SWAP ACTION ON MODULI SPACES OF POLYGONAL LINKAGES

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ABSTRACT. The basic object of the paper is the moduli space $M_{2,3}(L)$ of a closed polygonal linkage either in $\mathbb{R}^2$ or in $\mathbb{R}^3$. As was originally suggested by G. Khimshiashvili, the space $M_2(L)$ is equipped with the oriented area function A, whereas (as is suggested in the paper) $M_3(L)$ is equipped with the vector area function S. The latter are generically Morse functions, whose critical points have a nice description. In the preprint, we define a *swap action* (that is, the action of some group generated by edge transpositions) on the space $M_{2,3}(L)$ which preserves the functions A and S and the Morse points. We prove that the commutant of the group acts trivially, present some computer experiments and formulate a conjecture.

1. Introduction

We study the moduli space $M_2(L)$ and $M_3(L)$ of a closed polygonal linkage either in $\mathbb{R}^2$ or in $\mathbb{R}^3$. These spaces attract special attention firstly because of practical applications, and secondly because they can be equipped by additional structures. In this respect we briefly mention the papers by A. Klyachko [4], and by M. Kapovich, J. Millson [2].

In the paper, we consider the *oriented area function* A defined on the space $M_2(L)$, and the *vector area function* S defined on the space $M_3(L)$. Generically, these are Morse functions, whose critical points have a nice description. In the preprint, we enrich this structure by defining a *swap action* (that is, the action of some group generated by edge transpositions) on the space $M_{2,3}(L)$ which preserves the functions A and S and the Morse points. We show that the action can be visualized by a factor group of *the group of pure balanced annular braids*. Besides, we prove that commutant of the group acts trivially, present some computer experiments and formulate a natural conjecture.

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2. Moduli space and oriented area

A polygonal *n-linkage* is a sequence of positive numbers $l_1, \dots, l_n$. It should be interpreted as a collection of rigid bars of lengths l_i joined consecutively by revolving joints in a closed chain. We study its flexes with allowed self-intersections. This is formalized in the following definition:

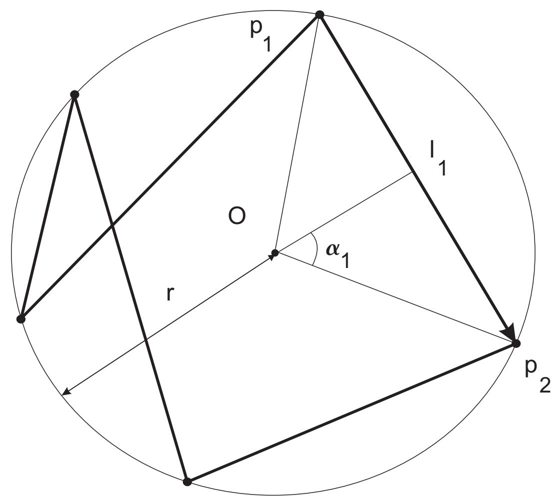

FIGURE 1. Basic notation for a pentagonal cyclic configuration with $E = (-1, -1, -1, 1, -1)$

Definition 2.1. For a linkage L, *a configuration* in the Euclidean space $\mathbb{R}^d$ is a sequence of points $R = (p_1, \dots, p_{n+1}),\ p_i \in \mathbb{R}^d$ with $l_i = |p_i, p_{i+1}|$.

The *the moduli space of* L is the set $M_d(L)$ of all such configurations modulo the action of orientation preserving isometries.

In the paper we make use of the signed area function as the Morse function on $M_2(L)$ and of the vector area function on $M_3(L)$.

We start with **2D**.

Definition 2.2. The *signed area* of a polygon $P \subset \mathbb{R}^2$ with the vertices $p_i = (x_i, y_i)$ is defined by

$$2A(P) = (x_1y_2 - x_2y_1) + \ldots + (x_ny_1 - x_1y_n).$$

Definition 2.3. A polygon P is called *cyclic* if all its vertices p_i lie on a circle.

Cyclic polygons arise in the framework of the paper as critical points of the signed area:

Theorem 2.4. [5] *Generically, a polygon P is a critical point of the signed area function A iff P is a cyclic configuration.* □

The following notation (see Fig.1) is used throughout the paper for closed cyclic configurations:

$r = r(P)$ is the radius of the circumscribed circle.

A cyclic configuration is called *central* if one of its edges contains O.

For a non-central configuration, ε_i is the *orientation* of the edge p_ip_{i+1}:

$$\varepsilon_i = \begin{cases} 1, & \text{if the center } O \text{ lies to the left of } p_ip_{i+1}; \\ -1, & \text{if the center } O \text{ lies to the right of } p_ip_{i+1}. \end{cases}$$

$E = E(P) = (\varepsilon_1, \dots, \varepsilon_n)$ is the string of orientations of all the edges.

Now we pass to the **3D**.

Definition 2.5. The *vector area* of a polygon $P \subset \mathbb{R}^3$ with the vertices $p_i = (x_i, y_i)$ is defined by

$$2\overrightarrow{S(P)} = p_1 \times p_2 + p_2 \times p_3 + \cdots + p_n \times p_1,$$
$$2S(P) = |p_1 \times p_2 + p_2 \times p_3 + \cdots + p_n \times p_1|.$$

Theorem 2.6. *Assume that $S(P) \neq 0$ for a configuration $P \in M_3(L)$. Generically, P is a critical point of the vector area function S if and only if the two following conditions hold:*

(1) *The orthogonal projection of P onto the plane $\overrightarrow{S(P)}^{\perp}$ is a cyclic polygon.*
(2) *For every i, the vectors $\overrightarrow{T_i}$, $\overrightarrow{S}$, and $\overrightarrow{d_i}$ are coplanar.*

Here $\overrightarrow{d_i}$ is the i-th short diagonal, $\overrightarrow{T_i}$ is the vector area of the triangle $p_{i-1}p_ip_{i+1}$, see Fig. 2, right.

Proof. We list $(2n-6)$ flexes that generate $(2n-6)$ elements of the tangent space $T_P(M_3(L))$. Generically, these vectors are linearly independent. Therefore, the point $P \in M_3(L)$ is critical if and only if the function S has a non-zero derivative in all these directions.

(1) Denote by $pr\ P$ the orthogonal projection of P onto $\overrightarrow{S(P)}^{\perp}$. Each flex of $pr\ P$ in the plane $\overrightarrow{S(P)}^{\perp}$ generates a flex of P in the space $\mathbb{R}^3$. During the flex, we maintain the slopes of the edges with respect to the plane $\overrightarrow{S(P)}^{\perp}$. Since $dim\ M_2(pr\ P) = n-3$, we can choose $(n-3)$ linearly independent tangent vectors of this type.
(2) Let us bend the triangle T_i around the diagonal d_i keeping the rest of configuration P frozen. We choose $(n-3)$ linearly independent tangent vectors of this type.

The flexes of the first (respectively, second) type provide the statement 1 (respectively, statement 2) of the theorem. □

3. Swap action

We assume that a polygonal linkage L with n edges and with all l_i different is fixed. We make a convention that the numbering is modulo n, that is, for instance, $n+1=1$.

Definition 3.1. Let $P \in M_{2,3}(L)$ be a polygon. For $i = 1, \ldots, n$, denote by $s_i(P)$ the polygon obtained from P by transposing of the two edges adjacent to the vertex p_i (see Fig. 3). For the dimension three, we assume that the new pair of edges lies in the plane spanned by the old one.

We get a homeomorphism

$$s_i : M_{2,3}(L) \to M_{2,3}(\sigma_i L),$$

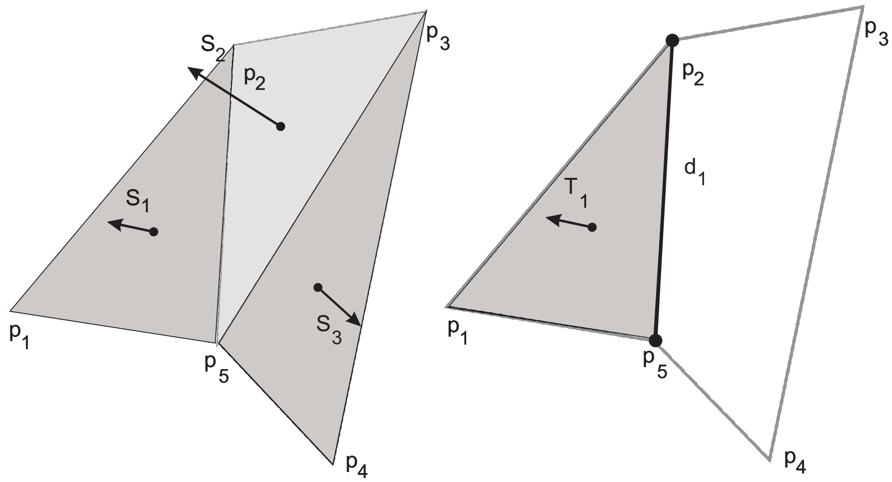

FIGURE 2.

where the element of the symmetric group $\sigma_i \in S_n$ is a transposition induced by s_i.

Define by F_n the free group whose generators are the abstract symbols s_i.

The group F_n acts on the disjoint union of the moduli spaces

$$\bigsqcup_{\sigma_i \in S_n} M_{2,3}(\sigma_i L).$$

Lemma 3.2. (1) *The action of F_n preserves the functions A and S.*

(2) *For $n = 4$, the action of F_4 preserves the volume of the convex hull $V(Conv(P))$.* □

However, we wish to restrict ourselves by just one linkage and just one moduli space. This means that we take only those elements that take a configuration to the same moduli space. We formalize this as follows: There is a natural mapping to the symmetrical group

$$\pi : F_n \to S_n,$$

which maps s_i to σ_i. We are interested in the action of its kernel F_n^0 on the moduli space $M_{2,3}(L)$.

Lemma 3.3. *For a 4-linkage L, the group F_4^0 acts trivially on $M_{2,3}(L)$.*

Proof. (2D). For a 4-gon $P = (p_1, p_2, p_3, p_4)$ denote by $O = O(P)$ the intersection point of perpendicular bisectors to the segments p_1p_3 and p_2p_4. Denote also

$$r_i(P) = |Op_i|, \quad \beta_i(P) = \angle p_i O p_{i_1}.$$

The lemma follows from the three geometrical observations:

(1) A 4-gon is completely defined by

$$r(P) = ((r_1(P), r_2(P), r_3(P), r_4(P)), \text{ and } \beta(P) = (\beta_1(P), \beta_2(P), \beta_3(P), \beta_4(P)).$$

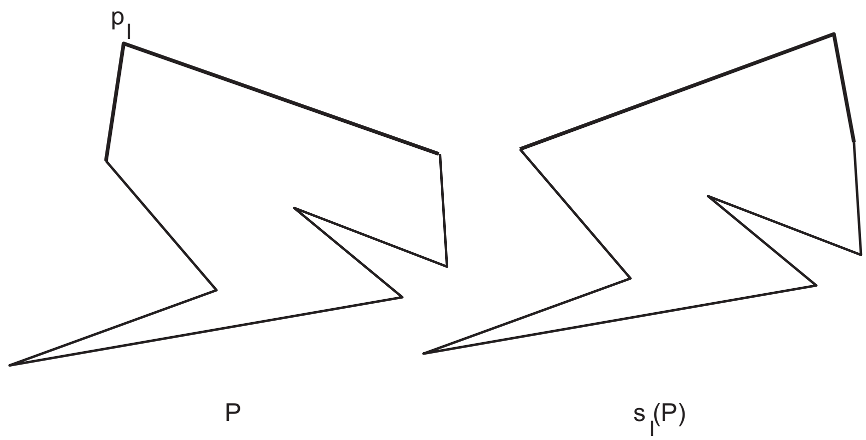

FIGURE 3.

(2) The action of F_n preserves the point $O(P)$ and the vector $r(P)$.
(3) The group F_n acts on $\beta(P)$ by permutations: $\beta(s(P)) = \pi(s)\beta(P)$.

(3D). By analyticity reasons it is enough to prove that $s = (s_1 s_2)^3$ acts trivially on some open subset U of the space of all 4-gons.

Take an equilateral 4-gon P_0 (that is, a rhombus but not a square). The swap s obviously takes P_0 to itself. Now, let P be a quadrilateral close to P_0. Its image sP is close to P and has the same values of $A(P)$ and $V(Conv(P))$. By continuity reasons, $sP = P$. In other words, $s = (s_1 s_2)^3$ acts trivially on a neighborhood of P which is an open set. □

Definition 3.4. Denote by $Stab = Stab(M_{2,3}(L)) \subset F_n^0$ the pointwise stabilizer of the space $M_{2,3}(L)$, that is, the the group of all elements with the trivial action. Denote also the factor $F_n^0/Stab$ by $SW_n = SW_n(L)$.

Proposition 3.5. *Generically, the group Stab does not depend on L.* □

Definition 3.6. Define $R \subset F_n^0$ as the subgroup generated by the elements of the following three types:

(1) s_i^2,
(2) $s_i s_j s_i^{-1} s_j^{-1}$, whenever $|i - j| > 1$, and
(3) $s_i s_{i+1} s_i s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1}$.

Proposition 3.7. *The group R is a subgroup of the stabilizer Stab.*

Proof. The first two items are obvious. The third one follows from Lemma 3.3. □

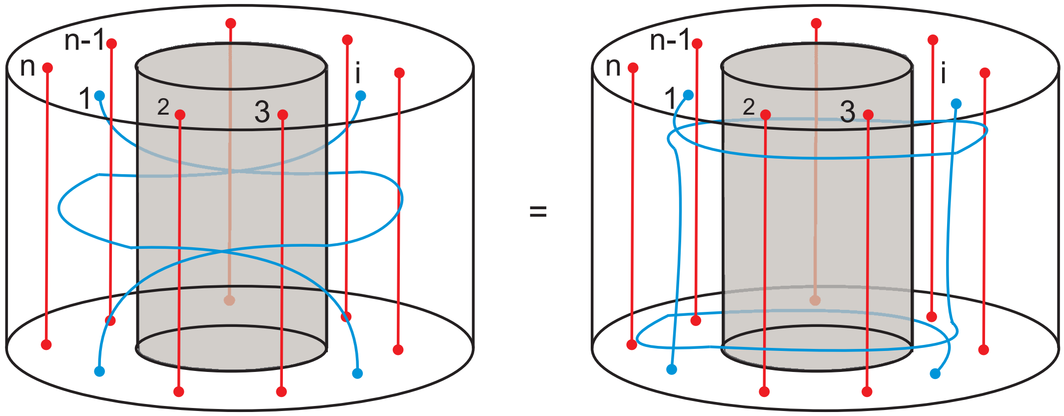

FIGURE 4. The i-th generator of the group F_n^0/R represented by a balanced annular braid ($i = 2, \dots, n$).

Theorem 3.8.
- *The group F_n^0/R acts on the moduli spaces $M_{2,3}(L)$.*
- *The group F_n^0/R is isomorphic to $\mathbb{Z}^{n-1}$, and is therefore commutative.*
- *The elements of the group F_n^0/R can be represented by balanced annular braids. In particular, Fig. 4 depicts the generators of the group.*

Proof.

The first statement follows from the above discussion. To prove the second statement, we construct an explicit homomorphism

$$\phi\colon F_n^0/R \to \mathbb{Z}^{n-1} \cong \{(w_1, w_2, \dots, w_n) \in \mathbb{Z}^n : \sum_{i=1}^n w_i = 0.\}$$

The core idea is to represent the two groups by one and the same braid group. We start with the *balanced annular braid group* which is defined as follows:

$$B_n = \langle \Sigma_1, \Sigma_2, \dots \Sigma_n \mid \Sigma_i\Sigma_j = \Sigma_j\Sigma_i,$$
$$\Sigma_i\Sigma_{i'}\Sigma_i = \Sigma_{i'}\Sigma_i\Sigma_{i'} \text{ whereas } i-j \neq \pm 1,\ i-i' = \pm 1\rangle.$$

Next, we take the group B_n^0 of *pure braids*, that is, the kernel of the natural map $B_n \to S_n$ which maps Σ_i to σ_i.

As usual, we visualize a braid as n non-intersecting strands living in a "thick" cylinder and going from the top to the bottom, see Fig. 4.

Finally, we introduce the group $\overline{B_n^0}$, that is, the group B_n^0 factorized by all relations of type $(\Sigma_i)^2 = 1$. The factorization means that the strands can pass freely through each other, but not through the central part of the cylinder.

There is a natural isomorphism

$$\psi : F_n^0/R \to \overline{B_n^0}$$

which maps s_i to Σ_i.

Besides, there is a homomorphism

$$w\colon \overline{B_n^0} \to \mathbb{Z}^n,\ b \mapsto w(b) = (w_1(b), w_2(b), \dots, w_n(b))$$

where $w_i(b)$ is a winding number of the i-th strut of the braid b around the central part of the cylinder. It is easy to check that for any pure braid b, we have

$$\sum_{i=1}^{n} w_i(b) = 0.$$

Taken together, the two maps give the homomorphism

$$w \circ \psi : F_n^0/R \to \{(w_1, w_2, \dots, w_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} w_i = 0.\},$$

which is obviously bijective.

Figure 4 depicts the preimage of the vector $(1, 0, 0, \dots, 0, 0, -1, 0, 0, \dots, 0)$ with just two non-zero entries. The preimage of the vector in the group F_n^0/R is represented by

$$s_{i+1}s_{i+2}\dots s_{i-1}s_{n-1}s_{n-2}\dots s_2s_1. \quad \square$$

Proposition 3.9. *The critical points of the function A and S are stable under the action of F_n^0.*

Proof.

(2D). Critical points of the function A are known to be cyclic polygons (see Theorem 2.4). A cyclic polygon P is completely determined by $r(P)$, L and $E(P)$. The action of F_n^0 preserves them all.

(3D). Assume that P is a critical point such that $S(P) \neq 0$. Fix a polygon P and a plane $\overrightarrow{S}^{\perp}$. First observe that a critical point is uniquely determined by radius $r(prP)$ of the circumscribing circle, the edge orientations $E(prP)$, and the heights $h_i = dist(p_i, \overrightarrow{S}^{\perp}),\ i = 1, \dots, n$.

Let g be an element of F_n^0/R. Theorem 2.6 implies that the swap s_i permutes the height differences $h_{i+1} - h_i$ and $h_i - h_{i-1}$.Therefore, g maintains the height differences $h_{i+1} - h_i$. Besides, g maintains both $E(prP)$ and $r(prP)$. By the above observation, g maps P to itself. $\square$

Computer experiments show the following:

Example 3.10. *Let $g = s_4s_3s_2s_1$. There exists a pentagon P such that g^kP are all different for $k = 1, 2, \cdots, 8!$ (see Fig. 5). By analyticity reasons this means that they are different for a generic pentagon.*

Example 3.11. *For the pentagon P depicted in Fig. 6, the pentagons $s_4s_3s_2s_1(P)$ and $s_5s_4s_3s_2(P)$ are different. This means that they are different for a generic pentagon.*

These two examples motivate the following conjecture:

Conjecture 1. For a generic polygonal linkage, the groups $Stab$ and R coincide, i.e., $SW_n = F_n^0/R$.

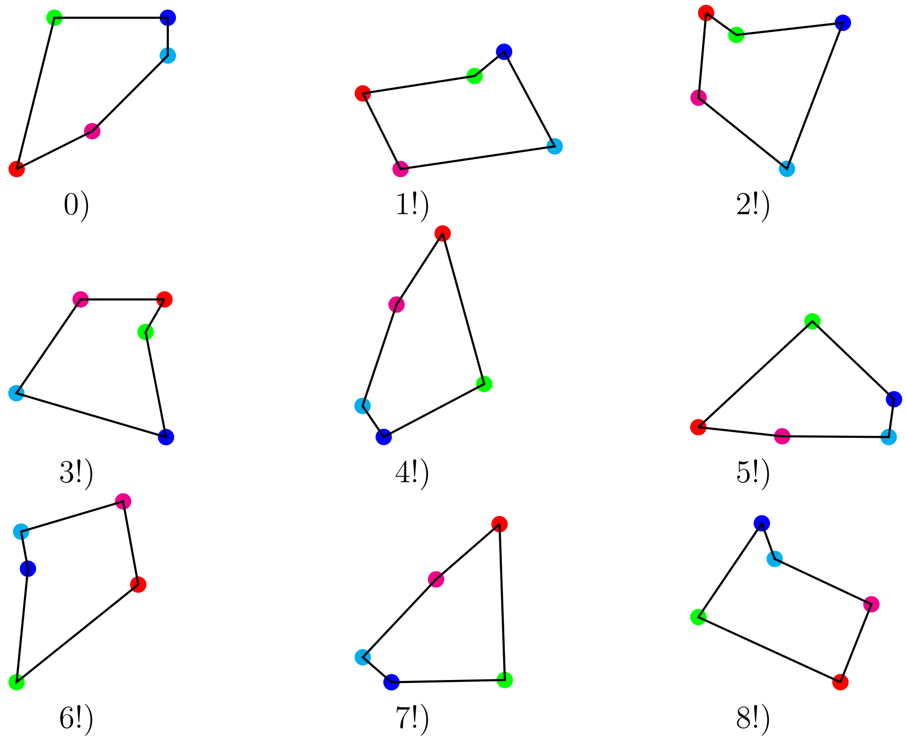

FIGURE 5. We depict here the polygons (0) P and the iterated actions $g^{k!}(P)$

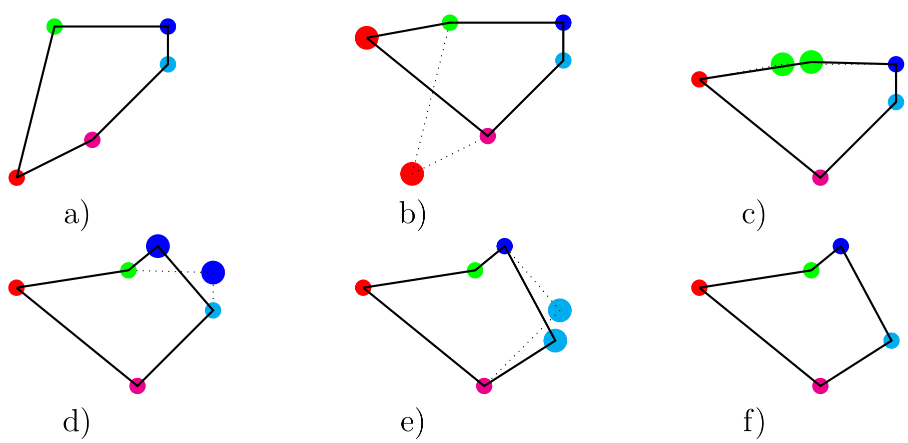

FIGURE 6. The action of the first generator. We depict here a) P, b) $P \to s_1(P)$, c) $s_1(P) \to s_2s_1(P)$, $\dots$, e) $s_3s_2s_1(P) \to s_4s_3s_2s_1(P)$, f) $s_4s_3s_2s_1(P)$.

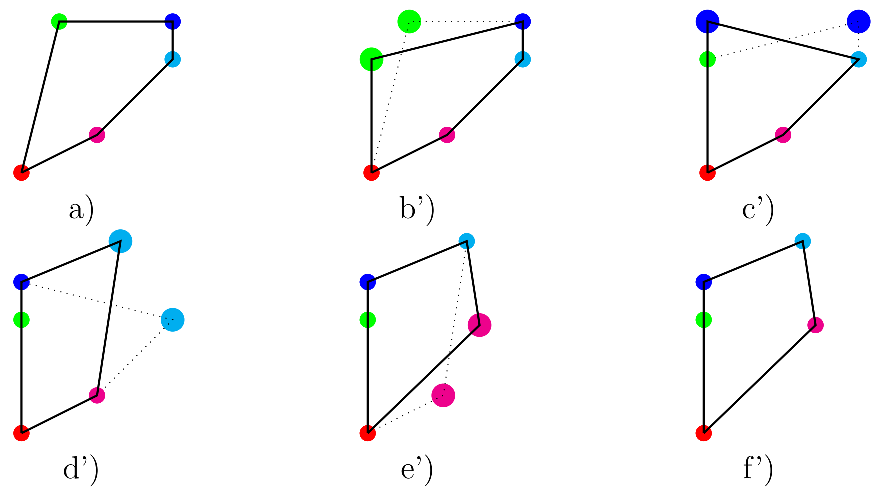

FIGURE 7. The action of the second generator. We depict a) P, b') $P \to s_2(P)$, c') $s_1(P) \to s_3 s_2(P)$, ..., e') $s_4 s_3 s_2(P) \to s_5 s_4 s_3 s_2(P)$, f') $s_5 s_4 s_3 s_2(P)$.

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