On a problem of Dobrowolski–Williams.

D. A. Frolenkov*

Abstract

In this paper we prove new upper bounds for the sum $\sum_{n=a+1}^{a+N} f(n)$, for a certain class of arithmetic functions f. Our results improve the previous results of G. Bachman and L. Rachakonda.

Keywords: arithmetic functions.

2000 Mathematics Subject Classification: 11A25, 26D15.

1 Introduction

For any positive real numbers A, B and a natural number q, denote by $F = F_{A,B}(q)$ the class of all functions $f : \mathbb{Z} \to \mathbb{C}$ satisfying the conditions

$$|f(n)| \leqslant A \quad \text{for all} \quad n \in \mathbb{Z}, \tag{1}$$

$$f(n+q) = f(n) \quad \text{for all} \quad n \in \mathbb{Z}, \tag{2}$$

$$\sum_{n=1}^{q} \left| \sum_{k=1}^{K} f(n+k) \right|^2 \leqslant BqK \quad \text{for all natural numbers} \quad K. \tag{3}$$

Dobrowolski and Williams [1] proved that the estimate

$$\left| \sum_{n=a+1}^{a+N} f(n) \right| \leqslant \frac{\sqrt{B}}{2\log 2} \sqrt{q} \log q + 3A\sqrt{q} \tag{4}$$

holds for all $f \in F_{A,B}(q)$. Bachman and Rachakonda [2] improved their result and obtained that

$$\left| \sum_{n=a+1}^{a+N} f(n) \right| \leqslant \frac{\sqrt{B}}{3\log 3} \sqrt{q} \log q + \left(5\sqrt{B} + \frac{3}{2}A\right) \sqrt{q}. \tag{5}$$

In this paper we improve (5).

Let $\{q_n\}_{n=-2}^{\infty}$ be a sequence of integers such that

$$q_{-2} = 1, \ q_{-1} = 1, \ q_n = 2q_{n-1} + q_{n-2} \quad \text{for } n \geqslant 0. \tag{6}$$

*The research was supported by the grant RFBR 11-01-00759-a

Then

$$q_n = \left(\frac{3}{2}+\sqrt{2}\right)\lambda_1^n + \left(\frac{3}{2}-\sqrt{2}\right)\lambda_2^n \tag{7}$$

where

$$\lambda_1 = 1+\sqrt{2},\ \lambda_2 = 1-\sqrt{2}. \tag{8}$$

Let $\{p_n\}_{n=-2}^{\infty}$ be a sequence of integers such that

$$p_{-2}=0,\ p_{-1}=0,\ p_n = 2p_{n-1}+p_{n-2}+\frac{q_{n-1}+q_{n-2}}{2} \quad \text{for } n \geqslant 0. \tag{9}$$

Then

$$p_n = \left(\frac{1}{2}+\frac{5\sqrt{2}}{16}\right)\lambda_1^n + \left(\frac{1}{2}-\frac{5\sqrt{2}}{16}\right)\lambda_2^n + \frac{n}{8}\lambda_1^{n+2} + \frac{n}{8}\lambda_2^{n+2}. \tag{10}$$

Define the quantity $\delta_n, n \geqslant 0$ by

$$\delta_n = \frac{p_n}{q_n \log q_n}. \tag{11}$$

Theorem 1. *For any $n \geqslant 0$, $q \geqslant q_n^6$ and any $f \in F_{A,B}(q)$, we have*

$$\left|\sum_{m=a+1}^{a+N} f(m)\right| \leqslant \sqrt{B}\delta_n\sqrt{q}\log q + \left(\sqrt{B}\sqrt{q_n+\frac{1}{q_n^2}} + \sqrt{B}(\sqrt{2}-1)\frac{2p_n}{q_n-1} + \frac{1}{2}A\right)\sqrt{q} + \psi_n(q,B),$$

where

$$\psi_n(q,B) = \sqrt{B}\left(q_n+\frac{1}{q_n^2}\right)\left(\delta_n\log q + (\sqrt{2}-1)\frac{2p_n}{q_n-1}\right).$$

The aim of the Theorem 1 is to improve the constant $\delta_0 = \frac{1}{3\log 3}$ in (5) (see Table 1).

n	q_n	p_n	δ_n
0	3	1	0.303413
1	7	4	0.293656
2	17	14	0.290670
3	41	44	0.288986
4	99	131	0.287965

Table 1

By (7), (10), (11) one has

$$\lim_{n\to\infty}\delta_n = \frac{1}{4\log(1+\sqrt{2})} = 0.283676\ldots.$$

To prove Theorem 1 we extend the method of Bachman and Rachakonda.

2 Proof of the theorem

If $f \in \mathcal{F}_{A,B}(q)$ then $\sum\limits_{n=1}^{q} f(n) = 0$. So, we may assume that

$$N < \frac{q}{2}. \tag{12}$$

If $N \leqslant \frac{\sqrt{q}}{2}$ then

$$\left| \sum_{n=a+1}^{a+N} f(n) \right| \leqslant A\frac{\sqrt{q}}{2}$$

and Theorem 1 is proved. So, we may assume that

$$N \geqslant \frac{\sqrt{q}}{2} \geqslant \frac{q_n^3}{2}. \tag{13}$$

As in [2] we define triangular sums $T_-(x,y), T_+(x,y)$ by

$$T_-(x,y) = \sum_{i=x+1}^{x+y} \sum_{j=x+1-i}^{0} f(i+j) = \sum_{k=1}^{y} (y+1-k) f(x+k). \tag{14}$$

$$T_+(x,y) = \sum_{i=x+1}^{x+y} \sum_{j=0}^{x+y-i} f(i+j) = \sum_{k=1}^{y} k f(x+k) \tag{15}$$

and the square sum $S(x,y)$ by

$$S(x,y) = \sum_{i=x+1}^{x+y} \sum_{j=0}^{y-1} f(i+j). \tag{16}$$

Observe that for any integer numbers x, y, u, v, k one has

$$\sum_{i=x}^{x+y} \sum_{j=u}^{u+v} f(i+j) = \sum_{i=x-k}^{x+y-k} \sum_{j=u+k}^{u+v+k} f(i+j). \tag{17}$$

Let $K, 1 \leqslant K \leqslant N$. It was shown in [2, (2.2),(2.3)] that

$$\left| \sum_{n=a+1}^{a+N} f(n) \right| \leqslant \sqrt{Bq}\sqrt{\frac{N}{K}} + \frac{1}{K} \left| -T_-(a,K) + T_+(a+N,K) \right|. \tag{18}$$

Let τ be the integer satisfying

$$\frac{q_n^{\tau}-1}{2} \leqslant N < \frac{q_n^{\tau+1}-1}{2}. \tag{19}$$

By (13) we have $\tau \geqslant 3$. Let $\{K_i\}_{i=0}^{\tau}$ be a sequence of integers such that

$$K_i = \frac{q_n^{\tau-i}-1}{2} \tag{20}$$

and

$$K = K_0 = \frac{q_n^{\tau} - 1}{2}.$$

Let $\{c_n\}_{n=-2}^{\infty}$ be a sequence of integers such that

$$c_n = \frac{q_n - 1}{2}. \tag{21}$$

Note that

$$c_j - c_{j-1} = \frac{q_{j-1} + q_{j-2}}{2}. \tag{22}$$

By (6), we have

$$c_n = 2c_{n-1} + c_{n-2} + 1. \tag{23}$$

Then for $0 \leqslant i \leqslant \tau - 1$ one has

$$K_i = q_n K_{i+1} + c_n \tag{24}$$

and

$$N < q_n K_0 + c_n. \tag{25}$$

So

$$\frac{N}{K_0} < q_n + \frac{q_n - 1}{q_n^{\tau} - 1} < q_n + \frac{1}{q_n^{\tau-1}} < q_n + \frac{1}{q_n^2}. \tag{26}$$

Lemma 1. *For any* $1 \leqslant i \leqslant \tau$ *one has*

$$T_-(a, K_{i-1}) = 2T_-(a, q_{n-1}K_i + c_{n-1}) - T_+(a - q_{n-2}K_i - c_{n-2}, q_{n-2}K_i + c_{n-2}) +$$
$$+ S(a - q_{n-2}K_i - c_{n-2}, (q_{n-2} + q_{n-1})K_i + c_n - c_{n-1}).$$

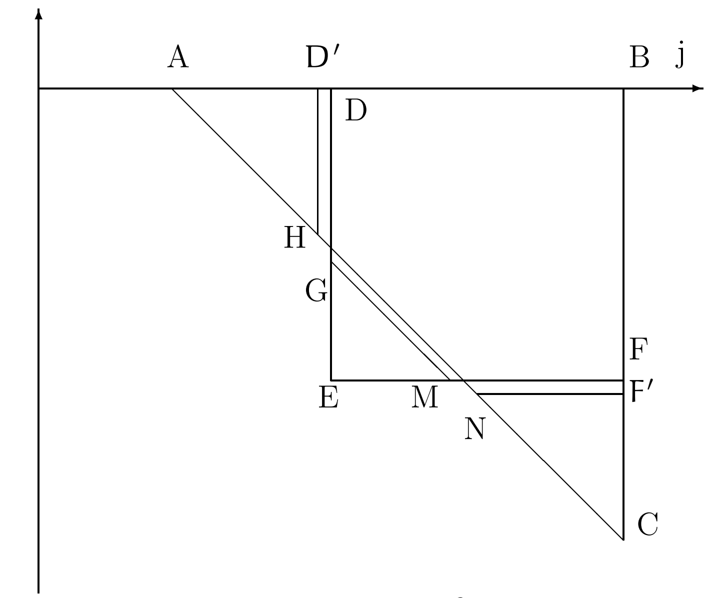

fig. 1

Proof.

Put

$$A(a+1,0);\ B(a+q_nK_i+c_n,0);\ C(a+q_nK_i+c_n,-q_nK_i-c_n+1).$$

Then

$$\sum_{ABC} f(i+j) = T_-(a, q_nK_i+c_n) = T_-(a, K_{i-1}). \tag{27}$$

To prove Lemma 1 we make a partition of the $\triangle ABC$ onto three triangles and the square (see fig. 1). So

$$\sum_{ABC} f(i+j) = \sum_{AD'H} f(i+j) + \sum_{NF'C} f(i+j) + \sum_{DEFB} f(i+j) - \sum_{EGM} f(i+j). \tag{28}$$

Let

$$s = (q_n - q_{n-1})K_i + (c_n - c_{n-1} - 1). \tag{29}$$

By (6), (23) we have

$$s = (q_{n-1}+q_{n-2})K_i + (c_n - c_{n-1} - 1) = (q_{n-1}+q_{n-2})K_i + (c_{n-1}+c_{n-2}). \tag{30}$$

Put

$$D(a+q_{n-1}K_i+c_{n-1}+1,0); E(a+q_{n-1}K_i+c_{n-1}+1,-s); F(a+q_nK_i+c_n,-s).$$

Applying (17), we have

$$\sum_{DEFB} f(i+j) = \sum_{D_1E_1F_1B_1} f(i+j), \tag{31}$$

where

$$D_1(a+q_{n-1}K_i+c_{n-1}+1-s,s);\ E_1(a+q_{n-1}K_i+c_{n-1}+1-s,0);$$
$$F_1(a+q_nK_i+c_n-s,0);\ B_1(a+q_nK_i+c_n-s,s).$$

By (29), (30)

$$D_1(a-q_{n-2}K_i-c_{n-2}+1,s);\ E_1(a-q_{n-2}K_i-c_{n-2}+1,0);$$
$$F_1(a+q_{n-1}K_i+c_{n-1}+1,0);\ B_1(a+q_{n-1}K_i+c_{n-1}+1,s).$$

Applying (16), (31), we have

$$\sum_{DEFB} f(i+j) = S(a-q_{n-2}K_i-c_{n-2}, (q_{n-2}+q_{n-1})K_i+c_n-c_{n-1}). \tag{32}$$

Put

$$D'(a+q_{n-1}K_i+c_{n-1},0);\ H(a+q_{n-1}K_i+c_{n-1},-q_{n-1}K_i-c_{n-1}+1).$$

By (14), (17), we have

$$\sum_{AD'H} f(i+j) = T_-(a, q_{n-1}K_i+c_{n-1}). \tag{33}$$

Put

$$F'(a+q_nK_i+c_n,-s-1); N(a+s+2,-s-1).$$

Applying (17), we have

$$\sum_{NF'C} f(i+j) = \sum_{N_1F_1'C_1} f(i+j), \tag{34}$$

where

$$F_1'(a+q_nK_i+c_n-s-1,0);\ N_1(a+1,0);\ C_1(a+q_nK_i+c_n-s-1,-q_nK_i-c_n+s+2).$$

Note that $\triangle N_1F_1'C_1 = \triangle AD'H$. By (34), (33), we have

$$\sum_{NF'C} f(i+j) = T_-(a, q_{n-1}K_i+c_{n-1}). \tag{35}$$

Put

$$G(a+q_{n-1}K_i+c_{n-1}+1,-q_{n-1}K_i-c_{n-1}-1);\ M(a+s,-s).$$

Applying (17), we have

$$\sum_{EGM} f(i+j) = \sum_{E_1G_1M_1} f(i+j), \tag{36}$$

where

$$E_1(a+q_{n-1}K_i+c_{n-1}+1-s,0);\ G_1(a+q_{n-1}K_i+c_{n-1}+1-s,-q_{n-1}K_i-c_{n-1}-1+s);\ M_1(a,0).$$

By (15), (30), we have

$$\sum_{EGM} f(i+j) = T_+(a-q_{n-2}K_i-c_{n-2}, q_{n-2}K_i+c_{n-2}). \tag{37}$$

Applying (27), (32), (33), (35), (37) to (28) one has

$$\begin{aligned} T_-(a,K_{i-1}) = 2T_-(a, q_{n-1}K_i+c_{n-1}) - T_+(a-q_{n-2}K_i-c_{n-2}, q_{n-2}K_i+c_{n-2})+ \\ +S(a-q_{n-2}K_i-c_{n-2}, (q_{n-2}+q_{n-1})K_i+c_n-c_{n-1}). \end{aligned}$$

Lemma is proved. □

Lemma 2. *For any* $1 \leqslant i \leqslant \tau$ *one has*

$$\begin{aligned} T_+(a-K_{i-1},K_{i-1}) = 2T_+(a-q_{n-1}K_i-c_{n-1}, q_{n-1}K_i+c_{n-1}) - T_-(a, q_{n-2}K_i+c_{n-2})+ \\ +S(a-q_nK_i-c_n, (q_{n-2}+q_{n-1})K_i+c_n-c_{n-1}). \end{aligned}$$

Proof. This Lemma can be proved in the same way as Lemma 1. □

Let

$$S_+^{(n)}(a, K_i) = S(a - q_n K_i - c_n, (q_{n-2} + q_{n-1})K_i + c_n - c_{n-1}),$$
$$S_-^{(n)}(a, K_i) = S(a - q_{n-2} K_i - c_{n-2}, (q_{n-2} + q_{n-1})K_i + c_n - c_{n-1}). \quad (38)$$

Let

$$T_+^{(n)}(a, K_i) = T_+(a - q_n K_i - c_n, q_n K_i + c_n), \quad T_-^{(n)}(a, K_i) = T_-(a, q_n K_i + c_n).$$

Note that

$$T_-^{(n)}(a, K_i) = T_-(a, K_{i-1}), \qquad T_+^{(n)}(a, K_i) = T_+(a - K_{i-1}, K_{i-1}). \quad (39)$$

Consecutive application of Lemma 1 and Lemma 2 gives the following result.

Lemma 3. *For $n \geqslant 0$, $1 \leqslant i \leqslant \tau - 1$ we have*

$$T_-^{(n)}(a, K_i) = \alpha_n T_-(a, K_i) - \beta_n T_+(a, K_i) + \sum_{j=0}^{n} a_{j,n} S_-^{(j)}(a, K_i) - \sum_{j=0}^{n-2} b_{j,n} S_+^{(j)}(a, K_i),$$

$$T_+^{(n)}(a, K_i) = \alpha_n T_+(a, K_i) - \beta_n T_-(a, K_i) + \sum_{j=0}^{n} a_{j,n} S_+^{(j)}(a, K_i) - \sum_{j=0}^{n-2} b_{j,n} S_-^{(j)}(a, K_i),$$

where

$$\alpha_n = \frac{q_n + 1}{2}, \quad \beta_n = \frac{q_n - 1}{2}$$

and $\{a_{j,n}\}_{j=0}^{n}$, $\{b_{j,n}\}_{j=0}^{n-2}$ *are integers.*

Proof. We prove this statement by induction. For $n = 0$ the result follows from

$$T_-(a, q_0 K_i + c_0) = T_-(a, 3K_i + 1) = 2T_-(a, K_i) - T_+(a - K_i, K_i) + S(a - K_i, 2K_i + 1), \quad (40)$$

$$T_+(a - 3K_i - 1, 3K_i + 1) = 2T_+(a - K_i, K_i) - T_-(a, K_i) + S(a - 3K_i - 1, 2K_i + 1), \quad (41)$$

(see [2]). For $n = 1$ by Lemma 1, we have

$$T_-^{(1)}(a, K_i) = T_-(a, K_{i-1}) = 2T_-(a, q_0 K_i + c_0) - T_+(a - q_{-1} K_i - c_{-1}, q_{-1} K_i + c_{-1}) +$$
$$+ S(a - q_{-1} K_i - c_{-1}, (q_{-1} + q_0) K_i + c_1 - c_0).$$

Applying (40), we obtain

$$T_-^{(1)}(a, K_i) = 4T_-(a, K_i) - 3T_+(a - K_i, K_i) + S(a - K_i, 4K_i + 2) + 2S(a - K_i, 2K_i + 1). \quad (42)$$

In the same way we can prove that

$$T_+^{(1)}(a, K_i) = 4T_+(a - K_i, K_i) - 3T_-(a, K_i) + S(a - 7K_i - 3, 4K_i + 2) +$$
$$+ 2S(a - 3K_i - 1, 2K_i + 1). \quad (43)$$

If our formulas are proved for $k \leqslant n-1$ then by Lemma 1 and Lemma 2, we have

$$T_-^{(n)}(a,K_i) = 2T_-^{(n-1)}(a,K_i) - T_+^{(n-2)}(a,K_i) + S_-^{(n)}(a,K_i) =$$
$$= (2\alpha_{n-1}+\beta_{n-2})T_-^(a,K_i) - (\alpha_{n-2}+2\beta_{n-1})T_+(a,K_i) +$$
$$+\Bigg(S_-^{(n)}(a,K_i) + 2a_{n-1,n-1}S_-^{(n-1)}(a,K_i) + 2a_{n-2,n-1}S_-^{(n-2)}(a,K_i) + 2a_{n-3,n-1}S_-^{(n-3)}(a,K_i) +$$
$$+\sum_{j=0}^{n-4}(2a_{j,n-1}+b_{j,n-2})S_-^{(j)}(a,K_i)\Bigg) -$$
$$-\left(a_{n-2,n-2}S_+^{(n-2)}(a,K_i) + \sum_{j=0}^{n-3}(a_{j,n-2}+2b_{j,n-1})S_+^{(j)}(a,K_i)\right).$$

So

$$\begin{cases} \alpha_n = 2\alpha_{n-1}+\beta_{n-2}, \\ \beta_n = \alpha_{n-2}+2\beta_{n-1}. \end{cases}$$

By (40), (42), we have

$$\alpha_0 = 2,\ \beta_0 = 1,\ \alpha_1 = 4,\ \beta_1 = 3.$$

By the definition of $q_n(6)$, we have

$$\alpha_n = \frac{q_n+1}{2}, \quad \beta_n = \frac{q_n-1}{2}. \tag{44}$$

For sequences $\{a_{j,n}\}_{j=0}^{n}$, $\{b_{j,n}\}_{j=0}^{n-2}$ we have

$$\begin{cases} a_{n,n} = 1, & \\ a_{j,n} = 2a_{j,n-1} & \text{for} \quad n-3 \leqslant j \leqslant n-1, \\ a_{j,n} = 2a_{j,n-1}+b_{j,n-2} & \text{for} \quad 0 \leqslant j \leqslant n-4, \\ b_{n-2,n} = a_{n-2,n-2}, & \\ b_{j,n} = a_{j,n-2}+2b_{j,n-1} & \text{for} \quad 0 \leqslant j \leqslant n-3. \end{cases} \tag{45}$$

□

Lemma 4. *For $n \geqslant 0$ we have*

$$2p_n = \sum_{j=0}^{n} a_{j,n}(q_{j-1}+q_{j-2}) + \sum_{j=0}^{n-2} b_{j,n}(q_{j-1}+q_{j-2}).$$

Proof. We prove this statement by induction. For $n=0$ we have $2p_0 = 2$ and by (45), (6) one has

$$a_{0,0}(q_{-1}+q_{-2}) = 2a_{0,0} = 2.$$

It follows from (45) that

$$\sum_{j=0}^{n} a_{j,n}(q_{j-1}+q_{j-2}) + \sum_{j=0}^{n-2} b_{j,n}(q_{j-1}+q_{j-2}) = \sum_{j=0}^{n-1} 2a_{j,n-1}(q_{j-1}+q_{j-2}) + \sum_{j=0}^{n-3} 2b_{j,n-1}(q_{j-1}+q_{j-2}) +$$
$$+\sum_{j=0}^{n-2} a_{j,n-2}(q_{j-1}+q_{j-2}) + \sum_{j=0}^{n-4} b_{j,n-2}(q_{j-1}+q_{j-2}) + q_{n-1}+q_{n-2}. \tag{46}$$

If the statement is proved for $k \leqslant n-1$ then by (46), (9) we have

$$\sum_{j=0}^{n} a_{j,n}(q_{j-1}+q_{j-2}) + \sum_{j=0}^{n-2} b_{j,n}(q_{j-1}+q_{j-2}) = 4p_{n-1} + 2p_{n-2} + q_{n-1} + q_{n-2} = 2p_n.$$

This completes the proof. □

Lemma 5. *Let* $0 < y < q$ *be an integer and* $\lambda_i \in \{-1, 1\}$. *Let*

$$0 < x_1 < x_2 < \ldots < x_m$$

be a sequence of integers such that $x_i + y < x_{i+1}$ *for* $1 \leqslant i < m$, $x_m + y - x_1 \leqslant q$ *then*

$$\left|\sum_{i=1}^{m} \lambda_i S(x_i, y)\right| \leqslant y\sqrt{Bqm}.$$

Proof. Let $J_i = [x_i + 1, x_i + y]$ for $1 \leqslant i \leqslant m$ then $\bigcup J_i \subseteq [x_1 + 1, x_1 + q]$ and $J_i \bigcap J_j = \emptyset$ for any $1 \leqslant i, j \leqslant m$ Let

$$\lambda_i(x) = \begin{cases} \lambda_i, & \text{if } x \in J_i; \\ 0, & \text{else.} \end{cases}$$

By (16), the Cauchy-Schwarz inequality and the assumption $f \in \mathcal{F}_{A,B}(q)$ we have

$$\left|\sum_{i=1}^{m} \lambda_i S(x_i, y)\right| = \left|\sum_{n \in \bigcup J_i} \sum_{k=0}^{y-1} \lambda_i(n) f(n+k)\right| \leqslant \sqrt{\sum_{n \in \bigcup J_i} \lambda_i^2(n)} \sqrt{\sum_{n \in \bigcup J_i} \left|\sum_{k=0}^{y-1} f(n+k)\right|^2} \leqslant$$

$$\leqslant \sqrt{my} \sqrt{\sum_{n=1}^{q} \left|\sum_{k=0}^{y-1} f(n+k)\right|^2} \leqslant y\sqrt{Bqm}.$$

□

To prove Theorem 1 we must estimate $\frac{1}{K_0}|-T_-(a, K_0) + T_+(a+N, K_0)|$ (see (18)). Put

$$\Sigma_-^{(n)}(a, K_i) = \sum_{j=0}^{n} a_{jn} S_-^{(j)}(a, K_i) - \sum_{j=0}^{n-2} b_{jn} S_+^{(j)}(a, K_i)$$

$$\Sigma_+^{(n)}(a, K_i) = \sum_{j=0}^{n} a_{jn} S_+^{(j)}(a, K_i) - \sum_{j=0}^{n-2} b_{jn} S_-^{(j)}(a, K_i).$$

Applying Lemma 3 and formula (39), we have

$$T_-(a, K_0) = T_-^{(n)}(a, K_1) = \alpha_n T_-^{(n)}(a, K_2) - \beta_n T_+^{(n)}(a, K_2) + \Sigma_-^{(n)}(a, K_1) = T_1 + S_1$$

with

$$T_1 = \alpha_n T_-^{(n)}(a, K_2) - \beta_n T_+^{(n)}(a, K_2), \quad S_1 = \Sigma_-^{(n)}(a, K_1). \tag{47}$$

Consecutive application of Lemma 3 will give us an upper bound for $T_-(a, K_0)$. If

$$T_i = A_i T_-^{(n)}(a, K_{i+1}) - B_i T_+^{(n)}(a, K_{i+1}), \quad 1 \leqslant i \leqslant \tau - 1,$$

then for $1 \leqslant i \leqslant \tau - 2$ by Lemma 3 one has

$$T_i = A_i \left(\alpha_n T_-^{(n)}(a, K_{i+2}) - \beta_n T_+^{(n)}(a, K_{i+2}) + \Sigma_-^{(n)}(a, K_{i+1}) \right) -$$
$$-B_i \left(\alpha_n T_+^{(n)}(a, K_{i+2}) - \beta_n T_-^{(n)}(a, K_{i+2}) + \Sigma_+^{(n)}(a, K_{i+1}) \right) = (\alpha_n A_i + \beta_n B_i) T_-^{(n)}(a, K_{i+2}) -$$
$$- (\alpha_n B_i + \beta_n A_i) T_+^{(n)}(a, K_{i+2}) + A_i \Sigma_-^{(n)}(a, K_{i+1}) - B_i \Sigma_+^{(n)}(a, K_{i+1}) = T_{i+1} + S_{i+1}.$$

with

$$S_{i+1} = A_i \Sigma_-^{(n)}(a, K_{i+1}) - B_i \Sigma_+^{(n)}(a, K_{i+1}).$$

So

$$\begin{cases} A_{i+1} = \alpha_n A_i + \beta_n B_i, \\ B_{i+1} = \alpha_n B_i + \beta_n A_i, \\ A_1 = \alpha_n, \ B_1 = \beta_n. \end{cases}$$

By (44) we have

$$A_i = \frac{q_n^i + 1}{2}, \quad B_i = \frac{q_n^i - 1}{2}. \tag{48}$$

Let

$$r = \left\lceil \frac{\tau}{2} \right\rceil \tag{49}$$

be the number of steps. As $\tau \geqslant 3$ then $r \leqslant \tau - 1$. So

$$T_-(a, K_0) = T_r + \Sigma_-^{(n)}(a, K_1) + \sum_{i=2}^{r} S_i =$$
$$= A_r T_-^{(n)}(a, K_{r+1}) - B_r T_+^{(n)}(a, K_{r+1}) + \Sigma_-^{(n)}(a, K_1) + \sum_{i=2}^{r} \left(A_{i-1} \Sigma_-^{(n)}(a, K_i) - B_{i-1} \Sigma_+^{(n)}(a, K_i) \right)$$

and

$$T_-(a + N, K_0) = A_r T_-^{(n)}(a + N, K_{r+1}) - B_r T_+^{(n)}(a + N, K_{r+1}) + \Sigma_-^{(n)}(a + N, K_1) +$$
$$+ \sum_{i=2}^{r} \left(A_{i-1} \Sigma_-^{(n)}(a + N, K_i) - B_{i-1} \Sigma_+^{(n)}(a + N, K_i) \right).$$

So

$$|T_-(a, K_0) - T_-(a + N, K_0)| \leqslant \Sigma_1 + \Sigma_2 + \sum_{i=2}^{r} \Sigma_3(i) \tag{50}$$

with

$$\Sigma_1 = \left|A_r T_-^{(n)}(a, K_{r+1}) - B_r T_+^{(n)}(a, K_{r+1}) - A_r T_-^{(n)}(a+N, K_{r+1}) + B_r T_+^{(n)}(a+N, K_{r+1})\right|,$$
$$\Sigma_2 = \left|\Sigma_-^{(n)}(a, K_1) - \Sigma_-^{(n)}(a+N, K_1)\right|,$$
$$\Sigma_3(i) = \left|A_{i-1}\Sigma_-^{(n)}(a, K_i) - B_{i-1}\Sigma_+^{(n)}(a, K_i) - A_{i-1}\Sigma_-^{(n)}(a+N, K_i) + B_{i-1}\Sigma_+^{(n)}(a+N, K_i)\right|.$$

Trivially we obtain by (39) and the definition of T_+, T_- (see (15), (14)) that

$$\Sigma_1 \leqslant 2A\frac{K_r(K_r+1)}{2}(A_r+B_r) = AK_r(K_r+1)(A_r+B_r). \tag{51}$$

By (38), (22), Lemma 5, Lemma 4 we have

$$\begin{aligned}\Sigma_2 &\leqslant \sum_{j=0}^{n} a_{j,n}\left|S_-^{(j)}(a, K_1) - S_-^{(j)}(a+N, K_1)\right| + \sum_{j=0}^{n-2} b_{j,n}\left|S_+^{(j)}(a, K_1) - S_+^{(j)}(a+N, K_1)\right| \leqslant \\ &\leqslant \sqrt{2Bq}\left(K_1+\frac{1}{2}\right)\left(\sum_{j=0}^{n} a_{j,n}(q_{j-1}+q_{j-2}) + \sum_{j=0}^{n-2} b_{j,n}(q_{j-1}+q_{j-2})\right) = \\ &= \sqrt{2Bq}\,(2K_1+1)\,p_n.\end{aligned} \tag{52}$$

Applying (48) we have $A_i = B_i + 1$ so

$$\begin{aligned}\Sigma_3(i) \leqslant B_{i-1}&\left|\Sigma_-^{(n)}(a, K_i) - \Sigma_+^{(n)}(a, K_i) - \Sigma_-^{(n)}(a+N, K_i) + \Sigma_+^{(n)}(a+N, K_i)\right| + \\ &+ \left|\Sigma_-^{(n)}(a, K_i) - \Sigma_-^{(n)}(a+N, K_i)\right|.\end{aligned}$$

By (38), (22), Lemma 5, Lemma 4 we have

$$\Sigma_3(i) \leqslant \sqrt{4Bq}B_{i-1}(2K_i+1)\,p_n + \sqrt{2Bq}\,(2K_i+1)\,p_n. \tag{53}$$

Using (50), (51), (52), (53) we have

$$\begin{aligned}&|T_-(a, K_0) - T_-(a+N, K_0)| \leqslant AK_r(K_r+1)(A_r+B_r) + \\ &+\sqrt{2Bq}p_n\sum_{i=1}^{r}(2K_i+1) + 2\sqrt{Bq}p_n\sum_{i=2}^{r}B_{i-1}(2K_i+1).\end{aligned} \tag{54}$$

By (20), (48) we have

$$\begin{aligned}K_r(K_r+1)(A_r+B_r) &= q_n^r\frac{q_n^{\tau-r}-1}{2}\frac{q_n^{\tau-r}+1}{2} = \left(K_0 - \frac{q_n^r-1}{2}\right)\frac{q_n^{\tau-r}+1}{2} = \\ &= K_0\frac{q_n^{\tau-r}+1}{2} - \frac{1}{2}K_0 - \frac{q_n^r - q_n^{\tau-r}}{4} = K_0\frac{q_n^{\tau-r}}{2} - \frac{q_n^r - q_n^{\tau-r}}{4}.\end{aligned}$$

And

$$B_{i-1}(2K_i+1) = \frac{q_n^{i-1}-1}{2}q_n^{\tau-i} = \frac{q_n^{\tau}-1-q_n^{\tau-i+1}+1}{2q_n} = \frac{2K_0+1}{2q_n} - \frac{q_n^{\tau}}{2q_n^{i}}.$$

So we obtain

$$|T_-(a,K_0)-T_-(a+N,K_0)| \leqslant A\left(K_0\frac{q_n^{\tau-r}}{2}-\frac{q_n^r-q_n^{\tau-r}}{4}\right)+2\sqrt{Bq}p_n\frac{2K_0+1}{2q_n}(r-1)+$$
$$+\sqrt{Bq}p_nq_n^\tau\left(\sqrt{2}\sum_{i=1}^r\frac{1}{q_n^i}-\sum_{i=2}^r\frac{1}{q_n^i}\right).$$

Applying (20) we have

$$\sqrt{Bq}p_nq_n^\tau\left(\sqrt{2}\sum_{i=1}^r\frac{1}{q_n^i}-\sum_{i=2}^r\frac{1}{q_n^i}\right)\leqslant\sqrt{Bq}\,(2K_0+1)\left(\sqrt{2}-1\right)\frac{p_n}{q_n-1}+\sqrt{Bq}\,(2K_0+1)\,\frac{p_n}{q_n}.$$

So by (49) we get

$$|T_-(a,K_0)-T_-(a+N,K_0)| \leqslant AK_0\frac{q_n^{\tau-r}}{2}+\sqrt{Bq}\,(2K_0+1)\left(\frac{p_n}{q_n}r+\left(\sqrt{2}-1\right)\frac{p_n}{q_n-1}\right).$$

Using (18) we have

$$\left|\sum_{n=a+1}^{a+N}f(n)\right|\leqslant\sqrt{Bq}\sqrt{\frac{N}{K_0}}+A\frac{q_n^{\tau-r}}{2}+\sqrt{Bq}\left(2+\frac{1}{K_0}\right)\left(\frac{p_n}{q_n}r+\left(\sqrt{2}-1\right)\frac{p_n}{q_n-1}\right). \quad (55)$$

By (49), (19), (12) we get

$$q_n^{\tau-r}=q_n^{\tau-\lceil\frac{\tau}{2}\rceil}\leqslant\sqrt{q_n^\tau}\leqslant\sqrt{q}.$$

So by (26) we obtain

$$\left|\sum_{n=a+1}^{a+N}f(n)\right|\leqslant\sqrt{Bq}\sqrt{q_n+\frac{1}{q_n^2}}+A\frac{\sqrt{q}}{2}+\sqrt{Bq}\left(2+\frac{q_n+\frac{1}{q_n^2}}{N}\right)\left(\frac{p_n}{q_n}r+\left(\sqrt{2}-1\right)\frac{p_n}{q_n-1}\right). \quad (56)$$

By the definition of τ, r we have

$$r\leqslant\frac{\log q}{2\log q_n}.$$

So

$$\left|\sum_{n=a+1}^{a+N}f(n)\right|\leqslant\sqrt{B}\delta_n\sqrt{q}\log q+\sqrt{Bq}\sqrt{q_n+\frac{1}{q_n^2}}+A\frac{\sqrt{q}}{2}+2\sqrt{Bq}\left(\sqrt{2}-1\right)\frac{p_n}{q_n-1}+$$
$$+\sqrt{Bq}\frac{1}{N}\left(q_n+\frac{1}{q_n^2}\right)\left(\frac{p_n}{q_n}\frac{\log q}{2\log q_n}+\left(\sqrt{2}-1\right)\frac{p_n}{q_n-1}\right). \quad (57)$$

Applying (13), we have

$$\left|\sum_{m=a+1}^{a+N}f(m)\right|\leqslant\sqrt{B}\delta_n\sqrt{q}\log q+\left(\sqrt{B}\sqrt{q_n+\frac{1}{q_n^2}}+\sqrt{B}(\sqrt{2}-1)\frac{2p_n}{q_n-1}+\frac{1}{2}A\right)\sqrt{q}+$$
$$+,2\sqrt{B}\left(q_n+\frac{1}{q_n^2}\right)\left(\frac{p_n}{q_n}\frac{\log q}{2\log q_n}+\left(\sqrt{2}-1\right)\frac{p_n}{q_n-1}\right).$$

Theorem 1 is proved.

3 On the constant in the Pólya–Vinogradov inequality

Burgess [3] proved that for a nonprincipal character $\chi \pmod{q}$ one has $\chi \in F_{1,1}(q)$. Applying this result to (4) and (5), we have

$$\left|\sum_{n=a+1}^{a+N} \chi(n)\right| \leqslant \frac{1}{2\log 2}\sqrt{q}\log q + 3\sqrt{q}, \tag{58}$$

$$\left|\sum_{n=a+1}^{a+N} \chi(n)\right| \leqslant \frac{1}{3\log 3}\sqrt{q}\log q + 6.5\sqrt{q}. \tag{59}$$

But this result is not the best one. Let

$$S_\chi = \max_{0\leqslant M<N\leqslant q}\left|\sum_{n=M}^{N} \chi(n)\right|, \qquad T_\chi = \max_{N}\left|\sum_{a=0}^{N} \chi(a)\right|.$$

Granville and Soundararajan [4] obtained two inequalities

$$T_\chi \leqslant \left(\frac{69}{70}\frac{c}{\pi\sqrt{3}} + o(1)\right)\sqrt{q}\log q \quad \text{if} \quad \chi(-1) = 1, \tag{60}$$

and

$$T_\chi \leqslant \left(\frac{c}{\pi} + o(1)\right)\sqrt{q}\log q \quad \text{if} \quad \chi(-1) = -1, \tag{61}$$

where

$$c = \begin{cases} \frac{1}{4}, & \text{if q is a cubefree;} \\ \frac{1}{3}, & \text{else.} \end{cases}$$

Up to now this result is the best-known one. Pomerance proved (see [5]) numerically explicit version of the Pólya–Vinogradov inequality

$$S_\chi \leqslant \frac{2}{\pi^2}\sqrt{q}\log q + \frac{4}{\pi^2}\sqrt{q}\log\log q + \frac{3}{2}\sqrt{q} \quad \text{if} \quad \chi(-1) = 1$$

and

$$S_\chi \leqslant \frac{1}{2\pi}\sqrt{q}\log q + \frac{1}{\pi}\sqrt{q}\log\log q + \sqrt{q} \quad \text{if} \quad \chi(-1) = -1.$$

Up to now these bounds are the best-known numerically explicit versions of the Pólya–Vinogradov inequality. These inequalities are weaker then (60), (61) but better then (59). Applying Theorem 1 we improve (59).

Corollary 1. *For any* $n \geqslant 0$, $q \geqslant q_n^6$ *and any nonprincipal character* $\chi \pmod{q}$, *we have*

$$\left|\sum_{m=a+1}^{a+N} \chi(m)\right| \leqslant \delta_n\sqrt{q}\log q + \left(\sqrt{q_n + \frac{1}{q_n^2}} + (\sqrt{2}-1)\frac{2p_n}{q_n-1} + \frac{1}{2}\right)\sqrt{q} + \psi_n(q, 1),$$

References

[1] Dobrowolski E. and Williams K. S. An upper bound for the sum $\sum_{n=a+1}^{a+H} f(n)$ for a certain class of functions f, Proc. Amer. Math. Soc. 114(1992), 29-35

[2] Bachman G. and Rachakonda L. On a problem of Dobrowolski and Williams and the Pólya–Vinogradov inequality, Ramanujan J. 5 (2001), 65-71.

[3] Burgess D. A. On a conjecture of Norton, Acta. Arith. 27 (1975) 265-267.

[4] Granville A. and Soundararajan K. Large character sums: pretentious characters and the Pólya–Vinogradov theorem, Jour. AMS Vol. 20, Number 2(2007), 357-384.

[5] Pomerance C. Remarks on the Pólya–Vinogradov inequality. Integers (Proceedings of the Integers Conference, October 2009), 11A (2011), Article 19, 11pp.

D.A. Frolenkov
Department of Number theory
Moscow State University
e-mail: frolenkov_adv@mail.ru