The Independence under Sublinear Expectations

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Abstract

We show that, for two non-trivial random variables X and Y under a sublinear expectation space, if X is independent from Y and Y is independent from X, then X and Y must be maximally distributed.

1 Introduction

Peng [7, 8, 9, 10] introduced the important notions of distributions and independence under the sublinear expectation framework. Like classical linear expectations, the independence play a key role in the sublinear analysis.

Unfortunately, Y is independent from X does not imply that X is independent from Y. But if X and Y are maximally distributed, this holds true. A natural problem is whether the maximal distribution is the only distribution? In this paper, we give an affirmative answer to this problem.

This paper is organized as follows: in Section 2, we recall some basic results of sublinear expectations. The main result is given and proved in Section 3.

2 Basic settings

We present some preliminaries in the theory of sublinear expectations. More details of this section can be found in [7-14].

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real functions defined on Ω such that $c \in \mathcal{H}$ for all constants c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. We further suppose that if $X_1, \ldots, X_n \in \mathcal{H}$, then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{b.Lip}(\mathbb{R}^n)$, where $C_{b.Lip}(\mathbb{R}^n)$ denotes the space of bounded and Lipschitz functions. $\mathcal{H}$ is considered as the space of random variables.

Definition 1 *A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) *Monotonicity:* $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ *if* $X \geq Y$*.*

(b) *Constant preserving:* $\hat{\mathbb{E}}[c] = c$ *for* $c \in \mathbb{R}$*.*

(c) *Sub-additivity:* $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$*.*

(d) *Positive homogeneity:* $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ *for* $\lambda \geq 0$*.*

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is called a sublinear expectation space (compare with a probability space* $(\Omega, \mathcal{F}, P)$*).*

Remark 2 *If the inequality in (c) is equality, then* $\hat{\mathbb{E}}$ *is a linear expectation on* $\mathcal{H}$*. We recall that the notion of the above sublinear expectations was systematically introduced by Artzner, Delbaen, Eber and Heath [1, 2], in the case where* Ω *is a finite set, and by Delbaen [3] for the general situation with the notation of risk measure:* $\rho(X) := \hat{\mathbb{E}}[-X]$*. See also Huber [5] for even earlier study of this notion* $\hat{\mathbb{E}}$ *(called the upper expectation* $\mathbf{E}^*$ *in Ch. 10 of [5]).*

Remark 3 *It is easy to deduce from (d) that*

$$\hat{\mathbb{E}}[\lambda X] = \lambda^+ \hat{\mathbb{E}}[X] + \lambda^- \hat{\mathbb{E}}[-X] \quad \text{for } \lambda \in \mathbb{R}.$$

Remark 4 *Let* $\{E_\theta : \theta \in \Theta\}$ *be a family of linear expectations defined on* $\mathcal{H}$*. Then*

$$\hat{\mathbb{E}}[X] := \sup_{\theta \in \Theta} E_\theta[X] \quad \text{for } X \in \mathcal{H}$$

is a sublinear expectation. In fact, every sublinear expectation has this kind of representation (see Peng [11, 12]).

Let $X = (X_1, \ldots, X_n)$, $X_i \in \mathcal{H}$, denoted by $X \in \mathcal{H}^n$, be a given n-dimensional random vector on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. We define a functional on $C_{b.Lip}(\mathbb{R}^n)$ by

$$\hat{\mathbb{F}}_X[\varphi] := \hat{\mathbb{E}}[\varphi(X)] \quad \text{for all } \varphi \in C_{b.Lip}(\mathbb{R}^n).$$

The triple $(\mathbb{R}^n, C_{b.Lip}(\mathbb{R}^n), \hat{\mathbb{F}}_X[\cdot])$ forms a sublinear expectation space. $\hat{\mathbb{F}}_X$ is called the distribution of X.

Definition 5 *A random vector* $X \in \mathcal{H}^n$ *is said to have distributional uncertainty if the distribution* $\hat{\mathbb{F}}_X$ *is not a linear expectation.*

The following simple property is very useful in sublinear analysis.

Proposition 6 *Let* X*,* $Y \in \mathcal{H}$ *be such that* $\hat{\mathbb{E}}[Y] = -\hat{\mathbb{E}}[-Y]$*. Then we have*

$$\hat{\mathbb{E}}[X + Y] = \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y].$$

In particular, if $\hat{\mathbb{E}}[Y] = \hat{\mathbb{E}}[-Y] = 0$*, then* $\hat{\mathbb{E}}[X + Y] = \hat{\mathbb{E}}[X]$*.*

Proof. It is simply because we have $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ and

$$\hat{\mathbb{E}}[X+Y] \geq \hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[-Y] = \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y].$$

□

Noting that $\hat{\mathbb{E}}[c] = -\hat{\mathbb{E}}[-c] = c$ for all $c \in \mathbb{R}$, we immediately have

$$\hat{\mathbb{E}}[X+c] = \hat{\mathbb{E}}[X] + c.$$

The following notion of independence plays an important role in the sublinear expectation theory.

Definition 7 *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. A random vector $Y = (Y_1, \cdots, Y_n) \in \mathcal{H}^n$ is said to be independent from another random vector $X = (X_1, \cdots, X_m) \in \mathcal{H}^m$ under $\hat{\mathbb{E}}[\cdot]$ if for each test function $\varphi \in C_{b.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have*

$$\hat{\mathbb{E}}[\varphi(X,Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x,Y)]_{x=X}].$$

Remark 8 *Under a sublinear expectation space, Y is independent from X means that the distributional uncertainty of Y does not change after the realization of $X = x$. Or, in other words, the "conditional sublinear expectation" of Y knowing X is $\hat{\mathbb{E}}[\varphi(x,Y)]_{x=X}$. In the case of linear expectation, this notion of independence is just the classical one.*

It is important to note that under sublinear expectations the condition "Y is independent from X" does not imply automatically that "X is independent from Y". See the following example:

Example 9 *We consider a case where $\hat{\mathbb{E}}$ is a sublinear expectation and $X, Y \in \mathcal{H}$ are identically distributed with $\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[-X] = 0$ and $\bar{\sigma}^2 = \hat{\mathbb{E}}[X^2] > \underline{\sigma}^2 = -\hat{\mathbb{E}}[-X^2]$. We also assume that $\hat{\mathbb{E}}[|X|] = \hat{\mathbb{E}}[X^+ + X^-] > 0$, thus $\hat{\mathbb{E}}[X^+] = \frac{1}{2}\hat{\mathbb{E}}[|X| + X] = \frac{1}{2}\hat{\mathbb{E}}[|X|] > 0$. In the case where Y is independent from X, we have*

$$\hat{\mathbb{E}}[XY^2] = \hat{\mathbb{E}}[X^+\bar{\sigma}^2 - X^-\underline{\sigma}^2] = (\bar{\sigma}^2 - \underline{\sigma}^2)\hat{\mathbb{E}}[X^+] > 0.$$

But if X is independent from Y we have

$$\hat{\mathbb{E}}[XY^2] = 0.$$

The following is a representation theorem of the distribution of a random vector (see [4, 6, 14]).

Theorem 10 *Let $X \in \mathcal{H}^n$ be a n-dimensional random vector on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Then there exists a weakly compact family of probability measures $\mathcal{P}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that*

$$\hat{\mathbb{F}}_X[\varphi] = \hat{\mathbb{E}}[\varphi(X)] = \max_{P \in \mathcal{P}} E_P[\varphi] \quad \text{for all } \varphi \in C_{b.Lip}(\mathbb{R}^n).$$

Definition 11 *A n-dimensional random vector $X \in \mathcal{H}^n$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called maximally distributed if there exists a closed set $\Gamma \subset \mathbb{R}^n$ such that*

$$\hat{\mathbb{F}}_X[\varphi] = \hat{\mathbb{E}}[\varphi(X)] = \sup_{x \in \Gamma} \varphi(x) \quad \text{for all } \varphi \in C_{b.Lip}(\mathbb{R}^n).$$

Remark 12 *In Peng [11, 12], the definition of maximal distribution demands the convexity of Γ. Here, we still call it the maximal distribution without the convexity of Γ for convenience.*

3 Main result

We now discuss some cases under which X is independent from Y and Y is independent from X. In this section, we do not consider the following two trivial cases:

(i) The distributions of X and Y are linear;

(ii) At least one of X and Y is constant.

The following example is a non-trivial case.

Example 13 *Let $\Omega = \mathbb{R}^2$, $\mathcal{H} = C_{b.Lip}(\mathbb{R}^2)$ and let K_1 and K_2 be two closed sets in $\mathbb{R}$. We define*

$$\hat{\mathbb{E}}[\varphi] = \sup_{(x,y) \in K_1 \times K_2} \varphi(x, y) \quad \text{for all } \varphi \in C_{b.Lip}(\mathbb{R}^2).$$

It is easy to check that $\xi(x, y) := x$ is independent from $\eta(x, y) := y$ and η is independent from ξ.

We will prove that this is the only case. The following theorem is the main theorem in this section.

Theorem 14 *Suppose that $X \in \mathcal{H}$ has distributional uncertainty and $Y \in \mathcal{H}$ is not a constant on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. If X is independent from Y and Y is independent from X, then X and Y must be maximally distributed.*

In order to prove this theorem, we need the following lemmas.

Lemma 15 *Suppose $X \in \mathcal{H}$ has distributional uncertainty on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Then there exists a $\varphi \geq 0$ such that $\hat{\mathbb{E}}[\varphi(X)] = 1$ and $-\hat{\mathbb{E}}[-\varphi(X)] < 1$.*

Proof. We first claim that there exists a $\varphi_0 \geq 0$ such that $-\hat{\mathbb{E}}[-\varphi_0(X)] < \hat{\mathbb{E}}[\varphi_0(X)]$. Otherwise, for each $\varphi \geq 0$, we have $\hat{\mathbb{E}}[\varphi(X)] = -\hat{\mathbb{E}}[-\varphi(X)]$. For each $\varphi \in C_{b.Lip}(\mathbb{R})$, let $M := \inf\{\varphi(x) : x \in \mathbb{R}\}$, then $M + \varphi \geq 0$ and

$$\hat{\mathbb{E}}[\varphi(X)] + M = \hat{\mathbb{E}}[\varphi(X) + M] = -\hat{\mathbb{E}}[-\varphi(X) - M] = -\hat{\mathbb{E}}[-\varphi(X)] + M,$$

which implies that $\hat{\mathbb{E}}[\varphi(X)] = -\hat{\mathbb{E}}[-\varphi(X)]$ for each $\varphi \in C_{b.Lip}(\mathbb{R})$. It follows from Proposition 6 that

$$\hat{\mathbb{E}}[\varphi(X) + \psi(X)] = \hat{\mathbb{E}}[\varphi(X)] + \hat{\mathbb{E}}[\psi(X)] \text{ for each } \varphi, \psi \in C_{b.Lip}(\mathbb{R}),$$

which contradics our assumption. We then take $\varphi^* = (\hat{\mathbb{E}}[\varphi_0(X)])^{-1}\varphi_0 \geq 0$. It is easy to verify that $\hat{\mathbb{E}}[\varphi^*(X)] = 1$ and $-\hat{\mathbb{E}}[-\varphi^*(X)] < 1$, the proof is complete. □

Lemma 16 *Suppose X and Y are as in Theorem 14. If X is independent from Y and Y is independent from X, then we have*

$$\hat{\mathbb{E}}[(\psi(Y) - \hat{\mathbb{E}}[\psi(Y)])^+] = 0 \quad \textit{for all } \psi \in C_{b.Lip}(\mathbb{R}).$$

Proof. It follows from Lemma 15 that there exists a $\varphi^* \geq 0$ such that $\hat{\mathbb{E}}[\varphi^*(X)] = 1$ and $-\hat{\mathbb{E}}[-\varphi^*(X)] < 1$. We set $\varepsilon = -\hat{\mathbb{E}}[-\varphi^*(X)] \in [0, 1)$ and define

$$G(a) = \hat{\mathbb{E}}[a\varphi^*(X)] = a^+\hat{\mathbb{E}}[\varphi^*(X)] + a^-\hat{\mathbb{E}}[-\varphi^*(X)] = a^+ - \varepsilon a^- \quad \text{for } a \in \mathbb{R}.$$

Note that Y is independent from X, then we have

$$\hat{\mathbb{E}}[\varphi^*(X)\psi(Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\psi(Y)]\varphi^*(X)] = G(\hat{\mathbb{E}}[\psi(Y)]) \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}). \quad (1)$$

On the other hand, X is independent from Y, then we get

$$\hat{\mathbb{E}}[\varphi^*(X)\psi(Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\psi(y)\varphi^*(X)]_{y=Y}] = \hat{\mathbb{E}}[G(\psi(Y))] \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}). \quad (2)$$

Combining (2) with (1), we obtain

$$\hat{\mathbb{E}}[G(\psi(Y))] = G(\hat{\mathbb{E}}[\psi(Y)]) \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}). \quad (3)$$

Noting that $G \circ \psi \in C_{b.Lip}(\mathbb{R})$ for each $\psi \in C_{b.Lip}(\mathbb{R})$, applying equation (3) to $G \circ \psi$, we have

$$\hat{\mathbb{E}}[G \circ G(\psi(Y))] = G \circ G(\hat{\mathbb{E}}[\psi(Y)]) \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}).$$

Denote

$$G^{\circ n} = \underbrace{G \circ G \circ \cdots \circ G}_{n},$$

continuing the above process, we can get

$$\hat{\mathbb{E}}[G^{\circ n}(\psi(Y))] = G^{\circ n}(\hat{\mathbb{E}}[\psi(Y)]) \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}). \quad (4)$$

It is easy to check that $G^{\circ n}(a) = a^+ - \varepsilon^n a^-$. By

$$|\hat{\mathbb{E}}[G^{\circ n}(\psi(Y))] - \hat{\mathbb{E}}[\psi^+(Y)]| = |\hat{\mathbb{E}}[\psi^+(Y) - \varepsilon^n \psi^-(Y)] - \hat{\mathbb{E}}[\psi^+(Y)]| \leq \varepsilon^n \hat{\mathbb{E}}[\psi^-(Y)]$$

and $G^{\circ n}(\hat{\mathbb{E}}[\psi(Y)]) = (\hat{\mathbb{E}}[\psi(Y)])^+ - \varepsilon^n(\hat{\mathbb{E}}[\psi(Y)])^-$, we can deduce by letting $n \to \infty$ that

$$\hat{\mathbb{E}}[\psi^+(Y)] = (\hat{\mathbb{E}}[\psi(Y)])^+ \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}). \tag{5}$$

For each $\psi \in C_{b.Lip}(\mathbb{R})$, applying equation (5) to $\tilde{\psi} := \psi - \hat{\mathbb{E}}[\psi(Y)]$, we obtain the result. The proof is complete. □

Proof of Theorem 14. It follows from Theorem 10 that there exists a weakly compact family of probability measures $\mathcal{P}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\hat{\mathbb{F}}_Y[\psi] = \hat{\mathbb{E}}[\psi(Y)] = \max_{P \in \mathcal{P}} E_P[\psi] \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}). \tag{6}$$

For this $\mathcal{P}$, we set

$$c(A) := \sup_{P \in \mathcal{P}} P(A) \text{ for all } A \in \mathcal{B}(\mathbb{R}). \tag{7}$$

By Lemma 16 and (6), we have

$$\hat{\mathbb{E}}[(\psi(Y) - \hat{\mathbb{E}}[\psi(Y)])^+] = \max_{P \in \mathcal{P}} E_P[(\psi - \hat{\mathbb{E}}[\psi(Y)])^+] = 0 \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}). \tag{8}$$

From this, it is easy to obtain that $c(\{y : \psi(y) > \hat{\mathbb{E}}[\psi(Y)]\}) = 0$ for each $\psi \in C_{b.Lip}(\mathbb{R})$. For each given $\psi_0 \in C_{b.Lip}(\mathbb{R})$, we set

$$A := \{y \in \mathbb{R} : \psi_0(y) = \hat{\mathbb{E}}[\psi_0(Y)]\}.$$

It is easy to verify that A is a closed set. We first assert that $c(A) > 0$. Otherwise,

$$c(\{y : \psi_0(y) \geq \hat{\mathbb{E}}[\psi_0(Y)]\}) \leq c(\{y : \psi_0(y) > \hat{\mathbb{E}}[\psi_0(Y)]\}) + c(A) = 0, \tag{9}$$

by (6) and (9), we get

$$\hat{\mathbb{E}}[\psi_0(Y)] = \max_{P \in \mathcal{P}} E_P[\psi_0] < \hat{\mathbb{E}}[\psi_0(Y)],$$

this is a contradiction, thus $c(A) > 0$. We then claim that there exists a $y_0 \in A$ such that

$$\psi(y_0) \leq \hat{\mathbb{E}}[\psi(Y)] \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}).$$

Otherwise, for each $\tilde{y} \in A$, there exists a $\tilde{\psi} \in C_{b.Lip}(\mathbb{R})$ such that $\tilde{\psi}(\tilde{y}) > \hat{\mathbb{E}}[\tilde{\psi}(Y)]$. Note that $c(\{y : \tilde{\psi}(y) > \hat{\mathbb{E}}[\tilde{\psi}(Y)]\}) = 0$, then there exists a $\tilde{\varepsilon} > 0$ such that $c([\tilde{y} - \tilde{\varepsilon}, \tilde{y} + \tilde{\varepsilon}]) = 0$. Noting that A is closed, by the Heine-Borel theorem, there exists a sequence $\{(y_n, \varepsilon_n) : n = 1, 2, \cdots\}$ such that

$$A \subset \cup_n [y_n - \varepsilon_n, y_n + \varepsilon_n] \text{ and } c([y_n - \varepsilon_n, y_n + \varepsilon_n]) = 0.$$

Thus, $c(A) \leq \sum_{n=1}^{\infty} c([y_n - \varepsilon_n, y_n + \varepsilon_n]) = 0$, which contradicts to $c(A) > 0$. Take $B = cl(\{y_0 : \psi_0 \in C_{b.Lip}(\mathbb{R})\})$ and $\mathcal{P}' = \{\delta_y : y \in B\}$, then

$$\hat{\mathbb{F}}_Y[\psi] = \hat{\mathbb{E}}[\psi(Y)] = \max_{P \in \mathcal{P}'} E_P[\psi] \text{ for all } \psi \in C_{b.Lip}(\mathbb{R}),$$

which implies that Y is maximally distributed. Similarly, we can prove that X is maximally distributed. The proof is complete.

Remark 17 *It is easy to check that (X, Y) is maximally distributed. Since $Y = (Y_1, \ldots, Y_m) \in \mathcal{H}^m$ independent from $X = (X_1, \ldots, X_n) \in \mathcal{H}^n$ implies Y_i independent from X_j for $i \leq m$ and $j \leq n$, the result of Theorem 14 still holds.*

Definition 18 *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. A random vector $Y \in \mathcal{H}^n$ is said to be weakly independent from another random vector $X \in \mathcal{H}^m$ under $\hat{\mathbb{E}}[\cdot]$ if*

$$\hat{\mathbb{E}}[\varphi(X)\psi(Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x)\psi(Y)]_{x=X}] \text{ for each } \varphi, \psi \in C_{b.Lip}(\mathbb{R}).$$

Remark 19 *It is easy to see from the proof that the result of Theorem 14 still holds under weak independence.*

Problem 20 *Whether weak independence is independence? Moreover, what kind of sets can determine sublinear expectations? Whether $\mathcal{H}_0 := \{\varphi(x)\psi(y) : \varphi, \psi \in C_{b.Lip}(\mathbb{R})\}$ is enough to determine sublinear expectations?*

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