ON TWISTED CONJUGACY CLASSES OF TYPE D IN SPORADIC SIMPLE GROUPS

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Abstract. We determine twisted conjugacy classes of type D associated to the sporadic simple groups. This is an important step in the program of the classification of finite-dimensional pointed Hopf algebras with non-abelian coradical. As a by-product we prove that every complex finite-dimensional pointed Hopf algebra over the group of automorphisms of M_{12}, J_2, Suz, He, HN, T is the group algebra.

1. Introduction

A fundamental step in the classification of finite-dimensional complex pointed Hopf algebras, in the context of the Lifting method [AS1] is the determination of all finite-dimensional Nichols algebras of braided vector spaces arising from Yetter-Drinfled modules over groups. This problem can be reformulated in other terms: to study finite-dimensional Nichols algebras of braided vector spaces arising from pairs (X, q), where X is a rack and q is a 2-cocycle of X.

A useful strategy to deal with this problem is to discard those pairs (X, q) whose associated Nichols algebra is infinite dimensional. A powerful tool to discard such pairs is the notion of rack of type D [AFGV1]. This notion is based on the theory of Weyl groupoids developed in [AHS] and [HS]. The ubiquity of racks of type D suggests a powerful approach for the classification problem of finite-dimensional pointed Hopf algebras. This program was described in [AFGaV, §2]. This paper is a contribution to this program.

Towards the classification of simple racks of type D, we study an important family of simple racks: the twisted conjugacy classes associated to L, where L is a sporadic simple group. Our aim is to classify which of these racks are of type D. For that purpose, we use the fact that these racks can be realized as conjugacy classes of the group of automorphisms of L. The main result of our work is the following theorem.

Theorem 1.1. *Let L be one of the following simple groups*

$$M_{12},\ M_{22},\ J_2,\ J_3,\ Suz,\ HS,\ McL,\ He,\ Fi_{22},\ ON,\ Fi'_{24},\ HN,\ T.$$

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TABLE 1.

Classes not of type D	
$\mathrm{Aut}(M_{22})$	2B
$\mathrm{Aut}(HS)$	2C
$\mathrm{Aut}(Fi_{22})$	2D
Classes not known to be of type D	
$\mathrm{Aut}(J_3)$	34A, 34B
$\mathrm{Aut}(ON)$	38A, 38B, 38C
$\mathrm{Aut}(McL)$	22A, 22B
$\mathrm{Aut}(Fi'_{24})$	2C, 2D, 46A, 46B

Let $\mathcal{O}$ be a conjugacy class of $\mathrm{Aut}(L)\setminus L$ which is not listed in Table 1. Then $\mathcal{O}$ is of type D.

Combining Theorem 1.1 with [AFGV3] and the lifting method [AS1] we obtain the following classification result.

Corollary 1.2. *Let L be one of the following simple groups*

$$M_{12},\ J_2,\ Suz,\ He,\ HN,\ T.$$

Then $\mathrm{Aut}(L)$ does not have non-trivial finite-dimensional complex pointed Hopf algebras. □

The strategy for proving Theorem 1.1 is the same as in [AFGV2, AFGV3]. We use the computer algebra system GAP to perform the computations [GAP] [B] [WPN+] [WWT+]. The main programs and logs of this paper can be found at our homepages.

2. Preliminaries

We refer to [AS2] for generalities about Nichols algebras and to [AG] for generalities about racks and their cohomologies in the context of Nichols algebras. We follow [CCNPW] for the notations concerning the sporadic simple groups.

A *rack* is a pair $(X,\triangleright)$, where X is a non-empty set and $\triangleright : X\times X \to X$ is a map (considered as a binary operation on X) such that the map $\varphi_x : X \to X$, $\varphi_x(y) = x\triangleright y$, is bijective for all $x \in X$, and $x\triangleright(y\triangleright z) = (x\triangleright y)\triangleright(x\triangleright z)$ for all $x, y, z \in X$. A *subrack* of a rack X is a non-empty subset $Y \subseteq X$ such that $(Y,\triangleright)$ is also a rack.

A rack $(X,\triangleright)$ is said to be of *type D* if it contains a decomposable subrack $Y = R \sqcup S$ such that $r\triangleright(s\triangleright(r\triangleright s)) \neq s$ for some $r \in R$, $s \in S$. Racks of type D have the following properties.

(i) If $Y \subseteq X$ is a subrack of type D, then X is of type D.
(ii) If Z is a finite rack and $p : Z \to X$ is an epimorphism, then X of type D implies Z is of type D.

The following result is the reason why we study racks of type D. This theorem is based on [AHS] and [HS].

Theorem 2.1. [AFGV1, Thm. 3.6] *Let X be a finite rack of type D. Then the Nichols algebra associated to the pair (X, q) is infinite-dimensional for all 2-cocycle q.* □

Recall that a rack X is *simple* if it has no quotients except itself and the one-element rack. By [AG, Prop. 3.1], every indecomposable rack has a projection onto a simple rack. The classification of finite simple racks is known, see [AG, Theorems 3.9 and 3.12], [J]. Twisted conjugacy classes of finite simple groups are examples of simple racks appearing in the classification.

2.1. **Twisted conjugacy classes.** Let G be a finite group and $u \in \mathrm{Aut}(G)$. The group G acts on itself by $y \rightharpoonup_u x = y\,x\,u(y^{-1})$ for all $x, y \in G$. The orbit of x under this action will be called the *u-twisted conjugacy class* of x and it will be denoted by $\mathcal{O}_x^{G,u}$. It is easy to prove that the orbit $\mathcal{O}_x^{G,u}$ is a rack with

$$y \triangleright_u z = y\,u(z\,y^{-1})$$

for all $y, z \in \mathcal{O}_x^{G,u}$. Notice that $\mathcal{O}_x^{G,\mathrm{id}}$ is a conjugacy class in G.

We write $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ for the group of outer automorphisms of G and $\pi : \mathrm{Aut}(G) \to \mathrm{Out}(G)$ for the canonical projection.

Assume that $\mathrm{Out}(G) \neq 1$. Let $u \in \mathrm{Aut}(G)$ such that $\pi(u) \neq 1$. Every u-twisted conjugacy class in G is isomorphic (as a rack) to a conjugacy class in the semidirect product $G \rtimes \langle u \rangle$. Indeed,

$$\mathcal{O}_{(x,u)}^{G\rtimes\langle u\rangle,\mathrm{id}} = \mathcal{O}_x^{G,u} \times \{u\}$$

for all $x \in G$. Therefore the problem of determining u-twisted conjugacy classes of type D in G can be reduced to study conjugacy classes of type D in $G \rtimes \langle u \rangle$ and contained in $G \times \{u\}$.

2.2. **Conjugacy classes to study.** Let L be one of the simple groups

$$M_{12},\ M_{22},\ J_2,\ J_3,\ Suz,\ HS,\ McL,\ He,\ Fi_{22},\ ON,\ Fi_{24}',\ HN,\ T.$$

It is well-known that $\mathrm{Aut}(L) \simeq L \rtimes \mathbb{Z}_2$ [CCNPW]. Hence since L is a normal subgroup of $L \rtimes \mathbb{Z}_2$, it is possible to compute the list of conjugacy classes of $\mathrm{Aut}(L) \setminus L$ from the character table of $\mathrm{Aut}(L)$, see for example [I]. For that purpose, we use the GAP function `ClassPositionsOfNormalSubgroups`. See the file `logs/tostudy.log` for the information concerning the conjugacy classes of $\mathrm{Aut}(L) \setminus L$.

2.3. **Strategy.** Our aim is the classification of twisted conjugacy classes of sporadic simple groups of type D. By Subsection 2.1, we need to consider the conjugacy classes of $\mathrm{Aut}(L) \setminus L$, where L is a sporadic simple group with $\mathrm{Out}(L) \neq 1$. The strategy for studying these conjugacy classes is essentially based on studying conjugacy classes of type D in maximal subgroups of $\mathrm{Aut}(L)$.

3. Proof of Theorem 1.1

The claim concerning the groups M_{12}, J_2 follows from the application of [AFGV3, Algorithm I]. The claim for the groups M_{22}, J_3, Suz, HS, McL, He, Fi_{22}, ON and T follows from the application of [AFGV3, Algorithm III]. There is one log file for each of these groups, see Table 2. The groups HN and Fi'_{24} are studied in Subsections 3.2 and 3.1, respectively.

Table 2. Logfiles

L	logfile
M_{12}	`M12.2.log`
M_{22}	`M22.2.log`
J_2	`J2.2.log`
J_3	`J3.2.log`
Suz	`Suz.2.log`
HS	`HS.2.log`
McL	`McL.2.log`
He	`He.2.log`
Fi_{22}	`Fi22.2.log`
ON	`ON.2.log`
T	`T.2.log`

Remark 3.1. *It seems that the conjugacy classes listed the second part of Table 1 are not of type D but our resources are not enough to perform the computations.*

3.1. The group $\mathrm{Aut}(Fi'_{24})$.

Lemma 3.2. *All the conjugacy classes in* $\mathrm{Aut}(Fi'_{24}) \setminus Fi'_{24}$ *are of type D, except* 2C, 2D, 46A, 46B *which are not known to be of type D.*

Proof. We use maximal subgroups of $\mathrm{Aut}(Fi'_{24})$. With GAP it is possible to recover the information related to the fusion of the conjugacy classes from the maximal subgroups into $\mathrm{Aut}(Fi'_{24})$. We split the proof into several steps. There is one logfile for each step. The fusion of conjugacy classes and all files concerning this proof belong to the folder `logs/F3+.2`.

Step 1. *The classes* 4D, 4E, 4F, 6L, 6M, 6N, 6O, 6P, 6Q, 6R, 6S, 6T, 6U, 8D, 8E, 8F, 10C, 10D, 12M, 12O, 12P, 12Q, 12R, 12S, 12T, 12W, 12X, 12Z, 14C, 14D, 16B, 18H, 18I, 18J, 18K, 18L, 18M, 18N, 18P, 18Q, 20C, 20D, 22B, 22C, 24F, 24G, 24I, 26B, 26C, 28B, 30C, 30D, 30E, 30F, 34A, 36E, 36F, 42C, 54A, 60B, 70A, 78A, 78B *are of type D.*

We use the maximal subgroup $\mathcal{M}_2 \simeq 2 \times Fi_{23}$. By [AFGV3, Thm. II], every conjugacy class of $\mathcal{M}_2$ with representative of order distinct from 2, 23, 46 is of type D.

Step 2. *The classes* 4G, 12N, 12U, 12V, 12Y, 24H, 40A *are of type D.*

We use the maximal subgroup $\mathcal{M}_5 \simeq O_{10}^{-}(2).2$. The conjugacy classes of elements of order 4, 12, 24, 40 in $\mathcal{M}_5$ are of type D.

Step 3. *The class* 30G *is of type D.*

We use the maximal subgroup $\mathcal{M}_{17} \simeq \mathbb{S}_5 \times \mathbb{S}_9$. The conjugacy classes of elements of order 30 in $\mathbb{S}_5 \times \mathbb{S}_9$ are of type D.

Step 4. *The classes* 6V, 42D, 84A *are of type D.*

We use the maximal subgroup $\mathcal{M}_{19} \simeq (\mathbb{Z}_7 \rtimes \mathbb{Z}_6) \times \mathbb{S}_7$. Notice that six conjugacy classes of $\mathcal{M}_{19}$ are contained in the class 6V. Further, three of them are of type D. On the other hand, the conjugacy classes of elements of order 42 and 84 in $\mathcal{M}_{19}$ are of type D.

Step 5. *The class* 18O *is of type D.*

We use the maximal subgroup $\mathcal{M}_{18}$. The conjugacy classes of elements of order 18 in $\mathcal{M}_{18}$ are of type D.

Step 6. *The classes* 66A, 66B *are of type D.*

We use the maximal subgroup $\mathcal{M}_9$. The two conjugacy classes of elements of order 66 in $\mathcal{M}_9$ are of type D.

Step 7. *The classes* 36D, 36G *are of type D.*

We use the maximal subgroup $\mathcal{M}_{12}$. The conjugacy classes of elements of order 36 in $\mathcal{M}_{12}$ are of type D.

Step 8. *The classes* 12A1, 28C, 28D *are of type D.*

We use the maximal subgroup $\mathcal{M}_{20}$. The conjugacy classes of elements of order 12 and 28 in $\mathcal{M}_{20}$ are of type D. ☐

3.2. **The group** $\mathrm{Aut}(HN)$.

Lemma 3.3. *All the conjugacy classes in* $\mathrm{Aut}(HN) \setminus HN$ *are of type D.*

Proof. We use maximal subgroups of $\mathrm{Aut}(HN)$. With `GAP` it is possible to recover the information related to the fusion of the conjugacy classes from the maximal subgroups into $\mathrm{Aut}(HN)$. We split the proof into several steps. There is one logfile for each step. The fusion of conjugacy classes and all files concerning this proof belong to the folder `logs/HN.2`.

Step 1. *The classes* 2C, 4D, 4E, 4F, 6D, 6E, 6F, 8C, 8D, 10G, 10H, 12D, 12E, 14B, 18A, 20F, 24A, 28A, 30C, 42A, 60A *are of type D.*

We use the maximal subgroup $\mathcal{M}_2 \simeq \mathbb{S}_{12}$. By [AFGV1, Thm. 4.1], every conjugacy class of $\mathbb{S}_{12}$ with representative of order distinct from 2, 3, 11 is of type D. It remains to study the class 2C. The class of transpositions in $\mathbb{S}_{12}$ is the unique class of involutions which is not of type D. But there are three different conjugacy classes of involutions of $\mathcal{M}_2$ contained in the class 2C of $\mathrm{Aut}(HN)$ and hence the latter is of type D.

Step 2. *The classes* 8F, 24B, 24C *are of type D.*

We use the maximal subgroup $\mathcal{M}_{13}$. The conjugacy classes of elements of order 8 and 24 in $\mathcal{M}_{13}$ are of type D.

Step 3. *The class* 8E *is of type D.*

We use the maximal subgroup $\mathcal{M}_9$. The conjugacy classes of elements of order 8 in $\mathcal{M}_9$ are of type D.

Step 4. *The classes* 20E, 20G, 20H, 20I, 40B, 40C, 40D *are of type D.*

We use the maximal subgroup $\mathcal{M}_7$. The conjugacy classes of elements of order 20 and 40 in $\mathcal{M}_7$ are of type D.

Step 5. *The classes* 44A, 44B *are of type D.*

Use the maximal subgroup $\mathcal{M}_3$. The conjugacy classes of elements of order 44 in $\mathcal{M}_3$ are of type D. $\square$

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References

[AFGaV] N. Andruskiewitsch, F. Fantino, G. A. Garcia, and L. Vendramin, *On Nichols algebras associated to simple racks*, Contemp. Math. **537** (2011) 31–56.

[AFGV1] N. Andruskiewitsch, F. Fantino, M. Graña, and L. Vendramin, *Finite-dimensional pointed Hopf algebras with alternating groups are trivial*, Ann. Mat. Pura Appl. (4) **190** (2011), no. 2, 225–245.

[AFGV2] ______, *The logbook of Pointed Hopf algebras over the sporadic simple groups*, J. Algebra **325** (2011), 282–304.

[AFGV3] ______, *Pointed Hopf algebras over the sporadic simple groups*, J. Algebra **325** (2011), 305–320.

[AG] N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math. **178** (2003), no. 2, 177–243.

[AHS] N. Andruskiewitsch, I. Heckenberger, and H.-J. Schneider, *The Nichols algebra of a semisimple Yetter-Drinfeld module*, Amer. J. Math. **132** (2010), no. 6, 1493–1547.

[AS1] N. Andruskiewitsch and H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3*, J. Algebra **209** (1998), no. 2, 658–691.

[AS2] ______,*Pointed Hopf algebras*, New directions in Hopf algebras, Math. Sci. Res. Inst. Publ., vol. 43, Cambridge Univ. Press, Cambridge, 2002, pp. 1–68.

[B] T. Breuer, *The GAP Character Table Library, Version 1.2 (unpublished)*; `http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib/`

[CCNPW] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[GAP] *The GAP Group*, 2008, GAP – Groups, Algorithms, and Programming, Version 4.4.12. Available at http://www.gap-system.org.

[HS] I. Heckenberger and H.-J. Schneider, *Root systems and Weyl groupoids for Nichols algebras*, Proc. Lond. Math. Soc. (3) **101** (2010), no. 3, 623–654.

[I] I. Martin Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006.

[J] D. Joyce, *Simple quandles*, J. Algebra **79** (1982), no. 2, 307–318.

[WPN+] R. A. Wilson, R. A. Parker, S. Nickerson, J. N. Bray and T. Breuer, *AtlasRep, A GAP Interface to the ATLAS of Group Representations, Version 1.4*, 2007, Refereed GAP package, http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasrep.

[WWT+] R. A. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray and R. Abbott, *A world-wide-web Atlas of finite group representations*, http://brauer.maths.qmul.ac.uk/Atlas/v3/.

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