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OPTIMAL STOPPING PROBLEMS FOR THE MAXIMUM PROCESS WITH UPPER AND LOWER CAPS

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This paper concerns optimal stopping problems driven by a spectrally negative Lévy process X. More precisely, we are interested in modifications of the Shepp-Shiryaev optimal stopping problem [2, 10, 11]. First, we consider a capped version of the latter and provide the solution explicitly in terms of scale function. In particular, the optimal stopping boundary is characterised by an ordinary differential equation involving scale function and changes according to the path variation of X. Secondly, in the spirit of [12], we consider a modification of the capped version of the Shepp-Shiryaev optimal stopping problem in the sense that the decision to stop has to be made before the process X falls below a given level.

1. Introduction. Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions. For $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the probability measure under which X starts at x and for simplicity write $\mathbb{P}_0 = \mathbb{P}$. We associate with X the maximum process $\overline{X} = \{\overline{X}_t : t \geq 0\}$ given by $\overline{X}_t := s \vee \sup_{0 \leq u \leq t} X_u$ for $t \geq 0, s \geq x$. The law under which $(X, \overline{X})$ starts at (x, s) is denoted by $\mathbb{P}_{x,s}$.

In this paper we are mainly interested in the following optimal stopping problem:

$$V^*_\epsilon(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau + \overline{X}_\tau \wedge \epsilon}\big], \tag{1}$$

where $\epsilon \in \mathbb{R}, q > 0, (x, s) \in E := \{(x, s) \in \mathbb{R}^2 \,|\, x \leq s\}$ and $\mathcal{M}$ is the set of all finite $\mathbb{F}$-stopping times. Since the constant ϵ bounds the process $\overline{X}$ from above, we refer to it as the upper cap. Due to the fact that the pair $(X, \overline{X})$ is a strong Markov process, (1) has also a Markovian structure and hence the general theory of optimal stopping suggests that the optimal stopping time is the first entry time of the process $(X, \overline{X})$ into some subset of E. Indeed, it turns out that the solution of (1) is given by

$$\tau^*_\epsilon := \inf\{t \geq 0 : \overline{X}_t - X_t \geq g(\overline{X})\}$$

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for some function g which is characterised as solution to a certain ordinary differential equation involving scale functions. The function $s \mapsto s - g(s)$ is sometimes referred to as the optimal stopping boundary. We will show that the shape of the optimal boundary changes according to the path variation of X. The solution of problem (1) is closely related to the solution of the Shepp-Shiryaev optimal stopping problem

$$V^*(x,s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau + \overline{X}_\tau}\big], \tag{2}$$

which was first studied by Shepp and Shiryaev [10–12] for the case when X is a linear Brownian motion and later by Avram, Kyprianou and Pistorius [2] for the case when X is a spectrally negative Lévy process. Shepp and Shiryaev [10] introduced the problem as a means to pricing Russian options, a topic we do not concern ourselves with in this paper. Our method for solving (1) consists of a verification technique, that is, we heuristically derive a candidate solution and then verify that it is indeed a solution. In particular, we will make use of the principle of smooth and continuous fit [1, 8] in a similar way to [8, 10].

It is also natural to ask for a modification of (1) with a lower cap. Whilst this is already included in the starting point of the maximum process $\overline{X}$, there is a stopping problem that captures this idea of lower cap in the sense that the decision to exercise has to be made before X drops below a certain level. Specifically, consider

$$V^*_{\epsilon_1,\epsilon_2}(x,s) = \sup_{\tau \in \mathcal{M}_{\epsilon_1}} \mathbb{E}_{x,s}\big[e^{-q\tau + \overline{X}_\tau \wedge \epsilon_2}\big], \tag{3}$$

where $\epsilon_1, \epsilon_2 \in \mathbb{R}$ such that $\epsilon_1 < \epsilon_2$, $q > 0, \mathcal{M}_{\epsilon_1} := \{\tau \in \mathcal{M} \,|\, \tau \leq T_{\epsilon_1}\}$ and $T_{\epsilon_1} := \inf\{t \geq 0 : X_t \leq \epsilon_1\}$. In the special case with no cap ($\epsilon_2 = \infty$), this problem was considered by Shepp, Shiryaev and Sulem [12] for the case where X is a linear Brownian motion. Inspired by their result we expect the optimal stopping time to be of the form $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$, where $\tau^*_{\epsilon_2}$ is the optimal stopping time in (1). Our main contribution here is that we find a closed form expression for the value function associated with the strategy $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$, thereby allowing us to verify that it is indeed an optimal strategy.

2. Notation and auxiliary results. The purpose of this section is to introduce some notation and collect some known results about spectrally negative Lévy processes. Moreover, we state the solution of the Shepp-Shiryaev optimal stopping problem (2) which will play an important role throughout this paper.

2.1. *Spectrally negative Lévy processes.* It is well known that a spectrally negative Lévy process X is characterised by its Lévy triplet (γ, σ, Π), where $\sigma \geq 0, \gamma \in \mathbb{R}$ and Π is a measure on $(-\infty, 0)$ satisfying the condition $\int_{(-\infty,0)} 1 \wedge x^2 \, \Pi(dx) < \infty$. By the Lévy-Itô decomposition, the latter may be represented in the form

$$X_t = \sigma B_t - \gamma t + X_t^{(1)} + X_t^{(2)} \tag{4}$$

where $\{B_t : t \geq 0\}$ is a standard Brownian motion, $\{X_t^{(1)} : t \geq 0\}$ is a compound Poisson process with discontinuities of magnitude bigger or equal to one and $\{X_t^{(2)} : t \geq 0\}$ is a square integrable martingale with discontinuities of magnitude strictly smaller than one and the three processes are mutually independent. In particular, if X is of bounded variation, the decomposition reduces to

$$X_t = \mathtt{d}t + S_t \tag{5}$$

where $\mathtt{d} > 0$ and $\{-S_t : t \geq 0\}$ is a driftless subordianator. Furthermore, the spectral negativity of X ensures existence of the Laplace exponent ψ of X, that is, $\mathbb{E}[e^{\theta X_1}] = e^{\psi(\theta)}$ for $\theta \geq 0$, which is known to take the form

$$\psi(\theta) = -\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} e^{\theta x} - 1 - \theta x 1_{\{x>-1\}} \, \Pi(dx).$$

Its right inverse is defined by

$$\Phi(q) := \inf\{\lambda \geq 0 : \psi(\lambda) = q\}$$

for $q \geq 0$.

For any spectrally negative Lévy process having $X_0 = 0$ we introduce the family of martingales

$$\exp(cX_t - \psi(c)t),$$

defined for any c for which $\psi(c) = \log \mathbb{E}[\exp(cX_1)] < \infty$, and further the corresponding family of measures $\{\mathbb{P}^c\}$ with Radon-Nikodym derivatives

$$\left.\frac{d\mathbb{P}^c}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp(cX_t - \psi(c)t). \tag{6}$$

For all such c the measure $\mathbb{P}_x^c$ will denote the translation of $\mathbb{P}^c$ under which $X_0 = x$.

2.2. *Scale functions.* A special family of functions associated with spectrally negative Lévy processes is that of scale functions (cf. [6]) which are defined as follows. For $q \geq 0$, the q-scale function $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\, dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

and is defined to be identically zero for $x \leq 0$. Equally important is the scale function $Z^{(q)} : \mathbb{R} \longrightarrow [1, \infty)$ defined by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z)\, dz.$$

The passage times of X above and below $k \in \mathbb{R}$ are denoted by

$$\tau_k^- = \inf\{t > 0 : X_t \leq k\} \quad \text{and} \quad \tau_k^+ = \inf\{t > 0 : X_t \geq k\}.$$

We will make use of the following four identities. For $q \geq 0$ and $x \in (a, b)$ it holds that

$$\mathbb{E}_x\left[e^{-q\tau_b^+} I_{\{\tau_b^+ < \tau_a^-\}}\right] = \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}, \tag{7}$$

$$\mathbb{E}_x\left[e^{-q\tau_a^-} I_{\{\tau_b^+ > \tau_a^-\}}\right] = Z^{(q)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}, \tag{8}$$

for $q > 0$ and $x \in \mathbb{R}$ it holds that

$$\mathbb{E}_x\left[e^{-q\tau_0^-} 1_{\{\tau_0^- < \infty\}}\right] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \tag{9}$$

and finally for $q > 0$ we have

$$\lim_{x \to \infty} \frac{Z^{(q)}(x)}{W^{(q)}(x)} = \frac{q}{\Phi(q)}. \tag{10}$$

The proofs of (7),(8) and (10) can be found in [2] and (9) is taken from [6]. For each $c \geq 0$ we denote by $W_c^{(q)}$ the q-scale function with respect to the measure $\mathbb{P}^c$. A useful formula (cf. [6]) linking the scale function under different measures is given by

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x) \tag{11}$$

for $x \geq 0$.

We conclude this subsection by stating some known regularity properties of scale functions [5].
Smoothness: For all $q \geq 0$,

$$W^{(q)}|_{(0,\infty)} \in \begin{cases} C^1(0,\infty), & \text{if } X \text{ is of bounded variation and } \Pi \text{ has no atoms,} \\ C^1(0,\infty), & \text{if } X \text{ is of unbounded variation and } \sigma = 0, \\ C^2(0,\infty), & \sigma > 0. \end{cases}$$

Continuity at the origin: For all $q \geq 0$,

$$W^{(q)}(0+) = \begin{cases} \mathtt{d}^{-1}, & \text{if } X \text{ is of bounded variation,} \\ 0, & \text{if } X \text{ is of unbounded variation.} \end{cases} \tag{12}$$

Derivative at the origin: For all $q \geq 0$,

$$W^{(q)\prime}(0+) = \begin{cases} \frac{q+\Pi(-\infty,0)}{\mathtt{d}^2}, & \text{if } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty, \\ \frac{2}{\sigma^2}, & \text{if } \sigma > 0 \text{ or } \Pi(-\infty,0) = \infty, \end{cases}$$

where we understand the second case to be $+\infty$ when $\sigma = 0$.

2.3. *Solution to the Shepp-Shiryaev optimal stopping problem.* In order to state the solution to the Shepp-Shiryaev optimal stopping problem, we give a reformulation of Lemma 2 in Section 6 of [2].

PROPOSITION 2.1. *Suppose that $W^{(q)}(0+) < q^{-1}$ and $q > \psi(1)$. Additionally assume that Π is atomless whenever X has paths of bounded variation. Then the function $h(s) := Z^{(q)}(s) - qW^{(q)}(s)$ is strictly decreasing on $(0,\infty)$ and satisfies $\lim_{s\downarrow 0} h(s) > 0$ and $\lim_{s\to\infty} h(s) = -\infty$. In particular, there exists a unique root $k^* \in (0,\infty)$ of the equation $h(s) = 0$.*

REMARK 2.2. *Note that the previous result is proved in [2] under the assumption that Π is absolutely continuous with respect to Lebesgue measure. An inspection of the proof shows that this is merely a technical assumption to ensure that $W^{(q)}$ is $C^1(0,\infty)$ which is known to be the case if X is of unbounded variation or if X is of bounded variation and Π is atomless.*

PROPOSITION 2.3.

(a) Suppose that $W^{(q)}(0+) < q^{-1}$ and $q > \psi(1)$. Assume additionally that Π is atomless whenever X has paths of bounded variation. Then the solution of (2) is given by

$$V^*(x,s) = e^s Z^{(q)}(k^* - s + x)$$

with optimal strategy

$$\tau^* := \inf\{t \geq 0 \,:\, \overline{X}_t - X_t \geq k^*\},$$

where $k^* \in (0,\infty)$ is given in Proposition 2.1.

(b) If $W^{(q)}(0+) < q^{-1}$ and $q \leq \psi(1)$, then $V^*(x,s) = \infty$.

(c) If $W^{(q)}(0+) \geq q^{-1}$, then the solution of (2) is given by $V^*(x,s) = e^s$ with associated optimal strategy $\tau^* = 0$.

Proof. Bearing in mind what was said in Remark 2.2, the first part can be extracted from Section 6 of [2].

As for the second part, first suppose that $q < \psi(1)$. By the martingale property of $\{\exp(X_t - \psi(1)t) : t \geq 0\}$, we obtain the estimate

$$\mathbb{E}_{x,s}\big[e^{-qt+\overline{X}_t}\big] \geq \mathbb{E}_{x,s}\big[e^{-qt+X_t}\big] = e^{t(\psi(1)-q)}$$

which implies the assertion. If $q = \psi(1)$, define $\tau_k := \inf\{t \geq 0 : \overline{X}_t - X_t \geq k\}$ for some $k > 0$ and write

$$\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau_k)+\overline{X}_{t\wedge\tau_k}}\big] \quad \geq \quad e^k\mathbb{E}_{x,s}\big[e^{-q\tau_k+\overline{X}_{\tau_k}}1_{\{\tau_k\leq t\}}\big] = e^k\mathbb{P}^1_{x,s}[\tau_k \leq t].$$

Since τ_k is $\mathbb{P}^1_{x,s}$-a.s. finite, the result follows.

As for the third part, note that $W^{(q)}(0+) \geq q^{-1}$ necessarily means that X is of bounded variation and in particular we have $\mathtt{d} = W^{(q)}(0+)^{-1} \leq q$. Integration by parts leads to

$$e^{-qt+\overline{X}_t} = e^s + \int_0^t e^{-qt+\overline{X}_u}(-q + \mathtt{d}1_{\{\overline{X}_u = X_u\}})\,du \leq e^s \quad \mathbb{P}_{x,s}\text{-a.s.}$$

This in conjunction with the Markov property shows that for $u \leq t$ we have

$$\mathbb{E}_{x,s}\big[e^{-qt+\overline{X}_t}\big|\mathcal{F}_u\big] = e^{-qu}\mathbb{E}_{X_u,\overline{X}_u}\big[e^{-q(t-u)+\overline{X}_{t-u}}\big] \leq e^{-qu+\overline{X}_u} \quad \mathbb{P}_{x,s}\text{-a.s.}$$

which shows that the process $\big\{\exp(-qt + \overline{X}_t) : t \geq 0\big\}$ is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for all $(x,s) \in E$. This together with Fatou's Lemma and Doob's stopping Theorem gives for any $\tau \in \mathcal{M}$,

$$\mathbb{E}_{x,s}\big[e^{-q\tau+\overline{X}_\tau}\big] \leq \liminf_{t\to\infty} \mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau)+\overline{X}_{t\wedge\tau}}\big] \leq e^s,$$

with equality for $\tau = 0$. This completes the proof. $\square$

3. Main results.

3.1. *Maximum process with upper cap.* Define

$$k^* := \inf\{z \in \mathbb{R} : Z^{(q)}(z) \le qW^{(q)}(z)\},$$

where we set $\inf \varnothing = \infty$. The next result ensures existence of a function g which will describe the optimal stopping boundary of problem (1).

LEMMA 3.1. *Let $\epsilon \in \mathbb{R}$ be given and suppose that $W^{(q)}(0+) < q^{-1}$. Additionally assume that Π is atomless whenever X has paths of bounded variation.*

a) If $q > \psi(1)$, then $k^* \in (0, \infty)$, otherwise $k^* = \infty$.
b) There exists a unique solution $g : (-\infty, \epsilon) \to (0, k^*)$ of the ordinary differential equation

$$g'(s) = 1 - \frac{Z^{(q)}(g(s))}{qW^{(q)}(g(s))} \quad \text{on } (-\infty, \epsilon) \tag{13}$$

satisfying the conditions $\lim_{s\uparrow\epsilon} g(s) = 0$ and $\lim_{s\to-\infty} g(s) = k^*$.

Next, extend g to the whole real line by setting $g(s) = 0$ for $s \ge \epsilon$. We now present the solution of (1).

THEOREM 3.2. *Let $\epsilon \in \mathbb{R}$ be given.*

a) Suppose that $W^{(q)}(0+) < q^{-1}$. Additionally assume that Π is atomless whenever X has paths of bounded variation. Then the solution of (2) is given by

$$V_\epsilon^*(x, s) = e^{s\wedge\epsilon} Z^{(q)}(x - s + g(s))$$

with corresponding optimal strategy

$$\tau_\epsilon^* := \inf\{t \ge 0 \, : \, \overline{X}_t - X_t \ge g(\overline{X}_t)\},$$

where g is given in Lemma 3.1.

b) If $W^{(q)}(0+) \ge q^{-1}$, then the solution of (2) is given by $V_\epsilon^*(x, s) = e^{s\wedge\epsilon}$ with corresponding optimal strategy $\tau_\epsilon^* = 0$.

Define the continuation region $C^* := \{(x, s) \in E \,|\, s < \epsilon, s - g(s) < x \le s\}$ and the stopping region $D^* := E \setminus C^*$. The shape of the boundary separating them, that is, the optimal stopping boundary, is of particular interest. Assuming that $W^{(q)}(0+) < q^{-1}$, Theorem 3.2 and (12) show that

$$\lim_{s\uparrow\epsilon} g'(s) = \begin{cases} -\infty, & \text{if } X \text{ is of unbounded variation,} \\ 1 - \mathtt{d}/q, & \text{if } X \text{ is of bounded variation.} \end{cases}$$

Also, using (10) we see that

$$\lim_{s\to-\infty} g'(s) = \begin{cases} 0, & \text{if } q > \psi(1), \\ 1-\Phi(q)^{-1}, & \text{if } q \le \psi(1). \end{cases}$$

This (qualitative) behaviour of g and the resulting shape of the continuation and stopping region are illustrated in Figure 1. Note in particular that the shape of g at ϵ (and consequently the optimal boundary) changes according to the path variation of X.

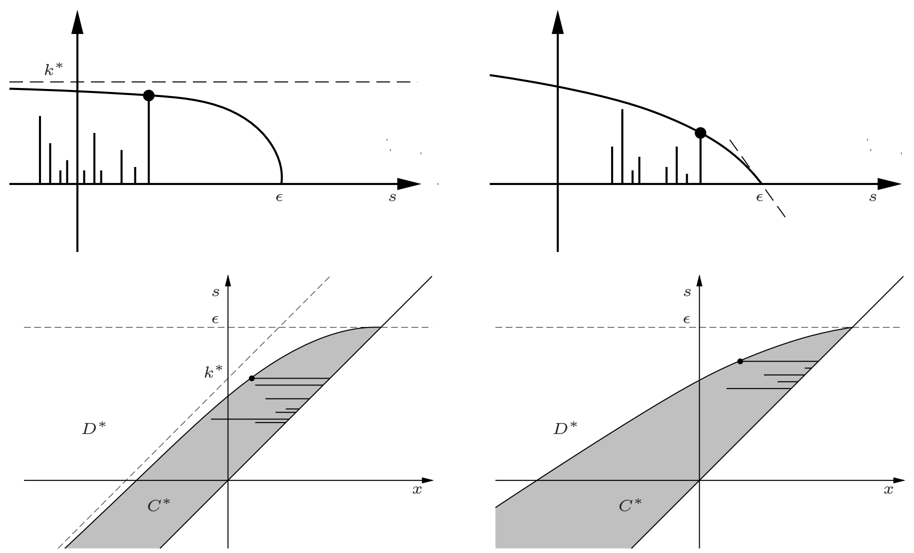

FIG 1. *For the two pictures on the left it is assumed that* $W^{(q)}(0+) = 0$ *and* $q > \psi(1)$ *whereas on the right it is assumed that* $W^{(q)}(0+) \in (0, q^{-1})$ *and* $q \le \psi(1)$.

The horizontal and vertical lines are meant to indicate a realisation of the trace of the process $(X, \overline{X})$ in the (x,s)-plane. Each horizontal line corresponds to the trace of an excursion away from the maximum. In other words, the optimal strategy consists of continuing if the height of the excursion away from the running supremum s does not exceed $g(s)$, otherwise we stop.

Finally, in the case $\mathtt{d}^{-1} = W^{(q)}(0+) \ge q^{-1}$ Theorem 3.2 asserts that the optimal strategy is to stop immediately. This is intuitively clear since $\mathtt{d} \le q$ means that even when the process $\overline{X}$ is increasing, the process $\exp(-qt + \overline{X}_t \wedge \epsilon)$ is decreasing, i.e., it is best to stop immediately.

3.2. *Maximum process with upper and lower cap.* Inspired by the result in [12], we expect the strategy $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$ to be optimal, where $\tau^*_{\epsilon_2}$ is given in Theorem 3.2 and $T_{\epsilon_1} := \inf\{t \geq 0 : X_t \leq \epsilon_1\}$. This means that the optimal boundary is expected to be a vertical line at ϵ_1 combined with the curve described by $g = g_{\epsilon_2}$ characterised in Lemma 3.1. If we define $A_{\epsilon_1,\epsilon_2} = A \in \mathbb{R}$ to be the unique solution to the equation $A - g(A) = \epsilon_1$, then our candidate optimal strategy splits the state space into the stopping regions

$$\begin{aligned} D^*_{I,\epsilon_1,\epsilon_2} = D^*_I &:= \{(x,s) \in E : x = \epsilon_1, \epsilon_1 \leq s \leq A\}, \\ D^*_{II,\epsilon_1,\epsilon_2} = D^*_{II} &:= \{(x,s) \in E : \epsilon_1 \leq x \leq s - g(s), s > A\}, \end{aligned}$$

and the continuation regions

$$\begin{aligned} C^*_{I,\epsilon_1,\epsilon_2} = C^*_I &:= \{(x,s) \in E : \epsilon_1 < x \leq s, \epsilon_1 < s < A\}, \\ C^*_{II,\epsilon_1,\epsilon_2} = C^*_{II} &:= \{(x,s) \in E : s - g(s) < x \leq s, A \leq s < \epsilon_2\}. \end{aligned}$$

Also let $E_{\epsilon_1} := \{(x,s) \in E \,|\, x \geq \epsilon_1\}$.

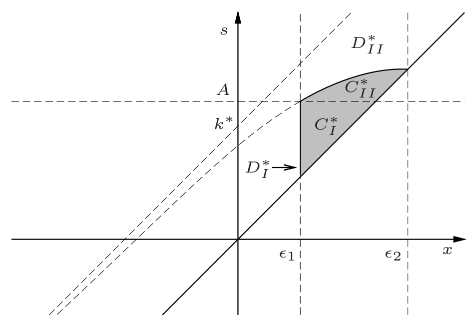

FIG 2. *A qualitative picture of the continuation and stopping region under the assumption that $W^{(q)}(0+) = 0$ and $q > \psi(1)$ (cf. Theorem 3.4).*

Clearly, if $(x,s) \in E \setminus E_{\epsilon_1}$, then the only stopping time in $\mathcal{M}_{\epsilon_1}$ is $\tau = 0$ and hence the optimal value function is given by $e^{s \wedge \epsilon_2}$. Furthermore, when $(x,s) \in C^*_{II} \cup D^*_{II}$ we have $\tau^*_{\epsilon_2} \leq T_{\epsilon_1}$, so that the optimality of $\tau^*_{\epsilon_2}$ in (1) implies $V^*_{\epsilon_1,\epsilon_2}(x,s) = V^*_{\epsilon_2}(x,s)$. Consequently, the interesting case is really $(x,s) \in C^*_I \cup D^*_I$. The key to verifying that $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$ is optimal, is to find the value function associated with it.

LEMMA 3.3. *Let $\epsilon_1 < \epsilon_2$ be given and suppose that $W^{(q)}(0+) < q^{-1}$. Additionally assume that Π is atomless whenever X has paths of bounded* *variation. We then have*

$$\begin{aligned}\mathbb{E}_{x,s}\Big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\wedge\epsilon_2}\Big] &= V_{\epsilon_1,\epsilon_2}(x,s)\\ &:= \begin{cases} V^*_{\epsilon_2}(x,s), & (x,s)\in C^*_{II}\cup D^*_{II},\\ U_{\epsilon_1,\epsilon_2}(x,s), & (x,s)\in C^*_{I}\cup D^*_{I},\\ e^{s\wedge\epsilon_2}, & \textit{otherwise,}\end{cases}\end{aligned}$$

where $V^*_{\epsilon_2}$ is given in Theorem 3.2 and

$$U_{\epsilon_1,\epsilon_2}(x,s) = e^s Z^{(q)}(x-\epsilon_1) + e^{\epsilon_1}W^{(q)}(x-\epsilon_1)\int_{s-\epsilon_1}^{g(A)} e^t \frac{Z^{(q)}(t)}{W^{(q)}(t)}\,dt.$$

Our main contribution here is the closed form expression for $U_{\epsilon_1,\epsilon_2}$, thereby allowing us to verify that the strategy $T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}$ is still optimal.

THEOREM 3.4. *Let $\epsilon_1 < \epsilon_2$ be given.*

a) Suppose that $W^{(q)}(0+) < q^{-1}$. Additionally assume that Π is atomless whenever X has paths of bounded variation. Then the solution to (3) is given by $V^*_{\epsilon_1,\epsilon_2} = V_{\epsilon_1,\epsilon_2}$ with corresponding optimal strategy $\tau^*_{\epsilon_1,\epsilon_2} = T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}$, where $\tau^*_{\epsilon_2}$ is given in Theorem 3.2.

b) If $W^{(q)}(0+)\geq q^{-1}$, then the solution to (3) is given by $V^*_{\epsilon_1,\epsilon_2}(x,s) = e^{s\wedge\epsilon_2}$ with corresponding optimal strategy $\tau^*_{\epsilon_1,\epsilon_2}=0$.

Moreover, going through the proof of Theorem 3.4 with $g\equiv k^*\in(0,\infty]$ and $A=\epsilon_1+k^*$, one obtains the solution to (3) with lower cap only.

COROLLARY 3.5. *Suppose that $\epsilon_2=\infty$, that is, there is no upper cap.*

a) Let $\epsilon_1\in\mathbb{R}$ and suppose that $W^{(q)}(0+) < q^{-1}$ and $q>\psi(1)$. Additionally assume that Π is atomless whenever X has paths of bounded variation. Then the solution to (3) is given by

$$V^*_{\epsilon_1,\infty}(x,s) = \begin{cases} V^*(x,s), & (x,s)\in C^*_{II}\cup D^*_{II},\\ U_{\epsilon_1,\infty}(x,s), & (x,s)\in C^*_{I}\cup D^*_{I},\\ e^{s}, & \textit{otherwise,}\end{cases}$$

where V^* is given in Proposition 2.3 and

$$U_{\epsilon_1,\infty}(x,s) = e^s Z^{(q)}(x-\epsilon_1) + e^{\epsilon_1}W^{(q)}(x-\epsilon_1)\int_{s-\epsilon_1}^{k^*} e^t \frac{Z^{(q)}(t)}{W^{(q)}(t)}\,dt.$$

The corresponding optimal strategy is given by $\tau^*_{\epsilon_1,\infty} = T_{\epsilon_1}\wedge\tau^*$, where τ^* is given in Proposition 2.3.

b) If $W^{(q)}(0+) < q^{-1}$ and $q \leq \psi(1)$, then

$$V^*_{\epsilon_1,\infty}(x,s) = \begin{cases} e^s, & \text{if } x \leq \epsilon_1, \\ \infty, & \text{otherwise.} \end{cases}$$

c) If $W^{(q)}(0+) \geq q^{-1}$, then the solution of (3) is given by $V^*_{\epsilon_1,\infty}(x,s) = e^s$ with associated optimal strategy $\tau^*_{\epsilon_1,\infty} = 0$.

In particular, if $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$, where $\mu \in \mathbb{R}, \sigma > 0$ and $(W_t)_{t\geq 0}$ is a standard Brownian motion, then Corollary 3.5 is nothing else than Theorem 3.1 of [12]. However, this is not immediately clear an requires a simple, but lengthy computation which is provided in Section 6.

4. Guess and verify via principle of smooth or continuous fit. Let us consider the solution to (1) from an intuitive point of view. We shall restrict ourselves to the case where $W^{(q)}(0+) < q^{-1}$ and $q > \psi(1)$. By Proposition 2.3, it follows in particular that $k^* \in (0,\infty)$.

It is clear that if $(x,s) \in E$ such that $x \geq \epsilon$, then it is optimal to stop immediately since one cannot obtain a higher payoff than ϵ and waiting is penalised by exponential discounting. If x is much smaller than ϵ, then the cap ϵ should not have too much influence and one expects that the optimal value function V^*_ϵ and the corresponding optimal strategy τ^*_ϵ look similar to the optimal value function V^* and optimal strategy τ^* of problem (2). However, if x is close to the cap (or the process gets close to the cap and has not yet been stopped), then the process X should be stopped "before" it is a distance k^* away from its running maximum. This can be explained as follows: The constant k^* in the solution to problem (2) quantifies the acceptable "waiting time" for a much higher running supremum at a later point in time. But if we impose a cap, there is no hope for a much higher supremum and therefore "waiting the acceptable time" for problem (2) does not pay off in the situation with cap. With exponential discounting we would therefore expect to exercise earlier. In other words, we expect an optimal strategy of the form

$$\tau_g = \inf\{t \geq 0 \,:\, \overline{X}_t - X_t \geq g(\overline{X})\}$$

for some function g satisfying $\lim_{s\to-\infty} g(s) = k^*$ and $\lim_{s\to\epsilon} g(s) = 0$.

This qualitative guess can be turned into a quantitative guess by appealing to the general theory of optimal stopping (cf. [8]). To this end, assume that X is of unbounded variation ($W^{(q)}(0+) = 0$). We will deal with the bounded variation case later. From the general theory we informally expect the value function $V_g(x,s)=\mathbb{E}_{x,s}\big[e^{-q\tau_g+\overline{X}_{\tau_g}}\big]$ to satisfy the system

$$\begin{aligned} \Gamma V_g(x,s) = qV_g(x,s) \quad & \text{for } s-g(s)<x<s \text{ with } s \text{ fixed,} \\ \tfrac{\partial V_g}{\partial s}(x,s)\big|_{x=s-} = 0 \quad & \text{(normal reflection),} \\ V_g(x,s)|_{x=(s-g(s))+} = e^s \quad & \text{(instantaneous stopping),} \end{aligned} \tag{14}$$

where Γ is the infinitesimal generator of the process X under $\mathbb{P}_0$. Moreover, the principle of smooth (cf. [1, 8]) fit suggests that this system should be complemented by

$$\frac{\partial V_g}{\partial x}(x,s)\big|_{x=(s-g(s))+} = 0 \quad \text{(smooth fit).} \tag{15}$$

Note that the smooth fit condition is not necessarily part of the general theory, it is imposed since we believe that it should hold in this setting. This belief will be vindicated when we show that system (14) with (15) leads to the solution of problem (1). Applying the strong Markov property at τ_s^+ and using (7) and (8) shows that

$$\begin{aligned} V_g(x,s) &= e^s\mathbb{E}_{x,s}\big[e^{-q\tau^-_{s-g(s)}}1_{\{\tau^-_{s-g(s)}<\tau_s^+\}}\big] \\ &\quad +\mathbb{E}_{x,s}\big[e^{-q\tau_s^+}1_{\{\tau^-_{s-g(s)}>\tau_s^+\}}\big]\mathbb{E}_{s,s}\big[e^{-q\tau_g+\overline{X}_{\tau_g}}\big] \\ &= e^s\bigg(Z^{(q)}(x-s+g(s))-W^{(q)}(x-s+g(s))\frac{Z^{(q)}(g(s))}{W^{(q)}(g(s))}\bigg) \\ &\quad +\frac{W^{(q)}(x-s+g(s))}{W^{(q)}(g(s))}V_g(s,s). \end{aligned}$$

Furthermore, the condition of smooth fit implies

$$0=\lim_{x\downarrow s-g(s)}\frac{\partial V_g}{\partial x}(x,s)=\frac{W^{(q)\prime}(0+)}{W^{(q)}(g(s))}\bigg(-e^sZ^{(q)}(g(s))+V_g(s,s)\bigg).$$

Since the first factor is strictly positive, this shows that $V_g(s,s)$ equals $e^sZ^{(q)}(g(s))$. This would mean that for $(x,s)\in E$ such that $s-g(s)<x<s$ we have

$$V_g(x,s)=e^sZ^{(q)}(x-s+g(s)). \tag{16}$$

Having derived the form of a candidate optimal value function V_g, we still need to do the same for g. Using the normal reflection condition shows that our candidate function g should satisfy the ordinary differential equation

$$Z^{(q)}(g(s))+qW^{(q)}(g(s))(g'(s)-1)=0.$$

If X is of bounded variation ($W^{(q)}(0+) \in (0, q^{-1})$) and $q > \psi(1)$, we informally expect from the general theory that V_g satisfies the first two equations of (14). Additionally, the principle of continuous fit suggests that the system should be complemented by

$$V_g(x,s)|_{x=(s-g(s))+} = e^s \quad \text{(continuous fit)}.$$

A very similar argument as above produces the same candidate value function and the same ordinary differential equation for g. One also sees that exactly the same argument leads to the same conclusion if $W^{(0+)}(q) < q^{-1}$ and $q \leq \psi(1)$.

5. Proofs of main results.

Proof of Lemma 3.1. The idea is to define a suitable bijection H from $(0, k^*)$ to $(-\infty, \epsilon)$ whose inverse satisfies the differential equation and the boundary conditions.

First consider the case $q > \psi(1)$. It follows from Proposition 2.1 that $k^* \in (0, \infty)$ and that the function $s \mapsto f(s) := 1 - \frac{Z^{(q)}(s)}{qW^{(q)}(s)}$ is negative on $(0, k^*)$ and satisfies $\lim_{s \downarrow 0} f(s) \in [-\infty, 0)$ and $\lim_{s \uparrow k^*} f(s) = 0$. These properties imply that the function $H : (0, k^*) \to (-\infty, \epsilon)$ defined by

$$H(s) := \epsilon + \int_0^s \left(1 - \frac{Z^{(q)}(\eta)}{qW^{(q)}(\eta)}\right)^{-1} d\eta = \epsilon + \int_0^s \frac{qW^{(q)}(\eta)}{qW^{(q)}(\eta) - Z^{(q)}(\eta)}\, d\eta \tag{17}$$

is strictly decreasing. If we can also show that the integral tends to $-\infty$ as s approaches k^* we could deduce that H is a bijection from $(0, k^*)$ to $(-\infty, \epsilon)$. Indeed, appealing to de l'Hospital's rule and using (9) we obtain

$$\begin{aligned}
\lim_{z \uparrow k^*} \frac{qW^{(q)}(z) - Z^{(q)}(z)}{k^* - z} &= \lim_{z \uparrow k^*} qW^{(q)}(z) - qW^{(q)\prime}(z) \\
&= \lim_{z \uparrow k^*} qe^{\Phi(q)z}\big((1 - \Phi(q))W_{\Phi(q)}(z) - W'_{\Phi(q)}(z)\big) \\
&= qe^{\Phi(q)k^*}\big((1 - \Phi(q))W_{\Phi(q)}(k^*) - W'_{\Phi(q)}(k^*)\big).
\end{aligned}$$

Denote the term on the right-hand side by c and note that $c < 0$ due to the fact that $W_{\Phi(q)}$ is strictly positive and increasing on $(0, \infty)$ and since $\Phi(q) > 1$ for $q > \psi(1)$. Hence, there exists a $\delta > 0$ and $0 < z_0 < k^*$ such that $c - \delta < \frac{qW^{(q)}(z) - Z^{(q)}(z)}{k^* - z}$ for all $z_0 < z < k^*$. Thus,

$$\frac{1}{qW^{(q)}(z) - Z^{(q)}(z)} < \frac{1}{(c - \delta)(k^* - z)} < 0 \quad \text{for } z_0 < z < k^*.$$

This shows that

$$\lim_{s\uparrow k^*} H(s) \leq \epsilon + \lim_{s\uparrow k^*} \int_{z_0}^{s} \frac{qW^{(q)}(\eta)}{(c-\delta)(k^*-\eta)}\, d\eta = -\infty.$$

The discussion above permits us to define $g := H^{-1} \in C^1((-\infty,\epsilon);(0,k^*))$. In particular, differentiating g gives

$$g'(s) = \frac{1}{H'(g(s))} = 1 - \frac{Z^{(q)}(g(s))}{qW^{(q)}(g(s))}$$

for $s \in (-\infty,\epsilon)$ and g also satisfies $\lim_{s\to-\infty} g(s) = k^*$ and $\lim_{s\uparrow\epsilon} g(s) = 0$ by construction.

As for the case $q \leq \psi(1)$, note that by (9) and (10) we have

$$Z^{(q)}(x) - qW^{(q)}(x) \geq Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x) > 0 \tag{18}$$

for $x \geq 0$ which shows that $k^* = \infty$. Moreover, (18) together with (10) implies that the map $s \mapsto f(s)$ is negative on $(0,\infty)$ and satisfies $\lim_{s\downarrow 0} f(s) \in [-\infty, 0)$ and $\lim_{s\to\infty} f(s) = 1 - \Phi(q)^{-1} \leq 0$. Defining $H : (0,\infty) \to (-\infty,\epsilon)$ as in (17), one deduces similarly as above that H is a continuously differentiable bijection whose inverse satisfies the requirements.

We finish the proof by addressing the question of uniqueness. To this end, assume that there is another solution $\tilde{g}$. In particular, $\tilde{g}'(s) = f(\tilde{g}(s))$ for $s \in (s_1,\epsilon) \subset (-\infty,\epsilon)$ and

$$s_1 = \epsilon - \int_{(s_1,\epsilon)} d\eta = \epsilon + \int_{(s_1,\epsilon)} \frac{|\tilde{g}'(s)|}{f(\tilde{g}(s))}\, ds = \epsilon + \int_0^{\tilde{g}(s_1)} \frac{1}{f(s)}\, ds = H(\tilde{g}(s_1))$$

which implies that $\tilde{g} = H^{-1} = g$. $\square$

Proof of Theorem 3.2 when $W^{(q)}(0+) < q^{-1}$. Define the function

$$V_\epsilon(x,s) := e^{s\wedge\epsilon} Z^{(q)}(x - s + g(s))$$

for $(x,s) \in E$ and let $\tau_g := \inf\{t \geq 0 : \overline{X}_t - X_t \geq g(\overline{X}_t)\}$, where g is as in Lemma 3.1. Because of the infinite horizon and Markovian claim structure of problem (1) it is enough to check the following conditions:

(i) $V_\epsilon(x,s) \geq e^{s\wedge\epsilon}$ for all $(x,s) \in E$,
(ii) $\{e^{-qt} V_\epsilon(X_t,\overline{X}_t)\}$ is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for $(x,s) \in E$,
(iii) $V_\epsilon(x,s) = \mathbb{E}_{x,s}\big[e^{-q\tau_g + \overline{X}_{\tau_g}\wedge\epsilon}\big]$ for all $(x,s) \in E$.

To see why these are sufficient conditions, note that (i) and (ii) together with Fatou's Lemma in the second inequality and Doob's stopping theorem in the third inequality show that for $\tau \in \mathcal{M}$,

$$\begin{aligned}\mathbb{E}_{x,s}\big[e^{-q\tau+\overline{X}_\tau\wedge\epsilon}\big] &\leq \mathbb{E}_{x,s}\big[e^{-q\tau}V_\epsilon(X_\tau,\overline{X}_\tau)\big]\\ &\leq \liminf_{t\to\infty}\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau)}V_\epsilon(X_{t\wedge\tau},\overline{X}_{t\wedge\tau})\big]\\ &\leq V_\epsilon(x,s),\end{aligned}$$

which in view of (iii) implies $V_\epsilon^* = V_\epsilon$ and $\tau_\epsilon^* = \tau_g$.

The remainder of this proof is devoted to checking conditions (i)-(iii). Clearly, condition (i) is satisfied since $Z^{(q)}(x,s) \geq 1$ by definition of $Z^{(q)}$.

Supermartingale property (ii). Given the inequality

$$\mathbb{E}_{x,s}\big[e^{-qt}V_\epsilon(X_t,\overline{X}_t)\big] \leq V_\epsilon(x,s), \quad (x,s)\in E, \tag{19}$$

the supermartingale property is a consequence of the Markov property of the process $(X,\overline{X})$. Indeed, for $u \leq t$ we have

$$\begin{aligned}\mathbb{E}_{x,s}\big[e^{-qt}V_\epsilon(X_t,\overline{X}_t)\big|\mathcal{F}_u\big] &= e^{-qu}\mathbb{E}_{X_u,\overline{X}_u}\big[e^{-q(t-u)}V_\epsilon(X_{t-u},\overline{X}_{t-u})\big]\\ &\leq e^{-qu}V_\epsilon(X_u,\overline{X}_u).\end{aligned}$$

We now prove (19), first under the assumption that $W^{(q)}(0+) = 0$, that is, X is of unbounded variation. Let Γ be the infinitesimal generator of X and formally define the function

$$\begin{aligned}\Gamma Z^{(q)}(x) &:= -\gamma Z^{(q)\prime}(x) + \frac{\sigma^2}{2}Z^{(q)\prime\prime}(x)\\ &\quad+\int_{(-\infty,0)} Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y\geq -1\}}\,\Pi(dy)\end{aligned}$$

It is shown in Lemma A.1 and A.2 in the Appendix that the function $x \mapsto \Gamma Z^{(q)}(x)$ is well-defined on $(0,\infty)$ and satisfies

$$\Gamma Z^{(q)}(x) = qZ^{(q)}(x) \quad x \in (0,\infty).$$

For $x < 0$ it is also well-defined and obviously $\Gamma Z^{(q)}(x) = 0$. At zero the second derivative of $Z^{(q)}$ does not exist. In this case we understand the second derivative as the left derivative and hence $\Gamma Z^{(q)}(0) = 0$.

Now fix $(x,s)\in E$ and define the semimartingale $Y_t := X_t - \overline{X}_t + g(\overline{X}_t)$. Applying an appropriate version of the Itô-Meyer formula (cf. Theorem 71, Ch. IV of [9]) to $Z^{(q)}(Y_t)$ yields $\mathbb{P}_{x,s}$-a.s.

$$\begin{aligned} Z^{(q)}(Y_t) &= Z^{(q)}(x-s+g(s)) + m_t + \int_0^t \Gamma Z^{(q)}(Y_u)\,du \\ &\quad + \int_0^t Z^{(q)\prime}(Y_u)(g'(\overline{X}_u)-1)\,d\overline{X}_u, \end{aligned}$$

where

$$\begin{aligned} &m_t = \int_{0+}^t \sigma Z^{(q)\prime}(Y_{u-})dB_u + \int_{0+}^t Z^{(q)\prime}(Y_{u-})dX^{(2)}_u \\ &+ \sum_{0<u\leq t} \Delta Z^{(q)}(Y_u) - \Delta X_u Z^{(q)\prime}(Y_{u-})1_{\{\Delta X_u\geq -1\}} \\ &- \int_0^t\int_{(-\infty,0)} Z^{(q)}(Y_{u-}+y) - Z^{(q)}(Y_{u-}) - yZ^{(q)\prime}(Y_{u-})1_{\{y\geq -1\}}\,\Pi(dy)du \end{aligned}$$

and $\Delta X_u = X_u - X_{u-}$, $\Delta Z^{(q)}(Y_u) = Z^{(q)}(Y_u) - Z^{(q)}(Y_{u-})$. By the boundedness of $Z^{(q)}$ on $(-\infty, g(s)]$ the first two stochastic integrals on the right are zero-mean martingales and by the compensation formula (cf. Corollary 4.6 of [6]) the third and fourth term constitute a zero-mean martingale. Next, recall that $V_\epsilon(x,s) = e^{s\wedge\epsilon}Z^{(q)}(x-s+g(s))$ and use stochastic integration by parts for semimartingales (cf. Corollary 2 of Theorem 22, Ch. II of [9]) to deduce that

$$\begin{aligned} &e^{-qt}V_\epsilon(X_t,\overline{X}_t) = V_\epsilon(x,s) + M_t + \int_0^t e^{-qu+\overline{X}_u\wedge\epsilon}(\Gamma - q)Z^{(q)}(Y_u)\,du \\ &+ \int_0^t e^{-qu+\overline{X}_t\wedge\epsilon}\big(Z^{(q)}(Y_u)1_{\{\overline{X}_u\leq\epsilon\}} + Z^{(q)\prime}(Y_u)(g'(\overline{X}_u)-1)\big)\,d\overline{X}_u \end{aligned}$$

where $M_t = \int_{0+}^t e^{-qu+\overline{X}_u\wedge\epsilon}dm_u$ is a zero-mean martingale. The first integral is nonpositive since $(\Gamma-q)Z^{(q)}(y)\leq 0$ for all $y\in\mathbb{R}$. The last integral vanishes since the process $\overline{X}_u$ only increments when $\overline{X}_u = X_u$ and by definition of g. Thus, taking expectations on both sides yields

$$\mathbb{E}_{x,s}\Big[e^{-qt}V_\epsilon(X_t,\overline{X}_t)\Big] \leq V_\epsilon(x,s).$$

If $W^{(q)}(0+)\in(0,q^{-1})$ (X has bounded variation), then the Itô-Meyer formula is nothing more than an appropriate version of the change of variable formula for Stieltjes' integrals and the rest of the proof follows the same line of reasoning as above. The only change worth mentioning is that the generator of X takes a different form. Specifically, one has to work with

$$\Gamma Z^{(q)}(x) = \mathtt{d} Z^{(q)\prime}(x) + \int_{(-\infty,0)} \big| Z^{(q)}(x+y) - Z^{(q)}(x) \big| \Pi(dy)$$

which satisfies all the required properties by Lemma A.1 and A.2 in the Appendix.

This completes the proof of the supermartingale property.

Verification of condition (iii). The assertion is clear for $(x,s) \in D^*$. Hence, suppose that $(x,s) \in C^*$. The assertion now follows from the proof of the supermartingale property (ii). More precisely, replacing t by $t \wedge \tau_g$ in (20) and recalling that $(\Gamma - q)Z^{(q)}(y) = 0$ for $y > 0$ shows that

$$\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau_g)+\overline{X}_{t\wedge\tau_g}\wedge\epsilon}\big] = V_\epsilon(x,s).$$

Using that $\tau_g < \infty$ a.s. and dominated convergence one obtains the desired equality. $\square$

Proof of Theorem 3.2 when $W^{(q)}(0+) \geq q^{-1}$. Recall that the condition $W^{(q)}(0+) \geq q^{-1}$ necessarily means that X is of bounded variation and $\mathtt{d} \leq q$ which after integrating by parts shows that $\mathbb{P}_{x,s}$-a.s.

$$e^{-qt+\overline{X}_t\wedge\epsilon} = e^{s\wedge\epsilon} + \int_0^t e^{-qu+\overline{X}_u\wedge\epsilon}\big(-q + \mathtt{d}1_{\{\overline{X}_u = X_u, \overline{X}_u \leq \epsilon\}}\big)\, du \leq e^{s\wedge\epsilon}.$$

The result now follows similarly to the proof of the second part of Proposition 2.3. $\square$

Proof of Lemma 3.3. For $(x,s) \in D_I^*$ we obviously have

$$\mathbb{E}_{x,s}\big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\wedge\epsilon_2}\big] = e^s = U_{\epsilon_1,\epsilon_2}(x,s).$$

As for the case $(x,s) \in C_I^*$, write

$$\begin{aligned}\mathbb{E}_{x,s}\big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\wedge\epsilon_2}\big] &= \mathbb{E}_{x,s}\Big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}} 1_{\{T_{\epsilon_1} > \tau_A^+\}}\Big] \\ &+ \mathbb{E}_{x,s}\Big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}} 1_{\{T_{\epsilon_1} < \tau_A^+\}}\Big]\end{aligned}$$

and denote the first expectation on the right by I_1 and the second expectation by I_2. An application of the strong Markov property at τ_A^+ and the definition of $V^*_{\epsilon_2}$ (see Theorem 3.2) give

$$\begin{aligned} I_1 &= \mathbb{E}_{x,s}\Big[e^{-q\tau_A^+}1_{\{T_{\epsilon_1}>\tau_A^+\}}\Big]\mathbb{E}_{A,A}\Big[e^{-q\tau^*_{\epsilon_2}+\overline{X}_{\tau^*_{\epsilon_2}}}\Big] \\ &= \frac{W^{(q)}(x-\epsilon_1)}{W^{(q)}(A-\epsilon_1)}e^A Z^{(q)}(g(A)). \end{aligned}$$

Recalling that $s<g(A)$ and using the strong Markov property at τ_s^+ yields

$$\begin{aligned} I_2 &= e^s\mathbb{E}_{x,s}\Big[e^{-qT_{\epsilon_1}}1_{\{T_{\epsilon_1}<\tau_s^+\}}\Big] \\ &\quad +\mathbb{E}_{x,s}\Big[e^{-q\tau_s^+}1_{\{T_{\epsilon_1}>\tau_s^+\}}\Big]\mathbb{E}_{s,s}\Big[e^{-qT_{\epsilon_1}+\overline{X}_{T_{\epsilon_1}}}1_{\{T_{\epsilon_1}<\tau_A^+\}}\Big] \\ &= e^s\bigg(Z^{(q)}(x-\epsilon_1)-W^{(q)}(x-\epsilon_1)\frac{Z^{(q)}(s-\epsilon_1)}{W^{(q)}(s-\epsilon_1)}\bigg) \\ &\quad +\frac{W^{(q)}(x-\epsilon_1)}{W^{(q)}(s-\epsilon_1)}\mathbb{E}_{s,s}\Big[e^{-qT_{\epsilon_1}+\overline{X}_{T_{\epsilon_1}}}1_{\{T_{\epsilon_1}<\tau_A^+\}}\Big] \\ &= e^s\bigg(Z^{(q)}(x-\epsilon_1)-W^{(q)}(x-\epsilon_1)\frac{Z^{(q)}(s-\epsilon_1)}{W^{(q)}(s-\epsilon_1)}\bigg) \\ &\quad +\frac{W^{(q)}(x-\epsilon_1)}{W^{(q)}(s-\epsilon_1)}e^s\mathbb{E}_{0,0}\Big[e^{-q\tau^-_{\epsilon_1-s}+L_{\tau^-_{\epsilon_1-s}}}1_{\{\tau^-_{\epsilon_1-s}<\tau^+_{A-s}\}}\Big]. \end{aligned} \tag{20}$$

Next, we compute the expectation on the right-hand side of (20) by excursion theory. To be more precise, we are going to make use of the compensation formula of excursion theory and hence we shall spend a moment setting up some necessary notation. We refer the reader to [3], Chapters 6 and 7 for background reading. The process $L_t:=\overline{X}$ serves as local time at 0 for the Markov process $\overline{X}-X$ under $\mathbb{P}_{0,0}$. Write $L^{-1}:=\{L_t^{-1}:t\geq 0\}$ for the right-continuous inverse of L. The Poisson point process of excursions indexed by local time shall be denoted by $\{(t,\varepsilon_t):t\geq 0\}$, where

$$\varepsilon_t=\{\varepsilon_t(s):=X_{L_t^{-1}}-X_{L_t^{-1}+s}:0<s\leq L_t^{-1}-L_{t-}^{-1}\}$$

whenever $L_t^{-1}-L_{t-}^{-1}>0$. Accordingly, we refer to a generic excursion as $\varepsilon(\cdot)$ (or just ε for short as appropriate) belonging to the space $\mathcal{E}$ of canonical excursions. The intensity measure of the process $\{(t,\varepsilon_t):t\geq 0\}$ is given by $dt\times dn$, where n is a measure on the space of excursions (the excursion measure). A functional of the canonical excursion which will be of interest is $\overline{\varepsilon}=\sup_{s\geq 0}\varepsilon(s)$. A useful formula for this functional that we shall make use of is the following (cf. [6], Equation (8.18)):

$$n(\overline{\varepsilon}>x)=\frac{W'(x)}{W(x)} \tag{21}$$

provided that x is not a discontinuity point in the derivative of W (which is only a concern when X is of bounded variation, but we have assumed that in this case Π is atomless and hence W is continuously differentiable on $(0,\infty)$). Another functional that we will also use is $\rho_a := \inf\{s>0 : \varepsilon(s)>a\}$, the first passage time above a of the canonical excursion ε. We now proceed with the promised calculation involving excursion theory. Specifically, an application of the compensation formula gives

$$\begin{aligned}
&\mathbb{E}\Big[e^{-q\tau^-_{\epsilon_1-s}+L_{\tau^-_{\epsilon_1-s}}}1_{\{\tau^-_{\epsilon_1-s}<\tau^+_{A-s}\}}\Big]\\
&=\mathbb{E}\left[\sum_{0<t<\infty} e^{-qL^{-1}_{t-}+t}1_{\left\{\substack{\overline{\varepsilon}_u\le u-\epsilon_1+s\,\forall\, u<t\\ t<A-s}\right\}}1_{\{\overline{\varepsilon}_t>t-\epsilon_1+s\}}e^{-q\rho_{t-\epsilon_1+s}(\varepsilon_t)}\right]\\
&=\mathbb{E}\left[\int_0^{A-s} dt\, e^{-qL^{-1}_t+t}1_{\{\overline{\varepsilon}_u\le u-\epsilon_1+s\,\forall\, u<t\}}\int_{\mathcal{E}}1_{\{\overline{\varepsilon}>t-\epsilon_1+s\}}e^{-q\rho_{t-\epsilon_1+s}(\varepsilon)}n(d\varepsilon)\right]\\
&=\int_0^{A-s} e^{t-\Phi(q)t}\mathbb{E}\Big[e^{-qL^{-1}_t+\Phi(q)t}1_{\{\overline{\varepsilon}_u\le u-\epsilon_1+s\,\forall\, u<t\}}\Big]f(t-\epsilon_1+s)\,dt,
\end{aligned}$$

where $f(u)=\frac{Z^{(q)}(u)W^{(q)\prime}(u)}{W^{(q)}(u)}-qW^{(q)}(u)$. The expression f is taken from Theorem 1 in [2]. Next, note that L^{-1}_t is a stopping time and hence a change of measure according to (6) shows that the expectation inside the integral can be written as

$$\mathbb{P}^{\Phi(q)}\big[\overline{\varepsilon}_u\le u-\epsilon_1+s \text{ for all } u<t\big].$$

Using the properties of the Poisson point process of excursions (indexed by local time) and with the help of (21) and (9) we may deduce

$$\begin{aligned}
\mathbb{P}^{\Phi(q)}\big[\overline{\varepsilon}_u\le u-\epsilon_1+s \text{ for all } u<t\big] &= \exp\left(-\int_0^t n_{\Phi(q)}(\overline{\varepsilon}>u-\epsilon_1+s)\,du\right)\\
&= e^{\phi(q)t}\frac{W^{(q)}(s-\epsilon_1)}{W^{(q)}(t-\epsilon_1+s)},
\end{aligned}$$

where $n_{\Phi(q)}$ denotes the excursion measure associated with X under $\mathbb{P}^{\Phi(q)}$. By a change of variables and the fact that $A-\epsilon_1=g(A)$ we further obtain

$$\begin{aligned}
&\mathbb{E}_{0,0}\Big[e^{-q\tau^-_{\epsilon_1-s}+L_{\tau^-_{\epsilon_1-s}}}1_{\{\tau^-_{\epsilon_1-s}<\tau^+_{A-s}\}}\Big]\\
&=W^{(q)}(s-\epsilon_1)e^{\epsilon_1-s}\int_{s-\epsilon_1}^{g(A)} e^t\frac{f(t)}{W^{(q)}(t)}\,dt\\
&=-W^{(q)}(s-\epsilon_1)e^{\epsilon_1-s}\int_{s-\epsilon_1}^{g(A)} e^t\left(\frac{Z^{(q)}}{W^{(q)}}\right)'(t)dt.
\end{aligned}$$

Integrating by parts on the right-hand side, plugging the resulting expression into (20) and finally adding I_1 and I_2 gives the result. $\square$

PROOF OF THEOREM 3.4. We only prove the first part, the second follows by a very similar argument as in the proof of Theorem 3.2 when $W^{(q)}(0+) \geq q^{-1}$. Recall that $T_{\epsilon_1} := \inf\{t \geq 0 : X_t \leq \epsilon_1\}$ and from Lemma 3.3 that

$$\mathbb{E}_{x,s}\big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\wedge\epsilon_2}\big] = V_{\epsilon_1}(x,s)$$
$$:= \begin{cases} V^*_{\epsilon_2}(x,s), & (x,s)\in C^*_{II}\cup D^*_{II}, \\ U_{\epsilon_1,\epsilon_2}(x,s), & (x,s)\in C^*_I\cup D^*_I, \\ e^{s\wedge\epsilon_2}, & (x,s)\in E\setminus E_{\epsilon_1}, \end{cases} \tag{22}$$

where $V^*_{\epsilon_2}$ and $\tau^*_{\epsilon_2}$ are given in Theorem 3.2. It is enough to prove

(i) $V_{\epsilon_1,\epsilon_2}(x,s) \geq e^{s\wedge\epsilon_2}$ for all $(x,s)\in E_{\epsilon_1}$,
(ii) $\{e^{-q(t\wedge T_{\epsilon_1})}V_{\epsilon_1,\epsilon_2}(X_{t\wedge T_{\epsilon_1}},\overline{X}_{t\wedge T_{\epsilon_1}})\}$ is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for all $(x,s)\in E_{\epsilon_1}$.

To see why this suffices, use (i) and (ii) together with Fatou's Lemma in the second inequality and Doob's optional stopping theorem in the third inequality to show that for $\tau\in\mathcal{M}_{\epsilon_1}$,

$$\begin{aligned}\mathbb{E}_{x,s}\big[e^{-q\tau+\overline{X}_\tau\wedge\epsilon_2}\big] &\leq \mathbb{E}_{x,s}\big[e^{-q\tau}V_{\epsilon_1,\epsilon_2}(X_\tau,\overline{X}_\tau)\big] \\ &\leq \liminf_{t\to\infty}\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau)}V_{\epsilon_1,\epsilon_2}(X_{t\wedge\tau},\overline{X}_{t\wedge\tau})\big] \\ &\leq V_{\epsilon_1,\epsilon_2}(x,s),\end{aligned}$$

which in view of (22) implies optimality of $V_{\epsilon_1,\epsilon_2}$ and $T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}$. Since condition (i) is obviously satisfied, we devote the remainder of this proof to checking condition (ii).

Supermartingale property (ii). Let $Y_t := e^{-qt}V_{\epsilon_1,\epsilon_2}(X_t,\overline{X}_t)$ and suppose that for all $(x,s)\in E_{\epsilon_1}$ we have the inequality

$$\mathbb{E}_{x,s}\big[Y_{t\wedge T_{\epsilon_1}}\big] \leq V_{\epsilon_1,\epsilon_2}(x,s). \tag{23}$$

The supermartingale property is then a consequence of the strong Markov property of $(X,\overline{X})$. Indeed, for $u\leq t$ we have

$$\begin{aligned}\mathbb{E}_{x,s}\big[Y_{t\wedge T_{\epsilon_1}}\big|\mathcal{F}_u\big] &= Y_{T_{\epsilon_1}}1_{\{T_{\epsilon_1}\leq u\}} + \mathbb{E}_{x,s}\big[Y_{t\wedge T_{\epsilon_1}}\big|\mathcal{F}_u\big]1_{\{T_{\epsilon_1}>u\}} \\ &= Y_{T_{\epsilon_1}}1_{\{T_{\epsilon_1}\leq u\}} + e^{-qu}\mathbb{E}_{X_u,\overline{X}_u}\big[Y_{(t-u)\wedge T_{\epsilon_1}}\big]1_{\{T_{\epsilon_1}>u\}} \\ &\leq Y_{T_{\epsilon_1}}1_{\{T_{\epsilon_1}\leq u\}} + e^{-qu}V_{\epsilon_1,\epsilon_2}(X_u,\overline{X}_u)1_{\{T_{\epsilon_1}>u\}} \\ &= Y_{u\wedge T_{\epsilon_1}}.\end{aligned}$$

Consequently, it boils down to proving (23). This is clear for $(x,s)\in D_I^*$. If $(x,s)\in C_{II}^*\cup D_{II}^*$ it can be extracted from the proof of Theorem 3.2 where it is shown that $\big(e^{-qt}V_{\epsilon_2}^*(X_t,\overline{X}_t)\big)_{t\geq 0}$ is a $P_{x,s}$-supermartinagle for all $(x,s)\in E$. In particular, the process $(Y_t)_{t\geq 0}$ is a $\mathbb{P}_{x,s}$-supermartingale for $(x,s)\in C_{II}^*\cup D_{II}^*$. The supermartingale property is preserved when stopping at T_{ϵ_1} and therefore we obtain, for $(x,s)\in C_{II}^*\cup D_{II}^*$,

$$\mathbb{E}_{x,s}\big[Y_{t\wedge T_{\epsilon_1}}\big]\leq V_{\epsilon_1,\epsilon_2}(x,s). \tag{24}$$

Thus, it remains to establish (23) for $(x,s)\in C_I^*$. To this end, we first prove that the process $\big(Y_{t\wedge T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\big)_{t\geq 0}$ is a $\mathbb{P}_{x,s}$-martingale. The strong Markov property gives

$$\begin{aligned}\mathbb{E}_{x,s}\big[Y_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\big|\mathcal{F}_t\big] &= Y_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}1_{\{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}\leq t\}}\\ &\quad +e^{-qt}\mathbb{E}_{X_t,\overline{X}_t}\big[Y_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\big]1_{\{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}>t\}}.\end{aligned} \tag{25}$$

By definition of $V_{\epsilon_1,\epsilon_2}$ we see that

$$Y_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}=\begin{cases}\exp\big(-qT_{\epsilon_1}+\overline{X}_{T_{\epsilon_1}}\big), & \text{on } \{T_{\epsilon_1}\leq\tau^*_{\epsilon_2}\},\\ \exp(-q\tau^*_{\epsilon_2}+\overline{X}_{\tau^*_{\epsilon_2}}), & \text{on } \{T_{\epsilon_1}>\tau^*_{\epsilon_2}\},\end{cases}$$

which shows that the second term on the right-hand side of (25) equals

$$\begin{aligned}&e^{-qt}\mathbb{E}_{X_t,\overline{X}_t}\Big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}}\Big](1_{\{t\leq\tau_A^+\}}+1_{\{t>\tau_A^+\}})1_{\{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}>t\}}\\ &=\Big(e^{-qt}U_{\epsilon_1,\epsilon_2}(X_t,\overline{X}_t)1_{\{t\leq\tau_A^+\}}+e^{-qt}V^*_{\epsilon_2}(X_t,\overline{X}_t)1_{\{t>\tau_A^+\}}\Big)1_{\{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}>t\}}\\ &=e^{-qt}V_{\epsilon_1,\epsilon_2}(X_t,\overline{X}_t)1_{\{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}>t\}}\\ &=Y_t1_{\{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}>t\}}.\end{aligned}$$

Thus,

$$\mathbb{E}_{x,s}\big[Y_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\big|\mathcal{F}_t\big]=Y_{t\wedge T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}$$

which implies the martingale property.

Again using the strong Markov property we obtain for $(x,s)\in C_I^*$,

$$\begin{aligned}\mathbb{E}_{x,s}\big[Y_{t\wedge T_{\epsilon_1}}\big|\mathcal{F}_{\tau^*_{\epsilon_2}}\big] &= Y_{t\wedge T_{\epsilon_1}}1_{\{t\wedge T_{\epsilon_1}\leq\tau^*_{\epsilon_2}\}}\\ &\quad +e^{-q\tau^*_{\epsilon_2}}\mathbb{E}_{X_{\tau^*_{\epsilon_2}},\overline{X}_{\tau^*_{\epsilon_2}}}\big[Y_{(t-u)\wedge T_{\epsilon_1}}\big]\Big|_{u=\tau^*_{\epsilon_2}}1_{\{t\wedge T_{\epsilon_1}>\tau^*_{\epsilon_2}\}}\\ &\leq Y_{t\wedge T_{\epsilon_1}}1_{\{t\wedge T_{\epsilon_1}\leq\tau^*_{\epsilon_2}\}}+e^{-q\tau^*_{\epsilon_2}}V_{\epsilon_1,\epsilon_2}(X_{\tau^*_{\epsilon_2}},\overline{X}_{\tau^*_{\epsilon_2}})1_{\{t\wedge T_{\epsilon_1}>\tau^*_{\epsilon_2}\}}\\ &= Y_{t\wedge T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}},\end{aligned}$$

where the inequality follows from (24) and the fact that $(X_{\tau^*_{\epsilon_2}}\overline{X}_{\tau^*_{\epsilon_2}})\in D^*_{II}$ on $\{t\wedge T_{\epsilon_1}>\tau^*_{\epsilon_2}\}$. Thus, $\mathbb{E}_{x,s}\big[Y_{t\wedge T_{\epsilon_1}}\big]\leq U_{\epsilon_1,\epsilon_2}(x,s)=V_{\epsilon_1,\epsilon_2}(x,s)$ for $(x,s)\in C_I^*$. This completes the proof. □

6. Examples. Suppose that $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$, where $\mu \in \mathbb{R}, \sigma > 0$ and $(W_t)_{t\geq 0}$ is a standard Brownian motion. It is well-known that in this case the scale functions are given by

$$W^{(q)}(x) = \frac{2}{\sigma^2\delta} e^{\gamma x}\sinh(\delta x) \quad \text{and} \quad Z^{(q)}(x) = e^{\gamma x}\cosh(\delta x) - \frac{\gamma}{\delta} e^{\gamma x}\sinh(\delta x),$$

on $x \geq 0$, where $\delta(q) = \delta = \sqrt{(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2q}{\sigma^2}}$ and $\gamma = \frac{1}{2} - \frac{\mu}{\sigma^2}$. Additionally, let $\gamma_1 = \gamma - \delta$ and $\gamma_2 = \gamma + \delta$ and note that γ_1 and γ_2 are the roots of the quadratic equation $\frac{\sigma^2}{2}\theta^2 + (\mu - \frac{\sigma^2}{2})\theta - q = 0$.

6.1. *Maximum process with upper cap.* Choose a cap $\epsilon \in \mathbb{R}$. Since X is of unbounded variation, we have $W^{(q)}(0+) = 0$. If $q > \psi(1)$ or, equivalently, $q > \mu$, solving $Z^{(q)}(z) - qW^{(q)}(z) = 0$ yields $k^* = \frac{1}{\gamma_2 - \gamma_1}\log\left(\frac{1-\gamma_1^{-1}}{1-\gamma_2^{-1}}\right) \in (0,\infty)$. Otherwise, we have $k^* = \infty$. The function g is then uniquely determined by the ordinary differential equation

$$g'(s) = 1 - \frac{\sigma^2\delta}{2} e^{\gamma g(s)}\coth(\delta g(s)) - \frac{\sigma^2\gamma}{2} e^{\gamma g(s)} \quad \text{on } (-\infty, \epsilon)$$

with boundary conditions $\lim_{s\to\epsilon} g(s) = 0$ and $\lim_{s\to-\infty} g(s) = k^*$.

6.2. *Maximum process with lower cap.*

Lemma 6.1. *Let $\epsilon_1 \in \mathbb{R}$. In the setting above, the V^* and $U_{\epsilon_1,\infty}$ part of the optimal value function $V^*_{\epsilon_1,\infty}$ are given by*

$$V^*(x,s) = \frac{1}{\gamma_2 - \gamma_1}\left(\gamma_2\left(\frac{e^x}{e^{s-k^*}}\right)^{\gamma_1} - \gamma_1\left(\frac{e^x}{e^{s-k^*}}\right)^{\gamma_2}\right)$$

and

$$\begin{aligned} U_{\epsilon_1,\infty}(x,s) \quad = \quad & \left(\frac{e^x}{e^{\epsilon_1}}\right)^{\gamma_1}\left[-\frac{e^{\epsilon_1}}{\beta}\left(\int_{\beta(s-\epsilon_1)}^{\beta k^*}\frac{e^{u(1+y)}}{e^u - 1}\,du - e^{k^*\gamma_2}\right)\right] \\ & + \left(\frac{e^x}{e^{\epsilon_1}}\right)^{\gamma_2}\left[\frac{e^{\epsilon_1}}{\beta}\left(\int_{\beta(s-\epsilon_1)}^{\beta k^*}\frac{e^{uy}}{e^u - 1}\,du - e^{k^*\gamma_1}\right)\right], \end{aligned}$$

where $\beta = \gamma_2 - \gamma_1 = 2\delta$ and $y = \beta^{-1}$.

In particular, if we set $\epsilon_1 = \epsilon$, $\mu = r$ for some $r \geq 0$ and $q = \lambda + r$ for some $\lambda > 0$ we recover Theorem 3.1 of [12].

PROOF. The first part is a short calculation using the definition of γ, δ, γ_1, γ_2 and the fact that $\cosh(z) = \frac{e^z+e^{-z}}{2}$ and $\sinh(z) = \frac{e^z-e^{-z}}{2}$. As for the second part, recall that

$$U_{\epsilon_1,\infty}(x,s) = e^s Z^{(q)}(x-\epsilon_1) + e^{\epsilon_1} W^{(q)}(x-\epsilon_1) \int_{s-\epsilon_1}^{k^*} e^t \frac{Z^{(q)}(t)}{W^{(q)}(t)}\,dt.$$

It is easy to see that

$$e^t \frac{Z^{(q)}(t)}{W^{(q)}(t)} = e^t \frac{\delta\sigma^2}{2}\left(\frac{1}{1-e^{-2\delta t}} + \frac{1}{e^{2\delta t}-1}\right) - e^t \frac{\gamma\sigma^2}{2}$$

which, after a change of variables, gives

$$\begin{aligned}\int_{s-\epsilon_1}^{k^*} e^t \frac{Z^{(q)}(t)}{W^{(q)}(t)}\,dt &= \frac{\sigma^2}{4}\left(\int_{\beta(s-\epsilon_1)}^{\beta k^*} \frac{e^{u(1+y)}}{e^u-1}\,du + \int_{\beta(s-\epsilon_1)}^{\beta k^*} \frac{e^{uy}}{e^u-1}\,du\right)\\ &\quad -\frac{\gamma\sigma^2}{2}(e^{s-\epsilon_1} - e^{k^*}).\end{aligned}$$

Denote the first integral on the right-hand side I_1 and the second integral I_2. After some algebra one sees that

$$\begin{aligned}&U_{\epsilon_1,\infty}(x,s) =\\ &\frac{e^s}{2}\big(e^{\gamma_2(x-\epsilon_1)} + e^{\gamma_1(x-\epsilon_1)}\big) + \frac{e^{\epsilon_1}}{2\beta}\big(e^{\gamma_2(x-\epsilon_1)} - e^{\gamma_1(x-\epsilon_1)}\big)\big(-2\gamma e^{k*} + I_1 + I_2\big)\\ &\frac{e^s}{2}\big(e^{\gamma_2(x-\epsilon_1)} + e^{\gamma_1(x-\epsilon_1)}\big) - \frac{e^{\epsilon_1+k^*}\gamma}{\beta}\big(e^{\gamma_2(x-\epsilon_1)} + e^{\gamma_1(x-\epsilon_1)}\big) + \frac{e^{\epsilon_1}}{2\beta}e^{\gamma_2(x-\epsilon_1)} I_1\\ &-\frac{e^{\epsilon_1}}{2\beta}e^{\gamma_1(x-\epsilon_1)} I_1 + \frac{e^{\epsilon_1}}{2\beta}e^{\gamma_2(x-\epsilon_1)} I_2 - \frac{e^{\epsilon_1}}{2\beta}e^{\gamma_1(x-\epsilon_1)} I_2.\end{aligned}$$

Next, note that

$$\begin{aligned}&\frac{e^{\epsilon_1}}{2\beta}e^{\gamma_2(x-\epsilon_1)} I_1 + \frac{e^{\epsilon_1}}{2\beta}e^{\gamma_1(x-\epsilon_1)} I_2\\ &= \frac{e^{\epsilon_1}}{2\beta}\big(e^{\gamma_2(x-\epsilon_1)} + e^{\gamma_1(x-\epsilon_1)}\big)(I_1 - I_2) - \frac{e^{\epsilon_1}}{2\beta}e^{\gamma_1(x-\epsilon_1)} I_1 + \frac{e^{\epsilon_1}}{2\beta}e^{\gamma_2(x-\epsilon_1)} I_2\\ &= \frac{e^{\epsilon_1}}{2}\big(e^{\gamma_2(x-\epsilon_1)} + e^{\gamma_1(x-\epsilon_1)}\big)(e^{k^*} - e^{s-\epsilon_1}) - \frac{e^{\epsilon_1}}{2\beta}e^{\gamma_1(x-\epsilon_1)} I_1 + \frac{e^{\epsilon_1}}{2\beta}e^{\gamma_2(x-\epsilon_1)} I_2\end{aligned}$$

where the second equality follows from evaluating $I_1 - I_2$. Plugging this into the expression for $U_{\epsilon_1,\infty}$ and simplifying yields

$$\begin{aligned}U_{\epsilon_1,\infty}(x,s) &= -e^{\gamma_1(x-\epsilon_1)} e^{\epsilon_1}\beta^{-1} I_1 + e^{\gamma_2(x-\epsilon_1)} e^{\epsilon_1}\beta^{-1} I_2\\ &\quad -e^{\epsilon_1+\gamma_2(x-\epsilon_1)} e^{k^*}\beta^{-1}\gamma_1 + e^{\epsilon_1+\gamma_1(x-\epsilon_1)} e^{k^*}\beta^{-1}\gamma_2.\end{aligned}$$

Rearranging the terms completes the proof. □

APPENDIX A: COMPLEMENTARY RESULTS ON THE INFINITESIMAL GENERATOR OF X

In this section we provide some results concerning the infinitesimal generator of X when applied to the scale function $Z^{(q)}$.

First assume that X is of unbounded variation and define an operator $(\Gamma, \mathcal{D}(\Gamma))$ as follows. $\mathcal{D}(\Gamma)$ stands for the family of functions $f \in C^2(0,\infty)$ such that the integral

$$\int_{(-\infty,0)} f(x-y) - f(x) - yf'(x)1_{\{y\geq -1\}}\,\Pi(dy)$$

is absolutely convergent for all $x > 0$. For any $f \in \mathcal{D}(\Gamma)$, we define

$$\Gamma f(x) = -\gamma f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{(-\infty,0)} \big(f(x+y) - f(x) - yf'(x)1_{\{y\geq -1\}}\big)\,\Pi(dy).$$

Similarly, if X is of bounded variation, then $\mathcal{D}(\Gamma)$ stands for the family of $f \in C^1(0,\infty)$ such that the integral

$$\int_{(-\infty,0)} f(x+y) - f(x)\,\Pi(dy)$$

is absolutely convergent for all $x > 0$ and, for $f \in \mathcal{D}(\Gamma)$, we define

$$\Gamma f(x) = \mathtt{d} f'(x) + \int_{(-\infty,0)} (f(x+y) - f(x))\Pi(dy).$$

In the sequel it should always be clear from the context in which of the two cases we are and therefore there should be no ambiguity when writing $\mathcal{D}(\Gamma)$ and Γ.

Lemma A.1. *We have that $Z^{(q)} \in \mathcal{D}(\Gamma)$ and the function $x \mapsto \Gamma Z^{(q)}(x)$ is continuous on $(0,\infty)$.*

Proof. We prove the unbounded and bounded variation case separately.

Unbounded variation: To show that $Z^{(q)} \in \mathcal{D}(\Gamma)$ it is enough the check that the integral part of $\Gamma Z^{(q)}$ is absolutely convergent since $Z^{(q)} \in C^2(0,\infty)$. Fix $x > 0$ and write the integral part of $\Gamma Z^{(q)}$ as

$$\int_{(-\infty,-\delta)} \big|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y\geq -1\}}\big|\Pi(dy)$$
$$+ \int_{(-\delta,0)} \big|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y\geq -1\}}\big|\Pi(dy)$$

where the value $\delta = \delta(x) \in (0,1)$ is chosen such that $x - \delta > 0$. For $y \in (-\infty, -\delta)$ the monotonicity of $Z^{(q)}$ implies

$$\left|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y \geq -1\}}\right| \leq 2Z^{(q)}(x) + Z^{(q)\prime}(x) \tag{26}$$

and for $y \in (-\delta, 0)$, using the mean value theorem, we have

$$\begin{aligned}
&|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)| \\
&= q|y||W^{(q)}(\xi(y)) - W^{(q)}(x)| \quad \text{where } \xi(y) \in (x+y, x) \\
&= q|y|\left|\int_{\xi(y)}^{x} W^{(q)\prime}(z)\,dz\right| \\
&\leq qy^2 \sup_{z \in [x-\delta, x]} W^{(q)\prime}(z). \qquad (27)
\end{aligned}$$

Using these two estimates and defining $C(\delta) = \int_{(-\delta,0)} y^2 \Pi(dy) < \infty$, we see that

$$\begin{aligned}
&\int_{(-\infty,0)} \left|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y \geq -1\}}\right| \Pi(dy) \\
&\leq \left(2Z^{(q)}(x) + Z^{(q)\prime}(x)\right)\Pi(-\infty, -\delta) + qC(\delta) \sup_{z \in [x-\delta, x]} W^{(q)\prime}(z) < \infty.
\end{aligned}$$

For continuity, let $x > 0$ and choose $\delta = \delta(x) \in (0,1)$ such that $x - 2\delta > 0$ as well as a sequence $(x_n)_{n \in \mathbb{N}}$ converging to x. Moreover, let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $|x_n - x| < \delta$. In particular, it holds that $x_n - \delta > 0$ for $n \geq n_0$ and hence, using the estimates in (26) and (27), we have for all $n \geq n_0$

$$\begin{aligned}
&|Z^{(q)}(x_n+y) - Z^{(q)}(x_n) - yZ^{(q)\prime}(x_n)1_{\{y \geq -1\}}| \\
&\leq qy^2 \sup_{z \in [x_n-\delta, x_n]} W^{(q)\prime}(z)1_{\{y \geq -\delta\}} + \left(2Z^{(q)}(x_n) + Z^{(q)\prime}(x_n)\right)1_{\{y < -\delta\}} \\
&\leq qy^2 \sup_{z \in [x-2\delta, x+\delta]} W^{(q)\prime}(z)1_{\{y \geq -\delta\}} + \left(2Z^{(q)}(x+\delta) + Z^{(q)\prime}(x+\delta)\right)1_{\{y < -\delta\}}.
\end{aligned}$$

Since the last term is Π-integrable, the continuity assertion follows by dominated convergence and the fact that $Z^{(q)} \in C^2(0,\infty)$.

Bounded variation: To show that $Z^{(q)} \in \mathcal{D}(\Gamma)$ it is enough to show that the integral part of $\Gamma Z^{(q)}$ is absolutely convergent since $Z^{(q)} \in C^1(0,\infty)$. Using the monotonicity and the definition of $Z^{(q)}$, it is easy to see that for fixed $x > 0$,

$$\begin{aligned}&\int_{(-\infty,0)} |Z^{(q)}(x+y) - Z^{(q)}(x)|\,\Pi(dy)\\&\leq 2Z^{(q)}(x)\Pi(-\infty,-1) + qW^{(q)}(x)\int_{(-1,0)} |y|\,\Pi(dy) < \infty.\end{aligned}$$

The continuity assertion follows in a straightforward manner from dominated convergence and the fact that $Z^{(q)} \in C^1(0,\infty)$. □

LEMMA A.2. *It holds that*

$$(\Gamma - q)Z^{(q)}(x) = 0, \quad x \in (0,\infty).$$

PROOF. The idea behind behind this proof is very similar to the idea behind the proof of Lemma 4.2 of [7]. Recall that for all $x \in (a,b)$ the process $e^{-q(t\wedge\tau_a^-\wedge\tau_b^+)}Z^{(q)}(X_{t\wedge\tau_a^-\wedge\tau_b^+})$ is a $\mathbb{P}_x$-martingale (cf. Remark 5 of [2]).

First assume that X is of unbounded variation. Let $x \in (a,b) \subset (0,\infty)$ and define $\tau := \tau_a^- \wedge \tau_b^+$. Applying the appropriate version of the Itô-Meyer formula (cf. Theorem 71, Ch. IV of [9]) to $Z^{(q)}(X_{t\wedge\tau})$ yields

$$Z^{(q)}(X_{t\wedge\tau}) = Z^{(q)}(x) + m_t + \int_0^{t\wedge\tau} \Gamma Z^{(q)}(X_u)\,du \quad \mathbb{P}_{x,s}\text{-a.s.}$$

where

$$\begin{aligned}m_t &= \int_{0+}^{t\wedge\tau} Z^{(q)\prime}(X_{u-})dB_u + \int_{0+}^{t\wedge\tau} Z^{(q)}(X_{u-})dX_u^{(2)}\\&+ \sum_{0<u\leq t\wedge\tau} \Delta Z^{(q)}(X_u) - \Delta X_u Z^{(q)\prime}(X_{u-})1_{\{\Delta X_u\geq -1\}}\\&- \int_0^{t\wedge\tau}\int_{(-\infty,0)} Z^{(q)}(X_{u-}+y) - Z^{(q)}(X_{u-}) - yZ^{(q)\prime}(X_{u-})1_{\{y\geq-1\}}\Pi(dy)du\end{aligned}$$

and $\Delta X_u = X_u - X_{u-}$, $\Delta Z^{(q)}(X_u) = Z^{(q)}(X_u) - Z^{(q)}(X_{u-})$. By the boundedness of $Z^{(q)}$ on $[a,b]$ the first two stochastic integrals on the right are zero-mean martingales and by the compensation formula (cf. Corollary 4.6 of [6]) the third and fourth term constitute a zero-mean martingale. Next, use stochastic integration by parts for semimartingales (cf. Corollary 2 of Theorem 22, Ch. II of [9]) to deduce that $\mathbb{P}_{x,s}$-a.s.

$$e^{-q(t\wedge\tau)}Z^{(q)}(X_{t\wedge\tau}) - Z^{(q)}(x) = \lambda_t + \int_0^{t\wedge\tau} e^{-qu}(\Gamma - q)Z^{(q)}(X_u)\,du,$$

where $\lambda_t = \int_{0+}^{t\wedge\tau} e^{-qu} dm_u$ is a zero-mean martingale. The claim now follows by exactly the same argument presented in the last part of the proof of Lemma 4.2 of [7].

The bounded variation case is very similar and is based on an application of the change of variable formula for Stiltjes' integrals. □

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