REMARKS ON MODULAR REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE IN NON-DEFINING CHARACTERISTIC

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Abstract. Let G be a finite group of Lie type and ℓ be a prime which is not equal to the defining characteristic of G. In this note we discuss some open problems concerning the ℓ-modular irreducible representations of G. We also establish a strengthening of the results in [13] on the classification of the ℓ-modular principal series representations of G.

1. Introduction

Let G be a finite group and ℓ be a prime number. Let K be a field of characteristic 0 and assume that K is "sufficiently large" (that is, K is a splitting field for G and all its subgroups). Let $\mathcal{O}$ be a discrete valuation ring in K, with residue field k of characteristic $\ell > 0$. Let $\mathrm{Irr}_K(G)$ denote the set of irreducible representations of G over K (up to isomorphism) and let $\mathrm{Irr}_k(G)$ denote the set of irreducible representations of G over k (up to isomorphism). In the setting of Brauer's classical modular representation theory (see, for example, Curtis–Reiner [6, §16]), we have a decomposition map

$$d_{\mathcal{O}} \colon \mathcal{R}_0(KG) \to \mathcal{R}_0(kG)$$

between the Grothendieck groups of finite-dimensional representations of KG and kG, respectively. Given $\rho \in \mathrm{Irr}_K(G)$ and $Y \in \mathrm{Irr}_k(G)$, we denote by $\langle \rho : Y \rangle_{\mathcal{O}}$ the corresponding decomposition number, that is, the multiplicity of the class of Y in the image of the class of ρ under the map $d_{\mathcal{O}}$. Assuming that $\mathrm{Irr}_K(G)$ is sufficiently well known, the decomposition numbers $\langle \rho : Y \rangle_{\mathcal{O}}$ provide a tool for using the available information in characteristic 0 to derive information concerning $\mathrm{Irr}_k(G)$.

We shall consider the situation where G is a finite group of Lie type and ℓ is a prime which is not equal to the defining characteristic of G. The work of Lusztig [25], [26] provides a complete picture about the classification and the dimensions of the irreducible representations of G over K. As far as ℓ-modular representations are concerned, the compatibility of ℓ-blocks with Lusztig series (see [3], [17]) suggests that $\mathrm{Irr}_k(G)$ is very closely related to $\mathrm{Irr}_K(G)$, where one might hope to quantify the degree of "closeness"

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in terms of suitable properties of the decomposition numbers of G. (This is in contrast to the situation for modular representations in the defining characteristic, which tend to be far away from $\mathrm{Irr}_K(G)$; see Remark 4.6.)

More precise information is available for groups of type A_n by the work of Fong–Srinivasan [10] and Dipper–James [9]. However, for groups of other types, much less is known except for special characteristics (see, for example, Gruber–Hiss [21]) or groups of small ranks where explicit computations are possible (see, for example, [24], [28], [29]).

In Section 2 we formulate a conjecture concerning the classification of the "unipotent" modular representations of G. There is considerable evidence that this conjecture holds in general; see Remark 2.4. In Section 3 we provide a partial proof as far as the unipotent modular principal series prepresentations of G are concerned; this strengthens the results obtained previously in [13]. Finally, in Section 4, we consider the dimensions of irreducible representations and state, as a challenge, a general "qualitative" conjecture for the set of all ℓ-modular representations of G.

Note that the conjectures that we state here are not variations of general conjectures on finite groups: Besides their potential immediate interest, they express properties of modular representations of finite groups of Lie type which, if true, would provide further evidence for the sharp distinction between the non-defining and the defining characteristic case.

To fix some notation, let p be a prime number and $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_p$ and $F\colon \mathbf{G} \to \mathbf{G}$ be a homomorphism of algebraic groups such that some power of F is the Frobenius map relative to a rational structure on $\mathbf{G}$ over some finite subfield of $\overline{\mathbb{F}}_p$. Then $\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$ is called a finite group of Lie type; we shall write $G := \mathbf{G}^F$. Similar conventions apply to F-stable subgroups of $\mathbf{G}$. Let $\mathbf{B} \subseteq \mathbf{G}$ be an F-stable Borel subgroup and $\mathbf{T}_0 \subseteq \mathbf{B}$ be an F-stable maximal torus. Then we write $B := \mathbf{B}^F$ and $T_0 := \mathbf{T}_0^F$. Let $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ be the Weyl group of $\mathbf{G}$. Then F induces an automorphism $\gamma\colon \mathbf{W} \to \mathbf{W}$.

Let $\delta \geqslant 1$ be minimal such that F^δ is the Frobenius map relative to a rational structure on $\mathbf{G}$ over a finite subfield $k_0 \subseteq \overline{\mathbb{F}}_p$. Define $q > 0$ to be the unique real number such that $|k_0| = q^\delta$. (If $\mathbf{G}$ is simple modulo its center, then $\delta = 1$ and q is a power of p, except when G is a Suzuki or Ree group in which case $\delta = 2$ and q is an odd power of $\sqrt{2}$ or $\sqrt{3}$.)

Throughout this paper (except for the final Remark 4.6), we assume that $K, \mathcal{O}, k$ as above are such that $\ell \neq p$.

2. On modular unipotent representations

Let $\mathrm{Unip}_K(G) \subseteq \mathrm{Irr}_K(G)$ be the set of unipotent representations of G, as defined by Deligne–Lusztig [7]. (Note that, in order to define $\mathrm{Unip}_K(G)$, one first needs to work over $\overline{\mathbb{Q}}_\ell$, where $\ell \neq p$, in order to construct the virtual representations $R_{T,1}$ of [7]; since the character values of $R_{T,1}$ are rational integers, the set $\mathrm{Unip}_K(G)$ is then unambiguously defined for any K of characteristic 0.) The *ℓ-modular unipotent representations* of G are defined to be

$$\mathrm{Unip}_k(G) := \{Y \in \mathrm{Irr}_k(G) \mid \langle \rho : Y \rangle_{\mathcal{O}} \neq 0 \text{ for some } \rho \in \mathrm{Unip}_K(G)\}.$$

We wish to state a conjecture about the classification of $\mathrm{Unip}_k(G)$. First, we recall some results about the characteristic 0 representations of G.

Let $\mathbf{O}$ be an F-stable unipotent conjugacy class of $\mathbf{G}$. Let $u_1 \ldots, u_r \in \mathbf{O}^F$ be representatives of the G-conjugacy classes contained in $\mathbf{O}^F$. For each j, let $A(u_j)$ be the group of connected components of the centraliser of u_j in $\mathbf{G}$. Since $F(u_j) = u_j$, there is an induced action of F on $A(u_j)$ which we denote by the same symbol. Now let $\rho \in \mathrm{Irr}_K(G)$. Then we define the *average value* of ρ on $\mathbf{O}^F$ by

$$\mathrm{AV}(\mathbf{O}, \rho) := \sum_{1 \leqslant j \leqslant r} [A(u_j) : A(u_j)^F]\, \mathrm{trace}(u_j, \rho).$$

(Note that $\mathrm{AV}(\mathbf{O}, \rho)$ does not depend on the choice of the representatives u_j.) Assuming that p, q are large enough, Lusztig [27] has shown that, given $\rho \in \mathrm{Irr}_K(G)$, there exists a *unique* F-stable unipotent class $\mathbf{O}_\rho$ satisfying the following two conditions:

- $\mathrm{AV}(\mathbf{O}_\rho, \rho) \neq 0$ and
- if $\mathbf{O}$ is any F-stable unipotent class $\mathbf{O}$ such that $\mathrm{AV}(\mathbf{O}, \rho) \neq 0$, then $\mathbf{O} = \mathbf{O}_\rho$ or $\dim \mathbf{O} < \dim \mathbf{O}_\rho$.

The class $\mathbf{O}_\rho$ is called the *unipotent support* of ρ. The assumptions on p, q have subsequently been removed in [20]. Thus, every $\rho \in \mathrm{Irr}_K(G)$ has a well-defined unipotent support $\mathbf{O}_\rho$. Using this concept, we can associate to every $\rho \in \mathrm{Irr}_K(G)$ a numerical invariant $\mathbf{a}_\rho$ by setting

$$\mathbf{a}_\rho := \dim \mathfrak{B}_u \qquad (u \in \mathbf{O}_\rho)$$

where $\mathfrak{B}_u$ is the variety of Borel subgroups of $\mathbf{G}$ containing u.

Now consider the unipotent representations $\mathrm{Unip}_K(G)$. By [25, Main Theorem 4.23], there is a bijection

$$\bar{X}(\mathbf{W}, \gamma) \quad \stackrel{1-1}{\longleftrightarrow} \quad \mathrm{Unip}_K(G), \qquad x \leftrightarrow \rho_x,$$

where $\bar{X}(\mathbf{W}, \gamma)$ is a finite set depending only on the Weyl group $\mathbf{W}$ of $\mathbf{G}$ and the automorphism $\gamma\colon \mathbf{W} \to \mathbf{W}$ induced by the action of F. (This bijection satisfies further properties as specified in [25, 4.23]; we shall not need to discuss these properties here.)

We shall need two further pieces of notation. Let $Z_{\mathbf{G}}$ be the center of $\mathbf{G}$. Then $(Z_{\mathbf{G}}/Z_{\mathbf{G}}^\circ)_F$ denotes the largest quotient of $Z_{\mathbf{G}}/Z_{\mathbf{G}}^\circ$ on which F acts trivially. Also recall (e.g., from [5, p. 28]) that a prime number is called *good* for $\mathbf{G}$ if it is good for each simple factor involved in $\mathbf{G}$; the conditions for the various simple types are as follows.

$$\begin{array}{rl} A_n: & \text{no condition}, \\ B_n, C_n, D_n: & \ell \neq 2, \\ G_2, F_4, E_6, E_7: & \ell \neq 2, 3, \\ E_8: & \ell \neq 2, 3, 5. \end{array}$$

Now we can state:

Conjecture 2.1 (Geck [11, §2.5], Geck–Hiss [18, §3])**.** *Assume that ℓ is good for $\mathbf{G}$ and that ℓ does not divide the order of $(Z_{\mathbf{G}}/Z_{\mathbf{G}}^\circ)_F$. Then there is a labelling $\mathrm{Unip}_k(G) = \{Y_x \mid x \in \bar{X}(\mathbf{W}, \gamma)\}$ such that the following conditions hold for all $x, x' \in \bar{X}(\mathbf{W}, \gamma)$:*

$$\langle \rho_x : Y_x \rangle_{\mathcal{O}} = 1,$$
$$\langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} \neq 0 \quad \Rightarrow \quad x = x' \quad \textit{or} \quad \mathbf{O}_{\rho_{x'}} \subsetneqq \overline{\mathbf{O}}_{\rho_x}.$$

(Note that, if such a labelling exists, then it is uniquely determined.)

Remark 2.2. Under the above assumption on ℓ, it is known by [17], [12] that

$$|\,\mathrm{Unip}_k(G)| = |\,\mathrm{Unip}_K(G)|.$$

If ℓ is not good for $\mathbf{G}$, or if ℓ divides the order of $(Z_{\mathbf{G}}/Z_{\mathbf{G}}^\circ)_F$, then we have $|\,\mathrm{Unip}_k(G)| \neq |\,\mathrm{Unip}_K(G)|$ in general. For further information on the cardinalities $|\,\mathrm{Unip}_k(G)|$ in such cases, see [18, 6.6].

Remark 2.3. The formulation of the above conjecture is somewhat stronger than that in [13, Conj. 1.3] where, instead of the "geometric" condition

$$\langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} \neq 0 \quad \Rightarrow \quad x = x' \quad \text{or} \quad \mathbf{O}_{\rho_{x'}} \subsetneqq \overline{\mathbf{O}}_{\rho_x},$$

we used the purely numerical condition:

$$\langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} \neq 0 \quad \Rightarrow \quad x = x' \quad \text{or} \quad \mathbf{a}_{\rho_{x'}} > \mathbf{a}_{\rho_x}.$$

The stronger version is known to hold for $G = \mathrm{GL}_n(\mathbb{F}_q)$ (see Dipper–James [9] and the references there) and $G = \mathrm{GU}_n(\mathbb{F}_q)$ (see [11, §2.5], [13, §2.5]). The argument for $\mathrm{GU}_n(\mathbb{F}_q)$ essentially relies on Kawanaka's theory [23] of generalised Gelfand–Graev representations; see also Remark 2.4 below. Further support will be provided by Proposition 3.1 below.

Remark 2.4. Using Brauer reciprocity, Conjecture 2.1 can be alternatively stated as follows. There should exist finitely generated, projective $\mathcal{O}G$-modules $\{\Phi_x \mid x \in \bar{X}(\mathbf{W}, \gamma)\}$ such that, for any $x \in \bar{X}(\mathbf{W}, \gamma)$, we have:

$$\begin{aligned} K \otimes_{\mathcal{O}} \Phi_x \cong \rho_x &\oplus \text{ (direct sum of various } \rho_{x'} \text{ where } \mathbf{O}_{\rho_{x'}} \subsetneqq \mathbf{O}_{\rho_x}) \\ &\oplus \text{ (direct sum of various non-unipotent } \rho' \in \mathrm{Irr}_K(G)). \end{aligned}$$

This formulation is particularly useful in connection with Kawanaka's theory [23] of generalised Gelfand–Graev representations. Assume that p, q are sufficiently large such that Lusztig's results [27] hold. Let $u \in G$ be a unipotent element and denote by Γ_u the corresponding generalised Gelfand–Graev representation of G over K. Since Γ_u is obtained by inducing a representation from a unipotent subgroup of G, we have $\Gamma_u \cong K \otimes_{\mathcal{O}} \Upsilon_u$ where Υ_u is a finitely generated, projective $\mathcal{O}G$-module.

Now let $x \in \bar{X}(\mathbf{W},\gamma)$. Then we can find a unipotent element $u \in G$ such that Γ_u is a linear combination of various $\rho' \in \mathrm{Irr}_K(G)$ where $\mathbf{O}_{\rho'} \subseteq \overline{\mathbf{O}}_{\rho_x}$, and where the coefficient of ρ_x is non-zero. This follows from the multiplicity formula in [27, Theorem 11.2], together with the refinement obtained by Achar–Aubert [1, Theorem 9.1]. Thus, Υ_u is a first approximisation to the hypothetical projective $\mathcal{O}G$-module Φ_x, more precisely, Φ_x should be a direct summand of Υ_u. This also shows that the closure relation among unipotent supports naturally appears in this context.

The special feature of the case where $G = \mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{GU}_n(\mathbb{F}_q)$ is that then $|\bar{X}(\mathbf{W},\gamma)|$ is equal to the number of unipotent classes of $\mathbf{G}$ and, using the above notation, we can just take Φ_x to be Υ_u.

In general, it seems to be necessary to work with certain modified generalised Gelfand–Graev representations, as proposed by Kawanaka [23, §2]. In this context, Conjecture 2.1 would follow from the conjecture in [23, (2.4.5)].

Finally, we give an alternative description of the closure relation among unipotent supports which will play a crucial role in Section 3.

Remark 2.5. Let $\leqslant_{\mathrm{LR}}$ be the two-sided Kazhdan–Lusztig pre-order relation on $\mathbf{W}$; see [25, Chap. 5]. Given $w, w' \in \mathbf{W}$ we write $w \sim_{\mathrm{LR}} w'$ if $w \leqslant_{\mathrm{LR}} w'$ and $w' \leqslant_{\mathrm{LR}} w$. The equivalence classes for this relation are called the two-sided cells of $\mathbf{W}$. By [25, 5.15], we have a natural partition

$$\mathrm{Irr}_K(\mathbf{W}) = \coprod_{\mathcal{F}} \mathrm{Irr}_K(\mathbf{W} \mid \mathcal{F})$$

where $\mathcal{F}$ runs over the two-sided cells of $\mathbf{W}$. Now the automorphism $\gamma\colon \mathbf{W} \to \mathbf{W}$ induces a permutation of the two-sided cells of $\mathbf{W}$. By [25, 6.17], we also have a natural partition

$$\mathrm{Unip}_K(G) = \coprod_{\mathcal{F}} \mathrm{Unip}_K(G \mid \mathcal{F})$$

where $\mathcal{F}$ runs over the γ-stable two-sided cells of $\mathbf{W}$. Given $x \in \bar{X}(\mathbf{W},\gamma)$, we denote by $\mathcal{F}_x$ the unique γ-stable two-sided cell of $\mathbf{W}$ such that $\rho_x \in \mathrm{Unip}_K(G \mid \mathcal{F}_x)$. By [27] and [20, Prop. 4.2], the above partition can be characterised in terms of unipotent supports as follows:

$$\mathcal{F}_x = \mathcal{F}_{x'} \qquad \Leftrightarrow \qquad \mathbf{O}_{\rho_x} = \mathbf{O}_{\rho_{x'}} \qquad \text{for all } x, x' \in \bar{X}(\mathbf{W},\gamma).$$

Now $\leqslant_{\mathrm{LR}}$ induces a partial order relation on the set of two-sided cells which we denote by the same symbol. Thus, given two-sided cells $\mathcal{F}, \mathcal{F}$ of $\mathbf{W}$, we write $\mathcal{F} \leqslant_{\mathrm{LR}} \mathcal{F}'$ if and only if $w \leqslant_{\mathrm{LR}} w'$ for all $w \in \mathcal{F}$ and $w' \in \mathcal{F}'$.

Proposition 2.6 (See [16, Cor. 5.6])**.** *In the above setting, we have*

$$\mathcal{F}_x \leqslant_{\mathrm{LR}} \mathcal{F}_{x'} \qquad \Leftrightarrow \qquad \mathbf{O}_{\rho_x} \subseteq \overline{\mathbf{O}}_{\rho_{x'}} \qquad \textit{for all } x, x' \in \bar{X}(\mathbf{W},\gamma).$$

Note that, in [16], we work in a slightly different setting where two-sided cells and unipotent classes are linked via the Springer correspondence. However, by [27] and [20, Theorem 3.7], the map which assigns to each

$\rho \in \mathrm{Unip}_K(G)$ its support support $\mathbf{O}_\rho$ can also be interpreted in terms of the Springer correspondence. Thus, indeed, Proposition 2.6 as formulated above is equivalent to [16, Cor. 5.6].

3. Principal series representations

Recall that $\mathbf{B} \subseteq \mathbf{G}$ is an F-stable Borel subgroup. Let $B := \mathbf{B}^F \subseteq G$ and consider the permutation module $K[G/B]$ on the cosets of B. Then the set $\mathrm{Irr}_K(G \mid B)$ of (unipotent) principal series representations is defined to be the set of all $\rho \in \mathrm{Irr}_K(G)$ such that ρ is an irreducible constituent of $K[G/B]$. The modular analogue of $\mathrm{Irr}_K(G \mid B)$ is more subtle since $k[G/B]$ is not semisimple in general. Following Dipper [8], we define $\mathrm{Irr}_k(G \mid B)$ to be the set of all $Y \in \mathrm{Irr}_k(G)$ such that $\mathrm{Hom}_{kG}(k[G/B], Y) \neq \{0\}$. By Frobenius reciprocity, we have $Y \in \mathrm{Irr}_k(G \mid B)$ if and only if Y admits non-zero vectors fixed by all elements of B. This fits with a general definition of Harish–Chandra series for $\mathrm{Irr}_k(G)$; see Hiss [22]. We have

$$\mathrm{Irr}_K(G \mid B) \subseteq \mathrm{Unip}_K(G) \quad \text{and} \quad \mathrm{Irr}_k(G \mid B) \subseteq \mathrm{Unip}_k(G).$$

Thus, there is a subset $\Lambda \subseteq \bar{X}(\mathbf{W}, \gamma)$ such that $\mathrm{Irr}_K(G \mid B) = \{\rho_x \mid x \in \Lambda\}$. This subset is explicitly described by [25, Chap. 4]. It is also known that this subset is in bijection with $\mathrm{Irr}_K(\mathbf{W}^\gamma)$; see [6, §68B].

Now assume that ℓ is good for $\mathbf{G}$. Then, by [13, Theorem 1.1] (see also [19, §4.4]), there exists a unique subset $\Lambda_k^\circ \subseteq \bar{X}(\mathbf{W}, \gamma)$ and a unique labelling $\mathrm{Irr}_k(G \mid B) = \{Y_x \mid x \in \Lambda_k^\circ\}$ such that the following conditions hold for all $x \in \Lambda_k^\circ$ and $x' \in \bar{X}(\mathbf{W}, \gamma)$:

$$\langle \rho_x : Y_x \rangle_{\mathcal{O}} = 1,$$
$$\langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} \neq 0 \quad \Rightarrow \quad x = x' \quad \text{or} \quad \mathbf{a}_{\rho_{x'}} > \mathbf{a}_{\rho_x}.$$

Furthermore, we have in fact $\Lambda_k^\circ \subseteq \Lambda$. So, here, we used the numerical condition in Remark 2.3. (This was the only condition available at the time of writing [13].) Our aim now is to show that we can replace this condition by the condition involving the closure relation among unipotent supports.

Proposition 3.1. *Assume that F acts as the identity on $\mathbf{W}$ (that is, $\gamma = \mathrm{id}$). Then, using the above notation, the following implication holds for all $x \in \Lambda_k^\circ$ and $x' \in \bar{X}(\mathbf{W}, \gamma)$:*

$$\langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} \neq 0 \quad \Rightarrow \quad x = x' \quad \textit{or} \quad \mathbf{O}_{\rho_{x'}} \subsetneqq \overline{\mathbf{O}}_{\rho_x}.$$

Proof. We go through the main steps of the proof of [13, Theorem 1.1]. For this purpose, consider the Hecke algebra $\mathcal{H}_{\mathcal{O}} = \mathrm{End}_{\mathcal{O}G}(\mathcal{O}[G/B])$ and let $\mathcal{H}_K := K \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}$ and $\mathcal{H}_k := k \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}$. Since $\gamma = \mathrm{id}$, the algebra $\mathcal{H}_{\mathcal{O}}$ has a standard basis $\{T_w \mid w \in \mathbf{W}\}$ where the multiplication is given as follows, where $s \in \mathbf{W}$ is a simple reflection, $w \in \mathbf{W}$ and l is the length function:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ qT_{sw} + (q-1)T_w & \text{if } l(sw) = l(w) - 1. \end{cases}$$

We also have a decomposition map between the Grothendieck groups of $\mathcal{H}_K$ and $\mathcal{H}_k$. Given $E \in \mathrm{Irr}(\mathcal{H}_K)$ and $M \in \mathrm{Irr}(\mathcal{H}_k)$, denote by $d_{E,M}$ the corresponding decomposition number. Now, by results of Dipper [8] (see [13, §2.2]), we have natural bijections

$$\begin{aligned} \mathrm{Irr}(\mathcal{H}_K) &\xrightarrow{1-1} \mathrm{Irr}_K(G \mid B), \qquad E \mapsto \rho_E, \\ \mathrm{Irr}(\mathcal{H}_k) &\xrightarrow{1-1} \mathrm{Irr}_k(G \mid B), \qquad M \mapsto Y_M; \end{aligned}$$

furthermore, for any $x \in \Lambda_k^\circ$ and $x' \in \bar{X}(\mathbf{W}, \gamma)$, we have

$$(*) \qquad \langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} = \begin{cases} d_{E,M} & \text{if } \rho_{x'} \cong \rho_E \text{ and } Y_x \cong Y_M, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we are reduced to a problem within the representation theory of $\mathcal{H}_{\mathcal{O}}$. Now, with every $E \in \mathrm{Irr}(\mathcal{H}_K)$ we can associate a two-sided Kazhdan–Lusztig cell $\mathcal{F}_E$ of $\mathbf{W}$; see [25, 5.15]. As in [13, §2.3], let us define $\mathbf{a}_E := \mathbf{a}_{\rho_E}$. Then, by [13, §2.4] (see also [19, Prop. 3.2.7]), there exists a unique injection

$$\mathrm{Irr}(\mathcal{H}_k) \hookrightarrow \mathrm{Irr}(\mathcal{H}_K), \qquad M \mapsto E_M,$$

satisfying the following conditions for all $M \in \mathrm{Irr}(\mathcal{H}_k)$ and $E \in \mathrm{Irr}(\mathcal{H}_K)$:

(a) $\qquad d_{E_M,M} = 1,$

(b) $\qquad d_{E,M} \neq 0 \quad \Rightarrow \quad E \cong E_M \quad \text{or} \quad \mathbf{a}_E > \mathbf{a}_{E_M}.$

Now we can argue as follows. Assume that $\langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} \neq 0$ where $x \in \Lambda_k^\circ$ and $x' \in \bar{X}(\mathbf{W}, \gamma)$. By $(*)$, we must have $\rho_{x'} \cong \rho_E$ for some $E \in \mathrm{Irr}(\mathcal{H}_K)$ and $Y_x \cong Y_M$ for some $M \in \mathrm{Irr}(\mathcal{H}_k)$; this also shows that $\rho_x \cong \rho_{E_M}$. Furthermore, $d_{E,M} = \langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} \neq 0$ and so, using (b), we obtain:

$$\langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} \neq 0 \quad \Rightarrow \quad x = x' \quad \text{or} \quad \mathbf{a}_{\rho_{x'}} > \mathbf{a}_{\rho_x},$$

as in the original version of [13, Theorem 1.1]. In order to prove the strengthening, we use the results in [14], [15] which show that the following variation of (b) holds:

$$d_{E,M} \neq 0 \quad \Rightarrow \quad E \cong E_M \quad \text{or} \quad \mathcal{F}_E \leqslant_{\mathrm{LR}} \mathcal{F}_{E_M},\ \mathcal{F}_E \neq \mathcal{F}_{E_M}.$$

Hence, arguing as above, we obtain

$$\langle \rho_{x'} : Y_x \rangle_{\mathcal{O}} \neq 0 \quad \Rightarrow \quad x = x' \quad \text{or} \quad \mathcal{F}_E \leqslant_{\mathrm{LR}} \mathcal{F}_{E_M},\ \mathcal{F}_E \neq \mathcal{F}_{E_M}.$$

Now, the multiplicity formula in [25, Main Theorem 4.23] shows that ρ_E appears with non-zero multiplicity in the "almost-representation" R_{E_0} of G associated with some $E_0 \in \mathrm{Irr}(\mathcal{H}_K)$ where $\mathcal{F}_E = \mathcal{F}_{E_0}$. Then [25, 6.17] shows that $\mathcal{F}_E = \mathcal{F}_{E_0} = \mathcal{F}_{\rho_E} = \mathcal{F}_{x'}$. Similarly, since $\rho_x \cong \rho_{E_M}$, we have $\mathcal{F}_{E_M} = \mathcal{F}_x$. Hence, it remains to use the equivalence in Proposition 2.6. $\square$

4. On the dimensions of irreducible representations

Finally, we wish to state a conjecture on the dimensions of the irreducible representations of G over k. Let us first consider the situation in characteristic 0. Then [25, Main Theorem 4.23] implies that there exists a collection of polynomials $\{D_x \mid x \in \bar{X}(\mathbf{W},\gamma)\} \subseteq \mathbb{K}[t]$ (where t is an indeterminate and $\mathbb{K}$ is a finite extension of $\mathbb{Q}$ of degree $[K:\mathbb{Q}]=\delta$) such that

$$\dim \rho_x = D_x(q) \qquad \text{for all } x \in \bar{X}(\mathbf{W},\gamma).$$

This set of polynomials only depends on $\mathbf{W}$ and γ. Using Lusztig's Jordan decomposition of representations and [26, Prop. 5.1], one can in fact formulate a global statement for all $\mathrm{Irr}_K(G)$, as follows.

Proposition 4.1 (Lusztig [25], [26]). *There exists a finite set of polynomials* $\mathcal{D}_0(\mathbf{W},\gamma) \subseteq \mathbb{K}[t]$*, depending only on* $\mathbf{W}$ *and* γ*, such that*

$$\{\dim \rho \mid \rho \in \mathrm{Irr}_K(G)\} \quad \subseteq \quad \{f(q) \mid f \in \mathcal{D}_0(\mathbf{W},\gamma)\}.$$

For example, if $\mathbf{W}$ is of type A_1 and $\gamma = \mathrm{id}$ (where, for example, $G = \mathrm{GL}_2(\mathbb{F}_q)$ or $\mathrm{SL}_2(\mathbb{F}_q)$ and q is any prime power), then we can take

$$\mathcal{D}_0(A_1,\mathrm{id}) = \Big\{1, t, t\pm 1, \frac{1}{2}(t\pm 1)\Big\}.$$

If $\mathbf{W}$ is of type C_2 and $\gamma \neq \mathrm{id}$ (where G is a Suzuki group and q is an odd power of $\sqrt{2}$), then we can take

$$\mathcal{D}_0(C_2,\gamma) = \Big\{1, t^4, t^4+1, \frac{1}{\sqrt{2}}t(t^2-1), (t^2-1)(t^2 \pm t\sqrt{2}+1)\Big\}.$$

Remark 4.2. The results in [25], [26] yield a precise and complete description of a set of polynomials which are needed to express $\dim \rho$ for all $\rho \in \mathrm{Irr}_K(G)$. However, here we will only be interested in a qualitative statement where it will be sufficient to find some, possibly much too large, but still *finite* set of polynomials $\mathcal{D}_0(\mathbf{W},\gamma)$ by which we can express $\dim \rho$ for all $\rho \in \mathrm{Irr}_K(G)$.

Note that it is actually not difficult to find such a set $\mathcal{D}_0(\mathbf{W},\gamma)$. Indeed, for any $w \in \mathbf{W}$, let $\mathbf{T}_w \subseteq \mathbf{G}$ be an F-stable maximal torus of type w and denote by $R_{\mathbf{T}_w,\theta}$ the virtual representation defined by Deligne and Lusztig [7], where $\theta \in \mathrm{Irr}_K(\mathbf{T}_w^F)$. By [5, §2.9, 3.3.8, 7.5.2], there exists a polynomial $f_w \in \mathbb{Z}[t]$ (depending only on $\mathbf{W}, \gamma$ and w) such that $\dim R_{\mathbf{T}_w,\theta} = f_w(q)$. Then the set

$$\mathcal{D}_0(\mathbf{W},\gamma) := \Big\{ \sum_{w\in\mathbf{W}} \frac{a_w}{b_w} f_w \;\Big|\; \begin{array}{l} a_w \in \mathbb{Z} \text{ and } b_w \in \mathbb{Z} \text{ such that} \\ |a_w| \leqslant |\mathbf{W}| \text{ and } 0 < |b_w| \leqslant |\mathbf{W}| \end{array} \Big\}$$

has the required properties. This follows using the scalar product formula for $R_{\mathbf{T}_w,\theta}$, the partition of $\mathrm{Irr}_K(G)$ into geometric conjugacy classes, and the uniformity of the regular representation of G. These results can be found in [5, 7.3.4, 7.3.8, 7.5.6]; note that these were all already available by [7].

Now consider the situation in characteristic $\ell > 0$.

Conjecture 4.3. *There exists a finite set of polynomials $\bar{\mathcal{D}}(\mathbf{W},\gamma) \subseteq \mathbb{K}[t]$, depending only on $\mathbf{W},\gamma$ (but not on q or ℓ), such that*

$$\{\dim Y \mid Y \in \mathrm{Irr}_k(G)\} \quad \subseteq \quad \{f(q) \mid f \in \bar{\mathcal{D}}(\mathbf{W},\gamma)\}.$$

Remark 4.4. If this conjecture holds then, in particular, the set $\bar{\mathcal{D}}(\mathbf{W},\gamma)$ will satisfy the condition in Proposition 4.1. (To see this just choose ℓ such that $\ell \nmid |G|$.) Thus, we may always assume that $\mathcal{D}_0(\mathbf{W},\gamma) \subseteq \bar{\mathcal{D}}(\mathbf{W},\gamma)$. This inclusion will be strict in general. For example, if $\mathbf{W}$ is of type C_2 and γ is non-trivial, then the results in [4] show that

$$\bar{\mathcal{D}}(C_2,\gamma) = \mathcal{D}_0(C_2,\gamma) \cup \{t^4-1\}.$$

The point of the conjecture is that, in general, it should be sufficient to add only finitely many polynomials to $\mathcal{D}_0(\mathbf{W},\gamma)$ in order to obtain $\bar{\mathcal{D}}(\mathbf{W},\gamma)$.

Example 4.5. Conjecture 4.3 is true for $G = \mathrm{GL}_n(\mathbb{F}_q)$. This seen as follows. Let D be the ℓ-modular decomposition matrix of G. Recall that this matrix has rows labelled by $\mathrm{Irr}_K(G)$ and columns labelled by $\mathrm{Irr}_k(G)$. By Fong–Srinivasan [10], there is a subset $\mathcal{S} \subseteq \mathrm{Irr}_K(G)$ such that the submatrix of D with rows labelled by $\mathcal{S}$ is square and invertible over $\mathbb{Z}$. Let D_0 denote this submatrix. Then we obtain each $\dim Y$ (where $Y \in \mathrm{Irr}_k(G)$) as an integral linear combination of $\{\dim \rho \mid \rho \in \mathcal{S}\}$ where the coefficients are entries of the inverse of D_0. The results in [10] show, furthermore, that D_0 is a block diagonal matrix where the sizes of the diagonal blocks are bounded by a constant which only depends on n (but not on q or ℓ). Now, Dipper–James [9] showed that these diagonal blocks of D_0 are given by the decomposition matrices of various q-Schur algebras. Each of these algebras is finite-dimensional where the dimension is bounded by a function in n. Hence, D_0 is a block diagonal matrix where both the sizes and the entries of the diagonal blocks are uniformly bounded by a constant which only depends on n. (But note that the total size of D_0 depends, of course, on q and ℓ.) Analogous statements then also hold for the inverse of D_0, with the only difference that the entries may be negative (but the absolute values will still be bounded by a constant which only depends on n). We conclude that each $\dim Y$ (where $Y \in \mathrm{Irr}_k(G)$) can be expressed as an integral linear combination of $\{f(q) \mid f \in \mathcal{D}_0(\mathbf{W},\gamma)\}$ where the absolute values of the coefficients are bounded by a constant which only depends on n. By taking all possible such linear combinations of the polynomials in $\mathcal{D}_0(\mathbf{W},\gamma)$, we obtain a finite set $\bar{\mathcal{D}}(\mathbf{W},\gamma)$ with the required property.

We note that no further examples are known except for some types of groups of small rank where explicit computations are possible and one can adopt the above arguments; see [24], [28], [29] and the references there. In particular, the problem is open for the groups $G = \mathrm{GU}_n(\mathbb{F}_q)$.

Remark 4.6. Recall that, throughout this paper, we have assumed that $\mathrm{char}(k) = \ell \neq p$. In this final remark, we drop this asumption and let

$\mathcal{O}$ be such that $\mathrm{char}(k) = \ell = p$. Consider the example $G = \mathrm{SL}_2(\mathbb{F}_p)$ where $\mathbf{W}$ is a cyclic group of order 2. Then

$$\{\dim Y \mid Y \in \mathrm{Irr}_k(G)\} = \{1, 2, \ldots, p\} \qquad \text{(see [2, §3])}.$$

So it is impossible that a statement like that in Conjecture 4.3 holds for $\mathrm{Irr}_k(G)$ where $\mathrm{char}(k) = \ell = p$. – Thus, Conjecture 4.3 is an indication of the sharp distinction between the modular representation theory of finite groups of Lie type in defining and non-defining characteristic.

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