CLOSURES OF K-ORBITS IN THE FLAG VARIETY FOR $GL(2n)$

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Abstract. We characterize the O_{2n}-orbits in the flag variety for GL_{2n} with rationally smooth closure via a graph-theoretic criterion. We also give a necessary pattern avoidance criterion for rational smoothness and conjecture its sufficiency.

1. Introduction

Let G be a complex reductive group with Borel subgroup B and let $K = G^\theta$ be the fixed point subgroup of an involution of G. In this paper we develop the program begun in [M09] and continued in [MT09, M10], seeking to characterize the K-orbits in G/B with rationally smooth closure via a combinatorial criterion. Here we treat the case $G = GL(2n, \mathbb{C}), K = O(2n, \mathbb{C})$ (and prove some results about the general case $G = GL(m, \mathbb{C}), K = O(m, \mathbb{C})$). We focus on the order ideal of orbits with closures contained in a fixed one $\bar{\mathcal{O}}$; this is an interval inside the poset of all K-orbits in G/B, ordered by containment of closures. Richardson and Springer have defined an action of the braid monoid corresponding to the Weyl group W of G on this poset [RS90], which we use to make it and its order ideals into graphs. Our criterion for rational smoothness of $\bar{\mathcal{O}}$ is given in terms of the degree of the bottom vertex of this order ideal; it is known that this degree condition is necessary for rational smoothness in general [Br99]. We also give a necessary pattern avoidance criterion for rational smoothness (for any $GL(m, \mathbb{C})$) and conjecture its sufficiency.

2. Preliminaries

Set $G = GL(m, \mathbb{C}), K = O(m, \mathbb{C})$. Let B be the subgroup of upper triangular matrices in G. The quotient G/B may be identified with the variety of complete flags $V_0 \subset V_1 \subset \cdots \subset V_m$ in $\mathbb{C}^m$. The group K acts on this variety with finitely many orbits; these are parametrized by the set I_m of involutions in the symmetric group S_m[MO88, RS90]. In more detail, let $(\cdot, \cdot)$ be the standard symmetric bilinear form on $\mathbb{C}^m$, with isometry group K. Then a flag $V_0 \subset \cdots \subset V_m$ lies in the orbit $\mathcal{O}_\pi$ corresponding to the involution π if and only if the rank of $(\cdot, \cdot)$ on $V_i \times V_j$ equals the cardinality $\#\{k : 1 \le k \le i, \pi(k) \le j\}$ for all $1 \le i, j \le m$.

We use the same definition of pattern avoidance for permutations as in [M10], decreeing that $\pi = \pi_1 \dots \pi_m$ (in one-line notation) includes the pattern $\mu = \mu_1 \dots \mu_r$ if there are indices $i_1 < i_2 < \cdots < i_r$ permuted by π such that $\pi_{i_j} > \pi_{i_k}$ if and only if $\mu_j > \mu_k$. We say that π avoids μ if it does not include it.

There are well-known poset- and graph-theoretic criteria for rational smoothness of complex Schubert varieties due to Carrell and Peterson. The poset criterion does not extend to our setting but the graph one does. To state it we first recall that

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the partial order on I_m corresponding to inclusion of orbit closures is the reverse Bruhat order [RS90]. Then I_m is graded via the rank function

$$r(\pi) = \lfloor m^2/4 \rfloor - \sum_{i<\pi(i)} (\pi(i) - i - \#\{k \in \mathbb{N} : i < k < \pi(i), \pi(k) < i\})$$

where $\lfloor m^2/4 \rfloor$ denotes the greatest integer to $m^2/4$ and $r(\pi)$ equals the difference in dimension between $\mathcal{O}_\pi$ and $\mathcal{O}_c$, the unique closed orbit, corresponding to the involution $w_0 = m \dots 1$ [RS90]. Let I_π be the interval consisting of all $\pi' \leq \pi$ in the reverse Bruhat order. We make I_π into a graph by decreeing that the vertices μ and ν in it are adjacent if and only if either $\nu = t\mu t \neq \mu$ for some transposition t in S_m, or $\nu = t\mu$ for some transposition t in S_m with $t\mu t = \mu$ and m is even; write $\nu = t \cdot \mu$ if either of these conditions holds. Then a necessary condition for $\bar{\mathcal{O}}_\pi$ to be rationally smooth is that the degree of w_0 in I_π must be $r(\pi)$; in fact, all vertices $w^{-1}w_0w$ conjugate to w_0 in W and lying in I_π must have this degree [Br99, 2.5].

3. Main result

Our first result is a pattern avoidance criterion valid for all m.

Theorem 1. *With notation as above, the orbit $\mathcal{O}_\pi$ has rationally singular closure whenever π contains one of the twenty-four bad patterns* 14325, 426153, 154326, 124356, 153624, 351426, 213654, 321465, 3614725, 1324657, 2137654, 4321576, 5276143, 5472163, 2135467, 1243576, 1657324, 4651327, 57681324, 65872143, 13247856, 34125768, 341258967, 749258163. *The same holds if π contains the pattern* 2143*, provided that there are an even number of fixed indices of π between* 21 *and* 43 *(e.g. $\pi = 21354687$, where the* 43 *occurs in the last two indices of π).*

Proof. One checks first that the degree of w_0 in I_π is greater than $r(\pi)$ for any π in the list above, *except* for 2137654 and 4321576. The closures of the orbits corresponding to these two permutations are also rationally singular, as can be shown by computing their intersection cohomology (via Kazhdan-Lusztig-Vogan polynomials) directly (e.g. via ATLAS, available at http://www.liegroups.org). If π is obtained from one of the bad patterns above other than 2143 by adding one or more fixed points, then one computes that the degree of w_0 is again more than $r(\pi)$, except for 2134765, 3214576, 2137564, and 4231576; these cases may again be checked directly via ATLAS, and any involution π' obtained from them by adding one fixed point has the degree of w_0 bigger than $r(\pi')$. The same holds if π is obtained from 2143 by adding two or more fixed points, with an even number of them lying between 21 and 43. If π is obtained from 2143 by adding just one fixed point not lying between 21 and 43, then the unique vertex conjugate to w_0 having this fixed point has degree larger than $r(\pi)$. Finally, if π is obtained from one of the bad patterns by adding fixed points as above and then pairs of flipped indices, then one argues as in the proof of the Lemma in [M10] that some vertex in I_π has degree greater than $r(\pi)$. Hence in all cases $\bar{\mathcal{O}}_\pi$ is rationally singular. □

Along these lines we also have the following

Conjecture. *The conditions of Theorem 1 are sufficient for rational smoothness of $\bar{\mathcal{O}}_\pi$; in addition, this orbit closure is smooth if and only if π avoids the twenty-four bad patterns above and* 2143 *(without qualification) and* 1324.

We specialize to the even case $m = 2n$ in our main result.

Theorem 2. *Assume that $m = 2n$ is even. The orbit $\mathcal{O}_\pi$ has rationally smooth closure if and only if the degree of w_0 in I_π is $r(\pi)$.*

Proof. Set $\mathcal{O} = \mathcal{O}_\pi$. We have already noted that the degree condition is necessary, so suppose that it is satisfied. We begin by constructing a slice of $\bar{\mathcal{O}}$ to $\mathcal{O}_c$ at a particular flag, as follows. Fix a basis (e_i) of $\mathbb{C}^{2n}$ such that $(e_i, e_j) = 1$ if $i + j = 2n + 1$ and $(e_i, e_j) = 0$ otherwise, where as above $(\cdot, \cdot)$ is the symmetric form. Let (a_{ij}) be a family of complex parameters indexed by ordered pairs (i, j) satisfying either $i \le n < j$ or $n < i < j$. We assume that $a_{ij} = a_{2n+1-j,2n+1-i}$ if $i \le n < j$ but otherwise put no restrictions on the a_{ij}. Define a basis (b_i) of $\mathbb{C}^{2n}$ via

$$b_i = \begin{cases} e_i + \displaystyle\sum_{j=n+1}^{2n} a_{ij} e_j & \text{if } i \le n \\ e_i + \displaystyle\sum_{j=i+1}^{2n} a_{ij} e_j & \text{otherwise} \end{cases}$$

Then the Gram matrix $G := (g_{ij} = (b_i, b_j)$ of the b_i relative to the form satisfies

$$g_{ij} = \begin{cases} 2a_{i,2n+1-j} & \text{if } i \le j \le n \\ g_{ji} & \text{if } j < i \le n \\ a_{j,2n+1-i} & \text{if } i < n < j < 2n+1-i \\ 1 & \text{if } i \le n < j = 2n+1-i \\ g_{ji} & \text{if } j \le n < i \\ 0 & \text{otherwise} \end{cases}$$

Thus the matrix G is symmetric and has zeroes below the antidiagonal from lower left to upper right. The antidiagonal entries are all 1. Now one checks that the set $\mathcal{S}'$ of all flags $V_0 \subset \ldots \subset V_{2n}$ where (b_i) runs through all bases obtained as above from the a_{ij} and V_i is the span of $b_1, \ldots b_i$ is a slice of G/B to $\mathcal{O}_c$ at the flag f_c corresponding to the basis (e_i), which in turn corresponds to the point P where all $a_{ij} = 0$ [Br99, 2.1]. Intersecting $\mathcal{S}'$ with $\bar{\mathcal{O}}$ we get a slice $\mathcal{S}$ of $\bar{\mathcal{O}}$ to $\mathcal{O}_c$ at P, defined by the vanishing of certain minors in the Gram matrix G. It is known that $\mathcal{S}$ is rationally smooth (resp. smooth) at P if and only if it is rationally smooth (resp. smooth) everywhere, or if and only if $\bar{\mathcal{O}}$ is rationally smooth (resp. smooth) everywhere [Br99, 2.1]. This construction works with minor modifications for odd m as well.

We now show that $\mathcal{S}$ is rationally smooth at P by verifying the conditions of [Br99, 1.4]. Define an action of the n-torus $T = \mathbb{T}^n$ on the matrix G by multiplying the first n rows and columns by $t_1, \ldots, t_n$, respectively, while multiplying the last n rows and columns by $t_n^{-1}, \ldots, t_1^{-1}$, respectively; this action preserves the 1s on the antidiagonal and the vanishing of the minors that define the slice $\mathcal{S}$. (T is just a maximal torus of K.) Then the weights of T occurring in the tangent space at P of the big slice $\mathcal{S}'$ are those of the form $2e_i, e_i + e_j$, or $e_i - e_j$ for some $1 \le i < j \le n$ and all occur with multiplicity one; each corresponds also to a variable in $\mathcal{S}$ and to a transposition t in S_{2n} such that the vertices $t \cdot w_0$ are distinct. In particular, they all lie on one side of a hyperplane and P is an attractive fixed point of both $\mathcal{S}'$ and $\mathcal{S}$. The subtori T' of T of codimension one such that the fixed point subvariety $(\mathcal{S}')^{T'}$ of the big slice $\mathcal{S}'$ under the T'-action contains more than one point correspond exactly to the above weights. Now the hypothesis on the degree of w_0 implies that

that there are $\lfloor m^2/4 \rfloor - r(\pi)$ distinct conjugates $c = t \cdot w_0$ of w_0 by a transposition t with $c \not\leq \pi$, where t corresponds to one of the above weights. Writing c as $c_1 \ldots c_{2n}$ in one-line notation, let i be the smallest index such that if $\pi_1 \ldots \pi_i$ is rearranged in increasing order as $\pi'_1 \ldots \pi'_i$ and similarly $c_1 \ldots c_i$ is rearranged as $c'_1 \ldots c'_i$, then $\pi'_j > c'_j$ for some j. Equating the minor of G corresponding to the first i rows and columns $c_1, \ldots, c_i$ to 0, we get a polynomial in the variables of $\mathcal{S}$ in which exactly one of the monomial terms is either the first or second power of the variable corresponding to (the weight of) t. Hence the dimension of the fixed point subvariety $\mathcal{S}^{T'}$ is 1 if T' corresponds to a t *not* arising from one of the conjugates c and 0 otherwise, whence Brion's conditions [Br99, 1.4(ii),(iii)] are satisfied.

Now we must verify condition [Br99, 1.4(i)]. By hypothesis there are c vertices $v = tw_0t$ or $v = tw_0$ adjacent to w_0 in I_π, where c is the codimension of $\mathcal{O}$ in G/B. Write each such v as $v_1 \ldots v_{2n}$ and let i be the smallest index with $\pi_1 \ldots \pi_i \not\leq v_1 \ldots v_i$ in the standard (entry by entry) partial order on sequences used to characterize Bruhat order [P82]. Thus if $\pi_1 \ldots \pi_i$ is rearranged in increasing order as $\pi'_1 \ldots \pi'_i$ and similarly $v_1 \ldots v_i$ is rearranged as $v'_1 \ldots v'_i$, then $\pi'_j > v'_j$ for some j. Choose the least such j and let $r_1, \ldots r_j$ be the positions of $v'_1, \ldots v'_j$ among $v_1 \ldots v_i$. Then $\mathcal{S}$ is defined by the vanishing of the minor M_v of G of the submatrix S_v corresponding to columns $v'_1, \ldots, v'_j$ and rows $r_1, \ldots r_j$, for all v. All but at most two rows of S_v contain a 1 and similarly for the columns. Use row and column operations to zero out all the entries other than 1 in any row or column of S_v with a 1, obtaining a new matrix S'_v with the same determinant M_v. Then requiring that all M_v vanish amounts to imposing a set of linear and quadratic conditions on a new set of variables, consisting of the old variables a_{ij} either not occurring in any S_v or originally occurring in a row or column with a 1 in some S_v, together with the entries p_{ij} of the new matrices S'_v not lying in the same row or column as a 1. Thus $\mathcal{S}$ may be realized as the product of a suitable affine space $\mathbb{C}^r$ and a variety with only one singular point (at the origin) and it satisfies [Br99, 1.4(i)], as desired.

□

We have already observed that Theorem 2 fails for odd m; indeed, the orbit closure $\bar{\mathcal{O}}_\pi$ indexed by $\pi = 2137654$ in $GL(7, \mathbb{C})/B$ is rationally singular and yet I_π satisfies the degree criterion at all vertices conjugate to the closed one. A smaller example has $\pi = 21435$; here the degree criterion holds at the closed orbit 54321, but fails at 43215, so again $\bar{\mathcal{O}}_\pi$ is rationally singular.

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