Fissioned triangular schemes via sharply 3-transitive groups

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Abstract

In [D. de Caen, E.R. van Dam. Fissioned triangular schemes via the cross-ratio, Europ. J. Combin., 22 (2001) 297-301], de Caen and van Dam constructed a fission scheme FT($q+1$) of the triangular scheme on PG$(1,q)$. This fission scheme comes from the naturally induced action of PGL$(2,q)$ on the 2-element subsets of PG$(1,q)$. The group PGL$(2,q)$ is one of two infinite families of finite sharply 3-transitive groups. The other such family M(q) is a "twisted" version of PGL$(2,q)$, where q is an even power of an odd prime. The group PSL$(2,q)$ is the intersection of PGL$(2,q)$ and M(q). In this paper, we investigate the association schemes coming from the actions of PSL$(2,q)$, M(q) and PΓL$(2,q)$, respectively. Through the conic model introduced in [H.D.L. Hollmann, Q. Xiang. Association schemes from the actions of PGL$(2,q)$ fixing a nonsingular conic, J. Algebraic Combin., 24 (2006) 157-193], we introduce an embedding of PΓL$(2,q)$ into PΓL$(3,q)$. For each of the three groups mentioned above, this embedding produces two more isomorphic association schemes: one on hyperbolic lines and the other on hyperbolic points (via a null parity) in a 3-dimensional orthogonal geometry. This embedding enables us to treat these three isomorphic association schemes simultaneously.

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1. Introduction

Let X be a finite set with cardinality $n \geq 2$ and $\mathbf{R} = \{R_0, R_1, \ldots, R_d\}$ be a set of binary relations on X. $\mathfrak{X} = (X, \mathbf{R})$ is called an *association scheme with d classes* (a *d-class association scheme*, or simply, a *scheme*) if the following axioms are satisfied:

(i) $\mathbf{R}$ is a partition of $X \times X$ and $R_0 = \{(x,x) \mid x \in X\}$ is the diagonal relation;

(ii) For $i = 0, 1, \ldots, d$, the inverse ${}^tR_i = \{(y,x) | (x,y) \in R_i\}$ of R_i is also among the relations: ${}^tR_i = R_{i'}$ for some i' $(0 \leq i' \leq d)$;

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(iii) For any triple of $i, j, k = 0, 1, \ldots, d$, there exists an integer p_{ij}^k such that for all $(x, y) \in R_k$,

$$|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| = p_{ij}^k.$$

The integers p_{ij}^k are called the *intersection numbers*. The integer $k_i = p_{ii'}^0$ is called the *valency* of R_i. In fact, for any $x \in X$, $k_i = |\{y \in X| \ (x, y) \in R_i\}|$.

If $p_{ij}^k = p_{ji}^k$ for all i, j, k, $\mathfrak{X}$ is called *commutative*. A relation R_i is said to be *symmetric* if $R_i = {}^tR_i$. The scheme $\mathfrak{X}$ is called *symmetric* if all relations R_i are symmetric. A partition $\Lambda_0, \Lambda_1, \ldots, \Lambda_e$ of the index set $\{0, 1, \ldots, d\}$ is called *admissible* [13] if $\Lambda_0 = \{0\}$, $\Lambda_i \neq \emptyset$ and $\Lambda_i' = \Lambda_j$ for some j $(1 \leq i, j \leq e)$, where $\Lambda' = \{\alpha'|\alpha \in \Lambda\}$. Let $R_{\Lambda_i} = \cup_{\alpha \in \Lambda_i} R_\alpha$. If $\mathfrak{Y} = (X, \{R_{\Lambda_i}\}_{i=0}^e)$ is an association scheme, it is called a *fusion scheme* of $\mathfrak{X}$, and $\mathfrak{X}$ is called a *fission scheme* of $\mathfrak{Y}$.

We mention a typical example of association scheme (see [1, Section 2.1] for details). If group G acts transitively on a finite set X, then the orbits of the induced diagonal action of G on $X \times X$ form an association scheme, denoted by $\mathfrak{X}(G, X)$. For any $x \in X$, the orbits of G_x on X are in one-to-one correspondence with those of G on $X \times X$, where G_x is the stabilizer of x in G. The scheme $\mathfrak{X}(G, X)$ is commutative if and only if the permutation representation of G on X is *multiplicity-free*, and it is symmetric if and only if G acts *generously transitively* on X.

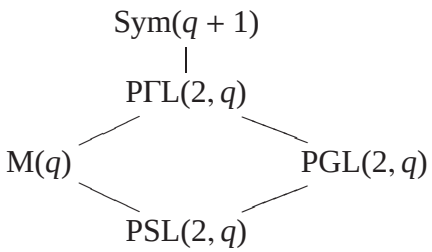
By a theorem of Zassenhaus [17, Section 7.1], there are two infinite families of finite sharply 3-transitive groups, and both are subgroups of the projective semilinear group $\mathrm{P\Gamma L}(2, q)$. If G is a sharply 3-transitive group of permutations on a finite set X, then X can be identified with the projective line $\mathrm{PG}(1, q)$ for some prime power q and G is one of the following:

(1) G is the projective general linear group $\mathrm{PGL}(2, q)$ in its natural action on $\mathrm{PG}(1, q)$;

(2) $q = p^{2f}$ for some odd prime p and a positive integer f, and if σ is the unique involution in $\mathrm{Aut}(\mathbb{F}_q)$, then G is the group $\mathrm{M}(q) = S \cup T$, where

$$S = \left\{ \lambda \mapsto \frac{a\lambda + b}{c\lambda + d}, \ \middle| \ ad - bc \text{ is a square in } \mathbb{F}_q^* \right\},$$
$$T = \left\{ \lambda \mapsto \frac{a\lambda^\sigma + b}{c\lambda^\sigma + d} \ \middle| \ ad - bc \text{ is a non-square in } \mathbb{F}_q^* \right\}.$$

In this paper, we shall investigate the association schemes determined by the following subgroups of the symmetric group $\mathrm{Sym}(q + 1)$:



These groups are permutation groups on $\mathrm{PG}(1, q)$ and they acts transitively on Ω, the collection of 2-element subsets of $\mathrm{PG}(1, q)$. Hence, each of these groups determines an association scheme on

Ω: Sym$(q+1)$ determines the triangular scheme T$(q+1)$, and PGL$(2,q)$ determines a symmetric fission scheme of T$(q+1)$. De Caen and van Dam [5] described $\mathfrak{X}(\mathrm{PGL}(2,q),\Omega)$ via the cross-ratio. See Subsection 2.3. The association schemes from the actions of the remaining three groups are the main object of the present paper.

The isomorphism $\mathrm{PGL}(2,q) \simeq \mathrm{SO}(3,q)$ introduces a conic model (see Subsection 2.1). The action PGL$(2,q)$ on Ω is equivalent to that of SO$(3,q)$ on the set of hyperbolic lines $\mathcal{L}_+$ in a 3-dimensional orthogonal geometry. This model has been used heavily in [6, 7, 22]. This conic model allows us to use an embedding ρ of PΓL$(2,q)$ into PΓL$(3,q)$ and the image of this embedding fixes this conic as a set. As a result, the image of PSL$(2,q)$ acts transitively on the hyperbolic lines $\mathcal{L}_+$ and so do the images of M(q) and PΓL$(2,q)$.

The conic above also introduces a (null) polarity $\perp$ such that the action of any subgroup of PΓL$(2,q)$ on the hyperbolic lines $\mathcal{L}_+$ is equivalent to that of its embedded image on the hyperbolic points $\mathcal{L}_+^\perp$; see Subsection 2.4. Therefore, we do not need to distinguish actions on hyperbolic lines and points in certain calculation. The advantage of this conic model allows us to treat (isomorphic) association schemes on Ω, $\mathcal{L}_+$ and $\mathcal{L}_+^\perp$ all at once.

This paper is organized as follows. In Section 2, we introduce the conic model, the polarity and the embedding of PΓL$(2,q)$ into PΓL$(3,q)$ mentioned above, and establish a few results on transitivity. In Section 3, we determine three fission schemes of the triangular scheme T$(q+1)$ via their isomorphic association schemes.

We refer to [1] for undefined terms and the basic theory of association schemes and to [4, 20, 21] for missing definitions and notation about various groups in this paper.

2. The 3-dimensional orthogonal geometry and an embedding of PΓL$(2, q)$

In this section, we introduce the 3-dimensional orthogonal geometry, a conic model, and various groups related to this conic.

2.1. The 3-dimensional orthogonal geometry

Let $\mathbb{F}_q$ be a finite field with q elements. Let $V = \mathbb{F}_q^3$ be a 3-dimensional vector space over $\mathbb{F}_q$ equipped with a non-degenerate quadratic form Q. *The general orthogonal group* GO(V) is the isometry group of V with respect to Q:

$$\mathrm{GO}(V) = \{A \in \mathrm{GL}(V) \mid Q(A(x_0,x_1,x_2)^T) = Q(x_0,x_1,x_2)\}.$$

The special linear orthogonal group SO(V) is the intersection of GO(V) and SL(V). It is also standard to write GO$(3,q)$ for GO(V), and SO$(3,q)$ for SO(V) when the underlying field is $\mathbb{F}_q$.

The projective plane PG$(2,q)$ on V has as points the 1-dimensional subspaces of V and as lines (hyperplanes) the 2-dimensional subspaces. Any point P is spanned by a nonzero vector $v = (v_0,v_1,v_2)$. Another vector $u = (u_0,u_1,u_2)$ spans P if and only if $v = \xi u$ for some $\xi \in \mathbb{F}_q^*$. So we use $(v_0 : v_1 : v_2)$ to denote the point P. A point $(v_0 : v_1 : v_2)$ is called *singular* if $Q(v_0,v_1,v_2) = 0$.

In the rest of this paper, we fix the quadratic form $Q(x_0,x_1,x_2) = x_1^2 - x_0x_2$. Let

$$\mathscr{O} = \{(\xi^2 : \xi : 1) \mid \xi \in \mathbb{F}_q\} \cup \{(1:0:0)\}. \tag{1}$$

Then $\mathscr{O}$ is a *conic*. Let $\infty = (1:0:0)$. We use P_ξ and P_∞ to denote the points $(\xi^2:\xi:1)$ and $(1:0:0)$, respectively. No three distinct points of $\mathscr{O}$ can be on a line. Therefore, any line ℓ intersects $\mathscr{O}$ at most 2 points. Accordingly, ℓ is called *hyperbolic* or *secant* if $|\ell \cap \mathscr{O}| = 2$, *tangent* if $|\ell \cap \mathscr{O}| = 1$, or *elliptic* or *exterior* if $|\ell \cap \mathscr{O}| = 0$. We denote by $\mathcal{L}_+$, $\mathcal{L}_0$ and $\mathcal{L}_-$ the set of all hyperbolic, tangent and elliptic lines, respectively.

The following result is well known (e.g. see [20, Theorem 11.6]), from which Theorem 2.2 follows immediately.

Theorem 2.1. *If V is an orthogonal geometry of dimension 3 and Witt index 1 over $\mathbb{F}_q$, then* $\mathrm{GO}(V) \simeq \{\pm 1\} \times \mathrm{SO}(V)$, $\mathrm{SO}(V) \simeq \mathrm{PGL}(2,q)$, *and* $\mathrm{SO}(V)$ *acts triply transitively on the set of all singular points.*

Theorem 2.2. *The group* $\mathrm{SO}(V)$ *acts generously transitively on $\mathcal{L}_\epsilon$ for $\epsilon \in \{+,0\}$.*

We remark that $\mathrm{SO}(V)$ also acts generously transitively on $\mathcal{L}_-$; see [18]. Therefore, the action $(\mathrm{SO}(V), \mathcal{L}_\epsilon)$ determines a symmetric association scheme, denoted by $\mathfrak{X}(\mathrm{SO}(3,q), \mathcal{L}_\epsilon)$, $\epsilon = 0, \pm$.

2.2. An embedding of $\mathrm{P\Gamma L}(2,q)$ *into* $\mathrm{P\Gamma L}(3,q)$

In this subsection, we describe an embedding of $\mathrm{P\Gamma L}(2,q)$ into $\mathrm{P\Gamma L}(3,q)$, which gives an isomorphism $\mathrm{PGL}(2,q) \simeq \mathrm{SO}(3,q)$ mentioned in Theorem 2.1 (cf. [4, Section 6.1] and [11, Section 3]). While this is a folklore, a detailed account is provided here to prepare for later sections.

We first describe the action of $\mathrm{P\Gamma L}(2,q)$ on the projective line $\mathrm{PG}(1,q)$. Let

$$\mathrm{PG}(1,q) = \{(\xi:1) \mid \xi \in \mathbb{F}_q\} \cup \{(1:0)\}$$

and $\mathbb{F}_q^{\cup\{\infty\}} = \mathbb{F}_q \cup \{\infty\}$, the union of $\mathbb{F}_q$ and $\{\infty\}$. Now we may identify $\mathrm{PG}(1,q)$ with $\mathbb{F}_q^{\cup\{\infty\}}$ by the map: $(\xi:1) \leftrightarrow \xi$, $(1:0) \leftrightarrow \infty$. The group $\Gamma\mathrm{L}(2,q)$ is the wreath product of $\mathrm{GL}(2,q)$ and $\mathrm{Aut}(\mathbb{F}_q)$, and $\mathrm{P\Gamma L}(2,q)$ is the group induced on the the projective line $\mathrm{PG}(1,q)$ by $\Gamma\mathrm{L}(2,q)$. Each element of $\mathrm{P\Gamma L}(2,q)$ is induced by a pair (A,τ) with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,q)$ and $\tau \in \mathrm{Aut}(\mathbb{F}_q)$, which acts on $\mathrm{PG}(1,q)$ as follows:

$$\xi \mapsto A(\xi^\tau) := \frac{a\xi^\tau + b}{c\xi^\tau + d} \quad \text{for all } \xi \in \mathbb{F}_q^{\cup\{\infty\}}.$$

The expressions involving ∞ are evaluated by standard limit rules: e.g., $\infty^\tau = \infty$, $A(\infty) = a/c$ if $c \neq 0$ and $A(\infty) = \infty$ if $c = 0$.

Consider the vector space W of quadratic forms on $\mathbb{F}_q^2$: $q(x,y) = ux^2 + vxy + wy^2$. So, $\dim W = 3$. The group $\Gamma\mathrm{L}(2,q)$ acts on W as follows: for each (A,τ) in $\Gamma\mathrm{L}(2,q)$,

$$q(x,y) \mapsto q(A(x^\tau, y^\tau)^T) \text{ for all } q \in W. \tag{2}$$

For brevity, we write $q(A(x^\tau, y^\tau)) = q(A(x^\tau, y^\tau)^T)$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,q)$, then

$$\begin{aligned} q(A(x^\tau, y^\tau)) =& u(ax^\tau + by^\tau)^2 + v(ax^\tau + by^\tau)(cx^\tau + dy^\tau) + w(cx^\tau + dy^\tau)^2 \\ =& x^{2\tau}(ua^2 + vac + wc^2) + x^\tau y^\tau(2uab + v(ad + bc) + 2wcd) + y^{2\tau}(ub^2 + vbd + wd^2) \\ =& \begin{pmatrix} x^{2\tau} & x^\tau y^\tau & y^{2\tau} \end{pmatrix} \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \end{aligned}$$

Let

$$\rho(A) = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}. \tag{3}$$

Since $\det \rho(A) = \det(A)^3$, it can be verified that (2) induces a homomorphism from $\Gamma\mathrm{L}(2,q)$ to $\Gamma\mathrm{L}(3,q)$. In particular, if τ is the identity automorphism, ρ is a homomorphism of $\mathrm{GL}(2,q)$ into $\mathrm{GL}(3,q)$. The kernel of ρ is $\{\pm I\}$, where I is the 2×2 identity matrix . Let $Z = \{cI | c \in \mathbb{F}_q^*\}$. Since $\mathrm{PGL}(2,q) = \mathrm{GL}(2,q)/Z$, ρ embeds $\mathrm{PGL}(2,q)$ into $\mathrm{PGL}(3,q)$. By abuse of notation, we also use ρ to denote the homomorphism induced by (2). Thus, ρ embeds $\mathrm{P\Gamma L}(2,q)$ into $\mathrm{P\Gamma L}(3,q)$.

Since

$$\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix} \begin{pmatrix} \xi^{2\tau} \\ \xi^\tau \\ 1 \end{pmatrix} = \begin{pmatrix} (a\xi^\tau + b)^2 \\ (a\xi^\tau + b)(c\xi^\tau + d) \\ (c\xi^\tau + d)^2 \end{pmatrix}, \tag{4}$$

$(\rho(A), \tau)$ maps $(\xi^2 : \xi : 1)$ to $\left(\left(\frac{a\xi^\tau + b}{c\xi^\tau + d}\right)^2, \frac{a\xi^\tau + b}{c\xi^\tau + d} : 1\right)$ if $c\xi^\tau + d \neq 0$ and to $(1 : 0 : 0)$ if $c\xi^\tau + d = 0$. Similarly, $(\rho(A), \tau)$ maps $(1 : 0 : 0)$ to $(1 : 0 : 0)$ if $c = 0$, and to $((a/c)^2, a/c, 1)$ if $c \neq 0$. Therefore, $\rho(\mathrm{P\Gamma L}(2,q))$ fixes the set $\mathscr{O}$.

Now we equip the space W with the quadratic form $Q(u,v,w) = v^2 - uw$. By (2), $\mathrm{GL}(2,q)$ acts on W. Note that

$$Q(q(A(x,y)) = \det(A)^2 Q(q(x,y)),$$

i.e.,Q is multiplied by a factor $\det(A)^2$ by the action of $\rho(A)$. Hence, we find a subgroup of $\mathrm{GO}(3,q)$ which is isomorphic to $\mathrm{SL}^\pm(2,q)/\{\pm I\}$, where $\mathrm{SL}^\pm(2,q)$ is the group of matrices with determinant ± 1. In general, $\mathrm{PSL}(2,q) \simeq \mathrm{P\Omega}(3,q)$ and they are simple if $q > 3$. As a matter of fact, $\rho(\mathrm{PSL}(2,q)) = \mathrm{P\Omega}(3,q)$.

Note that $\mathrm{SL}^\pm(2,q)/\{\pm I\}$ and $\mathrm{PGL}(2,q)$ have the same size. For q even, $\mathrm{SL}^\pm(2,q)/\{\pm I\} = \mathrm{SL}(2,q) = \mathrm{PGL}(2,q)$. For q odd, $\mathrm{SL}^\pm(2,q)/\{\pm I\}$ and $\mathrm{PGL}(2,q)$ are not isomorphic. In this case, $\rho(\mathrm{SL}^\pm(2,q)/\{\pm I\})$ is a subgroup $\mathrm{PGO}(3,q)$ of index 2, but it is not isomorphic to $\mathrm{PSO}(3,q)$.

Define a map f from $\mathrm{PG}(1,q)$ to $\mathscr{O}$ as follows:

$$f(\infty) = \infty \text{ and } f(\xi, 1) \mapsto (\xi^2 : \xi : 1) \text{ for all } \xi \in \mathbb{F}_q.$$

We have

$$f(A(\xi)) = \rho(A) f(\xi)^T. \tag{5}$$

The action of $\mathrm{PGL}(2,q)$ on $\mathrm{PG}(1,q)$ is equivalent to that of $\mathrm{SO}(3,q)$ on $\mathscr{O}$.

2.3. FT$(q+1)$ *and* $\mathfrak{X}(\mathrm{SO}(3,q),\mathcal{L}_\epsilon)_{\epsilon=\pm}$

The *triangular scheme* T(n) comes from the action of Sym(n) on the collection of 2-element subsets of an n-set ($n \geq 4$): for any such subsets x, y, $(x,y) \in R_i$ if their intersection has size $2-i$. T(n) is a 2-class Johnson scheme. It is well known that PGL$(2,q)$ acts sharply 3-transitively on PG$(1,q)$. So PGL$(2,q)$ acts generously transitively on the collection of 2-element subsets of PG$(1,q)$ and thus this action determines a symmetric association scheme. In the paper [5], de Caen and van Dam described this scheme as a fission scheme of the triangular scheme on PG$(1,q)$ using the cross ratio, denoted by FT$(q+1)$.

We know from the previous subsection that the action PGL$(2,q)$ on PG$(1,q)$ is equivalent to that of SO$(3,q)$ on $\mathscr{O}$. So the schemes FT$(q+1)$ is isomorphic to the scheme coming from the action of SO$(3,q)$ on the 2-element subsets of $\mathscr{O}$. By definition, the hyperbolic lines $\mathcal{L}_+$ are in one-to-one correspondence with the 2-element subsets of $\mathscr{O}$. Hence, FT$(q+1)$ is isomorphic to $\mathfrak{X}(\mathrm{SO}(3,q),\mathcal{L}_+)$. In [11], Hollmann and Xiang investigated $\mathfrak{X}(\mathrm{SO}(3,q),\mathcal{L}_\epsilon)$ for $\epsilon = \pm$ using the cross ratio. In fact, they studied the coherent configuration from the action of PGL$(2,q)$ ($\simeq$ SO$(3,q)$) on $\mathcal{L}_+ \cup \mathcal{L}_-$. For the case q being a power of 4, they calculated the intersection numbers for this coherent configuration and constructed another coherent configuration on $\mathcal{L}_+ \cup \mathcal{L}_-$ by merging certain relations. In this new coherent configuration, the fiber $\mathcal{L}_+$ supports a 4-class scheme and $\mathcal{L}_-$ supports a 3-class scheme.

The 4-class scheme on $\mathcal{L}_+$ has an interesting history. Its existence as a fusion scheme of FT$(q+1)$ was first conjectured by de Caen and van Dam [5]. Tanaka [19] proved this conjecture with character-theoretic method. Ebert et al. proved it with a geometric interpretation in [6] and with a direct calculation of intersection numbers in [7]. Xiang gave a summary of these work in [22].

The 3-class scheme is also very interesting. H. Tanaka observed that this 3-class scheme on $\mathcal{L}_-$ gives a family of primitive schemes having the same parameters as the first infinite family of Q-polynomial but not P-polynomial association schemes by Penttila and Williford [16]. (The reference to Hollmann and Xiang is incorrectly cited in [16], and it should be [11].)

We conclude this subsection with a description of FT$(q+1)$ following [5]. It has as vertices all 2-element subsets of PG$(1,q)$ as and its relations are as follows:

$$\left.\begin{aligned} R_0 &= \left\{(\{\xi,\gamma\},\{\xi,\gamma\}) \mid \xi,\gamma \in \mathbb{F}_q^{\cup\{\infty\}}\right\} \\ R_1 &= \left\{(\{\xi,\gamma\},\{\xi,\beta\}) \mid \xi,\gamma,\beta \in \mathbb{F}_q^{\cup\{\infty\}}\right\} \\ R_{-1} &= \left\{(\{\xi,\gamma\},\{\alpha,\beta\}) \mid \xi,\alpha,\gamma,\beta \in \mathbb{F}_q^{\cup\{\infty\}}, \mathrm{cr}(\xi,\gamma;\alpha,\beta) = -1\right\} \\ R_r &= \left\{(\{\xi,\gamma\},\{\alpha,\beta\}) \mid \xi,\gamma,\alpha,\beta \in \mathbb{F}_q^{\cup\{\infty\}}, \mathrm{cr}(\xi,\gamma;\alpha,\beta) = r \text{ or } r^{-1}\right\}, r \in \mathbb{F}_q \setminus \{0,\pm 1\}. \end{aligned}\right\} \quad (6)$$

where $\mathrm{cr}(x,y;z,w) = \frac{(x-z)(y-w)}{(x-w)(y-z)}$ is the cross ratio. R_1 has valency $2(q-1)$, and R_{-1} has valency $(q-1)/2$, which is half of that of R_r.

2.4. The case q odd

Throughout this subsection, q will be an odd prime power. The quadratic form $Q(x_0,x_1,x_2) = x_1^2 - x_0x_2$ polarizes to the symmetric bilinear form

$$B((x_0,x_1,x_2),(y_0,y_1,y_2)) = 2x_1y_1 - x_0y_2 - x_2y_0.$$

The form B (or the conic $\mathscr{O}$) defines the following *polarity* on PG$(2,q)$:

$$\perp\colon (x_0 : x_1 : x_2) \mapsto (x_0 : x_1, x_2)^{\perp} := \{(y_0 : y_1 : y_2) \mid B((x_0, x_1, x_2), (y_0, y_1, y_2)) = 0\}. \tag{7}$$

Denote by $L_{\xi,\gamma}$ the hyperbolic line to $\mathscr{O}$ through distinct points P_ξ and P_γ for $\xi, \gamma \in \mathbb{F}_q^{\cup\{\infty\}}$. It is not difficult to check the following:

$$L^{\perp}_{\xi,\gamma} = \begin{cases} (\gamma\xi : (\xi+\gamma)/2 : 1) & \text{if } \xi, \gamma \in \mathbb{F}_q,\ \xi \neq \gamma \\ (2\xi, : 1 : 0) & \text{if } \xi \in \mathbb{F}_q, \gamma = \infty, \end{cases} \tag{8}$$
$$Q(L^{\perp}_{\xi,\gamma}) = \left(\tfrac{\gamma-\xi}{2}\right)^2 \qquad \text{and} \qquad Q(L^{\perp}_{\xi,\infty}) = 1.$$

A point $P = (x_0 : x_1 : x_2)$ is called *hyperbolic* (respectively *elliptic*) if $(x_0, x_1, x_2)^{\perp}$ is is a hyperbolic (respectively elliptic) line. Because of (8), hyperbolic (elliptic) points are also referred to as *square type* (*non-square type*) in the literature. Since there are $q(q+1)/2$ secant lines, there are $q(q+1)/2$ hyperbolic points in total. Note that $\ell^{\perp}$ is a elliptic (respectively singular) point if ℓ is an elliptic (respectively tangent) line.

Denote by $\mathcal{L}^{\perp}_{\epsilon}$ the set of all hyperbolic, singular and elliptic points for $\epsilon = +, 0$, and $-$, respectively, Then the action of SO$(3,q)$ on $\mathcal{L}^{\perp}_{\epsilon}$ determines a symmetric association scheme, denoted by $\mathfrak{X}(\mathrm{SO}(3,q), \mathcal{L}^{\perp}_{\epsilon})$. We have the following result.

Theorem 2.3. *Let q be odd. Then $\mathfrak{X}(\mathrm{SO}(3,q), \mathcal{L}_{\epsilon})$ is isomorphic to $\mathfrak{X}(\mathrm{SO}(3,q), \mathcal{L}^{\perp}_{\epsilon})$.*

To our best knowledge, $\mathfrak{X}(\mathrm{SO}(3,q), \mathcal{L}^{\perp}_{\epsilon})$ for $\epsilon = \pm$ was first constructed by Shen [18]. In fact, he studied association schemes coming from the action of SO(n,q) on hyperbolic and elliptic points in PG$(n-1,q)$(also cf. [2]). Kwok [15] calculated the character tables of $\mathfrak{X}(\mathrm{SO}(3,q), \mathcal{L}^{\perp}_{\epsilon})_{\epsilon=\pm}$. We mention in passing that $\mathfrak{X}(\mathrm{SO}(3,q), \mathcal{L}_0)$ is a 2-class symmetric scheme.

3. Three fission schemes of T(q + 1)

This section we are concerned with the three fission schemes of T$(q+1)$ mentioned in Section 1. The next theorem holds for PSL$(2,q)$ regardless the parity of q (e.g, [20, Theorem 4.1]).

Theorem 3.1. *The group* PSL$(2,q)$ *acts doubly transitively on* PG$(1,q)$. *Hence,* P$\Omega(3,q)$ *acts doubly transitively on $\mathscr{O}$.*

Recall that Ω denotes the collection of 2-element subsets in PG$(1,q)$. By the above theorem, PSL$(2,q)$ acts transitively on Ω, and thus the action determines an association scheme, denoted by $\mathfrak{X}(\mathrm{PSL}(2,q), \Omega)$. This is a fission scheme of FT$(q+1)$.

In the rest of this paper, we shall describe association schemes from the actions of PSL$(2,q)$, M(q) and PΓL$(2,q)$ on Ω. In Section 2, we give a one-to-one correspondence of Ω with $\mathcal{L}_+$ (and $\mathcal{L}^{\perp}_+$ via polarity) and an embedding ρ of PΓL$(2,q)$. As result, for any transitive subgroup of PΓL$(2,q)$ on Ω, we obtain three isomorphic association schemes, e.g., $\mathfrak{X}(\mathrm{PSL}(2,q), \Omega)$, $\mathfrak{X}(\mathrm{P}\Omega(3,q), \mathcal{L}_+)$ and $\mathfrak{X}(\mathrm{P}\Omega(3,q), \mathcal{L}^{\perp}_+)$ in the case . of PSL$(2,q)$. So we shall use the action on $\mathcal{L}^{\perp}_+$ in subsequent calculation.

3.1. The association scheme $\mathfrak{X}(\mathrm{P}\Omega(3,q),\mathcal{L}_+)$

In this subsection, we consider the association scheme from the action of $\mathrm{P}\Omega(3,q)$. Namely, we shall prove the following result.

Theorem 3.2. *The association scheme* $\mathfrak{X}(\mathrm{P}\Omega(3,q),\mathcal{L}_+)$ *is non-symmetric with* $(3q+5)/4$ *classes if* $q \equiv 1 \bmod 4$ *and* $(3q+3)/4$ *classes if* $q \equiv 3 \bmod 4$.

Consider the stabilizer of $L_{0,\infty}^{\perp}$ in $\mathrm{P}\Omega(3,q)$, which is denoted by $\mathrm{P}\Omega(3,q)_{0,\infty}$. We know from Subsection 2.2 that $\mathrm{P}\Omega(3,q)$ is induced by elements of form (3) with $ad-bc=1$. Since $L_{\infty,0}^{\perp}$ has coordinates $(0:1:0)$ by (8),

$$\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2ab \\ ad+bc \\ 2cd \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ 0 \end{pmatrix}$$

for some λ in $\mathbb{F}_q^*$. Since $ad-bc=1$, we must have $a=d^{-1}, b=c=0$, or $a=d=0, b=-c^{-1}$. So $G := \mathrm{P}\Omega(3,q)_{0,\infty}$ is induced by the following elements

$$\begin{pmatrix} a^2 & & \\ & 1 & \\ & & a^{-2} \end{pmatrix}, \quad \begin{pmatrix} & & b^{-2} \\ & -1 & \\ b^2 & & \end{pmatrix}, \quad a,b \in \mathbb{F}_q^*. \tag{9}$$

For each relation R in (6), let

$$R^{\{0,\infty\}} = \left\{ L_{\xi,\gamma} \mid (L_{0,\infty}, L_{\xi,\gamma}) \in R \right\}.$$

These $R^{\{0,\infty\}}$ are all orbits of $\mathrm{SO}(3,q)$ on $\mathcal{L}_+$. Since $\mathrm{P}\Omega(3,q)$ is a subgroup of $\mathrm{SO}(3,q)$ of index 2. So any orbit of $\mathrm{SO}(3,q)$ either splits into a pair of orbits or remains to be an orbit of $\mathrm{P}\Omega(3,q)$.

In the rest of this paper, z is a fixed non-square element in $\mathbb{F}_q$*, s denotes a square element not in* $\{0,\pm 1\}$*, and t denotes a non-square element in* $\mathbb{F}_q$*. Let* $\mathbb{F}_q^{*2}$ *be the set of all nonzero square elements in* $\mathbb{F}_q$*. So* $z\mathbb{F}_q^{*2}$ *consists of all non-square elements of* $\mathbb{F}_q$*.*

By (8), the hyperbolic points come in with two forms:

$$(2\xi:1:0) \text{ with } \xi \in \mathbb{F}_q, \quad \text{or} \quad (\gamma\xi:(\xi+\gamma)/2:1) \text{ with } \xi,\gamma \in \mathbb{F}_q,\ \xi \neq \gamma. \tag{10}$$

First, we settle the G-orbits on $(2\xi:1:0)$. It is trivial that $\Gamma_0 = \{L_{0,\infty}\}$ is a G-orbit. Now assume $\xi \neq 0$. Note that

$$\begin{pmatrix} a^2 & & \\ & 1 & \\ & & a^{-2} \end{pmatrix}\begin{pmatrix} 2\xi \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a^2\xi \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} & & a^{-2} \\ & -1 & \\ a^2 & & \end{pmatrix}\begin{pmatrix} 2\xi \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2a^2\xi \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-1}{2a^2\xi} \\ 1 \end{pmatrix}, \tag{11}$$

where the last equality follows from that these two vectors represent the same (projective) point. Recall that hyperbolic points are in one-to-one correspondence with 2-element subsets in $\mathscr{O}$, which are again in one-to-correspondence with those of $\mathrm{PG}(1,q)$. Note that $(2a^2\xi:1:0)$ is determined

by $\{a^2\xi, \infty\}$, and $(0 : \frac{-1}{2a^2\xi} : 1)$ by $\{0, -(a^2\xi)^{-1}\}$. For a fixed square (respectively non-square) ξ in $\mathbb{F}_q^*$, if $-1 \in \mathbb{F}_q^{*2}$, then both $a^2\xi$ and $-(a^2\xi)^{-1}$ range over $\mathbb{F}_q^{*2}$ (respectively $z\mathbb{F}_q^{*2}$). On the other hand, if $-1 \notin \mathbb{F}_q^{*2}$, then $a^2\xi$ ranges over $\mathbb{F}_q^{*2}$ while $-(a^2\xi)^{-1}$ ranges over $z\mathbb{F}_q^{*2}$. Hence, $R_1^{\{0,\infty\}}$ splits into two orbits of length $q-1$ each:

$$\begin{cases} \Gamma_1^+ = \left\{L_{0,\xi}, L_{\infty,\xi} \mid \xi \in \mathbb{F}_q^{*2}\right\} & \Gamma_1^- = \left\{L_{0,\xi}, L_{\infty,\xi} \mid \xi \in z\mathbb{F}_q^{*2}\right\} & \text{if } q \equiv 1 \bmod 4 \\ \Gamma_1^+ = \left\{L_{0,\xi}, L_{\infty,\gamma} \mid \xi \in \mathbb{F}_q^{*2},\ \gamma \in z\mathbb{F}_q^{*2}\right\} & \Gamma_1^- = \left\{L_{0,\xi}, L_{\infty,\gamma} \mid \xi \in z\mathbb{F}_q^{*2},\ \gamma \in \mathbb{F}_q^{*2}\right\} & \text{if } q \equiv 3 \bmod 4. \end{cases} \tag{12}$$

The remaining hyperbolic points are $(\gamma\xi : (\xi+\gamma)/2 : 1)$ with $\xi\gamma \neq 0$. Note that

$$\begin{pmatrix} a^2 & & \\ & 1 & \\ & & a^{-2} \end{pmatrix}\begin{pmatrix} \gamma\xi \\ \frac{\xi+\gamma}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma_1\xi_1 \\ \frac{\xi_1+\gamma_1}{2} \\ 1 \end{pmatrix} \quad \text{if and only if} \quad \{a^2\gamma, a^2\xi\} = \{\gamma_1, \xi_1\} \tag{13}$$

and

$$\begin{pmatrix} & & b^{-2} \\ & -1 & \\ b^2 & & \end{pmatrix}\begin{pmatrix} \gamma\xi \\ \frac{\xi+\gamma}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma_1\xi_1 \\ \frac{\xi_1+\gamma_1}{2} \\ 1 \end{pmatrix} \quad \text{if and only if} \quad \left\{\frac{-1}{b^2\gamma}, \frac{-1}{b^2\xi}\right\} = \{\gamma_1, \xi_1\}. \tag{14}$$

If $\gamma/\xi = -1$, $R_{-1}^{\{0,\infty\}}$ remains to be a G-orbit for $q \equiv 3 \bmod 4$ and it splits into two orbits for $q \equiv 1 \bmod 4$:

$$\begin{cases} \Gamma_{-1} = \left\{L_{\xi,-\xi} \mid \xi \in \mathbb{F}_q^*\right\} & \text{if } q \equiv 3 \bmod 4 \\ \Gamma_{-1}^+ = \left\{L_{\xi,-\xi} \mid \xi \in \mathbb{F}_q^{*2}\right\}, \quad \Gamma_{-1}^- = \left\{L_{\xi,-\xi} \mid \xi \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}\right\} & \text{if } q \equiv 1 \bmod 4. \end{cases} \tag{15}$$

The orbit length is $(q-1)/2$ in the first case and $(q-1)/4$ in the second case.

Now suppose $r = \gamma/\xi \notin \{0, \pm 1\}$. We claim that $R_\gamma^{\{0,\infty\}}$ becomes the following G-orbit(s):

$$\begin{cases} \Gamma_r = \left\{L_{\xi,r\xi} \mid \xi \in \mathbb{F}_q^*\right\} & \text{if } -1/r \text{ is non-square} \\ \Gamma_r^+ = \left\{L_{\xi,r\xi} \mid \xi \in \mathbb{F}_q^{*2}\right\}, \quad \Gamma_r^- = \left\{L_{\xi,r\xi} \mid \xi \in z\mathbb{F}_q^{*2}\right\} & \text{if } -1/r \text{ is square.} \end{cases} \tag{16}$$

The orbit length is $(q-1)$ in the first case and $(q-1)/2$ in the second case.

Now suppose that $\gamma = r\xi$ and $\gamma_1 = r\xi_1$. If ξ and ξ_1 are both squares (non-squares), set $a^2 = \xi_1\xi^{-1} = a^2$. By (13), $L_{\xi,r\xi}$ and $L_{\xi_1,r\xi_1}$ are in the same G-orbit. Suppose that ξ is a square and ξ_1 is a non-square. If $-1/r$ is a non-square, set $b^2 = \frac{-1}{r\xi\xi_1}$. Then $L_{\xi,r\xi}$ and $L_{\xi_1,r\xi_1}$ are in the same G-orbit. On the other hand, if $-1/r$ is a square, then $\{\xi_1, r\xi_1\} = \left\{\frac{-1}{b^2\xi}, \frac{-1}{b^2r\xi}\right\}$ if and only if $r = \pm 1$. As $r \neq \pm 1$, $L_{\xi,r\xi}$ and $L_{\xi_1,r\xi_1}$ fall into different orbits. We prove the above claim.

Each orbit Γ determines a relation R of $\mathfrak{X}(\mathrm{P}\Omega(3,q), \mathcal{L}_+)$, and vice versa. So we may equip R with the same subscript/superscript of Γ, e.g., R_{-1}, R_r^+, etc. We will adopt this convention for the other two fission schemes to follow.

Now we determine the non-symmetric relations of $\mathfrak{X}(\mathrm{P}\Omega(3,q), \mathcal{L}_+)$. As an example, we illustrate with R_s^+. We shall use $\mathrm{PSL}(2,q)$, instead of $\mathrm{P}\Omega(3,q)$. Take $(L_{0,\infty}, L_{1,s}) \in R_s^+$. Note the function

$x \mapsto \frac{x-1}{x-s}$ maps $\{1, s\}$ to $\{0, \infty\}$. Now suppose that $\det\begin{pmatrix} 1 & -1 \\ 1 & -s \end{pmatrix} = 1 - s$ is a non-square, say z. The function $x \mapsto \frac{x-1}{z(x-s)}$ is in $\mathrm{PSL}(2, q)$. This function carries $\{1, s\}$ to $\{0, \infty\}$, and $\{0, \infty\}$ to $\{\frac{1}{z}, \frac{1}{sz}\}$. The hyperbolic line corresponding to $\{\frac{1}{z}, \frac{1}{sz}\}$ is in Γ_s^- and thus ${}^tR_s^+ = R_s^{-1}$. We can handle other relations in a similar way.

Suppose that $q \equiv 1 \bmod 4$. Then ${}^tR_{-1}^+ = R_{-1}^-$ if 2 is a non-square in $\mathbb{F}_q$, and ${}^tR_s^+ = R_s^{-1}$ if $1 - s$ is a non-square. It is well known that 2 is a square in $\mathbb{F}_q$ if and only if the Legendre symbol $\left(\frac{2}{q}\right) = \left[(-1)^{\frac{p^2-1}{8}}\right]^m = 1$, where $q = p^m$ for some odd prime p (e.g., see [12, Section 6.1]). Suppose that $q \equiv 3 \bmod 4$. Then ${}^tR_1^+ = R_1^-$, and ${}^tR_t^+ = R_t^{-1}$ if $1 - t$ is a square.

By counting orbits in (12), (15) and (16), we see that $\mathfrak{X}(\mathrm{P\Gamma L}(q), \mathcal{L}_+)$ has the asserted class number. The proof of Theorem 3.2 is complete.

3.2. The association scheme $\mathfrak{X}(\mathrm{M}(q), \mathcal{L}_+)$

In this subsection, we describe the association scheme $\mathfrak{X}(\mathrm{M}(q), \mathcal{L}_+)$, which is a fusion scheme of $\mathfrak{X}(\mathrm{P\Omega}(3, q), \mathcal{L}_+)$. Now we prove the following result.

Theorem 3.3. *Let q be an even power of an odd prime. The following holds:*

(i) For $q > 9$, $\mathfrak{X}(\mathrm{M}(q), \mathcal{L}_+)$ is a non-symmetric association scheme with $(3q + 5)/8$ classes ;

(ii) $\mathfrak{X}(\mathrm{M}(9), \mathcal{L}_+)$ is a symmetric P-polynomial association scheme.

Now consider the stabilizer $\mathrm{M}(q)_{0,\infty}$ of $L_{0,\infty}^{\perp}$ in $\mathrm{M}(q)$. Under the embedding ρ of $\mathrm{P\Gamma L}(2, q)$ into $\mathrm{P\Gamma L}(3, q)$ in Subsection 2.2, $\rho(S) = \mathrm{P\Omega}(3, q)$. The stabilizer of $L_{0,\infty}$ in $\mathrm{M}(q)$ is the union of $\mathrm{P\Omega}(3, q)_{0,\infty}$ and $\rho(T)_{0,\infty}$, where $\rho(T)_{0,\infty}$ consists of elements in $\rho(T)$ that fix $L_{0,\infty}$. Then $\rho(T)_{0,\infty}$ is induced by all the semilinear transformations of the following matrices and the unique involution σ:

$$\begin{pmatrix} a^2 & & \\ & z & \\ & & a^{-2}z^2 \end{pmatrix}, \quad \begin{pmatrix} & & b^2 \\ & -z & \\ b^{-2}z^2 & & \end{pmatrix}, \tag{17}$$

where a, b ranges over $\mathbb{F}_q^*$.

For typographical convenience, we write $\tilde{x} = x^\sigma$. Since $\tilde{x} = x$ for any x in the fixed field $\mathrm{Fix}(\sigma)$ of σ, we can check that, for any points $(2\theta : 1 : 0)$ and $(\gamma\theta : (\theta + \gamma)/2 : 1)$,

$$\begin{pmatrix} a^2 & & \\ & z & \\ & & a^{-2}z^2 \end{pmatrix}\begin{pmatrix} 2\tilde{\theta} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a^2\tilde{\theta}z^{-1} \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} & & b^2 \\ & -z & \\ b^{-2}z^2 & & \end{pmatrix}\begin{pmatrix} 2\tilde{\theta} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-b^2}{2z\tilde{\theta}} \\ 1 \end{pmatrix}, \tag{18}$$

We remark that $\mathrm{Fix}(\sigma)$ has $\sqrt{q}$ elements.

Note a transformation in (18) maps Γ_1^+ to Γ_1^-, and vice versa. So $\mathrm{M}(q)_{0,\infty}$ combines Γ_1^+ and Γ_1^-:

$$\Delta_1 = \Gamma_1^+ \cup \Gamma_1^-.$$

The following equations shall be used in determining the remaining orbits of $\mathrm{M}(q)_{0,\infty}$:

$$\begin{pmatrix} a^2 & & \\ & z & \\ & & a^{-2}z^2 \end{pmatrix}\begin{pmatrix} \tilde{\gamma}\tilde{\theta} \\ \frac{(\tilde{\theta}+\tilde{\gamma})}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a^4\tilde{\gamma}\tilde{\theta}}{z^2} \\ \frac{a^2(\tilde{\theta}+\tilde{\gamma})}{2z} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} & & b^2 \\ & -z & \\ b^{-2}z^2 & & \end{pmatrix}\begin{pmatrix} \tilde{\gamma}\tilde{\theta} \\ \frac{(\tilde{\theta}+\tilde{\gamma})}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{b^4}{\tilde{\theta}\tilde{\gamma}z^2} \\ -\frac{(\tilde{\theta}+\tilde{\gamma})b^2}{2\tilde{\theta}\tilde{\gamma}z} \\ 1 \end{pmatrix}. \tag{19}$$

We can deduce from (19) that $\mathrm{M}(q)_{0,\infty}$ combines Γ^+_{-1} and Γ^-_{-1}:

$$\Delta_{-1} = \Gamma^+_{-1} \cup \Gamma^-_{-1}.$$

Note that Δ_1 and Δ_{-1} have lengths $2(q-1)$ and $(q-1)/2$, respectively.

Now we consider the orbits of $\mathrm{M}(q)_{0,\infty}$ on the remaining points $(\theta\gamma : (\theta+\gamma)/2 : 1)$ with $\gamma + \theta \neq 0$. Let $r = \theta/\gamma$. These orbits depend on whether $\tilde{r}$ is in $\{r, r^{-1}\}$ or not. Note if $\tilde{r} = r$ (respectively r^{-1}) , then $r^{\sqrt{q}-1} = 1$ (respectively $r^{\sqrt{q}+1} = 1$) and thus r is a square in $\mathbb{F}_q$. In this case, we obtain $(\sqrt{q}-3)/2$ (respectively $(\sqrt{q}-1)/2$) orbits of length $(q-1)$ each. Now change r to s. So we can deduce from (19) that $\mathrm{M}(q)_{0,\infty}$ combines Γ^+_s and Γ^-_s if $\tilde{s} = s$ or s^{-1}:

$$\Delta_s = \Gamma^+_s \cup \Gamma^-_s.$$

If $s \in \mathbb{F}_q^{*2}$ and $\tilde{s} \notin \{s, s^{-1}\}$, then we can deduce again from (19) that $\mathrm{M}(q)_{0,\infty}$ combines Γ^+_s and $\Gamma^-_{\tilde{s}}$ (respectively Γ^-_s and $\Gamma^+_{\tilde{s}}$):

$$\Delta^+_s = \Gamma^+_s \cup \Gamma^-_{\tilde{s}}, \qquad \Delta^-_s = \Gamma^-_s \cup \Gamma^+_{\tilde{s}}.$$

We obtain $\frac{(\sqrt{q}-3)(\sqrt{q}-1)}{4}$ orbits of length $q-1$.

If t is a non-square in $\mathbb{F}_q$, then $\mathrm{M}(q)_{0,\infty}$ combines Γ_t and $\Gamma_{\tilde{t}}$:

$$\Delta_t = \Gamma_t \cup \Gamma_{\tilde{t}}.$$

In this case, we obtain $(q-1)/8$ orbits of length $2(q-1)$ each.

Now we write the relations of $\mathfrak{X}(\mathrm{M}(q), \mathcal{L}_+)$ according to these Δ. If $q \neq 9$, $\mathfrak{X}(\mathrm{M}(q), \mathcal{L}_+)$ is non-symmetric: ${}^tR^+_s = R^-_s$. So we prove (i) of Theorem 3.3. $\mathfrak{X}(\mathrm{M}(9), \mathcal{L}_+)$ is a symmetric scheme; see Example 3.1 below. In fact, it is a P-polynomial scheme on 45 vertices from a generalized octagon of order $(2,1)$ [3, p. 419]. We complete the proof of Theorem 3.3.

Remark 1. Limited computation with Gap [10] shows that $\mathfrak{X}(\mathrm{M}(q), \mathcal{L}_+)$ is not commutative for $q > 25$. it is not symmetric but commutative for $q = 25$.

3.3. The association scheme $\mathfrak{X}(\mathrm{P\Gamma L}(2,q), \mathcal{L}_+)$

In this subsection, we consider an association scheme that is a fusion scheme of both $\mathrm{FT}(q+1)$ and $\mathfrak{X}(\mathrm{M}(q), \Omega)$. Since both $\mathrm{PGL}(2,q)$ and $\mathrm{M}(q)$ are subgroups of $\mathrm{P\Gamma L}(2,q)$, the action of $\mathrm{P\Gamma L}(2,q)$ on Ω determines this fusion scheme.

The stabilizer of $\rho(\mathrm{P\Gamma L}(2,q))_{0,\infty}$ of $L^{\perp}_{0,\infty}$ induced by all the semilinear transformation of the following matrices associated with an automorphism τ of $\mathbb{F}_q$:

$$\begin{pmatrix} a^2 & & \\ & ad & \\ & & d^2 \end{pmatrix}, \quad \begin{pmatrix} & & b^2 \\ & bc & \\ c^2 & & \end{pmatrix}, \tag{20}$$

where a, b, c, d range over $\mathbb{F}_q^*$, and τ ranges over $\mathrm{Aut}(\mathbb{F}_q)$.

The group $\rho(\mathrm{P\Gamma L}(2,q))_{0,\infty}$ acts on $\mathcal{L}_+^{\perp}$ similarly to that in (18) and (19) with $\tilde{}$ replaced with τ. By a similar argument employed in the previous subsection, $\rho(\mathrm{P\Gamma L}(2,q))_{0,\infty}$ has the following orbits on $\mathcal{L}_+$:

$$\begin{gathered} \Lambda_0 = \{L_{0,\infty}\}, \quad \Lambda_1 = \{L_{0,\xi}, L_{\infty,\xi} \mid \xi \in \mathbb{F}_q^*\}, \quad \Lambda_{-1} = \{L_{\xi,-\xi} \mid \xi \in \mathbb{F}_q^*\}, \\ \Lambda_r = \{L_{\xi,\gamma} \mid \xi/\gamma \in \{r^\tau, r^{-\tau} \mid \tau \in \mathrm{Aut}\ \mathbb{F}_q\}\}, (r \in \mathbb{F}_q^* \setminus \{\pm 1\}). \end{gathered} \tag{21}$$

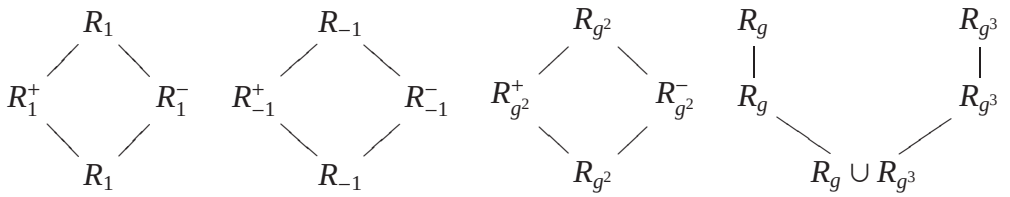
No we can write down the corresponding relations of $\mathfrak{X}(\mathrm{P\Gamma L}(2,q), \mathcal{L}_+)$:

$$\begin{gathered} R_0 = \{(L_{\xi,\gamma}, L_{\xi,\gamma}) \mid \{\xi,\gamma\} \subset \mathbb{F}_q^{\cup\{\infty\}}\}, \\ R_1 = \{(L_{\xi,\gamma}, L_{\xi,\beta}) \mid \{\xi,\gamma,\beta\} \subset \mathbb{F}_q^{\cup\{\infty\}}\}, \\ R_{-1} = \{(L_{\xi,\gamma}, L_{\alpha,\beta}) \mid \{\xi,\gamma,\alpha,\beta\} \subset \mathbb{F}_q^*,\ \mathrm{cr}(\xi,\gamma;\alpha,\beta) = -1\}, \\ R_r = \left\{(L_{\xi,\gamma}, L_{\alpha,\beta}) \mid \{\xi,\gamma,\alpha,\beta\} \subset \mathbb{F}_q^*,\ \mathrm{cr}(\xi,\gamma;\alpha,\beta) \in \{r^\tau, r^{-\tau} \mid \tau \in \mathrm{Aut}\ \mathbb{F}_q\}\right\}, \qquad r \in \mathbb{F}_q^* \setminus \{\pm 1\}. \end{gathered}$$

Theorem 3.4. *$\mathfrak{X}(\mathrm{P\Gamma L}(2,q), \mathcal{L}_+)$ is a symmetric association scheme with the above relations.*

Remark 2. Note that the relation R_1 is an original relation in $\mathrm{T}(q+1)$. For $q = 9$, $\mathfrak{X}(\mathrm{P\Gamma L}(2,q), \mathcal{L}_+)$ has 4 classes. It has the P-polynomial property given by R_{-1} but not the Q-polynomial property. For $q = 25$ and 49, $\mathfrak{X}(\mathrm{P\Gamma L}(2,q), \mathcal{L}_+)$ has 9 and 16 classes, respectively. These schemes don't have polynomial (P or Q) property. For the general case, the class number of $\mathfrak{X}(\mathrm{P\Gamma L}(2,q), \mathcal{L}_+)$ is reduced to counting the number of orbits of $\mathrm{Aut}(\mathbb{F}_q)$ on the collection $\{\{r^\tau, r^{-\tau}\} \mid r \in \mathbb{F}_q^* \setminus \{\pm 1\},\ \tau \in \mathrm{Aut}(\mathbb{F}_q)\}$. However, we are unable to give an explicit formula for this number.

Example 3.1. In this example, we illustrate the four fission schemes of $\mathrm{T}(10)$ for $q = 9$. Let g be a primitive element of $\mathbb{F}_9$. We have $\mathfrak{X}(\mathrm{P\Gamma L}(2,9), \mathcal{L}_+) = \mathfrak{X}(\mathrm{M}(9), \mathcal{L}_+)$. The relations of $\mathrm{FT}(10)$, $\mathfrak{X}(\mathrm{PSL}(2,9), \mathcal{L}_+)$ and $\mathfrak{X}(\mathrm{M}(9), \mathcal{L}_+)$ are shown in the following diagram:



The top, middle and bottom rows are the relations of $\mathrm{FT}(10)$, $\mathfrak{X}(\mathrm{PSL}(2,9), \mathcal{L}_+)$ and $\mathfrak{X}(\mathrm{M}(9), \mathcal{L}_+)$, respectively.

Remark 3. Primitive representations of $\mathrm{PSL}(2,q)$ were investigated by Sasha Ivanov and Loius Tchuda via the use of so-called Burnside marks [14]. A general method was outlined in [8], which allows one to obtain formulas for the rank and subdegrees for a (transitive) permutation representation [14].

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References

[1] E. Bannai, T. Ito. Algebraic Combinatorics I: Association Schemes, The Benjamin/Cummings Publishing Company, Inc., Menlo Park, CA, 1984.
[2] E. Bannai, H. Shen, S.Y. Song. Character tables of association schemes of finite orthogonal groups acting on the nonisotropic points. J. Combin. Theory Ser. A, 54 (1990) 164–200.
[3] A.E. Brouwer, A.M. Cohen, A. Neumaier. Distance-Regular Graphs, Springer-Verlag, Berlin, Heidelberg, 1989.
[4] P.J. Cameron. Classical Groups, Lecture notes, 2000, available at `www.maths.qmul.ac.uk/~pjc`
[5] D. de Caen, E.R. van Dam. Fissioned triangular schemes via the cross-ratio, Europ. J. Combin., 22 (2001) 297-301.
[6] G. Ebert, S. Egner, H.D. Hollmann, Q. Xiang. On a four-class association scheme, J. Combin. Theory Ser. A, 96 (2001) 180-191.
[7] G. Ebert, S. Egner, H.D. Hollmann, Q. Xiang. Proof of a conjecture of de Caen and van Dam, Europ. J. Combin., 23 (2002) 201-206.
[8] I.A. Faradžev, M.H. Klin, M.E. Muzichuk. Cellular rings and groups of automorphisms of graphs, in: I.A. Faradžev et al. (Eds.), Investigations in Algebraic Theory of Combinatorial Objects, Kluwer Academic Publishers, Dordrecht, 1994, pp. 1-152.
[9] T. Fujisaki. A four-class association scheme derived from a hyperbolic quadric in PG(3, q) Adv. Geom., 4 (2004) 105-117.
[10] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. (`http://www.gap-system.org`)
[11] H.D.L. Hollmann, Q. Xiang. Association schemes from the actions of PGL(2, q) fixing a nonsingular conic , J. Algebraic Combin., 24 (2006) 157-193.
[12] K.F. Ireland, M.I. Rosen. A classical introduction to modern number theory, Springer 1990.
[13] T. Ito, A. Munemasa, M. Yamada. Amorphous association schemes over the Galois rings of characteristic 4, European J. Combin., 12 (1991), 513–526.
[14] M. Klin, private communication, 2010.
[15] W.M. Kwok. Character table of a controlling association scheme defined by the general orthogonal group O(3, q), Graphs and Combin., 7 (1991) 39-52.
[16] T. Penttila, J. Williford. New families of Q -polynomial association schemes, J. Combin. Theory Ser. A, 118 (2011) 502-509.
[17] D. Robinson. A Course in the Theory of Groups, 2nd ed., Springer, New York, 1995.
[18] H. Shen. The construction of association schemes with 1-dimensional non-isotropic subspaces in the orthogonal geometry, Journal of Shanghai Jiaotong University, 18 (1984) 1-13 (in Chinese).
[19] H. Tanaka. A four-class subscheme of the association scheme coming from the action of PGL(2, 4^f), European J. Combin. 23 (2002) 121–129.
[20] D. E. Taylor. Geometry of the classical groups. Heldermann Verlag, Berlin, 1992.
[21] Z. Wan. Geometry of Classical Groups over Finite Fields, 2nd ed., Science Press, Beijing/New York, 2002.
[22] Q. Xiang. Association schemes from ovoids in PG(3, q), J. Geom., 76 (2003) 294-301.