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THE POWER QUANTUM CALCULUS AND VARIATIONAL PROBLEMS

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Abstract. We introduce the power difference calculus based on the operator $D_{n,q}f(t) = \frac{f(qt^n)-f(t)}{qt^n-t}$, where n is an odd positive integer and $0 < q < 1$. Properties of the new operator and its inverse — the $d_{n,q}$ integral — are proved. As an application, we consider power quantum Lagrangian systems and corresponding n,q-Euler–Lagrange equations.

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1 Introduction

Quantum derivatives and integrals play a leading role in the understanding of complex physical systems. In 1992 Nottale introduced the theory of scale-relativity without the hypothesis of space-time differentiability [37, 38]. A rigorous mathematical foundation to Nottale's scale-relativity theory is nowadays given by means of a quantum calculus [4, 6, 17, 26]. Roughly speaking, a quantum calculus substitute the classical derivative by a difference operator, which allows to deal with sets of non differentiable curves. For the motivation to study a non-differentiable quantum calculus we refer the reader to [4, 17, 26, 37].

Quantum calculus has several different dialects [13, 20, 26]. The most common tongue of quantum calculus is based on the q-operator (q stands for quantum), which is based on the Jackson q-difference operator and the associated Jackson q-integral [24, 25, 26]. The Jackson q-difference operator is defined by

$$D_q f(t) = \frac{f(qt) - f(t)}{t(q-1)}, \quad t \neq 0\,,$$

where q is a fixed number, normally taken to lie in $(0,1)$. Here f is supposed to be defined on a q-geometric set A, i.e., A is a subset of $\mathbb{R}$ (or $\mathbb{C}$) for which $qt \in A$ whenever $t \in A$. The derivative at zero is normally defined to be $f'(0)$, provided that $f'(0)$ exists [1, 9, 14, 15, 23, 24]. Jackson also introduced the q-integral

$$\int_0^a f(t)d_qt = a(1-q)\sum_{k=0}^{\infty} q^k f(aq^k) \tag{1}$$

provided that the series converges [8, 25, 26]. He then defined

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt. \tag{2}$$

There is no unique canonical choice for the q-integral from 0 to ∞. Following Jackson we will put

$$\int_0^{\infty} f(t)d_qt = (1-q)\sum_{k=-\infty}^{\infty} q^k f(q^k),$$

provided the sum converges absolutely [22, 27]. The other natural choices are then expressed by

$$\int_0^{s\cdot\infty} f(t)d_qt = s(1-q)\sum_{k=-\infty}^{\infty} q^k f(sq^k), \quad s > 0$$

(see [27]). The bilateral q-integral is defined by

$$\int_{-s\cdot\infty}^{s\cdot\infty} f(t)d_qt = s(1-q)\sum_{k=-\infty}^{\infty} \left[q^k f(sq^k) + q^k f(-sq^k)\right], \quad s > 0.$$

The main goal of this work is to generalize the important q-calculus in order to include also the quantum calculus that results from the n-power difference operator

$$D_n f(t) = \begin{cases} \frac{f(t^n)-f(t)}{t^n-t} & \text{if } t \in \mathbb{R} \setminus \{-1,0,1\}, \\ f'(t) & \text{if } t \in \{-1,0,1\}, \end{cases} \tag{3}$$

where n is a fixed odd positive integer [2]. For that we develop a calculus based on the new and more general proposed operator $D_{n,q}$ (see Definition 2). The class of quantum systems thus obtained has two parameters and is wider than the standard class of quantum dynamical systems studied in the literature. We claim that the n,q-calculus here introduced offers a better mathematical modeling technique to deal with quantum physical systems of time-varying graininess. We trust that our n,q quantum calculus will become a useful tool to investigate more about non-conservative dynamical systems in physics [12, 18, 19, 21].

The paper is organized as follows. Our results are given in Section 2 and Section 3: in §2.1 we introduce the notion of power quantum differentiation and prove its main properties; in §2.2 we develop the notion of power quantum integration as the inverse operation of power quantum differentiation; and in §3.1 we obtain the Euler–Lagrange equation for functionals defined by n, q-derivatives and integrals, generalizing the Euler–Lagrange equations presented in [10, 11]. We also provide (see §3.2) an additional tool for solving power quantum variational problems, by showing that the direct method introduced by Leitmann in the sixties of the XX century [28], and recently extended to different contexts [7, 29, 30, 34, 39], remains effective here.

2 The power quantum calculus

For a fixed $0 < q < 1$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and a fixed odd positive integer n, let us denote

$$\theta := \begin{cases} \infty & \text{if} \quad n = 1, \\ q^{\frac{1}{1-n}} & \text{if} \quad n \in 2\mathbb{N}+1, \end{cases} \qquad S := \begin{cases} \{0\} & \text{if} \quad n = 1, \\ \{-\theta, 0, \theta\} & \text{if} \quad n \in 2\mathbb{N}+1, \end{cases}$$

$$\text{and} \quad [k]_n := \begin{cases} \sum_{i=0}^{k-1} n^i & \text{if} \quad k \in \mathbb{N}, \\ 0 & \text{if} \quad k = 0. \end{cases}$$

Lemma 1. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $h(t) := qt^n$. Then, h is one-to-one, onto, and $h^{-1}(t) = \sqrt[n]{\frac{t}{q}}$. Moreover,*

$$h^k(t) := \underbrace{h \circ h \circ \cdots \circ h}_{k-times}(t) = q^{[k]_n} t^{n^k}$$

and

$$h^{-k}(t) := \underbrace{h^{-1} \circ h^{-1} \circ \cdots \circ h^{-1}}_{k-times}(t) = q^{-n^{-k}[k]_n} t^{n^{-k}}$$

with

$$\lim_{k \longrightarrow \infty} h^k(t) = \begin{cases} \infty & \text{if} \quad t > \theta \\ 0 & \text{if} \quad -\theta < t < \theta \\ -\infty & \text{if} \quad t < -\theta \\ t & \text{if} \quad t \in S \end{cases}$$

and

$$\lim_{k \longrightarrow \infty} h^{-k}(t) = \begin{cases} \theta & \text{if} \quad 0 < t \\ -\theta & \text{if} \quad t < 0 \\ t & \text{if} \quad t \in S\,. \end{cases}$$

In Figure 1 we illustrate the behaviour of $h^k(t)$ of Lemma 1 in the case $-\theta < t < \theta$.

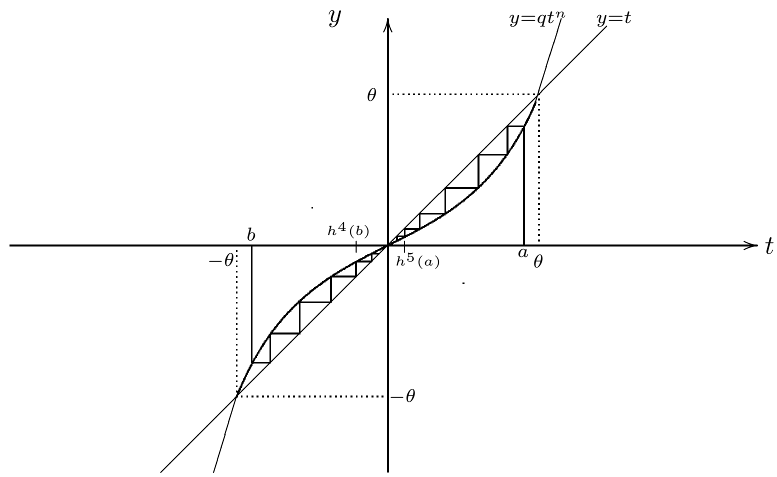

Figure 1: The iteration of $h(t) = qt^n,\ t \in \mathbb{R},\ n \in 2\mathbb{N}+1,\ 0 < q < 1$.

2.1 Power quantum differentiation

We introduce the n, q-power difference operator as follows:

Definition 2. Assume that f is a real function defined on $\mathbb{R}$. The n, q-power operator is given by

$$D_{n,q}f(t) := \begin{cases} \dfrac{f(qt^n) - f(t)}{qt^n - t} & \text{if} \quad t \in \mathbb{R} \setminus S, \\ f'(t) & \text{if} \quad t \in S, \end{cases}$$

provided f is differentiable at $t \in S$. If $D_{n,q}f(t)$ exists, we say that f is n, q-differentiable at t.

The following lemma is a direct consequence of Definition 2.

Lemma 3. *Let f be a real function and $t \in \mathbb{R}$.*

(i) If f is n, q-differentiable at t, $t \in S$, then f is continuous at t.

(ii) If f is n, q-differentiable on an interval $I \subset (-\theta, \theta)$, $0 \in I$, and

$$D_{n,q}f(t) = 0 \text{ for } t \in I,$$

then f is a constant function on I.

(iii) If f is n, q-differentiable at t, then $f(qt^n) = f(t) + (qt^n - t)D_{n,q}f(t)$.

The next theorem gives useful formulas for the computation of n, q-derivatives of sums, products, and quotients of n, q-differentiable functions.

Theorem 4. *Assume $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are n, q-differentiable at $t \in \mathbb{R}$. Then:*

(i) The sum $f + g : \mathbb{R} \longrightarrow \mathbb{R}$ is n, q-differentiable at t and

$$D_{n,q}(f + g)(t) = D_{n,q}f(t) + D_{n,q}g(t).$$

(ii) For any constant c, $cf : \mathbb{R} \longrightarrow \mathbb{R}$ is n, q-differentiable at t and

$$D_{n,q}(cf)(t) = cD_{n,q}f(t).$$

(iii) The product $fg : \mathbb{R} \longrightarrow \mathbb{R}$ is n, q-differentiable at t and

$$\begin{aligned} D_{n,q}(fg)(t) &= D_{n,q}f(t)g(t) + f(qt^n)D_{n,q}g(t) \\ &= f(t)D_{n,q}g(t) + D_{n,q}f(t)g(qt^n). \end{aligned}$$

(iv) If $g(t)g(qt^n) \neq 0$, then f/g is n, q-differentiable at t and

$$D_{n,q}\left(\frac{f}{g}\right)(t) = \frac{D_{n,q}f(t)g(t) - f(t)D_{n,q}g(t)}{g(t)g(qt^n)}.$$

Proof. The proof is done by direct calculations. □

Next example gives explicit formulas for the n, q-derivative of some simple functions.

Example 5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$.

(i) If $f(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is a constant, then $D_{n,q}f(t) = 0$.

(ii) If $f(t) = t$ for all $t \in \mathbb{R}$, then $D_{n,q}f(t) = 1$.

(iii) If $f(t) = at - b$ for all $t \in \mathbb{R}$, where a, b are real constants, then by Theorem 4 we have $D_{n,q}f(t) = a$.

(iv) If $f(t) = t^2$ for all $t \in \mathbb{R}$, then $D_{n,q}f(t) = t + qt^n$.

(v) If $f(t) = \frac{1}{t}$ for all $t \in \mathbb{R} \setminus \{0\}$, then $D_{n,q}f(t) = -\frac{1}{qt^{n+1}}$.

(vi) If $f(t) = (t + b)^m$ for all $t \in \mathbb{R}$, where $b \in \mathbb{R}$ is a constant and $m \in \mathbb{N}$, then, by induction on m, we obtain that

$$D_{n,q}f(t) = \sum_{k=0}^{m-1} (qt^n + b)^k (t + b)^{m-1-k}$$

for $t \neq S$.

We note that by definition of the n, q-difference operator, one has

$$D_{n,q}f(t) = f'(t), \quad t \in S,$$

for all functions f in (i)–(vi).

Definition 6. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$. We define the second n,q-derivative by $D_{n,q}^2 f := D_{n,q}(D_{n,q}f)$. More generally, we define $D_{n,q}^m f$ as follows:

$$\begin{aligned} D_{n,q}^0 f &= f,\\ D_{n,q}^m f &= D_{n,q}D_{n,q}^{m-1}f, \quad m\in\mathbb{N}. \end{aligned}$$

We now obtain, under certain conditions, the formula for the mth n,q-derivative of fg, $m\in\mathbb{N}$.

Let h be the function defined in Lemma 1 and let us write $h^\circ f$ to denote $f\circ h$. We will denote by $\mathcal{S}_k^m$ the set consisting of all possible strings of length m, containing exactly k times h° and $m-k$ times $D_{n,q}$.

Example 7. Let $k=2$ and $m=4$. Then,

$$\mathcal{S}_2^4=\Big\{D_{n,q}D_{n,q}h^\circ h^\circ,\ D_{n,q}h^\circ D_{n,q}h^\circ,\ D_{n,q}h^\circ h^\circ D_{n,q},\ h^\circ h^\circ D_{n,q}D_{n,q},\\ h^\circ D_{n,q}h^\circ D_{n,q},\ h^\circ D_{n,q}D_{n,q}h^\circ\Big\}.$$

Example 8. If $m=2$, then for $k=0,1,2$ we have

$$\mathcal{S}_0^2=\Big\{D_{n,q}D_{n,q}\Big\},\quad \mathcal{S}_1^2=\Big\{D_{n,q}h^\circ, h^\circ D_{n,q}\Big\},\quad \mathcal{S}_2^2=\Big\{h^\circ h^\circ\Big\}.$$

Let $f:\mathbb{R}\longrightarrow\mathbb{R}$. Then,

$$\begin{aligned} (D_{n,q}h^\circ f)(t) &= D_{n,q}(f\circ h)(t),\\ (h^\circ D_{n,q}f)(t) &= D_{n,q}f(h(t)), \end{aligned}$$

provided all these quantities exist. Observe that $D_{n,q}(f\circ h)(t)\neq D_{n,q}f(h(t))$. Indeed,

$$\begin{aligned} D_{n,q}f(h(t)) &= \frac{f(h(h(t)))-f(h(t))}{h(h(t))-h(t)}\\ &= \frac{f(h(h(t)))-f(h(t))}{(h(t)-t)(1+D_{n,q}(h(t)-t))} = \frac{D_{n,q}(f\circ h)(t)}{(1+D_{n,q}(h(t)-t))}. \end{aligned}$$

Theorem 9 (Leibniz formula)**.** *Let $\mathcal{S}_k^m$ be the set consisting of all possible strings of length m, containing exactly k times h° and $m-k$ times $D_{n,q}$. If f is a function for which Lf exist for all $L\in\mathcal{S}_k^m$, and function g is m times n,q-differentiable, then for all $m\in\mathbb{N}$ we have:*

$$D_{n,q}^m(fg)(t)=\sum_{k=0}^{m}\left(\sum_{L\in\mathcal{S}_k^m}Lf\right)(t)\,D_{n,q}^k g(t) \quad \text{for } t\in\mathbb{R}\setminus S, \tag{4}$$

and

$$D_{n,q}^m(fg)(t)=\sum_{k=0}^{m}\binom{m}{k}D_{n,q}^{m-k}f(t)\,D_{n,q}^k g(t) \quad \text{for } t\in S. \tag{5}$$

Proof. For $t \in S$ equality (5) yields

$$(fg)^{(m)}(t) = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) f^{(m-k)}(t)\; g^{(k)}(t),$$

which is true (the standard Leibniz formula of classical calculus). Assume $t \notin S$. The proof is done by induction on m. If $m = 1$, then by Theorem 4 we have $D_{n,q}(fg)(t) = D_{n,q}f(t)g(t) + h^{\circ} f(t)D_{n,q}g(t)$, i.e., (4) is true for $m = 1$. We now assume that (4) is true for $m = s$ and prove that it is also true for $m = s + 1$. First, we note that for $k \in \mathbb{N}$ and $t \notin S$

$$\begin{aligned}
D_{n,q}^{m+1}(fg)(t) &= D_{n,q}\left[\sum_{k=0}^{m}\left(\sum_{L\in\mathcal{S}_k^m} Lf\right)(t)\; D_{n,q}^{k}g(t)\right] \\
&= \sum_{k=0}^{m}\left[D_{n,q}\left(\sum_{L\in\mathcal{S}_k^m} Lf\right)(t)\; D_{n,q}^{k}g(t) + h^{\circ}\left(\sum_{L\in\mathcal{S}_k^m} Lf\right)(t)\; D_{n,q}^{k+1}g(t)\right] \\
&= \sum_{k=0}^{m}\left(\sum_{L\in\mathcal{S}_k^m} D_{n,q}Lf\right)(t)\; D_{n,q}^{k}g(t) + \sum_{k=1}^{m+1}\left(\sum_{L\in\mathcal{S}_{k-1}^m} h^{\circ}Lf\right)(t)\; D_{n,q}^{k}g(t) \\
&= \left(\sum_{L\in\mathcal{S}_m^m} h^{\circ}Lf\right)(t)\; D_{n,q}^{m+1}g(t) + \left(\sum_{L\in\mathcal{S}_0^m} D_{n,q}Lf\right)(t)\; g(t) \\
&\quad + \sum_{k=1}^{m}\left(\sum_{L\in\mathcal{S}_{k-1}^m} h^{\circ}L\,f(t) + \sum_{L\in\mathcal{S}_k^m} D_{n,q}L\,f\right)(t)\; D_{n,q}^{k}g(t) \\
&= \left(\sum_{L\in\mathcal{S}_{m+1}^{m+1}} Lf\right)(t)\; D_{n,q}^{m+1}g(t) + \left(\sum_{L\in\mathcal{S}_0^{m+1}} Lf\right)(t)\; g(t) \\
&\quad + \sum_{k=1}^{m}\left(\sum_{L\in\mathcal{S}_k^{m+1}} Lf\right)(t)\; D_{n,q}^{k}g(t) \\
&= \sum_{k=0}^{m+1}\left(\sum_{L\in\mathcal{S}_k^{m+1}} Lf\right)(t) D_{n,q}^{k}g(t).
\end{aligned}$$

We conclude that (4) is true for $m = s+1$. Hence, by mathematical induction, (4) holds for all $m \in \mathbb{N}$ and $t \in \mathbb{R} \setminus S$. ☐

The standard chain rule of classical calculus does not necessarily hold true for the n,q-quantum calculus. For example, if we assume that $f, g : \mathbb{R} \longrightarrow \mathbb{R}$

are defined by $f(t) = t^2$ and $g(t) = qt$, then we have

$$\begin{aligned} D_{n,q}(f \circ g)(t) = D_{n,q}(qt)^2 = D_{n,q}(q^2t^2) = q^2(t + qt^n) \\ \neq q^2(t + q^n t^n) = D_{n,q}f(g(t)) \cdot D_{n,q}g(t). \end{aligned}$$

However, we can derive an analogous formula of the chain rule for our power quantum calculus.

Theorem 10 (power chain rule)**.** *Assume $g : I \longrightarrow \mathbb{R}$ is continuous and n,q-differentiable, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable. Then there exists a constant c between qt^n and t with*

$$D_{n,q}(f \circ g)(t) = f'(g(c))D_{n,q}g(t). \tag{6}$$

Proof. For $t \notin S$ we have

$$D_{n,q}(f \circ g)(t) = \frac{f(g(qt^n)) - f(g(t))}{qt^n - t}.$$

We may assume that $g(qt^n) \neq g(t)$ (because if $g(qt^n) = g(t)$, then $D_{n,q}(f \circ g)(t) = D_{n,q}g(t) = 0$ and (6) holds for any c between qt^n and t). Then,

$$D_{n,q}(f \circ g)(t) = \frac{f(g(qt^n)) - f(g(t))}{g(qt^n) - g(t)} \cdot \frac{g(qt^n) - g(t)}{qt^n - t}. \tag{7}$$

By the mean value theorem, there exists a real number τ between $g(t)$ and $g(qt^n)$ with

$$\frac{f(g(qt^n)) - f(g(t))}{g(qt^n) - g(t)} = f'(\tau). \tag{8}$$

In view of the continuity of g, there exists c in the interval with end points qt^n and t such that $g(c) = \tau$. Thus from (7) and (8) we obtain (6). Relation (6) is true at t, $t \in S$, by the classical chain rule. □

2.2 Power quantum integration

In this section we are interested to study the inverse operation of $D_{n,q}f$. We call this inverse the n,q-integral of f (or the power quantum integral). We define the interval I to be $(-\theta, \theta)$.

Definition 11. Let $f : I \longrightarrow \mathbb{R}$ and $a, b \in I$. We say that F is a n,q-antiderivative of f on I if $D_{n,q}F(t) = f(t)$ for all $t \in I$.

¿From now on we assume that all series considered along the text are convergent.

Theorem 12. *Let $f : I \longrightarrow \mathbb{R}$ and $a, b \in I$. The function*

$$F(t) = -\sum_{k=0}^{\infty} q^{[k]_n} t^{n^k} \left(q^{n^k} t^{n^k(n-1)} - 1\right) f\left(q^{[k]_n} t^{n^k}\right)$$

is a n,q-antiderivative of f on I, provided f is continuous at 0.

Proof. For $t \neq 0$, we have

$$\begin{aligned} D_{n,q}F(t) &= \frac{F(qt^n) - F(t)}{qt^n - t} \\ &= \sum_{k=0}^{\infty} \Bigg[-\frac{q^{[k+1]_n} t^{n^{k+1}}}{qt^n - t} \left(q^{n^{k+1}} t^{n^{k+1}(n-1)} - 1 \right) f\left(q^{[k+1]_n} t^{n^{k+1}} \right) \\ &\quad + \frac{q^{[k]_n} t^{n^k} \left(q^{n^k} t^{n^k(n-1)} - 1 \right)}{qt^n - t} f\left(q^{[k]_n} t^{n^k} \right) \Bigg] \\ &= f(t). \end{aligned}$$

If $t = 0$, then the continuity of f at 0 implies that

$$\begin{aligned} D_{n,q}F(0) &= \lim_{s \longrightarrow 0} \frac{F(s) - F(0)}{s} \\ &= \lim_{s \longrightarrow 0} \frac{-\sum_{k=0}^{\infty} q^{[k]_n} s^{n^k} \left(q^{n^k} s^{n^k(n-1)} - 1 \right) f\left(q^{[k]_n} s^{n^k} \right)}{s} \\ &= \lim_{s \longrightarrow 0} -\sum_{k=0}^{\infty} q^{[k]_n} s^{n^k - 1} \left(q^{n^k} s^{n^k(n-1)} - 1 \right) f\left(q^{[k]_n} s^{n^k} \right) \\ &= \lim_{s \longrightarrow 0} -\sum_{k=0}^{\infty} \left(q^{[k+1]_n} s^{n^k + n^k(n-1) - 1} - q^{[k]_n} s^{n^k - 1} \right) f\left(q^{[k]_n} s^{n^k} \right) \\ &= \lim_{s \longrightarrow 0} -\sum_{k=0}^{\infty} \left(q^{[k+1]_n} s^{n^{k+1} - 1} - q^{[k]_n} s^{n^k - 1} \right) f\left(q^{[k]_n} s^{n^k} \right) \\ &= \lim_{s \longrightarrow 0} f(s) \\ &= f(0). \end{aligned}$$

This completes the proof. □

We then define the indefinite n,q-integral of f by

$$\int_I f(t)\, d_{n,q}t := F(t) + C,$$

where C is an arbitrary constant. The definite n,q-integral of f is defined as follows.

Definition 13. Let $f : I \longrightarrow \mathbb{R}$ and $a, b \in I$. We define the n,q-integral of f from a to b by

$$\int_a^b f(t)\, d_{n,q}t := \int_0^b f(t)\, d_{n,q}t - \int_0^a f(t)\, d_{n,q}t, \tag{9}$$

where

$$\int_0^x f(t)\, d_{n,q}t := -\sum_{k=0}^{\infty} q^{[k]_n} x^{n^k} \left(q^{n^k} x^{n^k(n-1)} - 1 \right) f\left(q^{[k]_n} x^{n^k} \right), \quad x \in I, \tag{10}$$

provided the series at the right-hand side of (10) converge at $x = a$ and $x = b$.

Definition 14. A function f is said to be n, q-integrable on a subinterval J of I if

$$\left| \int_a^b f(t)\, d_{n,q}t \right| < \infty \quad \text{for all } \ a, b \in J.$$

Remark 15. The integral formulas (9) and (10) yield (2) and (1) when $n = 1$; and yield the corresponding integral of the operator D_n defined in (3) when $q \longrightarrow 1$.

The following properties of the n, q-integral are direct consequences of the definition and provide extensions of analogous properties of the Jackson q-integral [24, 25, 26].

Lemma 16. *Let $f, g : I \longrightarrow \mathbb{R}$ be n, q-integrable, $k \in \mathbb{R}$, and $a, b, c \in I$. Then,*

(i) $\displaystyle\int_a^a f(t)\, d_{n,q}t = 0.$

(ii) $\displaystyle\int_a^b k f(t)\, d_{n,q}t = k \int_a^b f(t)\, d_{n,q}t.$

(iii) $\displaystyle\int_a^b f(t)\, d_{n,q}t = -\int_b^a f(t)\, d_{n,q}t.$

(iv) $\displaystyle\int_a^b f(t)\, d_{n,q}t = \int_a^c f(t)\, d_{n,q}t + \int_c^b f(t)\, d_{n,q}t$ *for* $a \leq c \leq b$.

(v) $\displaystyle\int_a^b \left(f(t) + g(t)\right)\, d_{n,q}t = \int_a^b f(t)\, d_{n,q}t + \int_a^b g(t)\, d_{n,q}t.$

Theorem 17. *Assume that $f : I \longrightarrow \mathbb{R}$ is continuous at* 0*. Then,*

$$\int_a^b D_{n,q} f(t) d_{n,q}t = f(b) - f(a) \quad \textit{for all } \ a, b \in I.$$

Proof. First, we note that $\lim_{r\longrightarrow\infty} q^{[r]_n} a^{n^r} = \lim_{r\longrightarrow\infty} q^{[r]_n} b^{n^r} = 0$. By the continuity of f at 0,

$$\lim_{r\longrightarrow 0} f(r) = \lim_{k\longrightarrow\infty} f(q^{[k]_n} a^{n^k}) = \lim_{k\longrightarrow\infty} f(q^{[k]_n} b^{n^k}) = f(0).$$

Thus,

$$\begin{aligned}\int_a^b D_{n,q}f(t)\,d_{n,q}t &= -\sum_{k=0}^{\infty} q^{[k]_n} b^{n^k}\left(q^{n^k} b^{n^k(n-1)}-1\right) D_{n,q}f\left(q^{[k]_n} b^{n^k}\right)\\ &\quad+\sum_{k=0}^{\infty} q^{[k]_n} a^{n^k}\left(q^{n^k} a^{n^k(n-1)}-1\right) D_{n,q}f\left(q^{[k]_n} a^{n^k}\right)\\ &=\sum_{k=0}^{\infty}\left[-q^{[k]_n} b^{n^k}\left(q^{n^k} b^{n^k(n-1)}-1\right)\frac{f\left(q^{[k+1]_n} b^{n^{k+1}}\right)-f\left(q^{[k]_n} b^{n^k}\right)}{q^{[k+1]_n} b^{n^{k+1}}-q^{[k]_n} b^{n^k}}\right]\\ &\quad+\sum_{k=0}^{\infty}\left[q^{[k]_n} a^{n^k}\left(q^{n^k} a^{n^k(n-1)}-1\right)\frac{f\left(q^{[k+1]_n} a^{n^{k+1}}\right)-f\left(q^{[k]_n} a^{n^k}\right)}{q^{[k+1]_n} a^{n^{k+1}}-q^{[k]_n} a^{n^k}}\right]\\ &=f(b)-f(a).\end{aligned}$$

This completes the proof. □

Lemma 18. *Let $s \in J \subseteq [0,\theta)$ and g be n,q-integrable on J. If $0 \leq |f(t)| \leq g(t)$ for all $t \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$, then*

$$\left|\int_0^b f(t)d_{n,q}t\right| \leq \int_0^b g(t)d_{n,q}t \qquad \text{and} \qquad \left|\int_a^b f(t)d_{n,q}t\right| \leq \int_a^b g(t)d_{n,q}t \tag{11}$$

for $a, b \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$ with $a < b$. Consequently, if $g(t) \geq 0$ for all $t \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$, then

$$\int_0^b g(t)d_{n,q}t \geq 0 \qquad \text{and} \qquad \int_a^b g(t)d_{n,q}t \geq 0 \tag{12}$$

for all $a, b \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$ such that $a < b$.

Proof. If $b \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$, then we can write $b = q^{[k_2]_n} s^{n^{k_2}}$ for some $k_2 \in \mathbb{N}_0$. Observe that, for all $k \in \mathbb{N}_0$,

$$q^{[k]_n} b^{n^k} = q^{[k+k_2]_n} s^{n^{k+k_2}} \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}.$$

Therefore, by assumption, we have $0 \leq |f(q^{[k]_n} b^{n^k})| \leq g(q^{[k]_n} b^{n^k})$ and $-q^{[k]_n} b^{n^k}\left(q^{n^k} b^{n^k(n-1)}-1\right) > 0$ for all $k \in \mathbb{N}_0$. Since g is n,q-integrable on J, it follows that the series

$$\sum_{k=0}^{\infty} -q^{[k]_n} b^{n^k}\left(q^{n^k} b^{n^k(n-1)}-1\right) f\left(q^{[k]_n} b^{n^k}\right)$$

is absolutely convergent. Therefore,

$$\begin{aligned}\left|\int_0^b f(t)d_{n,q}t\right| &= \left|\sum_{k=0}^{\infty} -q^{[k]_n} b^{n^k}\left(q^{n^k} b^{n^k(n-1)}-1\right) f\left(q^{[k]_n} b^{n^k}\right)\right| \\ &\leq \sum_{k=0}^{\infty} -q^{[k]_n} b^{n^k}\left(q^{n^k} b^{n^k(n-1)}-1\right)\left|f\left(q^{[k]_n} b^{n^k}\right)\right| \\ &\leq \sum_{k=0}^{\infty} -q^{[k]_n} b^{n^k}\left(q^{n^k} b^{n^k(n-1)}-1\right) g\left(q^{[k]_n} b^{n^k}\right) \\ &= \int_0^b g(t)d_{n,q}t.\end{aligned}$$

Now, if $a, b \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$ and $a < b$, then we can write $a = q^{[k_1]_n} s^{n^{k_1}}$ and $b = q^{[k_2]_n} s^{n^{k_2}}$ for some $k_1, k_2 \in \mathbb{N}$, $k_1 > k_2$. Hence,

$$\begin{aligned}&\left|\int_a^b f(t)d_{n,q}t\right| \\ &\quad= \left|-\sum_{k=0}^{\infty} q^{[k+k_2]_n} s^{n^{k+k_2}}\left(q^{n^{k+k_2}} s^{n^{k+k_2}(n-1)}-1\right) f\left(q^{[k+k_2]_n} s^{n^{k+k_2}}\right)\right. \\ &\qquad\left.+\sum_{k=0}^{\infty} q^{[k+k_1]_n} s^{n^{k+k_1}}\left(q^{n^{k+k_1}} s^{n^{k+k_1}(n-1)}-1\right) f\left(q^{[k+k_1]_n} s^{n^{k+k_1}}\right)\right| \\ &\quad= \left|\sum_{k=k_2}^{\infty} q^{[k]_n} s^{n^k}\left(1-q^{n^k} s^{n^k(n-1)}\right) f\left(q^{[k]_n} s^{n^k}\right)\right. \\ &\qquad\left.-\sum_{k=k_1}^{\infty} q^{[k]_n} s^{n^k}\left(1-q^{n^k} s^{n^k(n-1)}\right) f\left(q^{[k]_n} s^{n^k}\right)\right| \\ &\quad\leq \sum_{k=k_2}^{k_1-1} q^{[k]_n} s^{n^k}\left(1-q^{n^k} s^{n^k(n-1)}\right)\left|f\left(q^{[k]_n} s^{n^k}\right)\right| \\ &\quad\leq \sum_{k=k_2}^{k_1-1} q^{[k]_n} s^{n^k}\left(1-q^{n^k} s^{n^k(n-1)}\right) g\left(q^{[k]_n} s^{n^k}\right) \\ &\qquad\mp \sum_{k=k_1}^{\infty} q^{[k]_n} s^{n^k}\left(1-q^{n^k} s^{n^k(n-1)}\right) g\left(q^{[k]_n} s^{n^k}\right) \\ &\quad= \sum_{k=k_2}^{\infty} q^{[k]_n} s^{n^k}\left(1-q^{n^k} s^{n^k(n-1)}\right) g\left(q^{[k]_n} s^{n^k}\right) \\ &\qquad-\sum_{k=k_1}^{\infty} q^{[k]_n} s^{n^k}\left(1-q^{n^k} s^{n^k(n-1)}\right) g\left(q^{[k]_n} s^{n^k}\right)\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=0}^{\infty} q^{[k+k_2]_n} s^{n^{k+k_2}} \left(q^{n^{k+k_2}} s^{n^{k+k_2}(n-1)} - 1\right) g\left(q^{[k+k_2]_n} s^{n^{k+k_2}}\right) \\
&\quad + \sum_{k=0}^{\infty} q^{[k+k_1]_n} s^{n^{k+k_1}} \left(q^{n^{k+k_1}} s^{n^{k+k_1}(n-1)} - 1\right) g\left(q^{[k+k_1]_n} s^{n^{k+k_1}}\right) \\
&= \int_a^b g(t) d_{n,q} t.
\end{aligned}
$$

To show that (12) is true, we just put $f = 0$ in (11). $\square$

Lemma 19. *Let $s \in J \subseteq (-\theta, 0]$ and g be n,q-integrable on J. If $0 \leq |f(t)| \leq g(t)$ for all $t \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$, then*

$$
\left|\int_b^0 f(t) d_{n,q} t\right| \leq \int_b^0 g(t) d_{n,q} t \qquad \textit{and} \qquad \left|\int_a^b f(t) d_{n,q} t\right| \leq \int_a^b g(t) d_{n,q} t
$$

for $a, b \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$ such that $a < b$. Consequently, if $g(t) \geq 0$ for all $t \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$, then

$$
\int_b^0 g(t) d_{n,q} t \geq 0 \qquad \textit{and} \qquad \int_a^b g(t) d_{n,q} t \geq 0
$$

for all $a, b \in \left\{q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0\right\}$ such that $a < b$.

Proof. Arguing as in the proof of Lemma 18, we can show that the series

$$
\sum_{k=0}^{\infty} -q^{[k]_n} b^{n^k} \left(q^{n^k} b^{n^k(n-1)} - 1\right) f\left(q^{[k]_n} b^{n^k}\right)
$$

is absolutely convergent. Therefore, we have

$$
\begin{aligned}
\left|\int_b^0 f(t) d_{n,q} t\right| &= \left|\int_0^b f(t) d_{n,q} t\right| \\
&= \left|-\sum_{k=0}^{\infty} q^{[k]_n} b^{n^k} \left(q^{n^k} b^{n^k(n-1)} - 1\right) f\left(q^{[k]_n} b^{n^k}\right)\right| \\
&\leq \sum_{k=0}^{\infty} \left|-q^{[k]_n} b^{n^k} \left(q^{n^k} b^{n^k(n-1)} - 1\right)\right| \left|f\left(q^{[k]_n} b^{n^k}\right)\right| \\
&\leq \sum_{k=0}^{\infty} q^{[k]_n} b^{n^k} \left(q^{n^k} b^{n^k(n-1)} - 1\right) g\left(q^{[k]_n} b^{n^k}\right) \\
&= -\int_0^b g(t) d_{n,q} t = \int_b^0 g(t) d_{n,q} t.
\end{aligned}
$$

The rest of the proof can be done similarly to the proof of Lemma 18. $\square$

It should be noted that the inequality

$$\left|\int_a^b f(t)d_{n,q}t\right| \leq \int_a^b |f(t)|d_{n,q}t \quad \text{for all } a,b\in I$$

is not always true. For example, fix $n=1$, $0<q<1$, and define the function $f:[0,1]\longrightarrow\mathbb{R}$ by

$$f(x)=\begin{cases} \frac{1}{1-q}\left(4q^{-n}x-(1+3q)\right), & q^{n+1}\leq x\leq \frac{q^n(1+q)}{2},\ n\in\mathbb{N},\\ \frac{4}{1-q}\left(-xq^{-n}+1\right)-1, & \frac{q^n(1+q)}{2}\leqslant x\leqslant q^n,\ n\in\mathbb{N},\\ 0, & x=0,\end{cases}$$

(see [1]). It follows that f is n,q-integrable on $[0,1]$, $f(q^n)=-1$, and $f\left(\frac{1+q}{2}q^n\right)=1$ for all $n\in\mathbb{N}$. By a direct calculation one can see that

$$\int_{\frac{1+q}{2}}^1 f(t)\,d_{n,q}t=-\frac{3+q}{2} \quad \text{and} \quad \int_{\frac{1+q}{2}}^1 |f(t)|\,d_{n,q}t=\frac{1-q}{2}.$$

Thus,

$$\left|\int_{\frac{1+q}{2}}^1 f(t)\,d_{n,q}t\right| > \int_{\frac{1+q}{2}}^1 |f(t)|\,d_{n,q}t.$$

Lemma 20. *Let $f,g:I\longrightarrow\mathbb{R}$.*

(i) If functions f and g are n,q-differentiable, then the following integration by parts formula holds:

$$\int_a^b f(t)D_{n,q}g(t)\,d_{n,q}t = \left. f(t)g(t)\right|_a^b - \int_a^b D_{n,q}f(t)g(qt^n)\,d_{n,q}t, \quad a,b\in I. \tag{13}$$

(ii) If f is continuous at 0, then for $t\in I$

$$\int_t^{qt^n} f(r)\,d_{n,q}r=(qt^n-t)f(t).$$

Proof. (i) By Theorem 17 we have

$$\int_a^b D_{n,q}(fg)(t)d_{n,q}t=(fg)(b)-(fg)(a).$$

On the other hand, by (iii) of Theorem 4 and (v) of Theorem 16,

$$\int_a^b D_{n,q}(fg)(t)=\int_a^b f(t)D_{n,q}g(t)d_{n,q}t+\int_a^b D_{n,q}f(t)g(qt^n)d_{n,q}t.$$

Combining these two equalities we get the desired formula.

(ii)

$$\int_t^{qt^n} f(s)\,d_{n,q}s = \int_0^{qt^n} f(s)\,d_{n,q}s - \int_0^t f(s)\,d_{n,q}s$$
$$= \sum_{k=0}^{\infty}\Bigg[q^{[k]_n}t^{n^k}\left(q^{n^k}t^{n^k(n-1)}-1\right)f\left(q^{[k]_n}t^{n^k}\right)$$
$$- q^{[k+1]_n}t^{n^{k+1}}\left(q^{n^{k+1}}t^{n^{k+1}(n-1)}-1\right)f\left(q^{[k+1]_n}t^{n^{k+1}}\right)\Bigg]$$
$$= (qt^n - t)f(t).$$

□

3 The power quantum variational calculus

The calculus of variations is a classical subject of mathematics with many applications in physics, economics, biology, and engineering [31, 32, 40]. Although an old theory, is very much alive and still evolving — see, e.g., [3, 5, 33, 36].

Several quantum variational problems have been recently posed and studied [4, 10, 11, 16, 17]. Here we give one application of the power quantum calculus which we derived in Section 2, introducing the power quantum variational calculus and proving a quantum analog of the Euler–Lagrange equation (Section 3.1). This provides a necessary optimality condition for local minimizers. Direct methods can also be developed for our power quantum variational calculus, allowing to obtain global minimizers for certain classes of problems (Section 3.2).

As in Section 2.2, we define the interval I to be $(-\theta,\theta)$. Let $a,b\in I$ with $a<b$. We define the n,q-interval by

$$[a,b]_{n,q} := \left\{q^{[k]_n}a^{n^k} : k\in\mathbb{N}_0\right\}\cup\left\{q^{[k]_n}b^{n^k} : k\in\mathbb{N}_0\right\}\cup\{0\}.$$

Let $\mathcal{D}([a,b]_{n,q},\mathbb{R})$ be the set of all real valued functions continuous and bounded on $[a,b]_{n,q}$.

Lemma 21 (Fundamental Lemma of the power quantum variational calculus)**.** *Let $f\in\mathcal{D}([a,b]_{n,q},\mathbb{R})$. One has $\int_a^b f(t)g(qt^n)\,d_{n,q}t = 0$ for all functions $g\in\mathcal{D}([a,b]_{n,q},\mathbb{R})$ with $g(a)=g(b)=0$ if and only if $f(t)=0$ for all $t\in[a,b]_{n,q}$.*

Proof. The implication "$\Leftarrow$" is obvious. Let us prove the implication "$\Rightarrow$". Suppose, by contradiction, that $f(c)\neq 0$ for some $c\in[a,b]_{n,q}$.
Case I. If $c\neq 0$, then without loss of generality we can assume that $c=$

$q^{[k]_n}a^{n^k}$ for $k \in \mathbb{N}_0$. Define

$$g(t) = \begin{cases} f\left(q^{[k]_n}a^{n^k}\right) & \text{if } t = q^{[k+1]_n}a^{n^{k+1}} \\ 0 & \text{otherwise}. \end{cases}$$

Since $a \neq 0$, we see that

$$\int_a^b f(t)g(qt^n)\,d_{n,q}t = q^{[k]_n}a^{n^k}\left(q^{n^k}a^{n^k(n-1)} - 1\right)\left(f\left(q^{[k]_n}a^{n^k}\right)\right)^2 \neq 0,$$

which is a contradiction.

Case II. If $c = 0$, then without loss of generality we can assume that $f(0) > 0$. We know that (see Lemma 1)

$$\lim_{k\to\infty} q^{[k]_n}a^{n^k} = \lim_{k\to\infty} q^{[k]_n}b^{n^k} = 0.$$

As f is continuous at 0,

$$\lim_{k\to\infty} f\left(q^{[k]_n}a^{n^k}\right) = \lim_{k\to\infty} f\left(q^{[k]_n}b^{n^k}\right) = f(0).$$

Therefore, there exists $N \in \mathbb{N}$ such that for all $l > N$ the inequalities

$$f\left(q^{[l]_n}a^{n^l}\right) > 0, \quad f\left(q^{[l]_n}b^{n^l}\right) > 0,$$

hold. Let us fix $k > N$. If $a \neq 0$, then we define

$$g(t) = \begin{cases} f\left(q^{[k]_n}a^{n^k}\right) & \text{if } t = q^{[k+1]_n}a^{n^{k+1}} \\ 0 & \text{otherwise}. \end{cases}$$

Since $a \neq 0$, we see that

$$\int_a^b f(t)g(qt^n)\,d_{n,q}t = q^{[k]_n}a^{n^k}\left(q^{n^k}a^{n^k(n-1)} - 1\right)\left(f\left(q^{[k]_n}a^{n^k}\right)\right)^2 \neq 0,$$

which is a contradiction. If $a = 0$, then we define

$$g(t) = \begin{cases} f\left(q^{[k]_n}b^{n^k}\right) & \text{if } t = q^{[k+1]_n}b^{n^{k+1}} \\ 0 & \text{otherwise}. \end{cases}$$

Since $b \neq 0$, we obtain

$$\begin{aligned}\int_a^b f(t)g(qt^n)\,d_{n,q}t &= \int_0^b f(t)g(qt^n)\,d_{n,q}t \\ &= -q^{[k]_n}b^{n^k}\left(q^{n^k}b^{n^k(n-1)} - 1\right)\left(f\left(q^{[k]_n}b^{n^k}\right)\right)^2 \neq 0,\end{aligned}$$

which is a contradiction. □

Let $\mathbb{E}([a,b]_{n,q},\mathbb{R})$ be the linear space of functions $y \in \mathcal{D}([a,b]_{n,q},\mathbb{R})$ for which the n,q-derivative is continuous and bounded on $[a,b]_{n,q}$. We equip $\mathbb{E}([a,b]_{n,q},\mathbb{R})$ with the norm

$$\|y\|_1 = \sup_{t\in[a,b]_{n,q}} |y(t)| + \sup_{t\in[a,b]_{n,q}} |D_{n,q}y(t)|.$$

The following definition and lemma are similar to those of [35].

Definition 22. Let $g : [s]_{n,q} \times (-\bar{\epsilon}, \bar{\epsilon}) \to \mathbb{R}$, where

$$[s]_{n,q} := \left\{ q^{[k]_n} s^{n^k} : k \in \mathbb{N}_0 \right\}.$$

We say that $g(t,\cdot)$ is continuous in ϵ_0, uniformly in t, if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\epsilon - \epsilon_0| < \delta$ implies $|g(t,\epsilon) - g(t,\epsilon_0)| < \varepsilon$ for all $t \in [s]_{n,q}$. Furthermore, we say that $g(t,\cdot)$ is differentiable at ϵ_0, uniformly in t, if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |\epsilon - \epsilon_0| < \delta$ implies

$$\left| \frac{g(t,\epsilon) - g(t,\epsilon_0)}{\epsilon - \epsilon_0} - \partial_2 g(t,\epsilon_0) \right| < \varepsilon,$$

where $\partial_2 g = \frac{\partial g}{\partial \epsilon}$ for all $t \in [s]_{n,q}$.

Lemma 23. *Let $s \in I$. Assume $g(t,\cdot)$ is differentiable at ϵ_0, uniformly in t in $[s]_{n,q}$, and that $G(\epsilon) := \int_0^s g(t,\epsilon) d_{q,\omega} t$, for ϵ near ϵ_0, and $\int_0^s \partial_2 g(t,\epsilon_0) d_{q,\omega}$ exist. Then, $G(\epsilon)$ is differentiable at ϵ_0 with $G'(\epsilon_0) = \int_0^s \partial_2 g(t,\epsilon_0)\, d_{n,q}t$.*

Proof. Without loss of generality we can assume that $s > 0$. Let $\varepsilon > 0$ be arbitrary. Since $g(t,\cdot)$ is differentiable at ϵ_0, uniformly in t, there exists $\delta > 0$ such that, for all $t \in [s]_{n,q}$ and for $0 < |\epsilon - \epsilon_0| < \delta$, the following inequality holds:

$$\left| \frac{g(t,\epsilon) - g(t,\epsilon_0)}{\epsilon - \epsilon_0} - \partial_2 g(t,\epsilon_0) \right| < \frac{\varepsilon}{s}.$$

Applying Lemma 16 and Lemma 18, for $0 < |\epsilon - \epsilon_0| < \delta$, we have

$$\begin{aligned}
&\left| \frac{G(\epsilon) - G(\epsilon_0)}{\epsilon - \epsilon_0} - G'(\epsilon_0) \right| \\
&\quad = \left| \frac{\int_0^s g(t,\epsilon)\, d_{n,q}t - \int_0^s g(t,\epsilon_0)\, d_{n,q}t}{\epsilon - \epsilon_0} - \int_0^s \partial_2 g(t,\epsilon_0)\, d_{n,q}t \right| \\
&\quad = \left| \int_0^s \left[\frac{g(t,\epsilon) - g(t,\epsilon_0)}{\epsilon - \epsilon_0} - \partial_2 g(t,\epsilon_0) \right] d_{n,q}t \right| \\
&\quad < \int_0^s \frac{\varepsilon}{s}\, d_{n,q}t = \frac{\varepsilon}{s} \int_0^s 1\, d_{n,q}t = \varepsilon.
\end{aligned}$$

Hence, $G(\cdot)$ is differentiable at ϵ_0 and $G'(\epsilon_0) = \int_0^s \partial_2 g(t,\epsilon_0)\, d_{n,q}t$. $\square$

3.1 The power quantum Euler–Lagrange equation

We consider the variational problem of finding minimizers (or maximizers) of a functional

$$\mathcal{L}[y] = \int_a^b f(t, y(qt^n), D_{n,q}y(t))\, d_{n,q}t, \tag{14}$$

over all $y \in \mathbb{E}([a,b]_{n,q}, \mathbb{R})$ satisfying the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \quad \alpha, \beta \in \mathbb{R}, \tag{15}$$

where $f : [a,b]_{n,q} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function. A function $y \in \mathbb{E}([a,b]_{n,q}, \mathbb{R})$ is said to be admissible if it satisfies endpoint conditions (15). Let us denote by $\partial_2 f$ and $\partial_3 f$, respectively, the partial derivatives of $f(\cdot,\cdot,\cdot)$ with respect to its second and third argument. In the sequel, we assume that $(u,v) \to f(t,u,v)$ is a $C^1(\mathbb{R}^2, \mathbb{R})$ function for any $t \in [a,b]_{n,q}$ and $f(\cdot, y(\cdot), D_{n,q}y(\cdot))$, $\partial_2 f(\cdot, y(\cdot), D_{n,q}y(\cdot))$, and $\partial_3 f(\cdot, y(\cdot), D_{n,q}y(\cdot))$ are continuous and bounded on $[a,b]_{n,q}$ for all admissible functions $y(\cdot)$. We say that $p \in \mathbb{E}([a,b]_{n,q}, \mathbb{R})$ is an admissible variation for (14)–(15) if $p(a) = p(b) = 0$.

For an admissible variation p, we define function $\phi : (-\bar{\varepsilon}, \bar{\varepsilon}) \to \mathbb{R}$ by

$$\phi(\varepsilon) = \phi(\varepsilon; y, p) := \mathcal{L}[y + \varepsilon p].$$

The first variation of problem (14)–(15) is defined by

$$\delta\mathcal{L}[y,p] := \phi(0; y, p) = \phi'(0).$$

Observe that,

$$\begin{aligned}
\mathcal{L}[y + \varepsilon p] &= \int_a^b f(t, y(qt^n) + \varepsilon p(qt^n), D_{n,q}y(t) + \varepsilon D_{n,q}p(t))\, d_{n,q}t \\
&= \int_0^b f(t, y(qt^n) + \varepsilon p(qt^n), D_{n,q}y(t) + \varepsilon D_{n,q}p(t))\, d_{n,q}t \\
&\quad - \int_0^a f(t, y(qt^n) + \varepsilon p(qt^n), D_{n,q}y(t) + \varepsilon D_{n,q}p(t))\, d_{n,q}t.
\end{aligned}$$

Writing

$$\mathcal{L}_b[y + \varepsilon p] = \int_0^b f(t, y(qt^n) + \varepsilon p(qt^n), D_{n,q}y(t) + \varepsilon D_{n,q}p(t))\, d_{n,q}t$$

and

$$\mathcal{L}_a[y + \varepsilon p] = \int_0^a f(t, y(qt^n) + \varepsilon p(qt^n), D_{n,q}y(t) + \varepsilon D_{n,q}p(t))\, d_{n,q}t,$$

we have

$$\mathcal{L}[y + \varepsilon p] = \mathcal{L}_b[y + \varepsilon p] - \mathcal{L}_a[y + \varepsilon p].$$

Therefore,

$$\delta\mathcal{L}[y,p] = \delta\mathcal{L}_b[y,p] - \delta\mathcal{L}_a[y,p]. \tag{16}$$

Knowing (16), the following lemma is a direct consequence of Lemma 23.

Lemma 24. *Put* $g(t,\varepsilon) = f\left(t, y(qt^n) + \varepsilon p(qt^n), D_{n,q}y(t) + \varepsilon D_{n,q}p(t)\right)$ *for* $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$. *Assume that:*

(i) $g(t,\cdot)$ *is differentiable at* 0*, uniformly in* $t \in [a]_{n,q}$*, and* $g(t,\cdot)$ *is differentiable at* 0*, uniformly in* $t \in [b]_{n,q}$*;*

(ii) $\mathcal{L}_a[y+\varepsilon p]$ *and* $\mathcal{L}_b[y+\varepsilon p]$*, for* ε *near* 0*, exist;*

(iii) $\int_0^a \partial_2 g(t,0)\, d_{n,q}t$ *and* $\int_0^b \partial_2 g(t,0)\, d_{n,q}t$ *exist.*

Then,

$$\delta\mathcal{L}[y,h] = \int_a^b \Bigg[\partial_2 f(t, y(qt^n), D_{n,q}y(t))p(qt^n) \\ + \partial_3 f(t, y(qt^n), D_{n,q}y(t))D_{n,q}p(t)\Bigg]\, d_{n,q}t.$$

In the sequel, we always assume, without mentioning it explicitly, that variational problems satisfy the assumptions of Lemma 24.

Definition 25. An admissible function $\tilde{y}$ is said to be a local minimizer (resp. a local maximizer) to problem (14)–(15) if there exists $\delta > 0$ such that $\mathcal{L}[\tilde{y}] \leq \mathcal{L}[y]$ (resp. $\mathcal{L}[\tilde{y}] \geq \mathcal{L}[y]$) for all admissible y with $\|y - \tilde{y}\|_1 < \delta$.

The following result offers a necessary condition for local extremizer.

Theorem 26 (A necessary optimality condition for problem (14)–(15))**.** *Suppose that the optimal path to problem* (14)–(15) *exists and is given by* $\tilde{y}$*. Then,* $\delta\mathcal{L}[\tilde{y}, p] = 0$*.*

Proof. The proof is similar to the one found in [35]. ☐

Theorem 27 (Euler–Lagrange equation for problem (14)–(15))**.** *Suppose that* $\tilde{y}$ *is an optimal path to problem* (14)–(15)*. Then,*

$$D_{n,q}\partial_3 f(t, \tilde{y}(qt^n), D_{n,q}\tilde{y}(t)) = \partial_2 f(t, \tilde{y}(qt^n), D_{n,q}\tilde{y}(t)) \tag{17}$$

for all $t \in [a,b]_{n,q}$*.*

Proof. Suppose that $\mathcal{L}$ has a local extremizer $\tilde{y}$. Consider the value of $\mathcal{L}$ at a nearby function $y = \tilde{y} + \varepsilon p$, where $\varepsilon \in \mathbb{R}$ is a small parameter, $p \in \mathbb{E}$, and $p(a) = p(b) = 0$. Let

$$\phi(\varepsilon) = \mathcal{L}[\tilde{y} + \varepsilon p] = \int_a^b f(t, \tilde{y}(qt^n) + \varepsilon p(qt^n), D_{n,q}\tilde{y}(t) + \varepsilon D_{n,q}p(t))\, d_{n,q}t.$$

By Theorem 26, a necessary condition for $\tilde{y}$ to be an extremizer is given by

$$\phi'(\varepsilon)\big|_{\varepsilon=0} = 0 \Leftrightarrow \int_a^b \left[\partial_2 f(\cdots)p(qt^n) + \partial_3 f(\cdots)D_{n,q}p(t)\right]\, d_{n,q}t = 0\,, \tag{18}$$

where $(\cdots) = (t, \tilde{y}(qt^n), D_{n,q}\tilde{y}(t))$. Integration by parts (see (13)) gives

$$\int_a^b \partial_3 f(\cdots) D_{n,q}p(t)\, d_{n,q}t = \partial_3 f(\cdots)p(t)|_{t=a}^{t=b} - \int_a^b D_{n,q}\partial_3 f(\cdots)p(qt^n)\, d_{n,q}t.$$

Because $p(a) = p(b) = 0$, the necessary condition (18) can be written as

$$0 = \int_a^b \left(\partial_2 f(\cdots) - D_{n,q}\partial_3 f(\cdots)\right) p(qt^n)\, d_{n,q}t$$

for all p such that $p(a) = p(b) = 0$. Thus, by Lemma 21, we have

$$\partial_2 f(\cdots) - D_{n,q}\partial_3 f(\cdots) = 0$$

for all $t \in [a, b]_{n,q}$. □

Remark 28. Analogously to the classical calculus of variations [40], to the solutions of the Euler–Lagrange equation (17) we call *(power quantum) extremals*.

Remark 29. If the function under the sign of integration f (the Lagrangian) is given by $f = f(t, y_1, \ldots, y_m, D_{n,q}y_1, \ldots, D_{n,q}y_m)$, then the necessary optimality condition is given by m equations similar to (17), one equation for each variable.

Example 30. Let us fix n, q, such that $1 \in I$. Consider the problem

$$\text{minimize} \quad \mathcal{L}[y] = \int_0^1 \left(y(qt^n) + \frac{1}{2}(D_{n,q}y(t))^2 \right) d_{n,q}t \tag{19}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = \beta, \tag{20}$$

where $\beta \in \mathbb{R}$. If y is a local minimizer to problem (19)–(20), then by Theorem 27 it satisfies the Euler–Lagrange equation

$$D_{n,q}D_{n,q}y(t) = 1$$

for all $t \in \left\{q^{[k]_n} : k \in \mathbb{N}_0\right\} \cup \{0\}$. Applying Theorem 4 (see also Example 5) and Theorem 12, we obtain

$$y(t) = -\sum_{k=0}^{\infty} q^{[k]_n} t^{n^k} \left(q^{n^k} t^{n^k(n-1)} - 1\right)\left(q^{[k]_n} t^{n^k} + c\right),$$

where c satisfies equation

$$\beta = -\sum_{k=0}^{\infty} \left(q^{[k+1]_n} - q^{[k]_n}\right)\left(q^{[k]_n} + c\right),$$

as the power quantum extremal to problem (19)–(20). For example, choosing $n = 1$ and $\beta = \frac{1}{1+q}$ in (19)–(20), we get the extremal

$$y(t) = \frac{t^2}{1+q}.$$

3.2 Leitmann's direct optimization method

Let $\bar{f} : [a,b]_{n,q} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We assume that $(u,v) \to \bar{f}(t,u,v)$ is a $C^1(\mathbb{R}^2,\mathbb{R})$ function for any $t \in [a,b]_{n,q}$ and $\bar{f}(\cdot, y(\cdot), D_{n,q}y(\cdot))$, $\partial_2 \bar{f}(\cdot, y(\cdot), D_{n,q}y(\cdot))$, and $\partial_3 \bar{f}(\cdot, y(\cdot), D_{n,q}y(\cdot))$ are continuous and bounded on $[a,b]_{n,q}$ for all admissible functions $y(\cdot)$. Consider the integral

$$\bar{\mathcal{L}}[\bar{y}] = \int_a^b \bar{f}(t, \bar{y}(qt^n), D_{n,q}\bar{y}(t))\, d_{n,q}t.$$

Lemma 31 (Leitmann's power quantum fundamental lemma)**.** *Let $y = z(t,\bar{y})$ be a transformation having an unique inverse $\bar{y} = \bar{z}(t,y)$ for all $t \in [a,b]_{n,q}$ such that there is a one-to-one correspondence*

$$y(t) \Leftrightarrow \bar{y}(t),$$

for all functions $y \in \mathbb{E}$ satisfying (15) *and all functions $\bar{y} \in \mathbb{E}([a,b]_{n,q},\mathbb{R})$ satisfying*

$$\bar{y} = \bar{z}(a,\alpha), \quad \bar{y} = \bar{z}(b,\beta). \tag{21}$$

If the transformation $y = z(t,\bar{y})$ is such that there exists a function $G : [a,b]_{n,q} \times \mathbb{R} \to \mathbb{R}$ satisfying the functional identity

$$f(t, y(qt^n), D_{n,q}y(t)) - \bar{f}(t, \bar{y}(qt^n), D_{n,q}\bar{y}(t)) = D_{n,q}G(t, \bar{y}(t))\,, \tag{22}$$

then if $\bar{y}^*$ yields the extremum of $\bar{\mathcal{L}}$ with $\bar{y}^*$ satisfying (21)*, $y^* = z(t,\bar{y}^*)$ yields the extremum of $\mathcal{L}$ for y^* satisfying* (15)*.*

Proof. The proof is similar in spirit to Leitmann's proof [28, 29, 30]. Let $y \in \mathbb{E}([a,b]_{n,q},\mathbb{R})$ satisfies (15) and define functions $\bar{y} \in \mathbb{E}([a,b]_{n,q},\mathbb{R})$ through the formula $\bar{y} = \bar{z}(t,y)$, $t \in [a,b]_{n,q}$. Then $\bar{y} \in \mathbb{E}([a,b]_{n,q},\mathbb{R})$ and satisfies (21). Moreover, as a result of (22), it follows that

$$\begin{aligned}
\mathcal{L}[y] - \bar{\mathcal{L}}[\bar{y}] &= \int_a^b f(t, y(qt^n), D_{n,q}y(t))\, d_{n,q}t - \int_a^b \bar{f}(t, \bar{y}(qt^n), D_{n,q}\bar{y}(t))\, d_{n,q}t \\
&= \int_a^b D_{n,q}G(t, \bar{y}(t))\, d_{n,q}t = G(b, \bar{y}(b)) - G(a, \bar{y}(a)) \\
&= G(b, \bar{z}(b,\beta)) - G(a, \bar{z}(a,\alpha)),
\end{aligned}$$

from which the desired conclusion follows immediately: the right-hand side of the above equality is a constant depending only on the fixed-endpoint conditions (15). $\square$

Example 32. Let $a, b \in I$ with $a < b$, and α, β be two given reals, $\alpha \neq \beta$. We consider the following problem:

$$\begin{aligned}
\text{minimize} \quad & \mathcal{L}[y] = \int_a^b \left[D_{n,q}(y(t)g(t))\right]^2\, d_{n,q}t\,, \\
& y(a) = \alpha\,, \quad y(b) = \beta\,,
\end{aligned} \tag{23}$$

where g does not vanish on the interval $[a,b]_{n,q}$. Observe that $\bar{y}(t)=g^{-1}(t)$ minimizes $\mathcal{L}$ with end conditions $\bar{y}(a)=g^{-1}(a)$ and $\bar{y}(b)=g^{-1}(b)$. Let $y(t)=\bar{y}(t)+p(t)$. Then

$$\begin{aligned}\left[D_{n,q}(y(t)g(t))\right]^2 &= \left[D_{n,q}(\bar{y}(t)g(t))\right]^2\\ &\quad + D_{n,q}(p(t)g(t))D_{n,q}\left(2\bar{y}(t)g(t)+p(t)g(t)\right). \end{aligned} \tag{24}$$

Consequently, if $p(t)=(At+B)g^{-1}(t)$, where A and B are constants, then (24) is of the form (22), since $D_{n,q}(p(t)g(t))$ is constant. Thus, the function

$$y(t)=(At+C)g^{-1}(t)$$

with

$$A=\left[\alpha g(a)-\beta g(b)\right](a-b)^{-1}, \quad C=\left[a\beta g(b)-b\alpha g(a)\right](a-b)^{-1},$$

minimizes (23).

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