immersed in an arbitrarily given open subset of $T_1^*H^3$. By Darboux's rigidity theorem, this implies the existence of a complete Legendrian immersion $L : \mathbb{D}_1 \to B$, where B is an arbitrary ball in $\mathbb{R}^5$.

2. The Main Lemma

In this section, we state the main lemma, which is an analogue of [14, Main Lemma in page 121]. The main theorem in the introduction can be obtained as a direct conclusion of the main lemma in the same way as in [14].

2.1. **Preliminaries.**

We denote $\mathrm{i} = \sqrt{-1}$ and

$$\mathbb{D}_r := \{z \in \mathbb{C}\,;\, |z| < r\}, \qquad \overline{\mathbb{D}}_r := \mathbb{D}_r := \{z \in \mathbb{C}\,;\, |z| \le r\}$$

for a positive number r. Throughout this paper, the prime $'$ means the derivative with respect to the complex coordinate z on $\mathbb{C}$.

Proposition 2.1. *A holomorphic immersion $X \colon \overline{\mathbb{D}}_1 \to \mathrm{SL}(2, \mathbb{C})$ is Legendrian if and only if $X^{-1}X'$ is anti-diagonal;*

$$\psi_X\, dz := X^{-1}dX = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \varphi_1 + \mathrm{i}\varphi_2 \\ \varphi_1 - \mathrm{i}\varphi_2 & 0 \end{pmatrix} dz, \tag{2.1}$$

where φ_1 and φ_2 are holomorphic functions on $\overline{\mathbb{D}}_1$. The metric induced by X from the canonical Hermitian metric g_{SL} of $\mathrm{SL}(2, \mathbb{C})$ is represented as

$$ds_X^2 := |\omega|^2 + |\theta|^2 = \big(|\varphi_1|^2 + |\varphi_2|^2\big)|dz|^2. \tag{2.2}$$

In particular, φ_1 and φ_2 have no common zeros on $\overline{\mathbb{D}}_r$.

The holomorphic 1-forms ω and θ in (2.1) are called the *canonical one forms* for the flat front corresponding to X, see [12].

Definition 2.2. A pair of holomorphic functions $\varphi = (\varphi_1, \varphi_2)$ on $\overline{\mathbb{D}}_1$ is called *non-degenerate* if φ_1 and φ_2 have no common zeroes. The pair (φ_1, φ_2) given by (2.1) is called the *holomorphic data* of X. The matrix valued function

$$M_\varphi := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \varphi_1 + \mathrm{i}\varphi_2 \\ \varphi_1 - \mathrm{i}\varphi_2 & 0 \end{pmatrix} \tag{2.3}$$

is called *the matrix form* of the pair φ.

2.2. **The Main Lemma.** To state the lemma, we define the matrix norm $|A|$ of a 2×2-matrix A as

$$|A| := \sqrt{\mathrm{trace}(AA^*)} = \sqrt{\sum_{i,j=1,2} |A_{ij}|^2} \qquad \bigl(A = (A_{ij})_{i,j=1,2}\bigr). \tag{2.4}$$

Note that if $A \in \mathrm{SL}(2,\mathbb{C})$, then $|A| \geq \sqrt{2}$ holds. The equality holds if and only if A is the identity matrix.

For a vector $\boldsymbol{v} = (v_1, v_2) \in \mathbb{C}^2$, we set $|\boldsymbol{v}| = \sqrt{|v_1|^2 + |v_2|^2}$.

Main Lemma. *Let $X\colon \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C})$ be a holomorphic Legendrian immersion $X\colon \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C})$ satisfies the following properties*:

1. $X(0) = \mathrm{id}$, *where* id *is the identity matrix.*
2. $\overline{\mathbb{D}}_1$ *contains the geodesic disc of radius ρ centered at the origin with respect to the induced metric ds^2_X.*
3. *There exists a number $\tau > \sqrt{2}$ such that $|X| \leq \tau$ holds on $\overline{\mathbb{D}}_1$.*

Then, for any positive numbers ε and s, there exists a holomorphic Legendrian immersion $Y\colon \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C})$ such that

(i) $Y(0) = \mathrm{id}$,
(ii) $\overline{\mathbb{D}}_1$ *contains the geodesic disc of radius $\rho + s$ centered at the origin with respect to the induced metric ds^2_Y,*
(iii) $|Y| \leq \tau\sqrt{1 + 32s^2 + \varepsilon}$ *in* $\overline{\mathbb{D}}_1$,
(iv) $|Y - X| < \varepsilon$ *and* $|\varphi_Y - \varphi_X| < \varepsilon$ *in* $\mathbb{D}_{1-\varepsilon}$, *where φ_X and φ_Y denote holomorphic data of X and Y, respectively.*

The main theorem in the introduction is obtained by the same argument as [14, Section 3.4]).

2.3. **Key Lemma.** Now we state the key lemma, as an analogue of [14, Key Lemma in page 129]. The main lemma in the previous subsection can be obtained directly form the key lemma.

We work on the Nadirashvili's labyrinth [17]. Let us give a brief description of this labyrinth: Let N be a (sufficiently large) positive number. For $k = 0, 1, 2, \dots, 2N^2$, we set

$$r_k = 1 - \frac{k}{N^3} \qquad \left(r_0 = 1, r_1 = 1 - \frac{1}{N^3}, \dots, r_{2N^2} = 1 - \frac{2}{N}\right), \tag{2.5}$$

and let

$$\mathbb{D}_{r_k} = \{z \in \mathbb{C}\,;\, |z| < r_k\} \quad \text{and} \qquad S_{r_k} = \partial\mathbb{D}_{r_k} = \{z \in \mathbb{C}\,;\, |z| = r_k\}. \tag{2.6}$$

We define an annular domain $\mathcal{A}$ as

$$\mathcal{A} := \mathbb{D}_1 \setminus \mathbb{D}_{r_{2N^2}} = \mathbb{D}_1 \setminus \mathbb{D}_{1-\frac{2}{N}}, \tag{2.7}$$

and

$$A := \bigcup_{k=0}^{N^2-1} \mathbb{D}_{r_{2k}} \setminus \mathbb{D}_{r_{2k+1}}, \qquad \widetilde{A} := \bigcup_{k=0}^{N^2-1} \mathbb{D}_{r_{2k+1}} \setminus \mathbb{D}_{r_{2k+2}},$$

$$L = \bigcup_{k=0}^{N-1} l_{\frac{2k\pi}{N}}, \qquad \widetilde{L} = \bigcup_{k=0}^{N-1} l_{\frac{(2k+1)\pi}{N}},$$

where l_θ is the ray $l_\theta = \{re^{\mathrm{i}\theta}\,;\, r \geq 0\}$. Let Σ be a compact set defined as

$$\Sigma := L \cup \widetilde{L} \cup S, \qquad S = \bigcup_{j=0}^{2N^2} \partial \mathbb{D}_{r_j} = \bigcup_{j=0}^{2N^2} S_{r_j},$$

and define a compact set Ω by

$$\Omega = \mathcal{A} \setminus U_{1/(4N^3)}(\Sigma),$$

where $U_\varepsilon(\Sigma)$ denotes the ε-neighborhood (of the Euclidean plane $\mathbb{R}^2 = \mathbb{C}$) of Σ. Each connected component of Ω has width $1/(2N^3)$. For each number $j = 1, \ldots, 2N$, we set

$$\omega_j := \big(l_{\frac{j\pi}{N}} \cap \mathcal{A}\big) \cup \big(\text{ connected components of } \Omega \text{ which intersect with } l_{\frac{j\pi}{N}}\big),$$
$$\varpi_j := U_{1/(4N^3)}(\omega_j).$$

Then ω_j's are compact sets.

Key Lemma. *Assume that a holomorphic Legendrian immersion $\mathcal{L} = \mathcal{L}_0 \colon \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C})$ satisfies:*

(A-1) $\mathcal{L}(0) = \mathrm{id}$,

(A-2) $\overline{\mathbb{D}}_1$ *contains the geodesic disc of radius ρ centered at the origin with respect to the metric $ds^2_{\mathcal{L}}$.*

Then for any positive number ε and positive number $s \in (0, 1/3)$, there exists a sufficiently large integer N and a sequence of holomorphic Legendrian immersions $\mathcal{L}_0 = \mathcal{L}$, $\mathcal{L}_1$, $\ldots$, $\mathcal{L}_{2N}$ of $\overline{\mathbb{D}}_1$ such that

(C-1) $\mathcal{L}_j(0) = \mathrm{id}$ $(j = 0, \ldots, 2N)$,

(C-2) *for each $j = 1, \ldots, 2N$, $|\varphi_j - \varphi_{j-1}| < \varepsilon/(2N^2)$ holds on $\mathbb{D}_1 \setminus \varpi_j$, where φ_j is the non-degenerate holomorphic data of $\mathcal{L}_j$,*

(C-3) *for each $j = 1, \ldots, 2N$,*

$$|\varphi_j| \geq \begin{cases} cN^{3.5} & \text{on } \omega_j \\ cN^{-0.5} & \text{on } \varpi_j \end{cases}$$

holds, where c is a positive constant depending only on $\mathcal{L} = \mathcal{L}_0$,

(C-4) $\overline{\mathbb{D}}_1$ *contains the geodesic disc of radius $\rho + s$ centered at the origin with respect to the metric $ds^2_{\mathcal{L}_{2N}}$,*

(C-5) *on $\overline{\mathcal{D}}_g$ as in* (C-4), *it holds that*

$$|\mathcal{L}_{2N}| \leq \left(\max_{\overline{\mathbb{D}}_1} |\mathcal{L}_0|\right) \sqrt{1 + 32 s^2 + (b/\sqrt{N})},$$

where b is a positive constant depending only on $\mathcal{L} = \mathcal{L}_0$.

The proof is given in Section 3.

3. Proof of the Key Lemma

3.1. **Flat fronts in hyperbolic 3-space.** We denote by H^3 the hyperbolic 3-space, that is, the connected and simply connected 3-dimensional space form of constant sectional curvature -1, which is represented as

$$\text{(3.1)} \qquad H^3 = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2) = \{aa^*\,;\, a \in \mathrm{SL}(2,\mathbb{C})\}$$
$$= \{X \in \mathrm{Herm}(2)\,;\, \det X = 1, \mathrm{trace}\, X > 0\}, \qquad (a^* = {}^t\bar{a}).$$

where Herm(2) is the set of 2×2 Hermitian matrices. Identifying Herm(2) with the Lorentz-Minkowski 4-space $\mathbb{R}^4_1$ as

$$\text{Herm}(2) \ni \begin{pmatrix} x_0 + x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} \quad \longleftrightarrow \quad (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1, \tag{3.2}$$

the hyperbolic space H^3 can be considered as the connected component of the two-sheeted hyperboloid

$$\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1\,;\, -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2, x_0 > 0\}.$$

A Legendrian immersion $\mathcal{L}\colon \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C})$ induces a flat front

$$l = \mathcal{L}\mathcal{L}^*\colon \overline{\mathbb{D}}_1 \longrightarrow H^3.$$

Here flat fronts in H^3 are flat surfaces with certain kind of singularities, see Section 4.1. The pull-back of the metric of H^3 by l is computed as

$$ds_l^2 := |\omega|^2 + |\theta|^2 + \omega\theta + \bar{\omega}\bar{\theta} = |\omega + \bar{\theta}|^2 \qquad \left(\mathcal{L}^{-1}d\mathcal{L} = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}\right), \tag{3.3}$$

which is positive semi-definite and may degenerate. On the other hand, let $ds^2_{\mathcal{L}}$ be the pull-back of the canonical Hermitian metric of $\mathrm{SL}(2,\mathbb{C})$ by $\mathcal{L}$. Then by (2.2), we have

$$ds^2_{\mathcal{L}} - \frac{1}{2}ds_l^2 = \frac{1}{2}\bigl(|\omega|^2 + |\theta|^2 - \omega\theta - \bar{\omega}\bar{\theta}\bigr) = \frac{1}{2}|\omega - \bar{\theta}|^2 \geq 0,$$

and hence

$$ds_l^2 \leq 2ds^2_{\mathcal{L}} \tag{3.4}$$

holds. For any path γ in $\overline{\mathbb{D}}_1$ joining x and $y \in \mathbb{D}_1$, it holds that

$$\mathrm{Length}_{ds_l^2}\,\gamma := \int_\gamma ds_l \geq \mathrm{dist}_{H^3}\bigl(l(x), l(y)\bigr), \tag{3.5}$$

where dist_{H^3} denotes the distance in the hyperbolic 3-space. On the other hand,

$$2\cosh \mathrm{dist}_{H^3}\bigl(o, l(x)\bigr) = |\mathcal{L}(x)|^2 \tag{3.6}$$

holds (see [14, Lemma A.2]), where we set

$$o := \mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

namely o is the point on H^3 which corresponds to the origin of the Poincaré ball model.

3.2. **Inductive construction of $\mathcal{L}_j$'s.** In this section, we describe the recipe to construct a sequence $\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_{2N}$ in Key Lemma. Assume $\mathcal{L}_0, \dots, \mathcal{L}_{j-1}$ are already obtained, and we shall now construct $\mathcal{L}_j$ as follows: Let

$$\zeta_j := \left(1 - \frac{2}{N} - \frac{4}{N^3}\right)e^{\mathrm{i}\pi j/N} \tag{3.7}$$

be the base point of the compact set ω_j given in [14, Fig. 1]. We set

$$E_0(z) := \mathcal{L}_{j-1}(\zeta_j)^{-1}\,\mathcal{L}_{j-1}(z), \qquad f_0(z) := E_0(z)E_0^*(z).$$

That is, E_0 is the Legendrian immersion with the same holomorphic data as $\mathcal{L}_{j-1}$ such that $E_0(\zeta_j) = \mathrm{id}$, and f_0 the corresponding flat front. Here, if we write

$$f_0(0) = \begin{pmatrix} \xi_0 + \xi_3 & \xi_1 + \mathrm{i}\xi_2 \\ \xi_1 - \mathrm{i}\xi_2 & \xi_0 - \xi_3 \end{pmatrix} \in H^3, \tag{3.8}$$

then there exist real numbers t and $\hat{\xi}_1$ such that

$$(3.9)\quad af_0(0)a^* = \begin{pmatrix} \xi_0+\xi_3 & \hat{\xi}_1 \\ \hat{\xi}_1 & \xi_0-\xi_3 \end{pmatrix} \in H^3 \qquad \left(a = \begin{pmatrix} e^{\mathrm{i}t} & 0 \\ 0 & e^{-\mathrm{i}t} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \right).$$

We now set

$$(3.10)\qquad E := aE_0a^*, \qquad \psi := E^{-1}E'.$$

Then it holds that

$$\psi = a^*\psi_{j-1}a \qquad (\psi_{j-1} := \mathcal{L}_{j-1}^{-1}\mathcal{L}'_{j-1}).$$

Let $\varphi = (\varphi_1, \varphi_2)$ be the holomorphic data induced from E such that $\psi = M_\varphi$ (see (2.3)). Applying Lemma A.1 in the appendix to this φ, we get a new holomorphic data $\tilde{\varphi}$. Take $\tilde{E}$ such that

$$(3.11)\qquad \tilde{E}^{-1}\tilde{E}' = \tilde{\psi}, \qquad \tilde{E}(\zeta_j) = \mathrm{id} \qquad (\tilde{\psi} = M_{\tilde{\varphi}}).$$

Finally, we set

$$(3.12)\qquad \mathcal{L}_j := a^*\{\tilde{E}(0)\}^{-1}\tilde{E}a,$$

which gives the desired Legendrian immersion.

3.3. **Estimation of interior distance.** Applying the inductive constriction, we get a sequence of Legendrian immersion $\mathcal{L}_1, \cdots, \mathcal{L}_{2N}$. Then one can prove (C-1), (C-2), (C-3) and (C-4) by the exactly same argument as in [14, Page 129]. In fact, although the data φ (a pair of holomorphic functions) is different from that in [14] (a triple of holomorphic functions), the proof of the key lemma (as in [14, Page 129]) only needs the norm $|\varphi|$ of the data φ. In [14], we were working on the metric $ds^2_{\mathcal{B}_j} = |\psi_j|^2\, dz\, d\bar{z}$. In this paper, we now use the metric $ds^2_{\mathcal{L}_j} = |\psi_j|^2\, dz\, d\bar{z}$, where ψ_j is given in (3.10). If one replaces $|\psi_j|$ in [14] by this new $|\psi_j|$ as above, then the completely same argument as [14] works in our case.

3.4. **Extrinsic distance.** Thus, the only remaining assertion we should prove is (C-5) of Key Lemma. We shall now prove it: By (C-4), $(\overline{\mathbb{D}_1}, ds^2_{\mathcal{L}_{2N}})$ contains a geodesic disc $\overline{\mathcal{D}}_g$ centered at origin with radius $\rho + s$. By the maximum principle, it is sufficient to show that for each $p \in \partial\mathcal{D}_g$, it holds that

$$(3.13)\qquad |\mathcal{L}_{2N}(p)| \le \left(\max_{\mathbb{D}_1}|\mathcal{L}_0|\right)\sqrt{1 + 32s^2 + (b/\sqrt{N})} \qquad (p \in \partial\mathcal{D}_g),$$

where b is a positive constant depending only on the initial immersion $\mathcal{L}_0$.

If $p \in \partial\mathcal{D}_g$ is not in $\varpi_1 \cup \cdots \cup \varpi_{2N}$, the same argument as [14, Page 129] implies the conclusion. So, it is sufficient to consider the case that there exists $j \in \{1, ..., 2N\}$ such that

$$(3.14)\qquad p \in \partial\mathcal{D}_g \cap \varpi_j.$$

We fix such a p. *From now on, the symbols c_k $(k = 1, 2, \dots)$ denote suitable positive constants, which depend only on the initial data $\mathcal{L} = \mathcal{L}_0$.*

Like as in the proof of the inequality [14, (4.8)], we can apply [14, Corollary A.6] for $X := \mathcal{L}_j$ and $Y := \mathcal{L}_{j-1}$. Then we have

$$(3.15)\qquad \mathrm{dist}_{H^3}\big(l_j(z), l_{j-1}(z)\big) \le \frac{c_1\varepsilon}{2N^2} \qquad (\text{on } \mathbb{D}_1 \setminus \varpi_j),$$

where $l_j := \mathcal{L}_j\mathcal{L}_j^*$ and $l_{j-1} := \mathcal{L}_{j-1}\mathcal{L}_{j-1}^*$. Taking (3.15) into account, inequality (3.13) reduces to the following inequality

$$|\mathcal{L}_j(p)| \le \left(\max_{\overline{\mathbb{D}}_1} |\mathcal{L}_0|\right)\sqrt{1+32s^2+(c_2/\sqrt{N})} \qquad (p \in \partial\mathcal{D}_g \cap \varpi_j). \tag{3.16}$$

Take the $ds^2_{\mathcal{L}_{2N}}$-geodesic γ_0 joining p and the origin $0 \in \overline{\mathbb{D}}_1$ and denote by $\hat{p}$ the first point on γ_0 which meets $\partial\varpi_j$. Then we have

$$\operatorname{dist}_{ds^2_{\mathcal{L}_{2N}}}(\hat{p}, p) \le s + \frac{c_3}{\sqrt{N}}. \tag{3.17}$$

In fact, let γ_1 (resp. γ_2) be the subarc of γ_0 which joins 0 and $\hat{p}$ (resp. $\hat{p}$ and p). Since γ_0 is the geodesic and $\mathcal{D}_g$ is the disc with radius $\rho+s$, (C-2) and (C-3) yields that

$$\rho + s = \int_{\gamma_0} ds_{\mathcal{L}_{2N}} = \int_{\gamma_0} |\varphi_{2N}|\,|dz| \ge \frac{c_4}{\sqrt{N}} L_{\gamma_0},$$

where L_{γ_0} is the length of γ_0 with respect to the Euclidean metric of $\mathbb{C}$. Thus, we have $L_{\gamma_0} \le c_6\sqrt{N}(\rho+s)$. On the other hand, let σ be the shortest line segment on $\mathbb{C}$ joining $\hat{p}$ and $\partial\mathbb{D}_1$. Then the Euclidean length L_σ of σ satisfies $L_\sigma \le c_7/N$. Thus, by (A-2), we have

$$\begin{aligned}
\rho + s &= \int_{\gamma_0} |\varphi_{2N}|\,|dz| \ge \int_{\gamma_1} |\varphi_0|\,|dz| - \int_{\gamma_1} |\varphi_{2N}-\varphi_0|\,|dz| + \int_{\gamma_2} |\varphi_{2N}|\,|dz| \\
&\ge \int_{\gamma_1\cup\sigma} |\varphi_0|\,|dz| - \int_{\sigma} |\varphi_0|\,|dz| - L_{\gamma_0}\frac{\varepsilon}{N} + \operatorname{dist}_{ds^2\mathcal{L}_{2N}}(\hat{p}, p) \\
&\ge \rho - \frac{c_3}{\sqrt{N}} + \operatorname{dist}_{ds^2\mathcal{L}_{2N}}(\hat{p}, p)
\end{aligned}$$

which implies (3.17).

Moreover, since ζ_j and $\hat{p}$ can be joined by a path γ on $\overline{\mathbb{D}_1} \setminus (\varpi_1 \cup \cdots \cup \varpi_{2N})$ whose Euclidean length is not greater than c_8/N (see [14, Fig. 1]), we have

$$\operatorname{dist}_{ds^2_{\mathcal{L}_{2N}}}(\zeta_j, p) \le s + \frac{c_9}{\sqrt{N}} \tag{3.18}$$

(see [14, (4.9)]).

Lemma 3.1. *In the above setting, the following inequalities hold*:

$$|\mathcal{L}_{j-1}(\zeta_j)|^2 \le \left(\max_{z\in\overline{\mathbb{D}}_1} |\mathcal{L}_0(z)|^2\right)\left(1 + \frac{c_{10}}{N}\right), \tag{3.19}$$

$$\operatorname{dist}_{H^3}\big(l_{j-1}(\zeta_j), l_j(\zeta_j)\big) \le \frac{c_{11}}{N^2}. \tag{3.20}$$

Proof. The first inequality holds because

$$|\mathcal{L}_{j-1}(\zeta_j)|^2 = 2\cosh\big(\operatorname{dist}_{H^3}(o, l_{j-1}(\zeta_j)\big) \le 2\cosh\left(\operatorname{dist}_{H^3}\big(o, l_0(\zeta_j)\big) + \frac{c_{12}}{N}\right),$$

see (3.6). The second inequality directly follows from (3.15). □

Now, recall the procedure constructing $\mathcal{L}_j$ from $\mathcal{L}_{j-1}$: Two Legendrian immersions E, $\tilde{E}$ in (3.10) and (3.11) are congruent to $\mathcal{L}_{j-1}$, $\mathcal{L}_j$, respectively, and satisfy $E(\zeta_j) = \tilde{E}(\zeta_j) = \mathrm{id}$. Take flat fronts $f := EE^*$ and $\tilde{f} = \tilde{E}\tilde{E}^*$ associated to E and

$\tilde{E}$, respectively. By a choice of the matrix a in (3.9), the points $f(\zeta_j) = o(= \mathrm{id})$ and $f(0)$ lie on the "x_1x_3-plane" Π, i.e.

$$\Pi := \left\{ \begin{pmatrix} x_0 + x_3 & x_1 \\ x_1 & x_0 - x_3 \end{pmatrix} \in H^3 \,;\, x_0, x_1, x_3 \in \mathbb{R} \right\}.$$

Let Λ be the geodesic of H^3 passing through the origin o $(= \mathrm{id})$ and perpendicular to Π (i.e., the x_2-axis), and let q be the foot of the perpendicular from $\tilde{f}(p)$ to the line Λ.

Lemma 3.2. *In the above circumstances, one has*:

$$\mathrm{dist}_{H^3}\bigl(o, q\bigr) \le 2s + \frac{c_{13}}{\sqrt{N}}.$$

Proof. The triangle $\triangle oq\tilde{f}(p)$ is a right triangle such that the angle q is the right angle. Then by (3.20), (3.18) and (3.4), we have

$$\begin{aligned}
\mathrm{dist}_{H^3}\bigl(o, q\bigr) &\le \mathrm{dist}_{H^3}\bigl(o, \tilde{f}(p)\bigr) = \mathrm{dist}_{H^3}\bigl(l_{j-1}(\zeta_j), l_j(p)\bigr) \\
&\le \mathrm{dist}_{H^3}\bigl(l_{j-1}(\zeta_j), l_j(\zeta_j)\bigr) + \mathrm{dist}_{H^3}\bigl(l_j(\zeta_j), l_j(p)\bigr) \le \frac{c_{11}}{N} + \mathrm{dist}_{ds^2_{l_j}}(\zeta_j, p) \\
&\le \frac{c_{11}}{N} + 2\,\mathrm{dist}_{ds^2_{\mathcal{L}_j}}(\zeta_j, p) \le \frac{c_{14}}{N} + 2\,\mathrm{dist}_{ds^2_{\mathcal{L}_{2N}}}(\zeta_j, p) \\
&\le \frac{c_{15}}{N} + 2\left(s + \frac{c_8}{\sqrt{N}}\right) \le 2s + \frac{c_{13}}{\sqrt{N}}.
\end{aligned}$$

Thus we have the conclusion. □

Lemma 3.3. *Under the hypotheses above, one has*:

$$\mathrm{dist}_{H^3}\bigl(q, \tilde{f}(p)\bigr) \le 14s^2 + \frac{c_{10}}{\sqrt{N}}.$$

Proof. Let $\varphi = (\varphi_1, \varphi_2)$ and $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ be the holomorphic data of the Legendrian immersions E and $\tilde{E}$, respectively, that is,

$$\begin{aligned}
\psi &:= E^{-1}E' = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \varphi_1 + \mathrm{i}\varphi_2 \\ \varphi_1 - \mathrm{i}\varphi_2 & 0 \end{pmatrix} = M_{\varphi}, \\
\tilde{\psi} &:= \tilde{E}^{-1}\tilde{E} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \tilde{\varphi}_1 + \mathrm{i}\tilde{\varphi}_2 \\ \tilde{\varphi}_1 - \mathrm{i}\tilde{\varphi}_2 & 0 \end{pmatrix} = M_{\tilde{\varphi}}
\end{aligned}$$

hold. Set

$$F(z) := \int_{\zeta_j}^{z} \psi(z)\,dz, \qquad \text{and} \qquad \tilde{F}(z) := \int_{\zeta_j}^{z} \tilde{\psi}(z)\,dz.$$

We define two values Δ and $\tilde{\Delta}$ by

$$E(p) = \mathrm{id} + F(p) + \Delta, \qquad \tilde{E}(p) = \mathrm{id} + \tilde{F}(p) + \tilde{\Delta}.$$

Since $E(\zeta_j) = \tilde{E}(\zeta_j) = \mathrm{id}$, [14, Appendix A.4] yields that

$$|\Delta| \le \left[\bigl(\max_{\gamma} |E|\bigr) \int_{\gamma} |\psi|\,|dz|\right]^2, \quad |\tilde{\Delta}| \le \left[\bigl(\max_{\gamma} |\tilde{E}|\bigr) \int_{\gamma} |\tilde{\psi}|\,|dz|\right]^2.$$

Here γ is a path joining ζ_j and p as in [14, Fig. 1]. This argument is completely parallel to that in [14, Page 132]. Since the Euclidean length of γ in $\mathbb{D}_1$ is bounded by c_{17}/N, we have

$$|F(p)| \le \int_{\gamma} |\psi|\,|dz| \le \frac{c_{18}}{N}.$$

On the other hand, let $\tilde{\gamma}$ be the $ds^2_{\mathcal{L}_j}$-geodesic joining ζ_j and p. Then noticing that $ds^2_{\mathcal{L}_j} = ds^2_{\tilde{E}}$, (3.18) implies that

$$|\tilde{F}(p)| \le \int_\gamma |\tilde{\psi}|\,|dz| = \int_\gamma ds_{\tilde{F}} \le s + \frac{c_{19}}{\sqrt{N}}.$$

On the other hand, since

$$\begin{aligned}
\max_\gamma |E|^2 &= \max_\gamma \left\{2\cosh \operatorname{dist}_{H^3}\bigl(o, f(z)\bigr)\right\} \\
&= \max_\gamma \left\{2\cosh \operatorname{dist}_{H^3}\bigl(l_{j-1}(\zeta_j), l_{j-1}(z)\bigr)\right\} \le 2\cosh\frac{c_{20}}{N} \le 2\left(1 + \frac{c_{21}}{N}\right), \\
\max_\gamma |\tilde{E}|^2 &\le 2\cosh \operatorname{dist}_{H^3}\bigl(o, \tilde{f}(p)\bigr) = 2\cosh \operatorname{dist}_{H^3}\bigl(l_{j-1}(\zeta_j), l_j(p)\bigr) \\
&\le 2\cosh\{\operatorname{dist}_{H^3}\bigl(l_{j-1}(\zeta_j), l_j(\zeta_j)\bigr) + \operatorname{dist}_{H^3}\bigl(l_{j-1}(\zeta_j), l_j(p)\bigr)\} \\
&\le 2\cosh\left(2s + \frac{c_{22}}{\sqrt{N}}\right) \le 2\left(1 + 4s^2 + \frac{c_{23}}{\sqrt{N}}\right),
\end{aligned}$$

we have

$$|\Delta| \le \frac{c_{24}}{N}, \qquad |\tilde{\Delta}| \le 4s^2 + \frac{c_{25}}{\sqrt{N}},$$

whenever $s < 1/3$. Now, we set

$$\begin{aligned}
\begin{pmatrix} x_0 + x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} &:= \tilde{f}(p) = (\mathrm{id} + \tilde{F} + \tilde{\Delta})(\mathrm{id} + \tilde{F}^* + \tilde{\Delta}^*) \\
&= \mathrm{id} + \tilde{F} + \tilde{F}^* - F - F^* + \delta.
\end{aligned}$$

Then, reasoning as in [14, Page 133], we have

$$|\delta| = |F + F^* + \tilde{F}\tilde{F}^* + \Delta + \tilde{\Delta} + \tilde{\Delta}\tilde{F}^* + \tilde{F}\tilde{\Delta}^* + \tilde{\Delta}\tilde{\Delta}^*| < 14s^2 + \frac{c_{26}}{\sqrt{N}}.$$

Moreover, if we define

$$\begin{aligned}
h(z) &:= F(z) + F^*(z) = \sqrt{2}\int_{\zeta_j}^z \begin{pmatrix} 0 & \operatorname{Re}\varphi_1 + \mathrm{i}\operatorname{Re}\varphi_2 \\ \operatorname{Re}\varphi_1 - \mathrm{i}\operatorname{Re}\varphi_2 & 0 \end{pmatrix}, \\
\tilde{h}(z) &:= \tilde{F}(z) + \tilde{F}^*(z) = \sqrt{2}\int_{\zeta_j}^z \begin{pmatrix} 0 & \operatorname{Re}\tilde{\varphi}_1 + \mathrm{i}\operatorname{Re}\tilde{\varphi}_2 \\ \operatorname{Re}\tilde{\varphi}_1 - \mathrm{i}\operatorname{Re}\tilde{\varphi}_2 & 0 \end{pmatrix},
\end{aligned}$$

the x_3-component of $\tilde{h} - h$ vanishes, and the x_1-component is

$$\sqrt{2}\int \operatorname{Re}(\tilde{\varphi}_1 - \varphi_1) = |\tilde{h}(z) - h(z)|\frac{u_2}{u_1} \le \left(s + \frac{c_{27}}{\sqrt{N}}\right)\frac{u_2}{u_1} \le \frac{c_{28}}{N},$$

because of the property of $\boldsymbol{u}$ in Lemma A.1. Thus, we have

$$|x_1| \le 14s^2 + \frac{c_{29}}{\sqrt{N}}, \qquad |x_3| \le 14s^2 + \frac{c_{30}}{\sqrt{N}}.$$

Since $\operatorname{dist}_{H^3}\bigl(\tilde{f}(p), q\bigr)$ is the distance between $\tilde{f}(p)$ and x_2-axis, we have

$$\operatorname{dist}_{H^3}\bigl(\tilde{f}(p), q\bigr) = \sinh^{-1}\sqrt{x_1^2 + x_3^2} \le 14s^2 + \frac{c_{16}}{\sqrt{N}},$$

which proves the conclusion. □

3.5. **Proof of Key Lemma.** Note that for any positive numbers x and y, it holds that

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y \le \cosh x(\cosh y + \sinh y) = e^y \cosh x. \tag{3.21}$$

By (3.15), we have

$$\mathrm{dist}_{H^3}\big(o, l_{j-1}(\zeta_j)\big) \le \mathrm{dist}_{H^3}\big(o, l_0(\zeta_j)\big) + \frac{c_{31}}{N^2}. \tag{3.22}$$

Under the situation here, $f(\zeta_j)$ and $o(=\mathrm{id})$ lie on the x_1x_3-plane, and q is on the x_2-axis, the geodesic triangle $\triangle f(\zeta_j)oq$ in H^3 is a right triangle. Then by the hyperbolic Pythagorean theorem, we have

$$\begin{aligned}
\cosh \mathrm{dist}_{H^3}\big(f(0), q\big) &= \cosh \mathrm{dist}_{H^3}\big(f(0), o\big) \cosh \mathrm{dist}_{H^3}\big(o, q\big) \\
&= \cosh \mathrm{dist}_{H^3}\big(f(0), f(\zeta_j)\big) \cosh \mathrm{dist}_{H^3}\big(o, q\big) \\
&= \cosh \mathrm{dist}_{H^3}\big(l_{j-1}(0), l_{j-1}(\zeta_j)\big) \cosh \mathrm{dist}_{H^3}\big(o, q\big) \\
&= \cosh \mathrm{dist}_{H^3}\big(o, l_{j-1}(\zeta_j)\big) \cosh \mathrm{dist}_{H^3}\big(o, q\big) \\
&= \cosh\left(\mathrm{dist}_{H^3}\big(o, l_0(\zeta_j)\big) + \frac{c_{31}}{N^2}\right) \cosh \mathrm{dist}_{H^3}\big(o, q\big) \\
&\le \exp\left(\frac{c_{31}}{N^2}\right) \cosh \mathrm{dist}_{H^3}\big(o, l_0(\zeta_j)\big) \cosh \mathrm{dist}_{H^3}\big(o, q\big) \\
&= \frac{1}{2}|\mathcal{L}_0(\zeta_j)|^2 \exp\left(\frac{c_{31}}{N^2}\right) \cosh \mathrm{dist}_{H^3}\big(o, q\big) \\
&\le \left(\frac{1}{2}\max_{\mathbb{D}_1}|\mathcal{L}_0|^2\right) \cosh\left(2s + \frac{c_{13}}{\sqrt{N}}\right)\left(1 + \frac{c_{32}}{N^2}\right) \\
&\le \left(\frac{1}{2}\max_{\mathbb{D}_1}|\mathcal{L}_0|^2\right)\left(1 + \frac{c_{32}}{N^2}\right)\left(1 + 4s^2 + \frac{c_{33}}{\sqrt{N}}\right) \\
&\le \left(\frac{1}{2}\max_{\mathbb{D}_1}|\mathcal{L}_0|^2\right)\left(1 + 4s^2 + \frac{c_{34}}{\sqrt{N}}\right).
\end{aligned}$$

Thus, by using Lemmas 3.1, 3.2, and 3.3, we have

$$\begin{aligned}
\frac{1}{2}|\mathcal{L}_j(p)|^2 &= \cosh \mathrm{dist}_{H^3}\big(o, l_j(p)\big) = \cosh \mathrm{dist}_{H^3}\big(\tilde{f}(0), \tilde{f}(p)\big) \\
&\le \cosh\Big(\mathrm{dist}_{H^3}\big(\tilde{f}(0), f(0)\big) + \mathrm{dist}_{H^3}\big(f(0), q\big) + \mathrm{dist}_{H^3}\big(q, \tilde{f}(p)\big)\Big) \\
&\le \cosh\Big(\mathrm{dist}_{H^3}\big(l_j(\zeta_j), l_{j-1}(\zeta_j)\big) + \mathrm{dist}_{H^3}\big(f(0), q\big) + \mathrm{dist}_{H^3}\big(q, \tilde{f}(p)\big)\Big) \\
&\le \cosh\left(\frac{c_{11}}{N^2} + \mathrm{dist}_{H^3}\big(f(0), q\big) + \mathrm{dist}_{H^3}\big(q, \tilde{f}(p)\big)\right) \\
&\le \exp\left(\frac{c_{11}}{N^2}\right) \cosh\Big(\mathrm{dist}_{H^3}\big(f(0), q\big) + \mathrm{dist}_{H^3}\big(q, \tilde{f}(p)\big)\Big) \\
&\le \left(1 + \frac{c_{35}}{N^2}\right) \cosh\Big(\mathrm{dist}_{H^3}\big(f(0), q\big) + \mathrm{dist}_{H^3}\big(q, \tilde{f}(p)\big)\Big) \\
&\le \left(1 + \frac{c_{35}}{N^2}\right) \exp\Big(\mathrm{dist}_{H^3}\big(q, \tilde{f}(p)\big)\Big) \cosh \mathrm{dist}_{H^3}\big(f(0), q\big) \\
&\le \left(1 + \frac{c_{35}}{N^2}\right) \exp\left(14s^2 + \frac{c_{16}}{\sqrt{N}}\right) \cosh \mathrm{dist}_{H^3}\big(f(0), q\big) \\
&\le \left(1 + \frac{c_{35}}{N^2}\right)\left(1 + 2\left(14s^2 + \frac{c_{16}}{\sqrt{N}}\right)\right)\left(\frac{1}{2}\max_{\mathbb{D}_1}|\mathcal{L}_0|^2\right)\left(1 + 4s^2 + \frac{c_{36}}{\sqrt{N}}\right)
\end{aligned}$$

$$\leq \left(\frac{1}{2}\max_{\mathbb{D}_1}|\mathcal{L}_0|^2\right)\left(1+32s^2+\frac{c_2}{\sqrt{N}}\right)$$

which proves the conclusion.

4. Applications

This section is devoted to prove some applications of the main theorem as we stated in the introduction.

A smooth map $f\colon \mathbb{D}_1 \to M^3$ into a 3-manifold M^3 is called a (*wave*) *front* if there exists a Legendrian immersion $L_f\colon \mathbb{D}_1 \to P(T^*M^3)$ with respect to the canonical contact structure of the projective cotangent bundle $\pi : P(T^*M^3) \to M^3$ such that $\pi \circ L_f = f$.

4.1. **Flat fronts in hyperbolic 3-space.** Recall that

$$H^3 := \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) = \{aa^*\,;\, a \in \mathrm{SL}(2,\mathbb{C})\} \qquad (a^* = {}^t\bar{a}).$$

For a Legendrian immersion $\mathcal{L}\colon \mathbb{D}_1 \to \mathrm{SL}(2,\mathbb{C})$, the projection

$$f := \mathcal{L}\mathcal{L}^*\colon \mathbb{D}_1 \longrightarrow H^3 \tag{4.1}$$

gives a *flat front* in H^3 (see [11, 12] for the definition of flat fronts). We call $\mathcal{L}$ in (4.1) the *holomorphic lift* of f. A flat front f is called *weakly complete* if its holomorphic lift is complete with respect to the induced metric $ds^2_{\mathcal{L}}$ [13, 19].

Let $f\colon \mathbb{D}_1 \to H^3$ be a flat front and $\mathcal{L}$ its holomorphic lift. Take ω and θ as in (3.3) and set

$$\rho := \frac{\theta}{\omega}\colon \mathbb{D}_1 \longrightarrow \mathbb{C}\cup\{\infty\}. \tag{4.2}$$

Then a point $z \in \mathbb{D}_1$ is a singular point if and only if $|\rho(z)| = 1$. We denote by Σ_f the singular set of f;

$$\Sigma_f := \{z \in \mathbb{D}_1\,;\, |\rho(z)| = 1\}.$$

We have the following

Theorem 4.1. *There exists a weakly complete flat front $f\colon \mathbb{D}_1 \to H^3$ whose image is bounded in H^3 such that $\mathbb{D}_1 \setminus \Sigma_f$ is open dense in $\mathbb{D}_1$.*

Proof. Let $\mathcal{L}\colon \mathbb{D}_1 \to \mathrm{SL}(2,\mathbb{C})$ be a Legendrian immersion as in the main theorem, and set $f := \mathcal{L}\mathcal{L}^*\colon \mathbb{D}_1 \to H^3$, which gives a flat front. The boundedness of f follows from that of $\mathcal{L}$, and the weak completeness of f follows from the completeness of $\mathcal{L}$.

Finally, we shall prove that $\mathbb{D}_1 \setminus \Sigma_f$ is open dense: If Σ_f has an interior point, then ρ in (4.2) satisfies $|\rho| = 1$ identically because of the analyticity of ρ. However, if we take a initial immersion so that $|\rho|$ is not constant, then the resulting bounded weakly complete flat front has the same property, since our iteration can be taken to be small enough near the origin of $\mathbb{D}_1$. □

4.2. **Flat fronts in de Sitter 3-space.** The de Sitter 3-space S^3_1 is the connected and simply connected Lorentzian space form of constant curvature 1, which is represented as

$$S^3_1 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(1,1) = \{ae_3a^*\,;\, a \in \mathrm{SL}(2,\mathbb{C})\} \qquad \left(e_3 := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right).$$

Let $\mathcal{L}\colon \mathbb{D}_1 \to \mathrm{SL}(2,\mathbb{C})$ be a Legendrian immersion. Then the projection

$$f\colon \mathcal{L}e_3\mathcal{L}^*\colon \mathbb{D}_1 \longrightarrow S^3_1$$

gives a flat front. We remark that f is not an immersion at z if and only if $|\rho(z)| = 1$, where $\rho = \theta/\omega$ as in (4.2). The singular set Σ_f of f is characterized by $|\rho(z)| = 1$. Similar to the case of flat fronts in H^3, f is said to be *weakly complete* if the metric induced from the canonical Hermitian metric of $\mathrm{SL}(2,\mathbb{C})$ by $\mathcal{L}$ is complete. Then we have

Theorem 4.2. *There exists a weakly complete flat front $f\colon \mathbb{D}_1 \to H^3$ whose image is bounded in S^3_1 such that $\mathbb{D}_1 \setminus \Sigma_f$ is open dense in $\mathbb{D}_1$, where Σ_f is the set of singular points of f.*

Proof. Let $\mathcal{L}\colon \mathbb{D}_1 \to \mathrm{SL}(2,\mathbb{C})$ be a Legendrian immersion as in the main theorem, and set $f := \mathcal{L}e_3\mathcal{L}^*$. Then f is a weakly complete flat front in S^3_1. Moreover, one can see that $\mathbb{D}_1 \setminus \Sigma_f$ is open dense by the same argument as in Theorem 4.1. □

See [7], [10] and [3] for the relationships between flat surfaces and linear Weingarten surfaces in H^3 or S^3_1.

4.3. **Improper Affine front in affine 3-space.** A notion of *IA-maps* in the affine 3-space has been introduced by A. Martínez in [16]. IA-maps are improper affine spheres with a certain kind of singularities. Since all of IA-maps are wave fronts (see [18, 19]), we call them *improper affine fronts* (The terminology 'improper affine fronts' has been already used in Kawakami-Nakajo [8]). The precise definition of improper affine fronts is given in [19, Remark 4.3]. The set of singular points Σ_f of an improper affine front $f\colon \mathbb{D}_1 \to \mathbb{R}^3$ is represented as $\Sigma_f = \{z \in \mathbb{D}_1 \,|\, |\rho(z)| = 1\}$, where $\rho(z) := dG/dF$. In [19], *weak completeness* of improper affine fronts is introduced. Then we have

Theorem 4.3. *There exists a weakly complete improper affine front $f\colon \mathbb{D}_1 \to \mathbb{R}^3$ whose image is bounded such that $\mathbb{D}_1 \setminus \Sigma_f$ is open dense in $\mathbb{D}_1$, where Σ_f is the set of the singular points of f.*

Proof. Let $\mathcal{L} = (F, G, H)$ be a complete bounded Legendrian immersion into $\mathbb{C}^3$. Since $\mathcal{L}$ is Legendrian, (1.1) yields that $dH = -F\,dG$. Here, by completeness of $\mathcal{L}$, the induced metric

$$ds^2_{\mathcal{L}} = |dF|^2 + |dG|^2 + |dH|^2 = |dF|^2 + |dG|^2 + |F\,dG|^2 = |dF|^2 + (|F|^2 + 1)|dG|^2$$

is complete. Moreover, since the image of $\mathcal{L}$ is bounded, we have

$$ds^2_{\mathcal{L}} \le C(|dF|^2 + |dG|^2) \qquad (C > 0 \text{ is a constant}).$$

Thus, the metric

$$d\tau^2 := |dF|^2 + |dG|^2 \tag{4.3}$$

is complete. Hence, we have an improper affine front f using Martinez' representation formula [16] with respect to (F, G);

$$\begin{aligned} f &= \left(G + \overline{F}, \frac{1}{2}(|G|^2 - |F|^2) + \mathrm{Re}\left(GF - 2\int F\,dG\right)\right) \\ &= \left(G + \overline{F}, \frac{1}{2}(|G|^2 - |F|^2) + \mathrm{Re}\,(GF + 2H)\right) : \mathbb{D}_1 \to \mathbb{R}^3. \end{aligned} \tag{4.4}$$

Since $d\tau^2$ in (4.3) is complete, f is weakly complete, by definition of weak completeness given in [19]. Boundedness of f follows from that of $\mathcal{L}$.

We next prove that $\mathbb{D}_1 \setminus \Sigma_f$ is open dense: If we take a initial immersion so that $|dG/dF|$ is not constant, then the resulting bounded weakly complete improper affine front has the same property, since the singular set is characterized by $|dG/dF| = 1$. □

Remark 4.4. As mentioned in the introduction, there exist no compact flat fronts in H^3 (resp. S^3_1). In fact, [11, Proposition 3.6] implies non-existence of compact flat front in H^3. On the other hand, suppose that there exists a flat front $f\colon \Sigma^2 \to S^3_1$ where Σ^2 is a compact 2-manifold. Then the unit normal vector ν of f induces a flat front $\nu\colon \Sigma^2 \to H^3$, which makes a contradiction.

Next, we shall show the non-existence of compact improper affine fronts as mentioned in the introduction. An improper affine front $f\colon \Sigma^2 \to \mathbb{R}^3$ defined on a Riemann surface Σ^2 is represented as in (4.4), where F and G are holomorphic functions on Σ^2. If Σ^2 is compact, $G + \overline{F}$ is equal to a constant $c \in \mathbb{C}$ because it is harmonic. Then we have that $G = -\overline{F} + c$, where the left-hand side is holomorphic whereas the right-hand side is anti-holomorphic. Hence we can conclude that F and G are both constants. Hence $d\tau^2$ as in (4.3) vanishes identically, which contradicts to the definition of improper affine fronts.

Appendix A. An application of Runge's Theorem

To prove the Key Lemma, we prepare the following assertion, which is an analogue of [14, Lemma 4.1].

Lemma A.1. *Let $\varphi = (\varphi_1, \varphi_2)$ be a non-degenerate pair of holomorphic functions on $\overline{\mathbb{D}}_1$ (see Definition 2.2), and let $\varepsilon > 0$ and N be a positive number and a sufficiently large integer N, respectively. Then for each j $(j = 1, \dots, 2N)$, there exists a non-degenerate pair $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ of holomorphic functions satisfying the following three conditions:*

(a) $|\tilde{\varphi} - \varphi| < \frac{\varepsilon}{2N^2}$ *on* $\overline{\mathbb{D}}_1 \setminus \varpi_j$.

(b) $|\tilde{\varphi}| \geq \begin{cases} C\,N^{3.5} & (\text{on } \omega_j), \\ C\,N^{-0.5} & (\text{on } \varpi_j), \end{cases}$

where C is a constant depending only on φ.

(c) *There exists a unit vector $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ $(|\boldsymbol{u}| = 1)$ such that*

$$\boldsymbol{u} \cdot (\varphi - \tilde{\varphi}) = 0, \qquad |u_1| > 1 - \frac{2}{N},$$

where we set $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2$ for $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$.

As pointed out in Remark A.2 below, we can prove the lemma as a modification of [14, Lemma 4.1] for null holomorphic curves in $\mathbb{C}^3$. However, we believe that the theory of Legendrian curves should be established independently from the theory of null curves. So we give here a self-contained proof of the lemma, which might be convenient for the readers. In fact, our proof is easier than that of [14, Lemma 4.1].

Proof. In the proof, the symbols c_k $(k = 1, 2, \dots)$ denote suitable positive constants, which depend only on the initial data φ. Since φ has no zeroes, we can take ν and

m such that

$$0 < \nu \le |\varphi| \le m \qquad (\text{on } \overline{\mathbb{D}}_1). \tag{A.1}$$

Set

$$\hat\varphi = (\hat\varphi_1, \hat\varphi_2) = \big((\cos t)\varphi_1 + (\sin t)\varphi_2, -(\sin t)\varphi_1 + (\cos t)\varphi_2\big)$$

for $t \in \mathbb{R}$. To prove the lemma, we need to prove that one can choose $t \in [0, \frac{\pi}{2})$ such that

$$\sin t \le \sqrt{\frac{2}{N}} \qquad |\hat\varphi_k| \ge \frac{\nu}{2\sqrt{N}} \qquad (\text{on } \varpi_j, k = 1, 2). \tag{A.2}$$

In fact, if $|\varphi_k| \ge \nu/(2\sqrt{N})$ $(k = 1, 2)$ holds on ϖ_j, (A.2) holds obviously for $t = 0$. By exchanging the roles of φ_1 and φ_2, we may assume that there exists $x \in \varpi_j$ such that $|\varphi_1(x)| < \nu/(2\sqrt{N})$ without loss of generality. Here, notice that the diameter (as a subset of $\mathbb{C} = \mathbb{R}^2$) of ϖ_j satisfies $\mathrm{diam}_{\mathbb{R}^2}(\varpi_j) \le c_1/N$, where c_1 is a positive constant. Since the derivative of $\varphi \colon \overline{\mathbb{D}}_1 \to \mathbb{C}^2$ is bounded, it holds that, $\mathrm{diam}_{\mathbb{C}^2}(\varphi(\varpi_j)) \le c_2/N$ for $c_2 > 0$. Then it holds that

$$|\varphi_1(y)| \le |\varphi_1(x)| + |\varphi(y) - \varphi(x)| < \frac{\nu}{2\sqrt{N}} + \frac{c_2}{N}$$

for any $y \in \varpi_j$. Here, noticing $|\varphi|^2 \ge \nu^2$, we have

$$|\varphi_2(y)| \ge \sqrt{\nu^2 - |\varphi_1(y)|^2} \ge \sqrt{\nu^2 - \left(\frac{\nu}{2\sqrt{N}} + \frac{c_2}{N}\right)^2} \ge \nu\left(1 - \frac{1}{4N}\right)$$

since N is sufficiently large. We choose t as $\sin t = \sqrt{\frac{2}{N}}$. Then noticing

$$1 - \frac{2}{N} \le \cos t \le 1 - \frac{1}{N},$$

we have

$$\begin{aligned}
|\hat\varphi_1(y)| &= |(\cos t)\varphi_1(y) + (\sin t)\varphi_2(y)| \ge (\sin t)|\varphi_2(y)| - (\cos t)|\varphi_1(y)| \\
&\ge \frac{\sqrt{2}\nu}{\sqrt{N}}\left(1 - \frac{1}{4N}\right) - \left(1 - \frac{1}{N}\right)\left(\frac{\nu}{2\sqrt{N}} + \frac{c_2}{N}\right) \ge \frac{\nu}{2\sqrt{N}}, \\
|\hat\varphi_2(y)| &\ge (\cos t)|\varphi_2(y)| - (\sin t)|\varphi_1(y)| \\
&\ge \left(1 - \frac{2}{N}\right)\left(1 - \frac{1}{4N}\right)\nu - \sqrt{\frac{2}{N}}\left(\frac{\nu}{2\sqrt{N}} + \frac{c_2}{N}\right) \ge \frac{\nu}{2\sqrt{N}}.
\end{aligned} \tag{A.3}$$

Hence we have (A.2).

We now fix a real number t in (A.2) and will prove (a), (b) and (c): Since ω_j and $\mathbb{D}_1 \setminus \varpi_j$ are compact set such that $\mathbb{C} \setminus \big(\omega_j \cup (\mathbb{D}_1 \setminus \varpi_j)\big)$ is connect, Runge's theorem implies that there exists a holomorphic function h (cf. [17, (4)]) such that $h \ne 0$ on $\overline{\mathbb{D}}_1$ and

$$\begin{cases} |h - 2N^4| < \dfrac{1}{2N^2} & (\text{on } \omega_j) \\ \quad |h - 1| < \dfrac{\varepsilon}{2N^2 m} & (\text{on } \overline{\mathbb{D}}_1 \setminus \varpi_j), \end{cases} \tag{A.4}$$

where m is as in (A.1). We set

$$\check\varphi = (\check\varphi_1, \check\varphi_2) := (\hat\varphi_1, h\hat\varphi_2).$$

Then

$$\tilde{\varphi} := \bigl((\cos t)\check{\varphi}_1 - (\sin t)\check{\varphi}_2, (\sin t)\check{\varphi}_1 + (\cos t)\check{\varphi}_2\bigr)$$

satisfies the desired properties: In fact,

$$|\tilde{\varphi} - \varphi| = |\check{\varphi} - \hat{\varphi}| = |h-1|\,|\hat{\varphi}_2| < \frac{\varepsilon}{2N^2 m}|\varphi| \le \frac{\varepsilon}{2N^2}$$

which implies (a). On the other hand, by (A.3),

$$|\tilde{\varphi}| = |\check{\varphi}| \ge |\check{\varphi}_1| = |\hat{\varphi}_1| \ge \frac{\nu}{2\sqrt{N}}$$

holds on ϖ_j, which proves the first inequality of (b).

It holds on ω_j that

$$|\tilde{\varphi}| = |\check{\varphi}| \ge |h|\,|\hat{\varphi}_2| \ge \bigl|2N^4 - |h - 2N^4|\bigr|\,|\hat{\varphi}_2| \ge \left(2N^4 - \frac{1}{2N^2}\right)\frac{\nu}{2\sqrt{N}} \ge \frac{\nu}{2}N^{3.5}.$$

Hence we have the second inequality of (b). Finally, we set $\boldsymbol{u} = (\cos t, \sin t)$. Then (c) holds. □

Remark A.2. Lemma A.1 can be proved directly from the corresponding assertion for null curves in $\mathbb{C}^3$ given in [14, Lemma 4.1] as follows: Let $\varphi = (\varphi_1, \varphi_2)$ be as in Lemma A.1. Since φ_1, φ_2 have no common zeros, there exists a holomorphic function φ_3 defined on $\overline{\mathbb{D}}_1$ such that $(\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0$. We apply [14, Lemma 4.1] for $\Phi := (\varphi_1, \varphi_2, \varphi_3)$ and get a new Weierstrass data $\tilde{\Phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$. Then $\tilde{\varphi} := (\tilde{\varphi}_1, \tilde{\varphi}_2)$ satisfies (a) which follows immediately from (a) of [14, Lemma 4.1]. Next, by the proof of [14, Corollary B] it holds that

$$4(|\tilde{\varphi}_1|^2 + |\tilde{\varphi}_2|^2) \ge |\tilde{\varphi}_1|^2 + |\tilde{\varphi}_2|^2 + |\tilde{\varphi}_3|^2,$$

which implies that (b) of our lemma follows from (b) of [14, Lemma 4.1]. (c) of our lemma does not follows from (c) of [14, Lemma 4.1] directly. However, we can choose $u = (u_1, u_2, u_3)$ in the proof of [14, Lemma 4.1] in such a way that $u \in \mathbb{R}^3$ and $u_3 = 0$ without loss of generality. So this gives a alternative proof of the lemma.

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