Many collinear k-tuples with no $k+1$ collinear points

József Solymosi* Miloš Stojaković †

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Abstract

For every $k > 3$, we give a construction of planar point sets with many collinear k-tuples and no collinear $(k+1)$-tuples.

1 Introduction

In the early 60's Paul Erdős asked the following question about point-line incidences in the plane: *Is it possible that a planar point set contains many collinear four-tuples, but it contains no five points on a line?* There are various constructions for n-element point sets with $n^2/6 - O(n)$ collinear triples with no four on a line (see [3] or [10]). However, no similar construction is known for larger numbers.

Let us formulate Erdős' problem more precisely. For a finite set P of points in the plane and $k \geq 2$, let $t_k(P)$ be the number of lines meeting P in exactly k points, and let $T_k(P) := \sum_{k' \geq k} t_{k'}(P)$ be the number of lines meeting P in at least k points. For $r > k$ and n, we define

$$t_k^{(r)}(n) := \max_{\substack{|P|=n \\ T_r(P)=0}} t_k(P).$$

In plain words, $t_k^{(r)}(n)$ is the number of lines containing exactly k points from P, maximized over all n point sets P that do not contain r collinear points. In this paper we are concerned about bounding $t_k^{(k+1)}(n)$ from below for $k > 3$. Erdős conjectured that $t_k^{(k+1)}(n) = o(n^2)$ for $k > 3$ and offered \$100 for a proof or disproof [8] (the conjecture is listed as Conjecture 12 in the problem collection of Brass, Moser, and Pach [2]).

*Department of Mathematics, University of British Columbia, Vancouver, Canada, email: solymosi@math.ubc.ca

†Department of Mathematics and Informatics, University of Novi Sad, Serbia, email: milos.stojakovic@dmi.uns.ac.rs. Partly supported by Ministry of Science and Technological Development, Republic of Serbia, and Provincial Secretariat for Science, Province of Vojvodina.

1.1 Earlier Results

This problem was one of Erdős' favorite geometric problems, he frequently talked about it and listed it among the open problems in geometry, see [8, 7, 5, 6, 9]. It is not just a simple puzzle which might be hard to solve, it is related to some deep and difficult problems in other fields. It seems that the key to attack this question would be to understand the group structure behind point sets with many collinear triples. We will not investigate this direction in the present paper, our goal is to give a construction showing that Erdős conjecture, if true, is sharp – for $k > 3$, one can not replace the exponent 2 by $2 - c$, for any $c > 0$.

The first result was due to Kárteszi [13] who proved that $t_k^{(k+1)}(n) \geq c_k n \log n$ for all $k > 3$. In 1976 Grünbaum [11] showed that $t_k^{(k+1)}(n) \geq c_k n^{1+1/(k-2)}$. For some 30 years this was the best bound when Ismailescu [12], Brass [1], and Elkies [4] consecutively improved Grünbaum's bound for $k \geq 5$. However, similarly to Grünbaum's bound, the exponent was going to 1 as k went to infinity.

In what follows we are going to give a construction to show that for any $k > 3$ and $\delta > 0$ there is a threshold $n_0 = n_0(k, \delta)$ such that if $n \geq n_0$ then $t_k^{(k+1)}(n) \geq n^{2-\delta}$. On top of that, we note that each of the collinear k-tuples that we count in our construction has an additional property – the distance between every two consecutive points is the same.

1.2 Notation

For $r > 0$, a positive integer d and $x \in \mathbb{R}^d$, by $B_d(x, r)$ we denote the closed ball in $\mathbb{R}^d$ of radius r centered at x, and by $S_d(x, r)$ we denote the sphere in $\mathbb{R}^d$ of radius r centered at x. When $x = 0$, we will occasionally write just $B_d(r)$ and $S_d(r)$.

For a set $S \subseteq \mathbb{R}^d$, let $N(S)$ denote the number of points from the integer lattice $\mathbb{Z}^d$ that belong to S, i.e., $N(S) := \mathbb{Z}^d \cap S$.

2 A lower bound for $t_k^{(k+1)}(n)$

We will prove bounds for even and odd value of k separately, as the odd case needs a bit more attention.

2.1 k is even

Theorem 1 *For $k \geq 4$ even and $\varepsilon > 0$, there is a positive integer n_0 such that for $n > n_0$ we have $t_k^{(k+1)}(n) > n^{2-\varepsilon}$.*

Proof. We will give a construction of a point set P containing no $k+1$ collinear points, with a high value of $t_k(P)$.

Let d be a positive integer, and let $r_0 > 0$. It is known, see, e.g., [14], that for large enough r_0 we have

$$\begin{aligned} N(B_d(r_0)) &= (1+o(1))V(B_d(r_0)) \\ &= (1+o(1))C_d r_0^d \\ &\geq c_1 r_0^d, \end{aligned}$$

where $C_d = \frac{\pi^{d/2}}{\Gamma((n+2)/2)}$, and $c_1 = c_1(d)$ is a constant depending only on d.

For each integer point from $B_d(r_0)$, the square of its distance to the origin is at most r_0^2. As the square of that distance is an integer, we can apply pigeonhole principle to conclude that there exists r, with $0 < r \leq r_0$, such that the sphere $S_d(r)$ contains at least $1/r_0^2$ fraction of points from $B_d(r_0)$, i.e.,

$$N(S_d(r)) \geq \frac{1}{r_0^2} N(B_d(r_0)) \geq \frac{1}{r_0^2} c_1 r_0^d = c_1 r_0^{d-2}.$$

We now look at unordered pairs of different points from $\mathbb{Z}^d \cap S_d(r)$. The total number of such pairs is at least

$$\binom{N(S_d(r))}{2} \geq \binom{c_1 r_0^{d-2}}{2} \geq c_2 r_0^{2d-4},$$

for some constant $c_2 = c_2(d)$. On the other hand, for every $p, q \in \mathbb{Z}^d \cap S_d(r)$ we know that the Euclidean distance $d(p,q)$ between p and q is at most $2r$, and that the square of that distance is an integer. Hence, there are at most $4r^2$ different possible values for $d(p,q)$. Applying pigeonhole principle again, we get that there are at least

$$\frac{c_2 r_0^{2d-4}}{4r^2} \geq \frac{c_2}{4} r_0^{2d-6}$$

pairs of points from $\mathbb{Z}^d \cap S_d(r)$ that all have the same distance. We denote that distance by ℓ.

Let $p_1, q_1 \in \mathbb{Z}^d \cap S_d(r)$ with $d(p_1, q_1) = \ell$, and let s be the line going through p_1 and q_1. We define $k-2$ points $p_2, \ldots, p_{k/2}$, $q_2, \ldots, q_{k/2}$ on the line s such that $d(p_i, p_{i+1}) = \ell$ and $d(q_i, q_{i+1}) = \ell$, for all $1 \leq i < k/2$, and such that all k points $p_1, \ldots, p_{k/2}$, $q_1, \ldots, q_{k/2}$ are different, see Figure 1.

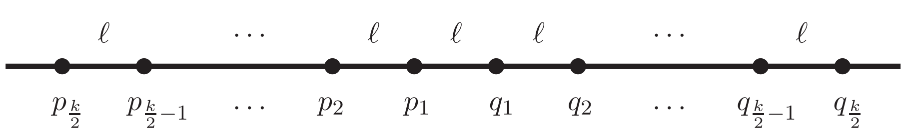

Figure 1: Line s with k points, for k even.

Knowing that p_1 and q_1 are points from $\mathbb{Z}^d$, the way we defined points $p_2, \ldots, p_{k/2}, q_2, \ldots, q_{k/2}$ implies that they have to be in $\mathbb{Z}^d$ as well. If we set

$r_i := \sqrt{r^2 + i(i-1)\ell^2}$, for all $i = 1, \ldots, k/2$, then the points p_i and q_i belong to the sphere $S_d(r_i)$, and hence, $p_i, q_i \in \mathbb{Z}^d \cap S_d(r_i)$, for all $i = 1, \ldots, k/2$, see Figure 2.

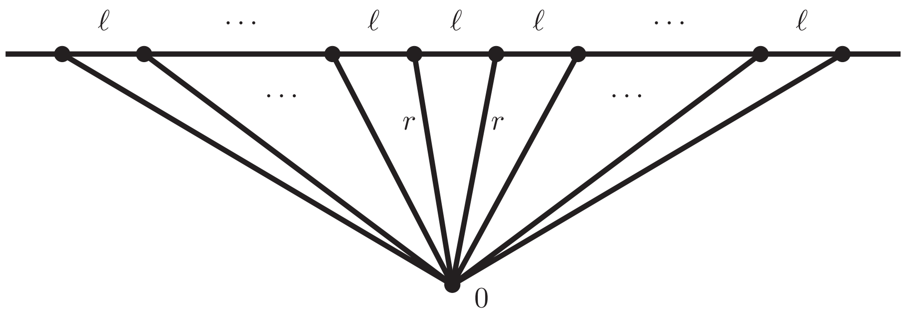

Figure 2: The position of the k points related to the origin, for k even.

We define the point set P to be the set of all integer points on spheres $S_d(r_i)$, for all $i = 1, \ldots, k/2$, i.e.,

$$P := \mathbb{Z}^d \cap \left(\cup_{i=1}^{k/2} S_d(r_i) \right).$$

Let $n := |P|$. Obviously, $P \subseteq B_d(r_{k/2})$, so we have

$$n \leq N\left(B_d(r_{k/2})\right) = (1 + o(1)) V(B_d(r_{k/2})) = c_1 r_{k/2}^d.$$

Plugging in the value of $r_{k/2}$ and having in mind that $\ell < 2r$, we obtain

$$n \leq c_1 \left(\sqrt{r^2 + k/2(k/2-1)4r^2} \right)^d \leq c_1 \sqrt{k^2+1}^d r^d \leq c_3 r^d \leq c_3 r_0^d,$$

where $c_3 = c_3(d,k)$ is a constant depending only on d and k.

As the point set P is contained in the union of $k/2$ spheres, there are obviously no $k+1$ collinear points in P. On the other hand, every pair of points $p_1, q_1 \in \mathbb{Z}^d \cap S_d(r)$ with $d(p_1, q_1) = \ell$ defines one line that contains k points from P. Hence, the number of lines containing exactly k points from P is

$$t_k(P) \geq \frac{c_2}{4} r_0^{2d-6} \geq \frac{c_2}{4} \frac{1}{c_3^{\frac{2d-6}{d}}} n^{\frac{2d-6}{d}} \geq c_4 n^{\frac{2d-6}{d}},$$

where $c_4 = c_4(d,k)$ is a constant depending only on d and k.

To obtain a point set in two dimensions, we can project our d dimensional point set to an arbitrary (two dimensional) plane in $\mathbb{R}^d$. The vector v along which we project should be chosen so that every two points from our point set are mapped to different points, and every three points that are not collinear are mapped to points that are still not collinear. Obviously, such vector can be found.

For ε given, we can pick d such that $\frac{2d-6}{d} > 2 - \varepsilon$. As we increase r_0, we obtain constructions with n growing to infinity. When n is large enough, the statement of the theorem will hold. $\square$

2.2 k is odd

Theorem 2 *For $k \geq 4$ odd and $\varepsilon > 0$, there is a positive integer n_0 such that for $n > n_0$ we have $t_k^{(k+1)}(n) > n^{2-\varepsilon}$.*

Proof. We will give a construction of a point set P containing no $k+1$ collinear points, with a high value of $t_k(P)$.

Let d be a positive integer, and let $r_0 > 0$. In the same way as in the proof of Theorem 1, we can find r with $0 < r \leq r_0$, such that the sphere $S_d(r)$ contains at least a $1/r_0^2$ fraction of the integer points from $B_d(r_0)$, and hence, $N(S_d(r)) \geq c_1 r_0^{d-2}$, for some constant $c_1 = c_1(d)$ and r_0 large enough.

Now, for every point $p \in \mathbb{Z}^d \cap S_d(r)$ there is a corresponding point p' on the sphere $S_d(2r)$ that belongs to the half-line from the origin to p. It is not hard to see that all coordinates of p' are even integers, so $p' \in (2\mathbb{Z})^d \cap S_d(2r)$. Hence, the number of points in $(2\mathbb{Z})^d \cap S_d(2r)$ is at least $c_1 r_0^{d-2}$.

We look at unordered pairs of different points from $(2\mathbb{Z})^d \cap S_d(2r)$. The total number of such pairs is at least $\binom{c_1 r_0^{d-2}}{2}$. If we just look at such pairs of points that have different first coordinate, we have at least $c_2 r_0^{2d-4}$ of those, for some constant $c_2 = c_2(d)$. To see that, observe that for every point $p \in (2\mathbb{Z})^d \cap S_d(2r)$, a point obtained from p by changing the sign of any number of coordinates of p and/or permuting the coordinates is still in $(2\mathbb{Z})^d \cap S_d(2r)$.

On the other hand, for every $p, q \in (2\mathbb{Z})^d \cap S_d(2r)$ we know that the Euclidean distance $d(p,q)$ between p and q is at most $4r$, and that the square of that distance is an integer. Hence, there are at most $16r^2$ different possible values for $d(p,q)$. Applying pigeonhole principle again, we get that there are at least

$$\frac{c_2 r_0^{2d-4}}{16r^2} \geq \frac{c_2}{16} r_0^{2d-6}$$

pairs of points with different first coordinate, from $(2\mathbb{Z})^d \cap S_d(2r)$, that have the same distance. We denote that distance by 2ℓ. Note that since both p and q are contained in $(2\mathbb{Z})^d$, we have that the middle point m of the segment pq belongs to $\mathbb{Z}^d$, and $d(p,m) = d(q,m) = \ell$.

Let $p_1, q_1 \in (2\mathbb{Z})^d \cap S_d(2r)$ with $d(p_1, q_1) = 2\ell$, let m_0 be the middle point of the segment $p_1 q_1$, and let s be the line going through p_1 and q_1. We define $k-3$ points $p_2, \ldots, p_{(k-1)/2}, q_2, \ldots, q_{(k-1)/2}$ on the line s such that $d(p_i, p_{i+1}) = \ell$ and $d(q_i, q_{i+1}) = \ell$, for all $1 \leq i < (k-1)/2$, and all k points $m_0, p_1, \ldots, p_{(k-1)/2}$, $q_1, \ldots, q_{(k-1)/2}$ are different, see Figure 3.

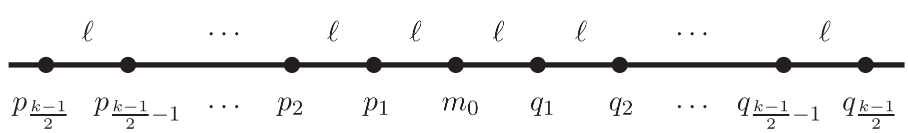

Figure 3: Line s with k points, for k odd.

Knowing that p_1 and q_1 are points from $(2\mathbb{Z})^d$, the way we defined points $m_0, p_2, \dots, p_{(k-1)/2}$, $q_2, \dots, q_{(k-1)/2}$ implies that they have to be in $\mathbb{Z}^d$. If we set $r_i := \sqrt{4r^2 + (i+1)(i-1)\ell^2}$, for all $i = 0, \dots, (k-1)/2$, the points p_i and q_i belong to the sphere $S_d(r_i)$, and the point m_0 belongs to $S_d(r_0)$. Hence, $p_i, q_i \in \mathbb{Z}^d \cap S_d(r_i)$, for all $i = 1, \dots, (k-1)/2$, and $m_0 \in \mathbb{Z}^d \cap S_d(r_0)$, see Figure 4.

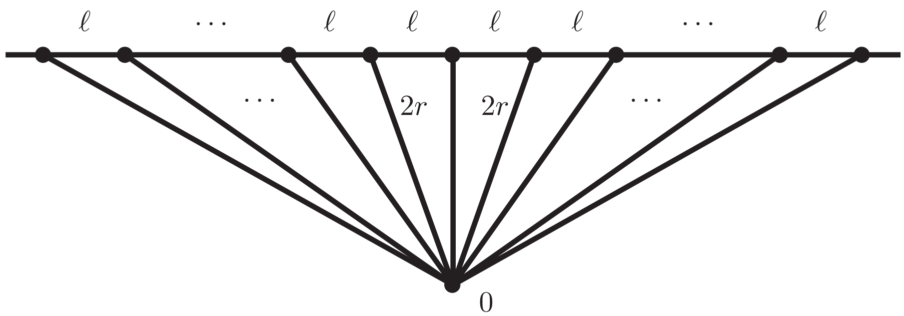

Figure 4: The position of the k points related to the origin, for k odd.

By α_x we denote the hyperplane containing all points in $\mathbb{R}^d$ with first coordinate equal to x. Let M be the multiset of points m such that there exist points $p, q \in (2\mathbb{Z})^d \cap S_d(2r)$ having different first coordinate, with $d(p,q) = 2\ell$, and with m being the middle point of segment pq. In this multiset, we include the point m once for every such p and q. From our previous calculations it follows that $|M| = \frac{c_2}{16} r_0^{2d-6}$ and $M \subseteq \mathbb{Z}^d \cap S_d(r_0)$. Each point from $\mathbb{Z}^d \cap S_d(r_0)$ is contained in α_x for some $-r_0 \le x \le r_0$, and hence, by the pigeonhole principle, there exists $-r_0 \le x_0 \le r_0$ such that $\alpha_{x_0} \cap M$ contains at least $|M|/(2r_0) \ge \frac{c_2}{32} r_0^{2d-7}$ points.

We define the point set P to be the set of all integer points on spheres $S_d(r_i)$, for all $i = 1, \dots, (k-3)/2$, all integer points on $S_d(r_{(k-1)/2})$ that do not belong to α_{x_0}, and all integer points on $S_d(r_0)$ that belong to α_{x_0}. I.e., we have

$$P := \mathbb{Z}^d \cap \left(\left(\cup_{i=1}^{(k-3)/2} S_d(r_i) \right) \cup \left(S_d(r_{(k-1)/2}) \setminus \alpha_{x_0} \right) \cup \left(S_d(r_0) \cap \alpha_{x_0} \right) \right).$$

Let $n := |P|$. Obviously, $P \subseteq B_d(r_{(k-1)/2})$, so, as before, we have

$$n \le N\left(B_d(r_{k/2})\right) = (1 + o(1)) V(B_d(r_{(k-1)/2})) = c_1 r_{(k-1)/2}^d.$$

Plugging in the value of $r_{(k-1)/2}$ and having in mind that $\ell < r$, we obtain

$$n \le c_1 \left(\sqrt{4r^2 + (k/2+1)(k/2-1)r^2} \right)^d \le c_3 r^d \le c_3 r_0^d,$$

where $c_3 = c_3(d,k)$ is a constant depending only on d and k.

Let us first prove that the point set P does not contain $k+1$ collinear points. As P is contained in the union of $(k-1)/2$ spheres and a hyperplane, any line that is not contained in that hyperplane cannot contain more than k

points from P. But the point set P restricted to the hyperplane α_{x_0} belongs to the union of $(k-1)/2$ spheres $S_d(r_i)$, for $i = 0, \ldots, (k-3)/2$, so we can also conclude that there are no $k+1$ collinear points in $P \cap \alpha_{x_0}$.

On the other hand, every pair of points $p_1, q_1 \in \mathbb{Z}^d \cap S_d(r)$ with different first coordinate, with $d(p_1, q_1) = 2\ell$, and with the middle point that belongs to $\alpha_{x_0} \cap M$, defines one line that contains k points from P. Note that such line cannot belong to α_{x_0}, as the first coordinates of p_1 and q_1 cannot be x_0 simultaneously.

Hence, the number of lines containing exactly k points from P is

$$t_k(P) \geq \frac{c_2}{32} r_0^{2d-7} \geq \frac{c_2}{32} \frac{1}{c_3^{\frac{2d-7}{d}}} n^{\frac{2d-7}{d}} \geq c_4 n^{\frac{2d-7}{d}},$$

where $c_4 = c_4(d, k)$ is a constant depending only on d and k.

To obtain a point set in two dimensions, we will project our d dimensional point set to a generic (two dimensional) plane in $\mathbb{R}^d$. The vector v along which we project should be chosen so that every two points from our point set are mapped to different points, and every three points that are not collinear are mapped to points that are still not collinear. Obviously, such vector can be found.

For ε given, we can pick d such that $\frac{2d-7}{d} > 2 - \varepsilon$. By increasing r_0, one can obtain constructions for arbitrary large n. $\square$

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