

ON THE EFFECTIVE MEMBERSHIP PROBLEM ON SINGULAR VARIETIES

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ABSTRACT. We present some extensions to singular varieties of effective membership results previously known in \mathbb{C}^n or smooth varieties. In particular, we extend a geometric effective Nullstellensatz due to Ein and Lazarsfeld, and a closely related result of Hickel for polynomial ideals on \mathbb{C}^n , to singular varieties. We also have extensions to singular varieties of classical theorems of Max Noether and Macaulay.

1. INTRODUCTION

Let V be a reduced n -dimensional subvariety of \mathbb{C}^N . If F_1, \dots, F_m are polynomials in \mathbb{C}^N with no common zeros on V , then by the Nullstellensatz there are polynomials Q_j such that $\sum F_j Q_j = 1$ on V . It was proved by Jelonek, [23], that if F_j have degree at most d , then one can find Q_j such that

$$\deg(F_j Q_j) \leq c_m d^\mu \deg V,$$

where $c_m = 1$ if $m \leq n$, $c_m = 2$ if $m > n$, and, throughout this paper,

$$\mu := \min(m, n).$$

Here $\deg X$ means the degree of the closure X of V in \mathbb{P}^N . This theorem generalizes Kollár's result¹, [24], for $V = \mathbb{C}^n$ and does not require any smoothness assumptions on V . The bound is optimal² when $m \leq n$ and almost optimal when $m > n$. However, in view of various known results in the case when $V = \mathbb{C}^n$, one can expect sharper degree estimates if the common zero set of the polynomials F_j behaves nicely at infinity in \mathbb{P}^N .

More generally one can take arbitrary polynomials F_j of degree at most d and look for a solution Q_j to

$$(1.1) \quad F_1 Q_1 + \dots + F_m Q_m = \Phi$$

with good degree estimates, provided that the polynomial Φ belongs to the ideal (F_j) generated by F_j on V . It follows from a result of Hermann, [20], that one can choose Q_j such that $\deg(F_j Q_j) \leq \deg \Phi + C(d, N)$, where $C(d, N)$ is like $2(2d)^{2^N - 1}$ for large d , thus doubly exponential. It is shown in [28] that this estimate cannot be substantially improved. However, under additional hypotheses on the singularities of the common zero set of the F_j and V , much sharper estimates are possible. For

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¹In Kollár's theorem $c_m = 1$ even for $m > n$ and this estimate is optimal.

²In Kollár's and Jelonek's theorems, as well as in [21], there are more precise results that take into account different degree bounds d_j of F_j , but for simplicity, in this paper we always keep all $d_j = d$.

instance, the classical $AF + BG$ theorem of Max Noether reads (originally for $n = 2$): *If F_1, \dots, F_n are polynomials in \mathbb{C}^n such that their common zero set is discrete and does not intersect the hyperplane at infinity, and Φ belongs to the ideal (F_j) , then one can find polynomials Q_j such (1.1) holds and $\deg F_j Q_j \leq \deg \Phi$.* This result is clearly optimal.

By homogenization, this kind of effective results can be reformulated as geometrical statements: Let $z = (z_0, \dots, z_N)$, $z' = (z_1, \dots, z_N)$, let $f_i(z) := z_0^d F_i(z'/z_0)$ be the d -homogenizations of F_i , and let $\varphi(z) := z_0^{\deg \Phi} \Phi(z'/z_0)$. Then there is a representation (1.1) on V with $\deg(F_j Q_j) \leq \rho$ if and only if there are $(\rho - d)$ -homogeneous forms q_i on \mathbb{P}^N such that

$$(1.2) \quad f_1 q_1 + \dots + f_m q_m = z_0^{\rho - \deg \Phi} \varphi$$

on the closure X of V in \mathbb{P}^N . As usual, we can consider f_j as holomorphic sections of the (restriction to X of the) line bundle $\mathcal{O}(d) \rightarrow \mathbb{P}^N$, $z_0^{\rho - \deg \Phi} \varphi$ a section of $\mathcal{O}(\rho)$, etc, so that (1.2) is a statement about sections of line bundles.

In this paper we present generalizations to singular varieties of some effective membership results that are previously known for $V = \mathbb{C}^n$ or smooth varieties. In particular we present some generalizations in Section 2 of results of Max Noether type, but our main results are global effective versions of the Briançon-Skoda-Huneke theorem:

Given $x \in V$ there is a number μ_0 such that if F_1, \dots, F_m, Φ are any holomorphic functions at x , $\ell \geq 1$, and $|\Phi| \leq C|F|^{\mu + \mu_0 + \ell - 1}$ in a neighborhood of x , where $|F|^2 = |F_1|^2 + \dots + |F_m|^2$ and C is a positive constant, then Φ belongs to the local ideal $(F_j)_x^\ell$ at x .

If x is a smooth point, then one can take $\mu_0 = 0$; this is the classical Briançon-Skoda theorem, [13]. The general case was proved by Huneke, [22], by purely algebraic methods. An analytic proof appeared in [7].

Our global formulation involves the following two numbers: Given $V \subset \mathbb{C}^N$, $\kappa = \kappa(V)$ is the least number such that all holomorphic sections of $\mathcal{O}(s)$ over X extend to global holomorphic sections over \mathbb{P}^N for $s \geq \kappa$, and $\nu = \nu(V)$ is the least number such that $H^{0,i}(X, \mathcal{O}(s)) = 0$ for $i \geq 1$ and $s \geq \nu$. See Section 6 for upper bounds of κ and ν ; in particular they are bounded by the so-called *regularity* $\text{reg } X$ of X . When X is smooth, by Kodaira's vanishing theorem, ν is less than or equal to the least number σ such that $\mathcal{O}(\sigma)|_X \otimes K_X^{-1}$ is strictly positive, where K_X^{-1} is the dual of the canonical bundle. In particular, if $V = \mathbb{C}^n$, i.e., $X = \mathbb{P}^n$ then $\nu = -n$.

Given polynomials F_1, \dots, F_m on V , let f_j denote the corresponding sections of $\mathcal{O}(d)|_X$, and let \mathcal{J}_f be the coherent analytic sheaf over X generated by f_j . Furthermore, let c_∞ be the maximal codimension of the so-called *distinguished varieties* of the sheaf \mathcal{J}_f , in the sense of Fulton-MacPherson, that are contained in

$$X_\infty := X \setminus V,$$

see Section 8. If there are no distinguished varieties contained in X_∞ , then we interpret c_∞ as $-\infty$. It is well-known that the codimension of a distinguished variety cannot exceed the number m , see, e.g., Proposition 2.6 in [15], and thus

$$(1.3) \quad c_\infty \leq \mu.$$

We let Z^f denote the zero variety of \mathcal{J}_f in X .

Theorem 1.1. *Assume that V is a reduced n -dimensional algebraic subvariety of \mathbb{C}^N .*

(i) *There exists a number μ_0 such that if F_1, \dots, F_m are polynomials of degree $\leq d$ and Φ is a polynomial such that*

$$(1.4) \quad |\Phi| \leq C|F|^{\mu+\mu_0}$$

locally on V , then one can solve (1.1) on V with

$$\deg(F_j Q_j) \leq \max(\deg \Phi + (\mu + \mu_0)d^{c_\infty} \deg X, d \min(m, n+1) + \nu, d + \kappa).$$

If X is smooth one can take $\mu_0 = 0$.

(ii) *If just V is smooth, then there is a number μ' such that if F_1, \dots, F_m are polynomials of degree $\leq d$ and Φ is a polynomial such that*

$$(1.5) \quad |\Phi| \leq C|F|^\mu$$

locally on V , then one can solve (1.1) on V with

$$\deg(F_j Q_j) \leq \max(\deg \Phi + \mu d^{c_\infty} \deg X + \mu', d \min(m, n+1) + \nu, d + \kappa).$$

There is an analogous result for powers $(F_j)^\ell$ of (F_j) , see Theorem 9.4. The number μ' in (ii) is related to the singularities of V at infinity, in particular one can take $\mu' = 0$ if X is smooth. If V is arbitrary and $Z^f \cap X_{\text{sing}} = \emptyset$, then (i) holds with $\mu_0 = 0$; for a slightly stronger statement, see Remark 9.2. Notice that if there are no distinguished varieties contained in X_∞ , then $d^{c_\infty} = 0$.

Remark 1.2. For $V = \mathbb{C}^n$ we thus get from Theorem 1.1 the estimate

$$(1.6) \quad \deg(F_j Q_j) \leq \max(\deg \Phi + \mu d^{c_\infty}, d \min(m, n+1) - n).$$

If $m \leq n$ it follows from the proof that we have in fact the sharper estimate

$$(1.7) \quad \deg(F_j Q_j) \leq \deg \Phi + m d^{c_\infty}.$$

The estimate (1.6) was proved by Hickel, [21], but with the term $\min(m, n+1)d^\mu$ rather than our μd^{c_∞} . The ideas in [21] are similar to the ones used in [15]. If one applies the geometric estimate in [15], rather than the (closely related) so-called refined Bezout estimate by Fulton-MacPherson that is used in [21], one can replace the power μ by c_∞ . This refinement was pointed out already in Example 1 in [15]. The number $(n+1)$ in the factor $\min(m, n+1)$ comes from an application of a global Briançon-Skoda type theorem. In our approach we have to annihilate a certain current, which is a purely local matter, and therefore it is enough with the local Briançon-Skoda power μ . \square

Remark 1.3. If we apply Theorem 1.1 to Nullstellensatz data, i.e., F_j with no common zero on V and $\Phi = 1$, we get back the optimal result of Jelonek, except for the annoying factor $\mu + \mu_0$ in front of d^{c_∞} . On the other hand, $(\mu + \mu_0)d^{c_\infty} < d^\mu$ if $c_\infty < \mu$ and d is large enough. \square

We have the following more abstract variant of Theorem 1.1. It is a generalization to nonsmooth varieties of the geometric effective Nullstellensatz of Ein-Lazarsfeld in [15] (Theorem 10.1 below). Let X be a reduced projective variety. Recall that if $L \rightarrow X$ is ample, then there is a (smallest) number ν_L such that $H^i(X, L^{\otimes s}) = 0$ for $i \geq 1$ and $s \geq \nu_L$, cf., [26, Ch. 1.2].

Theorem 1.4. *Let X be a reduced projective variety. There is a number μ_0 , only depending on X , such that the following holds: Let f_1, \dots, f_m be global holomorphic sections of an ample Hermitian line bundle $L \rightarrow X$, and let ϕ be a section of $L^{\otimes s}$, where $s \geq \nu_L + \min(m, n + 1)$. If*

$$(1.8) \quad |\phi| \leq C|f|^{\mu+\mu_0},$$

then there are holomorphic sections q_j of $L^{\otimes(s-1)}$ such that

$$(1.9) \quad f_1 q_1 + \dots + f_m q_m = \phi.$$

If X is smooth we can choose $\mu_0 = 0$, and we then get back (a slightly sharpened version of) one of the main results in [15], see Section 10 for the precise formulation.

Let \mathcal{J}_f be the ideal sheaf generated by f_j and assume that the associated distinguished varieties Z_k have multiplicities r_k , cf., Section 8. If we assume that ϕ is in $\bigcap_k \mathcal{J}(Z_k)^{r_k(\mu+\mu_0)}$, where $\mathcal{J}(Z_k)$ is the radical ideal associated with the distinguished variety Z_k , then (1.8) holds, and hence we have a representation (1.9).

The starting point for the proofs is the framework introduced in [2], and further developed in [5], for polynomial ideals in \mathbb{C}^n . In [31] and [32] these ideas are adapted to toric compactifications of \mathbb{C}^n other than \mathbb{P}^n , which leads to “sparse” effective membership results. In our case, let us first assume that X is a smooth projective variety and that f_1, \dots, f_m are sections of an ample line bundle $L \rightarrow X$. From the Koszul complex generated by f_j we define a current R^f with support on Z^f and taking values in (a direct sum of) negative powers of L . If ϕ is a section of $L^{\otimes s}$, vanishing enough on Z^f so that the current $R^f \phi$ vanishes, and if in addition $L^{\otimes s}$ is positive enough so that a certain sequence of $\bar{\partial}$ -equations can be solved on X , we end up with a holomorphic solution $q = (q_1, \dots, q_m)$ to (1.9). The tricky point here is to verify, by means of multivariable residue calculus, that ϕ annihilates R^f , i.e., $R^f \phi = 0$.

The main novelty in this paper is an extension of this framework to singular X . In this case we need an embedding $i: X \rightarrow Y$ of X into a smooth manifold Y . In Y we define a residue current $R^{\mathcal{J}^X}$ obtained from a free resolution of the ideal sheaf \mathcal{J}_X associated to X , and the “product” current $R^f \wedge R^{\mathcal{J}^X}$. If all data, i.e., L , f_j , and ϕ , admit extensions to Y , this is the case in Theorem 1.1, then we can proceed basically as before, showing that ϕ annihilates $R^f \wedge R^{\mathcal{J}^X}$, solving $\bar{\partial}$ -equations in Y and ending up with a holomorphic solution to (1.9). Via an additional trick it is in fact enough to solve $\bar{\partial}$ -equations on X , so that we can avoid relying on vanishing theorems on Y . This is described in Section 5. In the proof of Theorem 1.4 additional difficulties arise, since we have no a priori extensions to a smooth manifold Y . In the next section we present some generalizations of the Max Noether theorem. The proofs of all these theorems are gathered in Sections 7 to 10 together with some further results and comments.

2. RESULTS OF MAX NOETHER AND MACAULAY TYPE

Again let V be an algebraic subvariety of \mathbb{C}^N of pure dimension n . In the results discussed so far we have assumed that Φ is in the ideal (F_j) by virtue of a Briançon-Skoda-Huneke condition. In this section we consider results where the polynomial Φ a priori is just assumed to belong to the ideal (F_j) on V . To get good degree estimates

one then must impose geometrical restrictions on Z^f , as well as on X when it is non-smooth. Classical examples of such results are Max Noether's theorem, [29], already mentioned in Section 1 and Macaulay's theorem, [27]:

If the polynomials F_1, \dots, F_m of degree $\leq d$ in \mathbb{C}^n have no common zeros in \mathbb{P}^n , then there are polynomials Q_j such that $F_1Q_1 + \dots + F_mQ_m = 1$ and $\deg F_jQ_j \leq d(n+1) - n$.

As before, let X be the closure of V in \mathbb{P}^N . For the formulation of our first result of this type we have to introduce certain intrinsic subvarieties

$$X^{n-1} \subset \dots \subset X^1 \subset X^0 \subset X,$$

of X that reflect the complexity of the singularities of X ; for the definition, see Section 3.6. In fact, $X^0 = X_{\text{sing}}$ and X^ℓ are BEF sets, see below, of the sheaf \mathcal{J}_X in \mathbb{P}^N . In particular, $\text{codim } X^\ell \geq \ell + 1$, $\ell \geq 0$. Our result can be seen as a generalization of Max Noether's (and Macaulay's) theorem. As in that theorem the dimension of Z^f is assumed to be as small as possible in V , i.e., $\text{codim } Z^f \geq m$, whereas at infinity we only require Z^f not to intersect X_{sing} too "much".

Theorem 2.1. *Assume that V is an algebraic subvariety of \mathbb{C}^N and let X be its closure in \mathbb{P}^N . Let F_1, \dots, F_m be given polynomials of degree $\leq d$ such that*

$$(2.1) \quad \text{codim } Z^f \cap V \geq m.$$

Furthermore, assume that

$$(2.2) \quad \text{codim } (Z^f \cap X^\ell) \geq m + \ell + 1, \quad \ell \geq 0.$$

If $\Phi \in (F_j)$ on V , then there is a representation (1.1) on V with

$$\deg(F_jQ_j) \leq \max(\deg \Phi + \mu d^{c_\infty} \deg X, d \min(m, n+1) + \nu, d + \kappa).$$

Notice that (2.2) forces that either $Z^f \cap X_{\text{sing}} = \emptyset$ or $m < n$. Recall that $c_\infty \leq m$. If X is smooth, then (2.2) is vacuous. If Z^f has no irreducible component in X_∞ at all, then $d^{c_\infty} = 0$.

Corollary 2.2. *Assume that X is smooth, $m \leq n$, $\text{codim } Z^f \geq m$, and Z^f has no irreducible component contained in X_∞ . If Φ is a polynomial that belongs to the ideal (F_j) in V , then there is a representation (1.1) with*

$$(2.3) \quad \deg(F_jQ_j) \leq \max(\deg \Phi, dm + \nu, d + \kappa).$$

If $V = \mathbb{C}^n$, and $m \leq n$, one can replace $dm + \nu$ by zero since then there are no \bar{d} -cohomology obstructions, and $\kappa = -\infty$, so the right hand side of (2.3) is just $\deg \Phi$. This case appeared already in [2]. If $m = n$, and thus Z^f is discrete, we get back the classical theorem of Max Noether.

Corollary 2.3. *If Z^f is empty, then there are polynomials Q_j such that $\sum F_jQ_j = 1$ on V and*

$$\deg(F_jQ_j) \leq \max(d(n+1) + \nu, d + \kappa).$$

Recall that if $V = \mathbb{C}^n$, then $\nu = -n$, cf., the introduction, and $\kappa = -\infty$ so we get back Macaulay's theorem.

The principal condition in the previous theorem was assumption (2.1). We have a similar result even when this condition is not fulfilled. For simplicity we restrict to the case when X is smooth.

Assume that the variety $Z^f \subset X$ associated to \mathcal{J}_f has codimension p . There are intrinsic analytic varieties

$$\cdots Z_{k+1}^{\text{bef}} \subset Z_k^{\text{bef}} \subset \cdots \subset Z_1^{\text{bef}} \subset Z_0^{\text{bef}} = Z^f,$$

that we call the *BEF varieties*³, that reflect the complexity of \mathcal{J}_f . The codimension of Z_k^{bef} is at least k , and if the local ideal $\mathcal{J}_{f,x}$ at x has codimension p , then $Z_{p,x}^{\text{bef}} = Z_x^{\text{bef}}$. Furthermore, $\mathcal{J}_{f,x}$ is Cohen-Macaulay if and only if $Z_{k,x}^{\text{bef}} = \emptyset$ for $k > p$, see [16].

Theorem 2.4. *Assume that V is an algebraic subvariety of \mathbb{C}^N such that its closure X is smooth in \mathbb{P}^N . Let F_1, \dots, F_m be polynomials on V , let Z_k^{bef} be the BEF varieties associated with \mathcal{J}_f , and assume that*

$$(2.4) \quad \text{codim}(X_\infty \cap Z_k^{\text{bef}}) \geq k + 1, \quad k \geq 1.$$

Then there is a constant $\beta = \beta(X, F_1, \dots, F_m)$ such that if $\Phi \in (F_j)$, then there is a representation (1.1) on V with

$$\deg(F_j Q_j) \leq \max(\deg \Phi, \beta).$$

This estimate is clearly sharp if $\deg \Phi \geq \beta$. If the ideal sheaf \mathcal{J}_f is locally Cohen-Macaulay, for instance locally a complete intersection, then (2.4) just means that no irreducible component of Z^f is contained in X_∞ . In special cases we have good control of β , see Section 7 below. In particular, we get back Corollary 2.2 and a smooth version of Corollary 2.3.

3. SOME PRELIMINARIES ON RESIDUE THEORY

Let X be a reduced projective variety of pure dimension n . The sheaf $\mathcal{C}_{\ell,k}$ of currents of bidegree (ℓ, k) on X is by definition the dual of the sheaf $\mathcal{E}_{n-\ell, n-k}$ of smooth $(n-\ell, n-k)$ -forms on X . If $i: X \rightarrow Y$ is an embedding in a smooth manifold Y of dimension N , then $\mathcal{E}_{n-\ell, n-k}$ can be identified with the quotient sheaf $\mathcal{E}_{n-\ell, n-k}^Y / \text{Ker } i^*$, where $\text{Ker } i^*$ is the sheaf of forms ξ on Y such that $i^* \xi$ vanish on X_{reg} . It follows that the currents τ in $\mathcal{C}_{\ell,k}$ can be identified with currents $\tau' = i_* \tau$ on Y of bidegree $(N-n+\ell, N-n+k)$ that vanish on $\text{Ker } i^*$.

Given a holomorphic function f on X , we have the principal value current $[1/f]$, defined for instance as the limit

$$\lim_{\epsilon \rightarrow 0} \chi(|f|^2/\epsilon) \frac{1}{f},$$

where $\chi(t)$ is the characteristic function of the interval $[1, \infty)$ or a smooth approximand of it. The existence of this limit for a general f relies on Hironaka's theorem that ensures that there is a modification $\pi: \tilde{X} \rightarrow X$ such that $\pi^* f$ is locally a monomial. It also follows that the function $\lambda \rightarrow |f|^{2\lambda}(1/f)$, a priori defined for $\text{Re } \lambda \gg 0$, has a current-valued analytic continuation to $\text{Re } \lambda > -\epsilon$, and that the value at $\lambda = 0$ is precisely the current $[1/f]$, see, for instance, [11] or [12]. Although less natural at first sight, this latter definition via analytic continuation turns out to be much more

³ The sets Z_k^{bef} are the zero varieties of certain Fitting ideals associated with a free resolution of $\mathcal{O}^X/\mathcal{J}_f$; the importance of these sets (ideals) was pointed out by Buchsbaum and Eisenbud in the 70'. We have not seen any notion for these important sets in the literature, and "Buchsbaum-Eisenbud varieties" is already occupied for another purpose, so we stick to BEF as an acronym for Buchsbaum-Eisenbud-Fitting.

convenient. The same idea will be used throughout this paper. For the rest of this paper we skip the brackets and write just $1/f$. It is readily checked that

$$(3.1) \quad f \frac{1}{f} = 1, \quad f \bar{\partial} \frac{1}{f} = 0.$$

3.1. Pseudomeromorphic currents. In [9] we introduced the sheaf \mathcal{PM} of *pseudomeromorphic currents* on a smooth manifold X . The definition when X is singular is identical. In this paper we will use the slightly extended definition introduced in [6]: We say that a current of the form

$$\frac{\xi}{s_1^{\alpha_1} \cdots s_{n-1}^{\alpha_{n-1}}} \wedge \bar{\partial} \frac{1}{s_n^{\alpha_n}},$$

where s is a local coordinate system and ξ is a smooth form with compact support, is an *elementary pseudomeromorphic current*. The sheaf \mathcal{PM} consists of all possible (locally finite sums of) push-forwards under a sequence of maps $X^m \rightarrow \cdots \rightarrow X^1 \rightarrow X$, of elementary pseudomeromorphic currents, where X^m is smooth, and each mapping is either a modification or a simple projection $\hat{X} \times Y \rightarrow \hat{X}$.

The sheaf \mathcal{PM} is closed under $\bar{\partial}$ (and ∂) and multiplication with smooth forms. If τ is in \mathcal{PM} and has support on a subvariety V and η is a holomorphic form that vanishes on V , then $\bar{\eta} \wedge \tau = 0$. We also have the

Dimension principle: If τ is a pseudomeromorphic current on X of bidegree $(, p)$ that has support on a variety V of codimension $> p$, then $\tau = 0$.*

If τ is in \mathcal{PM} and V is a subvariety of X , then the natural restriction of τ to the open set $X \setminus V$ has a canonical extension as a principal value to a pseudomeromorphic current $\mathbf{1}_{X \setminus V} \tau$ on X : Let h be a holomorphic tuple with common zero set V . The current-valued function $\lambda \mapsto |h|^{2\lambda} \tau$, a priori defined for $\operatorname{Re} \lambda \gg 0$, has an analytic continuation to $\operatorname{Re} \lambda > -\epsilon$ and its value at $\lambda = 0$ is by definition $\mathbf{1}_{X \setminus V} \tau$, see, e.g., [9]. One can also take a smooth approximand χ of the characteristic function of the interval $[1, \infty)$ and obtain $\mathbf{1}_{X \setminus V} \tau$ as the limit of $\chi(|h|/\epsilon) \tau$ when $\epsilon \rightarrow 0$. It follows that $\mathbf{1}_V \tau := \tau - \mathbf{1}_{X \setminus V} \tau$ is pseudomeromorphic and has support on V . Notice that if α is a smooth form, then $\mathbf{1}_V \alpha \tau = \alpha \mathbf{1}_V \tau$. Moreover, if $\pi: \tilde{X} \rightarrow X$ is a modification, $\tilde{\tau}$ is in $\mathcal{PM}(\tilde{X})$, and $\tau = \pi_* \tilde{\tau}$, then

$$(3.2) \quad \mathbf{1}_V \tau = \pi_* (\mathbf{1}_{\pi^{-1}V} \tilde{\tau})$$

for any subvariety $V \subset X$. There is actually a reasonable definition of $\mathbf{1}_W \tau$ for any constructible set W , and

$$(3.3) \quad \mathbf{1}_W \mathbf{1}_{W'} \tau = \mathbf{1}_{W \cap W'} \tau.$$

Recall that a current is *semimeromorphic* if it is the quotient of a smooth form and a holomorphic function. We say that a current τ is *almost semimeromorphic* in X if there is a modification $\pi: \tilde{X} \rightarrow X$ and a semimeromorphic current $\tilde{\tau}$ such that $\tau = \pi_* \tilde{\tau}$. Analogously we say that τ is *almost smooth* if $\tau = \pi_* \tilde{\tau}$ and $\tilde{\tau}$ is smooth. Any almost semimeromorphic (or smooth) τ is pseudomeromorphic.

3.2. Residues defined from Hermitian complexes. Assume that

$$(3.4) \quad 0 \rightarrow E_M \xrightarrow{f^M} \cdots \xrightarrow{f^3} E_2 \xrightarrow{f^2} E_1 \xrightarrow{f^1} E_0 \rightarrow 0$$

is a generically exact complex of Hermitian vector bundles over X and let Z be the subvariety where (3.4) is not pointwise exact. The bundle $E = \bigoplus E_k$ gets a

natural superbundle structure, i.e., a \mathbb{Z}_2 -grading, $E = E^+ \oplus E^-$, E^+ and E^- being the subspaces of even and odd elements, respectively, by letting $E^+ = \bigoplus_{2k} E_k$ and $E^- = \bigoplus_{2k+1} E_k$. This extends to a \mathbb{Z}_2 -grading of the sheaf $\mathcal{C}_\bullet(E)$ of E -valued currents, so that the degree of $\xi \otimes e$ is the sum of the current degree of ξ and the degree of e , modulo 2. An endomorphism on $\mathcal{C}_\bullet(E)$ is even if it preserves degree and odd if it switches degree. The mappings $f = \sum f_j$ and $\bar{\partial}$ are then odd mappings on $\mathcal{C}_\bullet(E)$. We introduce $\nabla = \nabla_f = f - \bar{\partial}$; it is just (minus) the $(0, 1)$ -part of Quillen's superconnection $D - \bar{\partial}$. Since the odd mappings f and $\bar{\partial}$ anticommute, $\nabla^2 = 0$. Moreover, ∇ extends to an odd mapping ∇_{End} on $\mathcal{C}_\bullet(\text{End}E)$ so that

$$(3.5) \quad \nabla(\alpha\xi) = \nabla_{\text{End}}\alpha \cdot \xi + (-1)^{\deg \alpha} \alpha \nabla\xi$$

for sections ξ and α of E and $\text{End}E$, respectively, and then $\nabla_{\text{End}}^2 = 0$. In $X \setminus Z$ we define, following [8, Section 2], a smooth $\text{End}E$ -valued form u such that

$$\nabla_{\text{End}}u = I,$$

where $I = I_E$ is the identity endomorphism on E . We have that

$$u = \sum_{\ell} u^{\ell} = \sum_{\ell} \sum_{k \geq \ell+1} u_k^{\ell},$$

where u_k^{ℓ} is in $\mathcal{E}_{0, k-\ell-1}(\text{Hom}(E_{\ell}, E_k))$ over $X \setminus Z$. Following [8]⁴ we define a pseudomeromorphic current extension U of u across Z , as the value at $\lambda = 0$ of the current-valued analytic function

$$\lambda \mapsto U^{\lambda} := |F|^{2\lambda}u,$$

a priori defined for $\text{Re } \lambda \gg 0$, where F is a holomorphic tuple that vanishes on Z . In the same way we define the *residue current* R associated to (3.4) as the value at $\lambda = 0$ of

$$\lambda \mapsto R^{\lambda} := I - |F|^{2\lambda} + \bar{\partial}|F|^{2\lambda} \wedge u.$$

This current clearly has its support on Z , and

$$R = \sum_{\ell} R^{\ell} = \sum_{\ell} \sum_{k \geq \ell+1} R_k^{\ell},$$

where R_k^{ℓ} is a $\text{Hom}(E_{\ell}, E_k)$ -valued $(0, k - \ell)$ -current. The currents U^{ℓ} and U_k^{ℓ} are defined analogously. Notice that U has odd degree and R has even degree. By the dimension principle, R_k^{ℓ} vanishes if $k - \ell < \text{codim } Z$. In particular, $R_0^0 = (I - |F|^{2\lambda})|_{\lambda=0}$ is zero, unless some components W of Z has codimension 0, in which case R_0^0 is the characteristic function for W times the identity I . However, when we define products of currents later on, all components of R^{λ} may play a role.

Since $\nabla_{\text{End}}U^{\lambda} = I - R^{\lambda}$ and $\nabla_{\text{End}}R^{\lambda} = 0$ when $\text{Re } \lambda \gg 0$, we conclude that

$$(3.6) \quad \nabla_{\text{End}}U = I - R, \quad \nabla_{\text{End}}R = 0.$$

In particular, (3.6) means that, cf. (3.5),

$$f^1 U_1^0 = I_{E_0}, \quad f^{k+1} U_{k+1}^0 - \bar{\partial} U_k^0 = R_k^0; \quad k \geq 1;$$

Notice that when ϕ is a section of E_0 , then $R^0 \phi = R\phi$ and $U^0 \phi = U\phi$, and we will often skip the upper indices.

⁴The definition is the same when X is singular.

Example 3.1 (The Koszul complex). Let f_1, \dots, f_m be holomorphic sections of a Hermitian line bundle $L \rightarrow X$. Let E^j be disjoint trivial line bundles with basis elements e_j and define the rank m bundle

$$E = L^{-1} \otimes E^1 \oplus \dots \oplus L^{-1} \otimes E^m$$

over X . Then $f = \sum f_j e_j^*$, where e_j^* is the dual basis, is a section of the dual bundle $E^* = L \otimes (E^1)^* \oplus \dots \oplus L \otimes (E^m)^*$. If $S \rightarrow X$ is a Hermitian line bundle we can form a complex (3.4) with

$$E_0 = S, \quad E_k = S \otimes \Lambda^k E,$$

where all the mappings f^k in (3.4) are interior multiplication δ_f with the section f . Notice that

$$E_k = S \otimes L^{-k} \otimes \Lambda^k (E^1 \oplus \dots \oplus E^m).$$

The superstructure of $\oplus_k E_k$ in this case coincides with the natural grading of the exterior algebra ΛE of E modulo 2.

Let us recall how the currents U^0 and R^0 are defined in this case. For simplicity we suppress the upper indices throughout this example. We have the natural norm

$$|f|^2 = \sum_j |f_j|_L^2$$

on E^* . Let σ be the section over $X \setminus Z$ of pointwise minimal norm such that $f \cdot \sigma = \delta_f \sigma = 1$, i.e.,

$$\sigma = \sum_j \frac{f_j^* e_j}{|f|^2},$$

where f_j^* are the sections of L^{-1} of minimal norm such that $f_j f_j^* = |f_j|_L^2$.

Let us consider the exterior algebra over $E \oplus T^*(X)$ so that $d\bar{z}_j \wedge e_\ell = -e_\ell \wedge d\bar{z}_j$ etc. Then, e.g., $\bar{\partial}\sigma$ is a form of positive degree. We have the smooth form $u = \sum u_k$, where $u_k = \sigma \wedge (\bar{\partial}\sigma)^{k-1}$ in $X \setminus Z$, and it turns out that it admits a natural current extension U across Z , e.g., defined as the analytic continuation of $U^\lambda = |f|^{2\lambda} u$ to $\lambda = 0$. Furthermore, the associated residue current R is obtained as the evaluation at $\lambda = 0$ of

$$\begin{aligned} R^\lambda &:= 1 - |f|^{2\lambda} + \bar{\partial}|f|^{2\lambda} \wedge u = \\ &1 - |f|^{2\lambda} + \bar{\partial}|f|^{2\lambda} \wedge u_1 + \dots + \bar{\partial}|f|^{2\lambda} \wedge u_{\min(m,n)} =: R_0^\lambda + R_1^\lambda + \dots + R_{\min(m,n)}^\lambda. \end{aligned}$$

The existence of the analytic continuations follows from a suitable resolution $\tilde{X} \rightarrow X$, see [1], see also Section 8 below. \square

3.3. The associated sheaf complex. Given the complex (3.4) we have the associated complex of locally free sheaves

$$(3.7) \quad 0 \rightarrow \mathcal{O}(E_M) \xrightarrow{f^M} \dots \xrightarrow{f^3} \mathcal{O}(E_2) \xrightarrow{f^2} \mathcal{O}(E_1) \xrightarrow{f^1} \mathcal{O}(E_0).$$

In this paper E_0 is always a line bundle so that $\mathcal{J} := \text{Im } f^1$ is a coherent ideal sheaf over X .

Consider the double sheaf complex $\mathcal{M}_{\ell,k} = \mathcal{C}_{0,k}(E_\ell)$ with mappings f and $\bar{\partial}$. We have the associated total complex

$$\dots \xrightarrow{\nabla_f} \mathcal{M}_j \xrightarrow{\nabla_f} \mathcal{M}_{j-1} \xrightarrow{\nabla_f} \dots \quad ,$$

where $\mathcal{M}_j = \bigoplus_{\ell-k=j} \mathcal{M}_{\ell,k}$. If X is smooth, then $\mathcal{M}_{\ell,k}$ is exact in the k -direction except at $k = 0$, and the kernels there are $\mathcal{O}(E_\ell)$. Notice that if ϕ is in $\mathcal{O}(E_\ell)$ and $f^\ell \phi = 0$, then also $\nabla_f \phi = 0$. We therefore have a natural mapping

$$(3.8) \quad H^j(\mathcal{O}(E_\bullet)) \rightarrow H^j(\mathcal{M}_\bullet).$$

By standard homological algebra, (3.8) is in fact an isomorphism. We can also consider the corresponding sheaf complexes $\mathcal{M}_{\ell,k}^\mathcal{E}$, $\mathcal{M}_j^\mathcal{E}$ of smooth sections, and the analogue of (3.8) is then an isomorphism as well.

Lemma 3.2. *If ϕ is a holomorphic section of E_0 that ϕ annihilates R , i.e., $R\phi = 0$, then ϕ is in \mathcal{J} .*

Proof. In fact, by (3.6) we have that

$$\nabla_f(U\phi) = \phi - R\phi = \phi.$$

Since X is smooth, (3.8) is an isomorphism, and thus locally $\phi = f_1\psi$ for some holomorphic ψ , i.e., ϕ is in \mathcal{J} . \square

The smoothness assumption is crucial, as the following example shows.

Example 3.3. Let f be one single function. Then the residue condition $R\phi = 0$ means that $\bar{\partial}(\phi/f) = 0$. Thus $\psi = \phi/f$ is in the Barlet-Henkin-Passare class, cf., [19] and [6]; however in general ψ is not (strongly) holomorphic, i.e., in general ϕ is not in $\mathcal{J} = (f)$. \square

We shall now see that if X is smooth and there is a global current solution to $\nabla W = \phi$, then there is also a smooth global solution. For further reference however we need a slightly more general statement about of the associated complex of global sections. Let $\mathcal{M}_{\ell,k}(X)$ and $\mathcal{M}_{\ell,k}^\mathcal{E}(X)$ be the double complexes of global current valued and smooth sections, respectively, and let $\mathcal{M}_\bullet(X)$ and $\mathcal{M}_\bullet^\mathcal{E}(X)$ be the associated total complexes. Notice that we have natural mappings

$$(3.9) \quad H^j(\mathcal{M}_\bullet^\mathcal{E}(X)) \rightarrow H^j(\mathcal{M}_\bullet(X)), \quad j \in \mathbb{Z}.$$

Proposition 3.4. *If X is smooth, then the mappings (3.9) are isomorphisms.*

Proof. By the de Rham theorem, the natural mappings

$$(3.10) \quad H^k(\mathcal{E}_{0,\bullet}(X, E_\ell)) \rightarrow H^k(\mathcal{C}_{0,\bullet}(X, E_\ell)), \quad k \in \mathbb{Z},$$

are isomorphisms; these spaces are in fact naturally isomorphic to the cohomology groups $H^k(X, \mathcal{O}(E_\ell))$. The short exact sequence

$$0 \rightarrow \mathcal{M}^\mathcal{E}(X) \rightarrow \mathcal{M}(X) \rightarrow \mathcal{M}(X)/\mathcal{M}^\mathcal{E}(X) \rightarrow 0$$

gives rise to, for each fixed ℓ , the long exact sequence

$$\begin{aligned} \dots \rightarrow H^{k-1}(\mathcal{E}_{0,\bullet}(X, E_\ell)) \rightarrow H^{k-1}(\mathcal{C}_{0,\bullet}(X, E_\ell)) \rightarrow \\ H^{k-1}(\mathcal{C}_{0,\bullet}(X, E_\ell)/\mathcal{E}_{0,\bullet}(X, E_\ell)) \rightarrow H^k(\mathcal{E}_{0,\bullet}(X, E_\ell)) \rightarrow \dots \end{aligned},$$

and in view of (3.10) therefore the cohomology in the k -direction of $\mathcal{M}_{\ell,k}(X)/\mathcal{M}_{\ell,k}^\mathcal{E}(X)$ is zero. By a simple homological algebra argument, using that the double complexes involved are bounded it follows that

$$H^k(\mathcal{M}_\bullet(X)/\mathcal{M}_\bullet^\mathcal{E}(X)) = 0$$

for each k . The proposition now follows from the long exact sequence

$$\begin{aligned} \dots \rightarrow H^{k-1}(\mathcal{M}_\bullet^\mathcal{E}(X)) \rightarrow H^{k-1}(\mathcal{M}_\bullet(X)) \rightarrow \\ H^{k-1}(\mathcal{M}_\bullet(X)/\mathcal{M}_\bullet^\mathcal{E}(X)) \rightarrow H^k(\mathcal{M}_\bullet^\mathcal{E}(X)) \rightarrow \dots \end{aligned}$$

□

3.4. Duality principle and BEF varieties. We now consider the case when the locally free complex (3.7) is exact, i.e., a resolution of the sheaf $\mathcal{O}(E_0)/\mathcal{J}$. Let Z_k^{bef} be the (algebraic) set where the mapping f^k does not have optimal rank. These sets Z_k^{bef} are independent of the choice of resolution; we call them the BEF-sets associated to \mathcal{J} , cf., Section 2. It follows from the Buchsbaum-Eisenbud theorem that $\text{codim } Z_k^{\text{bef}} \geq k$. If moreover \mathcal{J} has pure dimension, for instance \mathcal{J} is the radical ideal sheaf of a pure-dimensional subvariety, then $\text{codim } Z_k^{\text{bef}} \geq k + 1$ for $k \geq 1 + \text{codim } \mathcal{J}$, see [16].

Let us equip E_k with Hermitian metrics and let R be the associated residue current. We will refer to a (locally free) resolution $\mathcal{O}(E_0)/\mathcal{J}$ together with a choice of Hermitian metrics on the corresponding vector bundles E_k as a *(locally free) Hermitian resolution*. Then by [8, Theorem 3.1], we have that $R^\ell = 0$ for each $\ell \geq 1$, i.e., $R = R^0$. There are almost semimeromorphic $\text{Hom}(E_k, E_{k+1})$ -valued $(0, 1)$ -forms α_{k+1} , that are smooth outside Z_{k+1}^{bef} , such that

$$R_{k+1} = \alpha_{k+1} R_k$$

there, see [8]. From [8] we also have:

Duality principle: If X is smooth and (3.7) is a resolution of the sheaf $\mathcal{O}(E_0)/\mathcal{J}$, then $\phi \in \mathcal{J}$ if and only if $R\phi = 0$.

That is, the annihilator ideal sheaf of the residue current R is precisely the ideal sheaf \mathcal{J} generated by f^1 .

If for instance $f^1 = (f_1, \dots, f_m)$ defines a complete intersection, i.e, $\text{codim } Z^f = m$, then the Koszul complex is a resolution of \mathcal{J} and hence the duality principle states that the annihilator of the residue current in Example 3.1 is the ideal itself. This special case was proved already in [30].

3.5. Tensor product of complexes. Assume that E_\bullet^g, g and E_\bullet^h, h are Hermitian complexes. We can then define a complex $E_\bullet^f = E_\bullet^g \otimes E_\bullet^h, f$, where

$$E_k^f = \bigoplus_{i+j=k} E_i^g \otimes E_j^h,$$

and $f = g + h$, or more formally $f = g \otimes I_{E^h} + I_{E^g} \otimes h$, such that

$$(3.11) \quad f(\xi \otimes \eta) = g\xi \otimes \eta + (-1)^{\deg \xi} \xi \otimes h\eta.$$

Notice that E_0 is the line bundle $E_0^g \otimes E_0^h$. If $g^1 \mathcal{O}(E_1^g) = \mathcal{J}_g$ and $h^1 \mathcal{O}(E_1^h) = \mathcal{J}_h$, then $f^1 \mathcal{O}(E_1) = \mathcal{J}_g + \mathcal{J}_h$. One extends (3.11) to current-valued sections ξ and η , and $\deg \xi$ then means total degree. We write $\xi \cdot \eta$, or sometimes $\xi \wedge \eta$ to emphasize that the sections may be form- or current-valued, rather than $\xi \otimes \eta$, and define

$$(3.12) \quad \eta \cdot \xi = (-1)^{\deg \xi} \deg \eta \xi \cdot \eta.$$

Notice that

$$\nabla_f(\xi \cdot \eta) = \nabla_g \xi \cdot \eta + (-1)^{\deg \xi} \xi \cdot \nabla_h \eta.$$

Let u^g and u^h be the corresponding $\text{End}(E^g)$ -valued and $\text{End}(E^h)$ -valued forms, cf., Section 3.2. Then $u = u^h \wedge u^g$ is a $\text{End}(E^f)$ -valued form defined outside $Z^g \cup Z^h$. Following the proof of Proposition 2.1 in [9] we can define $\text{End}(E^f)$ -valued pseudomeromorphic currents

$$U^h \wedge R^g := U^{h,\lambda} \wedge R^g|_{\lambda=0}, \quad R^h \wedge R^g := R^{h,\lambda} \wedge R^g|_{\lambda=0}.$$

We have that, cf., (3.6) and [4, Section 4],

$$\nabla_{\text{End},f}(U^h \wedge R^g + U^g) = I_{E^f} - R^h \wedge R^g.$$

In general, the current $R^h \wedge R^g$ will change if we interchange the roles of g and h .

In particular we can form the product $E^\bullet \otimes E^\bullet$ of E^\bullet by itself. In this case we consider (3.12) as an identification, so that, for instance,

$$(E^\bullet \otimes E^\bullet)_1 = E_1^h \dot{\otimes} E_0^h, \quad (E^\bullet \otimes E^\bullet)_2 = E_2^h \dot{\otimes} E_0^h + \Lambda^2 E_1^h,$$

etc, where $\dot{\otimes}$ denotes symmetric tensor product. In general, $\xi \cdot \xi = 0$ if ξ has odd (total) degree.

We can just as well form a similar product of more than two complexes, and in particular, we can form the product $(E^h)^{\otimes k} = E^h \otimes E^h \otimes \dots \otimes E^h$ of a given complex by itself.

3.6. The structure form ω on a singular variety. Let $i: X \rightarrow Y$ be an embedding of X in a smooth projective manifold Y , let \mathcal{J}_X be the radical ideal sheaf associated to X in Y , and let $S \rightarrow Y$ be an ample Hermitian line bundle. Moreover, let E_k^j be disjoint trivial line bundles over Y with basis elements $e_{k,j}$. There is a (possibly infinite) resolution, see, e.g., [26, Ch.1, Example 1.2.21],

$$(3.13) \quad \dots \xrightarrow{g^3} \mathcal{O}(E_2) \xrightarrow{g^2} \mathcal{O}(E_1) \xrightarrow{g^1} \mathcal{O}(E_0)$$

of $\mathcal{O}(E_0)/\mathcal{J}_X = \mathcal{O}^X$, where

$$E_k = (E_k^1 \otimes S^{-d_k^1}) \oplus \dots \oplus (E_k^{r_k} \otimes S^{-d_k^{r_k}}), \quad E_0 = E_0^0 \simeq \mathbb{C},$$

E_k^i are trivial line bundles, and

$$g^k = \sum_{ij} g_{ij}^k e_{k-1,i} \otimes e_{k,j}^*,$$

are matrices of sections

$$g_{ij}^k \in \mathcal{O}(Y, S^{d_k^j - d_{k-1}^i});$$

here $e_{k,j}^*$ are the dual basis elements. There are natural induced norms on E_k . The associated residue current⁵ R is annihilated by all smooth forms ξ such that $i^*\xi = 0$. Let γ be a global non-vanishing $(\dim Y, 0)$ -form with values in K_Y^{-1} . By [6, Proposition 16] there is a (unique) almost semimeromorphic current ω on X , smooth on X_{reg} , such that

$$i_*\omega = R \wedge \gamma.$$

We say that ω is a *structure form* on X .

As an immediate consequence of the existence of ω , the product $\alpha \wedge R$ is well-defined for (sufficiently) smooth forms α on X . If $\alpha = i^*a$, we let $\alpha \wedge R := a \wedge R$. This product only depends on α , since if $i^*a = 0$, then $a \wedge R \wedge \gamma = i_*(i^*a \wedge \omega) = 0$ and hence $a \wedge R = 0$ since $\gamma \neq 0$.

⁵The fact that (3.13) may be infinite causes no problem, since, for degree reasons, U and R only contain a finite number of terms.

Let X_k be the BEF sets of \mathcal{J}_X , and define

$$X^0 = X_{\text{sing}}, \quad X^\ell = X_{N-n+\ell}, \quad \ell \geq 1.$$

Since \mathcal{J}_X has pure dimension it follows that $\text{codim } X^k \geq k + 1$, and in particular, $X^n = \emptyset$. These sets X^ℓ are actually independent of the choice of embedding of X , cf., the text after Lemma 4.2 in Section 4.

Let g_ℓ be the restriction to X of $g^{N-n+\ell}$, and let $\nabla^g = g_\bullet - \bar{\partial}$ on X . Let $E^\ell = E_{N-n+\ell}|_X$. Then $\omega = \omega_0 + \omega_1 + \dots + \omega_n$, where ω_ℓ is a (n, ℓ) -form on X taking values in E^ℓ , and $\nabla^g \omega = 0$ on X . There are almost semimeromorphic $\text{Hom}(E^\ell, E^{\ell+1})$ -valued $(0, 1)$ -forms $\alpha^{\ell+1}$ such that

$$(3.14) \quad \omega_{\ell+1} = \alpha^{\ell+1} \omega_\ell$$

there. In fact, α^ℓ is the pullback to X if the form $\alpha_{N-n+\ell}$ associated to a resolution of $\mathcal{O}^Y/\mathcal{J}_X$ in Y , cf., Section 3.4.

Since ω is almost semi-meromorphic, it has the *the standard extension property*, SEP on X , which means that $\mathbf{1}_W \omega = 0$ for all varieties $W \subset X$ of positive codimension.

The singularities of a structure form ω only depend on X , in the following sense:

Proposition 3.5. *Let X be a projective variety. There is a smooth modification $\tau: \tilde{X} \rightarrow X$ and a holomorphic section η of a line bundle $S \rightarrow \tilde{X}$ such that the following holds: If $X \rightarrow Y$ is an embedding of X in a smooth manifold Y , $\mathcal{O}(E_\bullet^g)$, g is a locally free Hermitian resolution of $\mathcal{O}^Y/\mathcal{J}_X$, and ω is the associated structure form on X , then $\eta\tau^*\omega$ is smooth on \tilde{X} .*

The proof is postponed to Section 4. Since ω is almost semimeromorphic, the pullback $\tau^*\omega$ is well-defined; this will follow from the proof below, cf., also the remark after Definition 12 in [6].

Example 3.6. See [8, Section 6]. If $Y = \mathbb{P}^N$ and $S = \mathcal{O}(1)$, then

$$E_k = (E_k^1 \otimes \mathcal{O}(-d_k^1)) \oplus \dots \oplus (E_k^{r_k} \otimes \mathcal{O}(-d_k^{r_k}))$$

and g^k are matrices of homogeneous forms with $\deg g_{ij}^k = d_k^j - d_{k-1}^i$, and

$$|\xi(z)|_{E_k}^2 = \sum_{j=1}^{r_k} |\xi_j(z)|^2 |z|^{2d_k^j},$$

if $\xi = (\xi_1, \dots, \xi_{r_k})$. Moreover,

$$\gamma = \text{const} \times \sum (-1)^j z_j dz_0 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_N$$

in \mathbb{P}^N .

Let J_X denote the homogeneous ideal in the graded ring $\mathcal{S} = \mathbb{C}[z_0, \dots, z_N]$ that corresponds to X , and let $\mathcal{S}(\ell)$ denote the module \mathcal{S} but where all degrees are shifted by ℓ . Then $\mathcal{O}(E_\bullet^g)$, g corresponds to a free resolution

$$\dots \rightarrow \oplus_i \mathcal{S}(-d_k^i) \rightarrow \dots \rightarrow \oplus_i \mathcal{S}(-d_2^i) \rightarrow \oplus_i \mathcal{S}(-d_1^i) \rightarrow \mathcal{S}$$

of the module \mathcal{S}/J_X . By Hilbert's Syzygy theorem one can assume that $E_k = 0$ for $k > N + 1$. \square

3.7. Local division problems on a singular variety. Still assume that we have the embedding $i: X \rightarrow Y$, where Y is smooth, and the complex E_\bullet^g, g over Y corresponding to a locally free Hermitian resolution of $\mathcal{O}^Y/\mathcal{I}_X$. If E_\bullet^f, f is an arbitrary Hermitian complex over Y we have the complex $E^F = E^f \otimes E^g$ with mappings $F = g + f$ as in Section 3.5. Let $F^k = F|_{E_k}$. Since $R^f \wedge R^g = R^{f,\lambda} \wedge R^g|_{\lambda=0}$ and $U^f \wedge R^g = U^{f,\lambda} \wedge R^g|_{\lambda=0}$ cf., Section 3.6, these currents only depend on the values of f on X . From Section 3.5 we also have that

$$(3.15) \quad \nabla_{\text{End}, F} U = I - R^f \wedge R^g$$

if $U = U^f \wedge R^g + U^g$. If Φ is a (locally defined) holomorphic function in Y and $R^f \wedge R^g \Phi = 0$, then, following the proof of Lemma 3.2, there is a local holomorphic solution $v = v_g + v_f$ in $E_1^F = E_1^f \otimes E_0^g + E_0^f \otimes E_1^g$ to $g^1 v_f + f^1 v_g = F^1 v = \Phi$. Notice that in fact $R^f \wedge R^g \Phi$ only depends on the class ϕ of Φ in $\mathcal{O}^Y/\mathcal{I}_X = \mathcal{O}^X$, so $R^f \wedge R^g \phi$ is well-defined for ϕ in \mathcal{O}^X . We can define the intrinsic residue current

$$R^f \wedge \omega := R^{f,\lambda} \wedge \omega|_{\lambda=0}$$

on X . Since $i_* R^{f,\lambda} \wedge \omega = R^{f,\lambda} \wedge R^g \wedge \gamma$ when $\text{Re } \lambda \gg 0$, we can conclude that

$$i_* R^f \wedge \omega = R^f \wedge R^g \wedge \gamma.$$

Since γ is non-vanishing, $R^f \wedge \omega \phi = 0$ implies that $R^f \wedge R^g \phi = 0$ and thus we have:

Proposition 3.7. *Assume that E_\bullet^f, f is a Hermitian complex on X . If ϕ is a holomorphic section of E_0^f on X such that $R^f \wedge \omega \phi = 0$, then locally ϕ is in the image of f^1 on X .*

3.8. A fine resolution of \mathcal{O} on X . It was proved in [6], see [6, Theorem 2], that there exist sheaves \mathcal{A}_k of $(0, k)$ -currents on X with the following properties:

- (i) \mathcal{A}_k is equal to $\mathcal{E}_{0,k}$ on X_{reg} ,
- (ii) $\mathcal{A} = \bigoplus_k \mathcal{A}_k$ is closed under multiplication with smooth $(0, *)$ -forms,
- (iii) $\bar{\partial}$ maps $\mathcal{A}_k \rightarrow \mathcal{A}_{k+1}$ and if E is any vector bundle over X , then the sheaf complex

$$0 \rightarrow \mathcal{O}(E) \rightarrow \mathcal{A}_0(E) \xrightarrow{\bar{\partial}} \mathcal{A}_1(E) \xrightarrow{\bar{\partial}} \mathcal{A}_2(E) \xrightarrow{\bar{\partial}} \dots$$

is exact.

By standard sheaf theory we have canonical isomorphisms

$$H^k(X, \mathcal{O}(E)) = \frac{\text{Ker}(\Gamma(X, \mathcal{A}_k(E)) \xrightarrow{\bar{\partial}} \Gamma(X, \mathcal{A}_{k+1}(E)))}{\text{Im}(\Gamma(X, \mathcal{A}_{k-1}(E)) \xrightarrow{\bar{\partial}} \Gamma(X, \mathcal{A}_k(E)))}, \quad k \geq 1.$$

4. SINGULARITIES OF THE STRUCTURE FORM

In this section we provide a proof of Proposition 3.5. For the first part of the proof let us fix an embedding $i: X \rightarrow Y$ where Y is projective and smooth of dimension N . Recall that the k th *Fitting ideal (sheaf)* of $\mathcal{O}^Y/\mathcal{I}_X$, $\text{Fitt}_0 g^k$, is the ideal generated by all r_k -minors of (the matrix) g^k in a locally free resolution $\mathcal{O}(E_\bullet^g), g$ of $\mathcal{O}^Y/\mathcal{I}_X$, where r_k is the generic rank of g^k , see, e.g., [16]. It is well-known that these ideals are independent of the resolution $\mathcal{O}(E_\bullet^g), g$, and the zero variety of $\text{Fitt}_0 g^k$ is just the BEF-variety Z_k^{bef} , cf., Section 3.4. Moreover, since X has pure dimension, $\text{Fitt}_0 g^k$ are trivial when $k \geq N$, see [16, Corollary 20.14]. Let $p = N - n$ be the codimension of X in Y . For $\ell = 1, \dots, n - 1$, let \mathbf{a}_ℓ be the pullback (restriction) of $\text{Fitt}_0 g^{p+\ell}$ to

X . It follows that these ideals only depend on the embedding $i: X \rightarrow Y$. Let us call them the *structure ideals* on X associated to the given embedding.

Let $\tau: \tilde{X} \rightarrow X$ be a smooth modification that satisfies:

- (i) all the ideals $\tau^* \mathfrak{a}_\ell$ are principal on \tilde{X} and have simple normal crossings,
- (ii) $\tau^{-1} X_{\text{sing}}$ is a hypersurface in \tilde{X} with simple normal crossings.

The simple normal crossing assumption means that the ideal $\tau^* \mathfrak{a}_\ell$ is generated by a section s_ℓ (of a line bundle) that is a monomial in suitable local coordinates.

Our goal is to find a holomorphic section η such that $\eta \tau^* \omega$ is smooth for all structure forms ω coming from locally free Hermitian resolutions $\mathcal{O}(E_\bullet^g), g$ of $\mathcal{O}^Y/\mathcal{I}_X$ in Y ; eventually we also want to vary the embedding i . We will start by looking for a section s such that $s \tau^* \omega_0$ is smooth. Let us fix a Hermitian resolution $\mathcal{O}(E_\bullet^g), g$ and let σ_k be the pointwise minimal inverse of g^k , cf., [8, Section 2]. The denominator of σ_k is the modulus square of a tuple of generators of $\text{Fitt}_0 g^k$, cf. [8, Section 4], and thus $i^* \sigma_{p+k} =: \sigma^k$ is well-defined and almost semimeromorphic on X . By [6, Lemma 17], for $x \in X$, in a neighborhood \mathcal{U} , ω_0 admits a representation $\omega_0 = \pi a$, where a is meromorphic and $\pi = I_{E_p} - g^{p+1} \sigma^1$. Furthermore, by [8, Lemma 2.1],

$$(4.1) \quad s_k \tau^* \sigma^k$$

is smooth in \tilde{X} . It follows that $\tau^* \omega_0$ is semimeromorphic in $\tau^{-1} \mathcal{U}$, and since ω_0 is indeed smooth outside X_{sing} it follows, see [8, Lemma 3.2], that $s_0 \tau^* \omega_0$ is smooth if s_0 is a holomorphic section on \tilde{X} that vanishes enough on the divisor $\tau^{-1} X_{\text{sing}}$.

Lemma 4.1. *Let $\mathcal{O}(E_\bullet^g), g$ and $\mathcal{O}(E_\bullet^{g'}), g'$ be locally free Hermitian resolutions of $\mathcal{O}^Y/\mathcal{I}_X$ with corresponding structure forms ω and ω' . If s is a holomorphic section such that $s \tau^* \omega_0$ is smooth, then*

$$s_1^2 s \tau^* \omega'_0$$

is smooth in \tilde{X} .

Proof. Fix a point $x \in X$ and let $\mathcal{O}(E_\bullet^{\hat{g}}), \hat{g}$ be a Hermitian minimal resolution of $\mathcal{O}^Y/\mathcal{I}_X$ at x with corresponding structure form $\hat{\omega}$.

It is well-known that $E_\bullet^{\hat{g}}, \hat{g}$ is a direct summand of E_\bullet^g, g in a small neighborhood $\Omega \subset Y$ of x , and following the proof of Theorem 4.4 in [8], we see that $R^{\hat{g}} \oplus 0$ is the residue current obtained from the resolution $\mathcal{O}(E_\bullet^g), g$ if it is equipped with an appropriate metric. For simplicity, we can assume therefore that we have just one resolution $\mathcal{O}(E_\bullet^g), g$, but two different metrics, and write $R^{\hat{g}}$ rather than $R^{\hat{g}} \oplus 0$. Let $\hat{\sigma}_k$ be the pointwise minimal inverses with respect to the new metric on E_\bullet^g of the mappings g_k . In $\Omega \setminus X_{p+1}$ we have that

$$R_p^{\hat{g}} = (I_{E_p} - g^{p+1} \hat{\sigma}^1) R_p^g,$$

see the proof of Theorem 4.4 in [8], and thus

$$\hat{\omega}_0 = (I_{E_p} - g^{p+1} \hat{\sigma}^1) \omega_0,$$

in $X \setminus X^1$. Since $s_1 \tau^* \hat{\sigma}^1$ is smooth and $\hat{\omega}_0$ has the SEP we conclude that $s_1 s \tau^* \hat{\omega}_0$ is smooth in $\tau^{-1}(X \cap \Omega)$.

By the same arguments, since $E_\bullet^{\hat{g}}, \hat{g}$ is also a direct summand of $E_\bullet^{g'}, g'$, in a possibly smaller neighborhood Ω of x , we have that

$$\omega'_0 = (I_{E_p} - (g')^{p+1} (\sigma')^1) \hat{\omega}_0,$$

where $(\sigma')^1$ is (the pullback to X of) the pointwise minimal inverse of $(g')^{p+1}$. Thus $s_1^2 s \tau^* \omega_0$ is smooth in $\tau^{-1}(X \cap \Omega)$. Since x was arbitrary we conclude that $s_1^2 s \tau^* \omega_0$ is smooth in \tilde{X} . \square

Summing up so far we have found a holomorphic section $s := s_1^2 s_0$ such that $s \tau^* \omega_0$ is smooth for all structure forms ω associated with Hermitian resolutions $\mathcal{O}(E_\bullet^g), g$. Next, we will vary the embedding $i : X \rightarrow Y$.

Lemma 4.2. *Assume that \mathfrak{a}_ℓ and \mathfrak{a}'_ℓ are the structure ideals associated with the embeddings $i : X \rightarrow Y$ and $i' : X \rightarrow Y'$, respectively. Then for each $\ell \geq 1$,*

$$(4.2) \quad \mathfrak{a}_\ell \cdots \mathfrak{a}_{n-1} \subset \mathfrak{a}'_\ell.$$

Since the zero set of \mathfrak{a}_{k+1} is contained in the zero set of \mathfrak{a}_k it follows that the zero set X^ℓ of \mathfrak{a}_ℓ , cf., Section 3.6, coincides with the zero set of \mathfrak{a}'_ℓ . It follows that X^ℓ is independent of the embedding i .

Proof. Given $i : X \rightarrow Y$ and a point $x \in X$ there is a neighborhood $\mathcal{V} \subset X$ such that the restriction to \mathcal{V} of i factorizes as

$$\mathcal{V} \xrightarrow{j} \hat{\Omega} \xrightarrow{\iota} \hat{\Omega} \times \mathbb{B}^M =: \Omega,$$

where j is a minimal (and therefore basically unique) embedding at x , $\mathbb{B}^M \subset \mathbb{C}_w^M$ is a ball centered at 0, ι is the trivial embedding $z \mapsto (z, 0)$ if z are coordinates in $\hat{\Omega}$, and Ω is a neighborhood of x in Y . Let now $\mathcal{O}(\hat{E}_\bullet^g), \hat{g}$ be a Hermitian minimal resolution of $\mathcal{O}^{\hat{\Omega}}/\mathcal{J}_{\mathcal{V}}$ at x in $\hat{\Omega}$ and assume that \hat{p} is the codimension of \mathcal{V} in $\hat{\Omega}$. Thus $p = \hat{p} + M$, where as before p is the codimension of X in Y .

Let E^w, δ_w be the Koszul complex generated by $w = (w_1, \dots, w_M)$. The sheaf complex associated with the product complex $E^{\hat{g}} \otimes E^w$ with mappings $g = \hat{g}(z) + \delta_w$, cf., Section 3.5, provides a (minimal) resolution of $\mathcal{O}^\Omega/\mathcal{J}_X$ in Ω , see [4, Remark 8]. Notice that $g^{p+\ell}$ is the mapping

$$(E_{\hat{p}+\ell}^{\hat{g}} \otimes E_M^w) \oplus (E_{\hat{p}+\ell+1}^{\hat{g}} \otimes E_{M-1}^w) \oplus \cdots \oplus (E_{\hat{p}+\ell+M}^{\hat{g}} \otimes E_0^w) \xrightarrow{\hat{g}(z)+\delta_w} \\ (E_{\hat{p}+\ell-1}^{\hat{g}} \otimes E_M^w) \oplus (E_{\hat{p}+\ell}^{\hat{g}} \otimes E_{M-1}^w) \oplus \cdots \oplus (E_{\hat{p}+\ell+M-1}^{\hat{g}} \otimes E_0^w).$$

Since $w = 0$ on X , the restriction of $g^{p+\ell}$ to X splits into the direct sum of the separate mappings

$$\hat{g}^{\hat{p}+\ell+j} : E_{\hat{p}+\ell+j}^{\hat{g}} \otimes E_{M-j}^w \rightarrow E_{\hat{p}+\ell+j-1}^{\hat{g}} \otimes E_{M-j}^w, \quad j = 0, 1, \dots, M.$$

Since the optimal rank $r_{p+\ell}$ of $g^{p+\ell}$ is attained at every point on X_{reg} , it follows that $r_{p+\ell} = \hat{r}_{\hat{p}+\ell} + \hat{r}_{\hat{p}+\ell+1} + \cdots + \hat{r}_{\hat{p}+M}$, where \hat{r}_k is the generic rank of \hat{g}^k . Therefore, the restriction to X of $\text{Fitt}_0 g^{p+\ell}$ is equal to (the restriction to X of) the product ideal

$$\text{Fitt}_0 \hat{g}^{\hat{p}+\ell} \cdot \text{Fitt}_0 \hat{g}^{\hat{p}+\ell+1} \cdots \text{Fitt}_0 \hat{g}^{\hat{p}+\ell+M}.$$

Since X is of pure dimension $\text{Fitt}_0 \hat{g}^k$ are trivial for $k \geq \hat{p} + n = \dim \hat{\Omega}$, and thus if $\hat{\mathfrak{a}}_\ell$ are the structure ideals associated with $j : \mathcal{V} \rightarrow \hat{\Omega}$,

$$(4.3) \quad \mathfrak{a}_\ell = \hat{\mathfrak{a}}_\ell \cdots \hat{\mathfrak{a}}_{\min(n-1, \ell+M)}.$$

Hence

$$(4.4) \quad \hat{\mathfrak{a}}_\ell \cdots \hat{\mathfrak{a}}_{n-1} \subset \mathfrak{a}_\ell \subset \hat{\mathfrak{a}}_\ell.$$

By the same argument, since i' factorizes as $\mathcal{V} \xrightarrow{j} \hat{\Omega} \xrightarrow{i'} \hat{\Omega} \times \mathbb{B}^{M'}$, at least if \mathcal{V} is small enough,

$$(4.5) \quad \mathbf{a}'_\ell = \hat{\mathbf{a}}_\ell \cdots \hat{\mathbf{a}}_{\min(n-1, \ell+M')}$$

and so (4.4) holds at x for \mathbf{a}'_ℓ instead of \mathbf{a}_ℓ . Combining we see that (4.2) holds in a neighborhood of x . Since $x \in X$ is arbitrary, the inclusion holds globally on X . \square

If a product of local ideals is principal each of the factors must be principal. Since by assumption $\tau^*\mathbf{a}_\ell$ are principal, (4.3) thus implies that $\tau^*\hat{\mathbf{a}}_\ell$ are principal, and by (4.5) we conclude that $\tau^*\mathbf{a}'_\ell$ are principal. Let the corresponding generating sections be denoted by s'_ℓ . In fact, in light of (4.3) and (4.5), $\mathbf{a}_\ell = \mathbf{a}'_\ell$ whenever $M = M'$, i.e., $\dim Y = \dim Y'$.

Lemma 4.3. *Let $i : X \rightarrow Y$ and $i' : X \rightarrow Y'$ be embeddings of X and into smooth manifolds, let $\mathcal{O}(E_\bullet^g, g)$ and $\mathcal{O}(E_\bullet^{g'}, g')$ be locally free Hermitian resolutions of $\mathcal{O}^Y/\mathcal{J}_X$ and $\mathcal{O}^{Y'}/\mathcal{J}_X$ with corresponding structure forms ω and ω' , respectively, and let s_ℓ and s'_ℓ , $\ell \geq 1$, be sections generating the ideals $\tau^*\mathbf{a}_\ell$ and $\tau^*\mathbf{a}'_\ell$, respectively. If s is a holomorphic section such that $s\tau^*\omega_0$ is smooth, then $(s'_1)^2 s_1^2 \tau^*\omega'_0$ is smooth.*

Proof. Take a point $x \in X$, factorize $i : X \rightarrow Y$ at x as in the proof of Lemma 4.2, and let $\mathcal{O}(E_\bullet^{\hat{g}}, \hat{g})$ be a Hermitian minimal resolution of $\mathcal{O}^{\hat{\Omega}}/\mathcal{J}_X$. Using the notation from that proof, recall that $\mathcal{O}(E_\bullet^{\hat{g}} \otimes E_\bullet^w), \hat{g} + w$ is a minimal resolution of $\mathcal{O}^{\hat{\Omega}}/\mathcal{J}_X$. The associated residue current is equal to $R^{\hat{g}(z)} \wedge R^w$, see [4, Remark 4.6].

Since w are just the coordinate functions in \mathbb{C}^M , the Bochner-Martinelli formula asserts that $R_M^w \wedge dw_1 \wedge \dots \wedge dw_M = (2\pi i)^M [w = 0]$. Let $\hat{N} = \dim \hat{\Omega}$, and let $\hat{\omega}$ denote the structure form in $\hat{\Omega}$ associated with $R^{\hat{g}(z)}$, so that $j_*\hat{\omega} = R^{\hat{g}} \wedge dz_1 \wedge \dots \wedge dz_{\hat{N}}$. Then,

$$\begin{aligned} i_*\hat{\omega} &= \iota_* R^{\hat{g}} \wedge dz_1 \wedge \dots \wedge dz_{\hat{N}} = R^{\hat{g}} \wedge dz_1 \wedge \dots \wedge dz_{\hat{N}} \wedge [w = 0] \sim \\ &R^{\hat{g}} \wedge R^w \wedge dw_1 \wedge \dots \wedge dw_M \wedge dz_1 \wedge \dots \wedge dz_{\hat{N}}, \end{aligned}$$

where \sim denotes “equal to a nonzero constant times”. We conclude that $\hat{\omega}$ is also a structure form associated to a resolution of $\mathcal{O}^{\hat{\Omega}}/\mathcal{J}_X$ with a special choice of Hermitian metric. Now Lemma (4.1) implies that $s_1^2 \hat{\omega}_0$ is smooth.

With the same arguments it follows that $\hat{\omega}$ is the structure form obtained from a local embedding of $X \rightarrow \Omega'$ where Ω' is a neighborhood of x in Y' and a Hermitian resolution $\mathcal{O}(E_\bullet^{g'}, g')$ in Ω' . By another application of Lemma 4.1 we find that $(s'_1)^2 s_1^2 \tau^*\omega'_0$ is smooth. \square

We can now conclude the proof of Proposition 3.5. Fix an embedding $i : X \rightarrow Y$ and a resolution $\mathcal{O}(E_\bullet^g), g$ with structure form ω . Let s_1, \dots, s_{n-1} be associated sections on \tilde{X} , and let s_0 be a section such that $s_0\tau^*\omega_0$ is smooth.

If ω' is a structure form associated with an embedding $i' : X \rightarrow Y'$, with corresponding sections s'_1, \dots, s'_{n-1} , and a Hermitian resolution $\mathcal{O}(E_\bullet^{g'}), g'$, then Lemma 4.3 asserts that $(s'_1)^2 s_1^2 s_0 \tau^*\omega'_0$ is smooth. Outside X^ℓ , $\omega'_\ell = \alpha'_{p+\ell} \cdots \alpha'_{p+1} \omega'_0$, where $\alpha'_k = \mathbf{1}_{X_{\text{reg}}} \bar{\partial} \sigma'_k$, cf., (3.14) and [8, Section 2]. Thus, since ω'_ℓ has the SEP, in light of (4.1), $s'_\ell \cdots s'_1 (s'_1)^2 s_1^2 s_0 \tau^*\omega'_\ell$ is smooth, and so $s'_{n-1} \cdots s'_1 (s'_1)^2 s_1^2 s_0 \tau^*\omega'$ is smooth. Lemma 4.2 implies that $s_\ell \cdots s_{n-1}/s'_\ell$ is holomorphic in \tilde{X} . Hence

$$\eta := (s_{n-1} \cdots s_1)^{n+1} s_1^2 s_0$$

satisfies that $\eta\tau^*\omega'$ is smooth.

5. GLOBAL DIVISION PROBLEMS AND RESIDUES

Let (3.4) be a generically exact Hermitian complex over a smooth variety X . Moreover, let ϕ be a global holomorphic section of E_0 such that $R^f\phi = 0$. As we have seen, then $\nabla_f(U^f\phi) = \phi$. If the double complex $\mathcal{M}_{\ell,k} = \mathcal{C}_{0,k}(X, E_\ell)$ is exact in the k -direction except at $k = 0$, then it follows, cf., (3.8), that there is a global holomorphic solution to $f^1q = \phi$. Let us see more precisely what is needed: Notice that $U_{\min(M,n+1)}^f\phi$ is automatically $\bar{\partial}$ -closed. Since X is smooth then by the Dolbeault isomorphism for currents it is possible to solve successively the equations

$$\bar{\partial}w_{\min(M,n+1)} = U_{\min(M,n+1)}^f\phi, \quad \bar{\partial}w_k = U_k^f\phi - f^{k+1}w_{k+1}, \quad 1 \leq k < \min(M, n+1),$$

if

$$(5.1) \quad H^{k-1}(X, \mathcal{O}(E_k)) = 0, \quad 1 \leq k \leq \min(M, n+1).$$

Then

$$q := U_1^f\phi - f^2w_2$$

is a holomorphic solution to $f^1q = \phi$. To sum up we have

Proposition 5.1. *Assume that X is smooth and ϕ is a holomorphic section of E_0 . If $R^f\phi = 0$ and (5.1) holds, then there is a global holomorphic section q of E_1 such that $f^1q = \phi$.*

Remark 5.2. Assume that ϕ belongs to the sheaf $\mathcal{J}_f = \text{Im } f^1$. This means that locally we have a holomorphic solution q to $\nabla_f q = \phi$. However, this does *not* imply that there is a global (smooth or current) solution to $\nabla_f v = \phi$, unless the complex $\mathcal{O}(E_\bullet), f$ is exact.

For instance, take global sections f_j^1 of $\mathcal{O}(d) \rightarrow \mathbb{P}^n$, i.e., homogeneous forms f_j^1 of degree d on \mathbb{C}^{n+1} , and let $\mathcal{O}(E_\bullet), f$ be the Koszul complex generated by $f^1 = (f_1^1, \dots, f_m^1)$, cf., Example 3.1, tensorized by $\mathcal{O}(\rho)$. Assume that ϕ is a section of $\mathcal{O}(\rho)$ that is locally in the image of f^1 , i.e., ϕ is a global section of $\mathcal{J}_f \otimes \mathcal{O}(\rho)$. If there is a global solution to $\nabla_f v = \phi$ and $\rho \geq (n+1)d - n$ so that (5.1) is fulfilled, then, cf., the proof of Theorem 1.1 below, there are holomorphic forms q_j such that $\sum f_j^1 q_j = \phi$. However, in general the mapping

$$\oplus \Gamma(\mathbb{P}^n, \mathcal{O}(\rho - d)) \xrightarrow{f^1} \Gamma(\mathbb{P}^n, \mathcal{J}_f \otimes \mathcal{O}(\rho))$$

seems to be surjective only if ρ is much larger than $(n+1)d - n$, see, e.g., [10] and [17, Proposition 4.16].

If $\mathcal{O}(E_\bullet), f$ is exact, then by the duality principle, ϕ annihilates the residue R^f , and so we get a global solution to $\nabla_f v = \phi$. One can also piece together local holomorphic solutions to a global smooth solution elementarily, using the exactness of $\mathcal{O}(E_\bullet), f$, cf., the proof of Lemma 5.6 below. \square

We will now present an analogous result for a singular X . Since we have no access to a $\bar{\partial}$ -theory for currents on X , we must in first place assume that the complex E_\bullet, f is defined in a small neighborhood of X in some embedding in a larger smooth manifold. One can get rid of the hypothesis of extendability of f , but to the cost of a slightly more complicated residue current to annihilate; see the proof of Theorem 1.4 below.

Theorem 5.3. *Let $i: X \rightarrow Y$ be an embedding of X in a projective manifold Y , let $\mathcal{O}(E_\bullet^g), g$ be a locally free Hermitian resolution of $\mathcal{O}^Y/\mathcal{J}_X$ in Y , and let ω be the associated structure form on X .*

Let (3.4) be a Hermitian complex over (an open neighborhood \mathcal{U} of X in) Y , and let $R^f \wedge \omega$ be the associated residue current. Moreover let ϕ be a global section of E_0 on X .

(i) *If $R^f \wedge \omega \phi = 0$, then there is a global smooth solution W on X to*

$$(5.2) \quad \nabla_f W = \phi.$$

(ii) *If (5.2) has a global smooth solution on X and (5.1) holds, then there is a global holomorphic section q of $\mathcal{O}(E_1)$ such that $f^1 q = \phi$ on X .*

With minor modifications of the proof we get the following more general version of Theorem 5.3:

With the general hypotheses of Theorem 5.3, assume that ϕ is a global holomorphic section of E_ℓ such that $f^\ell \phi = 0$.

(i) *If $R^\ell \wedge \omega \phi = 0$ then there is a smooth global solution to (5.2).*

(ii) *If (5.2) has a smooth solution and*

$$H^{0,k-1-\ell}(X, \mathcal{O}(E_k)) = 0, \quad \ell + 1 \leq k \leq \min(M, n + 1 + \ell),$$

then there is a global holomorphic section q of $E_{\ell+1}$ such that $f^{\ell+1} q = \phi$.

Remark 5.4. Assume that we have an embedding $i: X \rightarrow Y$ and that the data E_\bullet^f, f and ϕ are defined globally on Y . If $\nabla_f v = \phi$ has a (current) solution on Y , cf., Section 3.7, and all the the occurring $\bar{\partial}$ -equations are solvable, we get a global holomorphic solution (q, η) on Y to $f^1 q + g^1 \eta = \phi$. If for instance, $Y = \mathbb{P}^N$ and all the bundles E_k^f are direct sums of line bundles $\mathcal{O}(\ell)$ for various ℓ , then q is a tuple of homogeneous forms q_j on \mathbb{C}^{N+1} such that $\sum f_j^1 q_j = \phi$ on X . However, we get slightly sharper degree estimates if we solve $f^1 q = \phi$ intrinsically on X by Theorem 5.3 and then extend to a global solution by Proposition 6.1 below. \square

Remark 5.5. If we just have a current solution to $\nabla_f V = \phi$ on X it does not follow that there is a holomorphic solution, not even locally. In fact, if X is non-normal, there are holomorphic f and ϕ such that $\bar{\partial}(\phi/f) = 0$ but $U = \phi/f$ is not holomorphic. Thus $(f - \bar{\partial})U = \phi$ but ϕ is not in the ideal (f) . If X is normal but non-smooth, there are similar examples with more generators, see [25]. \square

We cannot assume that the section ϕ in Theorem 5.3 has a holomorphic extension to (a neighborhood of X in) Y . However, as a substitute we can find what we will call a ∇_g -closed extension.

Lemma 5.6. *Let $i: X \rightarrow Y$ be an embedding of X in a projective manifold Y and let ϕ be a global holomorphic section on X of a line bundle $S \rightarrow Y$.*

(i) *There is a global smooth section $\Phi = \sum_{\ell \geq 0} \Phi_\ell$ of $\oplus_{\ell \geq 0, \ell} \mathcal{E}_{0, \ell}(E_\ell^g \otimes S)$ on Y such that $\nabla_g \Phi = 0$ on Y and $\Phi_0 = \phi$ on X , i.e., $i^* \Phi_0 = \phi$.*

(ii) *Φ is such an extension of ϕ if and only if*

$$(5.3) \quad \Phi - R^g \phi = \nabla_g w$$

for some current w .

Recall that $E_0^g \simeq \mathbb{C}$ is a trivial line bundle.

One can obtain a ∇_g -closed extension Φ of ϕ quite elementarily by piecing together local holomorphic extensions, due to the exactness of $\mathcal{O}(E_\bullet^g), g$. However, we prefer an argument that also relates to residue calculus as in (ii), and we also think that Lemma 5.6 (ii) may be of independent interest.

Proof. As noted in Section 3.7, $R^g\phi$ is a well-defined ∇_g -closed current in Y . In view of Proposition 3.4 there is a smooth ∇_g -closed Φ such that (5.3) holds for some current w . Thus (i) follows from (ii).

Assume that Φ is a smooth extension of ϕ as in (i). From (3.6) we have that $\nabla_g(U^g\wedge\Phi) = \Phi - R^g\wedge\Phi$. Since $\mathcal{O}(E_\bullet^g), g$ is exact, $(R^g)^\ell = 0$ for $\ell \geq 1$, cf., Section 3.4, and hence $R^g\wedge\Phi = R^g\Phi_0 = R^g\phi$, since $\Phi_0 = \phi$ on X . Thus

$$\nabla_g(U^g\wedge\Phi) = \Phi - R^g\phi.$$

Conversely, assume that Φ is smooth and (5.3) holds. Then clearly $\nabla_g\Phi = 0$. We have to prove that $\Phi_0 = \phi$ on X . Since this is a local statement, given a point on X there is a neighborhood \mathcal{U} where we have holomorphic extension $\hat{\phi}$ of ϕ . Then $\nabla_g(U^g\hat{\phi}) = \hat{\phi} - R^g\hat{\phi} = \hat{\phi} - R^g\phi$ in \mathcal{U} . Thus $\nabla_g(w - U^g\hat{\phi}) = \Phi - \hat{\phi}$. By Proposition 3.4 there is a smooth ξ such that $\nabla_g\xi = \Phi - \hat{\phi}$. It follows that $g^1\xi_1 = \Phi_0 - \hat{\phi}$ and hence $\Phi_0 = \hat{\phi} = \phi$ in \mathcal{U} . \square

Proof of Theorem 5.3. Recall from Section 3.7 that $R^f\wedge\omega\phi = 0$ implies that $R^f\wedge R^g\phi = 0$. Let Φ be a ∇_g -closed smooth extension of ϕ , as in Lemma 5.6 (i), to Y . We claim that

$$(5.4) \quad R^f\wedge R^g\wedge\Phi = 0.$$

In fact, cf., the proof of Lemma 5.6, $R^g\wedge\Phi = R^g\Phi_0 = R^g\phi$. Thus $R^f\wedge R^g\wedge\Phi = R^f\wedge R^g\phi = 0$.

From (5.4) and (3.15) we get, cf., Section 3.7,

$$\nabla_F[(U^f\wedge R^g + U^g)\wedge\Phi] = \Phi.$$

By Proposition 3.4 we have a smooth solution Ψ to $\nabla_F\Psi = \Phi$ in Y ; i.e.,

$$F^1\Psi_1 = \Phi_0, \quad F^{k+1}\Psi_{k+1} - \bar{\partial}\Psi_k = \Phi_k, \quad k \geq 1.$$

If we let lower indices (i, j) denote values in $E_i^f \otimes E_j^g$, and notice that $\Phi_k = \Phi_{0,k}$, we see that

$$(5.5) \quad f^1\Psi_{1,0} + g^1\Psi_{0,1} = \Phi_0, \quad f^{k+1}\Psi_{k+1,0} + g^1\Psi_{k,1} - \bar{\partial}\Psi_{k,0} = 0, \quad k \geq 1.$$

Since Ψ is smooth we can define the forms $W_k = i^*\Psi_{k,0}$ on X , and (5.5) then means that

$$f^1W_1 = \phi, \quad f^{k+1}W_{k+1} - \bar{\partial}W_k = 0, \quad k \geq 1.$$

Thus (i) follows.

Now (ii) follows as in the case when X is smooth, cf., the beginning of Section 5 (now W plays the role of $U^f\phi$), but using the sheaves \mathcal{A}_k over X , rather than $\mathcal{C}_{0,k}$, and the isomorphisms in Section 3.8. \square

It should be possible to express the ∇_F -exactness of Φ in Y by means of Čech cohomology, then make the restriction to X , and rely on the vanishing of the relevant Čech cohomology groups on X . In this way one could possibly avoid the reference to the sheaves \mathcal{A}_k over X .

6. UPPER BOUNDS FOR κ AND ν

The following simple (and most certainly well-known) interpolation result gives an estimate of the number κ introduced in Section 1.

Proposition 6.1. *Let $i: X \rightarrow \mathbb{P}^N$ be an embedding. Assume that $0 \rightarrow \mathcal{O}(E_M) \xrightarrow{g^M} \dots \xrightarrow{g^3} \mathcal{O}(E_2) \xrightarrow{g^2} \mathcal{O}(E_1) \xrightarrow{g^1} \mathcal{O}$ is a resolution of the sheaf $\mathcal{O}/\mathcal{J}_X$ over \mathbb{P}^N , where E_k are as in Example 3.6. The restriction mapping*

$$(6.1) \quad \mathcal{O}(\mathbb{P}^N, \mathcal{O}(\ell)) \rightarrow \mathcal{O}(X, \mathcal{O}(\ell))$$

is surjective if $\ell \geq \max_i d_N^i - N$. If $M < N$, then (6.1) is surjective for all ℓ .

In particular, $\kappa \leq \max_i d_N^i - N$. For the proof we need, see, e.g., [14], that

$$(6.2) \quad H^k(\mathbb{P}^N, \mathcal{O}(\ell)) = 0 \quad \text{if } \ell \geq -N \quad \text{or} \quad k < N.$$

Proof. Let Φ be a smooth ∇_g -closed extension of ϕ to \mathbb{P}^N as in Lemma 5.6. Notice that Φ_N is a $(0, N)$ -form that takes values in $\oplus_i \mathcal{O}(\ell - d_N^i)$ so by (6.2) there is a smooth solution to $\bar{\partial} w_N = \Phi_N$ if $\ell - d_N^i \geq -N$. We can then successively find smooth solutions to the equations $\bar{\partial} w_k = \Phi_k + g^{k+1} w_{k+1}$ for $k = N-1, \dots, 1$. Finally we obtain the holomorphic section $\hat{\phi} = \Phi_0 + g^1 w_1$ of $\mathcal{O}(\ell)$ over \mathbb{P}^N , which, since w_1 is smooth, coincides with Φ_0 on X . If $M < N$ no cohomological obstruction occurs, cf., (6.2). \square

Let $M_X = \mathbb{C}[z_0, \dots, z_N]/J_X$, where J_X is the homogeneous ideal associated with X , cf., Example 3.6. The *regularity* $\text{reg } X$ of X introduced by Eisenbud and Goto, [18], is defined as $\max_{i,k} (d_k^i - k)$ for a minimal free resolution of M_X as in Example 3.6. It follows from Proposition 6.1 that $\kappa \leq \text{reg } X$. This estimate also follows from [18], see [17, Proposition 4.16]. The same proposition also implies that $H^i(X, \mathcal{O}(\ell)) = 0$ for all $i \geq 1$ if $\ell \geq \text{reg } X - 1$; thus $\nu \leq \text{reg } X - 1$. It follows that

$$(6.3) \quad \max(d \min(m, n+1) + \nu, d + \kappa) \leq d \min(m, n+1) + \text{reg } X.$$

If $m > 1$ we actually have strict inequality in (6.3).

If M_X is Cohen-Macaulay and X is not contained in any hyperplane in \mathbb{P}^N , then $\text{reg } X \leq \text{deg } X - (N-n)$, see e.g., [17, Corollary 4.15]. In that case (if $n, d, \text{deg } X \geq 2$) the last two entries in the estimates in Theorem 1.1 can be omitted unless $c_\infty = -\infty$. A particular case is when X is a complete intersection:

Example 6.2. Assume that $X \subset \mathbb{P}^N$ is a reduced complete intersection not contained in a hyperplane, i.e., there are homogeneous forms g_1, \dots, g_{N-n} such that $(g_j) = \mathcal{J}_X$ and $d_j := \text{deg } g_j \geq 2$. Then the Koszul complex generated by g_j is a resolution, and it terminates before level N unless $n = 0$. As long as $n > 0$ thus (6.1) is surjective for all ℓ , i.e., $\kappa = -\infty$. If $n = 0$, then $d_N^1 = d_1 + d_2 + \dots + d_N$ and hence $\kappa \leq d_1 + d_2 + \dots + d_N - N$. It is easy to see that $\text{reg } X = d_1 + \dots + d_{N-n} - (N-n)$ and thus $\nu \leq d_1 + \dots + d_{N-n} - (N-n) - 1$. By Bezout's formula, $\text{deg } X = d_1 \cdots d_{N-n}$ and since $d_j \geq 2$ this is larger than or equal to $\geq d_1 + \dots + d_{N-n} = \text{reg } X + (N-n)$. \square

We do not know whether κ and ν always are small compared to $\text{deg } X$.

7. PROOF OF THEOREM 2.4

Let f_1, \dots, f_m be the d -homogenizations of the polynomials F_1, \dots, F_m , considered as sections of $\mathcal{O}(d)|_X \rightarrow X$. Let $\mathcal{O}(E_\bullet^f), f$ be a locally free resolution of $\mathcal{O}_X/\mathcal{J}_f$ over X , where E_k^f have the form $\oplus_i E_k^i \otimes \mathcal{O}(-d_k^i)$ and E_k^i are trivial line bundles, cf. Section 3.6, and let R^f be the associated residue current on X . Recall that, cf., Section 3.4, that $R^f = (R^f)^0$.

Claim. *The hypothesis (2.4) implies that $\mathbf{1}_{X_\infty} R^f = 0$.*

By the dimension principle, $\mathbf{1}_{X_\infty} R_0^f = 0$ since the current has bidegree $(0, 0)$ and support on X_∞ , which has codimension at least 1. We proceed by induction so assume that $\mathbf{1}_{X_\infty} R_k^f = 0$. Outside Z_{k+1}^{bef} we then have, cf., Section 3.4, that

$$\mathbf{1}_{X_\infty} R_{k+1}^f = \mathbf{1}_{X_\infty} \alpha_{k+1} R_k^f = \alpha_{k+1} \mathbf{1}_{X_\infty} R_k^f = 0$$

since α_{k+1} is smooth there. Thus $\mathbf{1}_{X_\infty} R_{k+1}^f$, which has bidegree $(0, k+1)$, has support on $X_\infty \cap Z_{k+1}^{\text{bef}}$ and in view of (2.4) and the dimension principle it must vanish. Thus the claim follows.

Fix an integer $\rho \geq \deg \Phi$. Notice that the complex $E_\bullet^f \otimes \mathcal{O}(\rho), f$ also has R^f as its associated residue current. Let ϕ be the ρ -homogenization of Φ , so that ϕ is a section of $\mathcal{O}(\rho) = E_0^f \otimes \mathcal{O}(\rho)$. By the duality principle, $R^f \phi = 0$ in V since Φ is in (F_j) , and thus

$$(7.1) \quad \mathbf{1}_V R^f \phi = 0.$$

Since $R^f \phi = \mathbf{1}_V R^f \phi + \mathbf{1}_{X_\infty} R^f \phi$, cf., Section 3.1, we conclude, from (7.1) and the claim, that $R^f \phi = 0$.

Assume that the complex E_\bullet^f, f ends at level M and let

$$r_f = \max_i d_{\min(n+1, M)}^i.$$

If also $\rho \geq r_f + \nu$, it follows from Proposition 5.1 that we have a global holomorphic q on X such that $f^1 q = \phi$. Since q_j take values in $\mathcal{O}(\rho - d)|_X$ they have holomorphic extensions to \mathbb{P}^N if $\rho - d \geq \kappa$. We then get the desired polynomial solution Q_j to (1.1) after dehomogenization. Thus Theorem 2.4 follows with

$$\beta = \max(r_f + \nu, d + \kappa).$$

If f_1, \dots, f_m form a complete intersection, i.e., $\text{codim}_X Z^f = m$ (so that $Z^f = \emptyset$ if $m \geq n+1$), then the Koszul complex generated by f_j provides a resolution of $\mathcal{O}_X/\mathcal{J}_f$. Using the notation from the preceding proof, we then have $r_f = d \min(m, n+1)$. Since a complete intersection is locally Cohen-Macaulay, $Z_k^{\text{bef}} = \emptyset$ for $k > m$. The condition (2.4) then means that Z^f has no irreducible component contained in X_∞ . In particular, Corollary 2.3 for smooth X and Corollary 2.2 follow.

8. INTEGRAL CLOSURE, DISTINGUISHED VARIETIES AND RESIDUES

Let f_1, \dots, f_m be global holomorphic sections of the ample Hermitian line bundle $L \rightarrow X$, and let \mathcal{J}_f be the coherent ideal sheaf they generate. Let

$$\nu: X_+ \rightarrow X$$

be the normalization of the blow-up of X along \mathcal{J}_f , and let $W = \sum r_j W_j$ be the exceptional divisor; here W_j are irreducible Cartier divisors. The images $Z_j = \nu(W_j)$

are called the Fulton-MacPherson *distinguished varieties* associated with \mathcal{J}_f . If $f = (f_1, \dots, f_m)$ is considered as a section of $E^* = \bigoplus_1^m L$, then $\nu^*f = f^0 f'$, where f^0 is a section of the line bundle $\mathcal{O}(-W)$ defined by W , and $f' = (f'_1, \dots, f'_m)$ is a non-vanishing section of $\nu^*E \otimes \mathcal{O}(W)$ where, $\mathcal{O}(W) = \mathcal{O}(-W)^{-1}$. Furthermore, $\omega_f := dd^c \log |f'|^2$ is a smooth first Chern form for $\nu^*L \otimes \mathcal{O}(W)$.

Recall that (a germ of) a holomorphic function ϕ belongs to the *integral closure* $\overline{\mathcal{J}_{f,x}}$ of $\mathcal{J}_{f,x}$ at x if $\nu^*\phi$ vanishes to order (at least) r_j on W_j for all j such that $x \in Z_j$. This holds if and only if $|\nu^*\phi| \leq C|f^0|$ (in a neighborhood of the relevant W_j), which in turn holds if and only if $|\phi| \leq C|f|$ in some neighborhood of x . Let $\overline{\mathcal{J}}_f$ denote the integral closure sheaf. It follows that

$$(8.1) \quad |\phi| \leq C|f|^\ell \quad \text{if and only if} \quad \phi \in \overline{\mathcal{J}}_f^\ell.$$

If X is smooth it follows that ϕ is in the integral closure, if for each j , ϕ vanishes to order r_j at a generic point on Z_j . See [26, Section 10.5] for more details (e.g., the proof of Lemma 10.5.2).

We will use the geometric estimate

$$(8.2) \quad \sum r_j \deg_L Z_j \leq \deg_L X$$

from [15], see also [26, (5.20)].

Lemma 8.1. *There is a number μ_0 , only depending on X , such that if*

$$(8.3) \quad |\phi| \leq C|f|^{\mu+\mu_0},$$

then $R^f \wedge \omega \phi = 0$ if ω is a structure form of X and R^f is the residue current obtained from the Koszul complex of f . If X is smooth one can take $\mu_0 = 0$.

This proposition (and its proof) is analogous to Proposition 4.1 in [7]; the important novelty here is that μ_0 can be chosen uniform in ω , which is ensured by Proposition 3.5. However, for the readers convenience and further reference we discuss the proof.

Proof. Let us first assume that X is smooth and $\mu_0 = 0$, and that ϕ satisfies (8.3). Then ω is smooth so we have to show that $R^f \phi = 0$. If $f \equiv 0$ on (a component of) X , then $R^f \equiv 1$ and $\phi \equiv 0$, and thus $\phi R^f = 0$. Let us now assume that $\text{codim } Z^f \geq 1$. Then $R_0^f = 0$ by the dimension principle. Let $\nu: X_+ \rightarrow X$ be the normalization of the blow-up along \mathcal{J}_f as above, so that $\nu^*f = f^0 f'$. Using the notation in Example 3.1, then $\nu^*\sigma = (1/f^0)\sigma'$, where $1/f^0$ is a meromorphic section of $\mathcal{O}(W)$ and σ' is a smooth section of $\nu^*E \otimes \mathcal{O}(-W)$. It follows that

$$\nu^*(\sigma \wedge (\bar{\partial}\sigma)^{k-1}) = \frac{1}{(f^0)^k} \sigma' \wedge (\bar{\partial}\sigma')^{k-1},$$

and hence

$$\nu^* R_k^{f,\lambda} = \bar{\partial} |f^0 f'|^{2\lambda} \wedge \frac{1}{(f^0)^k} \sigma' \wedge (\bar{\partial}\sigma')^{k-1},$$

when $k \geq 1$. Recall that $R_0^f = 0$ unless $f \equiv 0$ on some component, in which case $\phi \equiv 0$ as well and $R^f = 1$ so that $R^f \phi = 0$.

Since f' is nonvanishing, the value at $\lambda = 0$ is precisely, see, e.g., [1, Lemma 2.1],

$$(8.4) \quad R_k^+ := \bar{\partial} \frac{1}{(f^0)^k} \wedge \sigma' \wedge (\bar{\partial}\sigma')^{k-1}.$$

Notice that

$$(8.5) \quad \nu_* R_k^+ = R_k^f.$$

Assume that ϕ satisfies (8.3) for $\mu_0 = 0$. Then $|\nu^*\phi| \leq C|f^0|^\mu$ and since X_+ is normal it follows that $\nu^*\phi$ contains a factor $(f^0)^\mu$. Therefore,

$$(8.6) \quad \nu^*\phi \bar{\partial} \frac{1}{(f^0)^k} = 0, \quad k \leq \mu,$$

because of (3.1). Moreover, since $\sigma' \wedge (\bar{\partial}\sigma')^{k-1}$ is smooth on X_+ , it follows from (8.6) and (8.4) that $\nu^*\phi R_k^+ = 0$. Therefore, cf., (8.5), $R_k^f \phi = \nu_*(R_k^+ \nu^*\phi) = 0$.

Notice that we could have used any normal modification $\pi: \tilde{X} \rightarrow X$ such that π^*f is of the form $f^0 f'$ in the proof so far.

Now consider a general X . Let us take a smooth modification $\tau: \tilde{X} \rightarrow X$ as in Section 4, such that $\tau^*\omega$ is semimeromorphic with a denominator that is locally a monomial in suitable coordinates s_j . In this proof it is convenient to use the regularization

$$R^{f,\epsilon} := 1 - \chi(|f|^2/\epsilon) + \bar{\partial}\chi(|f|^2/\epsilon) \wedge u,$$

where u is the form from Example 3.1 and χ is a smooth approximand of the characteristic function of $[1, \infty)$, cf., the beginning of Section 3, so that all the approximands $R^{f,\epsilon}$ are smooth. If $f \equiv 0$ on a component \tilde{X}_j of \tilde{X} , then $R^{f,\epsilon} \equiv 1$ on \tilde{X}_j and if ϕ satisfies (8.3) for any μ_0 , then $\phi \equiv 0$ on \tilde{X}_j . Hence $\mathbf{1}_{\tilde{X}_j} R^{f,\epsilon} \wedge \omega \phi = 0$ and so $\mathbf{1}_{\tilde{X}_j} R^f \wedge \omega \phi = 0$. We can therefore assume that $f \not\equiv 0$ on \tilde{X} . Locally on \tilde{X} , the action of $R^{f,\epsilon} \wedge \omega \phi$ on a test form is a sum of integrals like (suppressing the notation τ^* for simplicity)

$$\int_{\tilde{X}} \frac{ds_1 \wedge \dots \wedge ds_n}{s_1^{\alpha_1+1} \dots s_n^{\alpha_n+1}} \wedge R^{f,\epsilon} \phi \wedge \xi,$$

where α_j are nonnegative integers and ξ is a smooth form. Following [7, Section 3] one can integrate by parts $|\alpha| = |\alpha_1| + \dots + |\alpha_n|$ times, and get a constant times

$$(8.7) \quad \int_{\tilde{X}} \frac{ds_1 \wedge \dots \wedge ds_n}{s_1 \dots s_n} \wedge \partial_s^\alpha (R^{f,\epsilon} \phi \wedge \xi),$$

where $\partial_s^\alpha = \frac{\partial^{|\alpha|}}{\partial s_1^{\alpha_1} \dots \partial s_n^{\alpha_n}}$. If $\mu_0 \geq \mu + |\alpha| + 1$ and ϕ satisfies (8.3), then by the smooth Briançon-Skoda theorem, locally in \tilde{X} , ϕ is in the ideal $(f)^{\mu+|\alpha|+1}$. Therefore,

$$|\partial_s^\ell \phi| \leq C|f|^{\mu+|\alpha|-|\ell|+1}.$$

Following [7, Section 4] one finds that $\partial_s^\ell R^{f,\epsilon}$ has a certain homogeneity with a singularity that increases with ℓ : if we take a smooth modification $\pi: \hat{X} \rightarrow \tilde{X}$ such that π^*f is principal, then $\partial_s^\ell R^{f,\epsilon}$ is like $1/(f^0)^{\mu+|\ell|+1}$ and with support where $|f|^2 \sim \epsilon$. It follows by dominated convergence that (8.7) tends to zero when $\epsilon \rightarrow 0$, and this means that $R^f \wedge \omega \phi = 0$.

In view of Proposition 3.5 we can thus choose μ_0 that is larger than or equal to any number $\mu + |\alpha| + 1$ that may appear here, and so the lemma follows. \square

9. PROOFS OF THEOREMS 1.1 AND 2.1 AND VARIATIONS

For the proof of Theorem 1.1, besides the basic Lemma 8.1 we also need

Lemma 9.1. *Assume that $V \subset \mathbb{C}^N$ is smooth, and let ω be a structure form on X . Then there is a number μ' such that $z_0^{\mu'}\omega$ is almost smooth on X .*

Proof. Let $\tau : \tilde{X} \rightarrow X$ be as in Section 4. Then $\tilde{\omega} := \tau^*\omega$ is a semimeromorphic form whose denominator locally is a monomial whose zeros are contained in $\tau^{-1}X_{\text{sing}}$. Since V is smooth, $X_{\text{sing}} \subset X_\infty \subset \{z_0 = 0\}$, and it follows that $\tau^*(z_0^{\mu'})\omega$ is smooth for some large enough number μ' . Hence $z_0^{\mu'}\omega$ is almost smooth. \square

Proof of Theorem 1.1. Let f_j be the d -homogenizations of F_j and let R^f be the residue current constructed from the Koszul complex E_\bullet^f, δ_f generated by f_1, \dots, f_m , and let ϕ be the ρ -homogenization of Φ , where

$$(9.1) \quad \rho = \max(\deg \Phi + (\mu + \mu_0)d^{c_\infty} \deg X, d \min(m, n+1) + \nu, d + \kappa),$$

where μ_0 is chosen as in Lemma 8.1; in particular, $\mu_0 = 0$ if X is smooth. Throughout this proof we will use the notation from Section 8.

The assumption (1.4) implies that $\nu^*\phi$ vanishes to order $(\mu + \mu_0)r_j$ on each W_j such that $\nu(W_j)$ is not contained in X_∞ . Now consider W_j such that $\nu(W_j) \subset X_\infty$. If Ω is a first Chern form for $\mathcal{O}(1)|_X$, e.g., $\Omega = dd^c \log |z|^2$, then $d\Omega$ is a first Chern form for $L = \mathcal{O}(d)|_X$ on X (notice that d is the degree and not the differential). By (8.2) we therefore have that

$$r_j \int_{Z_j} (d\Omega)^{\dim Z_j} \leq \int_X (d\Omega)^n.$$

which implies that

$$(9.2) \quad r_j \leq d^{\text{codim } Z_j} \deg X.$$

By the choice (9.1) of ρ , ϕ is of the form $z_0^{(\mu+\mu_0)d^{c_\infty} \deg X}$ times a holomorphic section, and thus $\nu^*\phi$ vanishes to order at least $(\mu + \mu_0)r_j$ on W_j . Thus ϕ vanishes to order $(\mu + \mu_0)r_j$ on W_j for each j and thus (8.3) holds, cf., (8.1). It thus follows from Lemma 8.1 that $\phi R^f \wedge \omega = 0$.

Since $\rho \geq d \min(m, n+1) + \nu$ it follows that $E_k^f \otimes \mathcal{O}(\rho)$ is a direct sum of line bundles $\mathcal{O}(\ell_i)$, where $\ell_i \geq \nu$. By Theorem 5.3 we therefore have a holomorphic solution to $f q = \phi$ on X . By the definition of κ , q extends to a global section over \mathbb{P}^N . After dehomogenization, part (i) of Theorem 1.1 follows.

For the second part choose $\rho = \max(\deg \Phi + \mu d^{c_\infty} \deg X + \mu', d \min(m, n+1) + \nu, d + \kappa)$, where μ' is chosen as in Lemma 9.1, and let ϕ and ϕ' be the ρ - and $(\deg \Phi + \mu d^{c_\infty} \deg X)$ -homogenizations of Φ , respectively. Then

$$\phi R^f \wedge \omega = \phi' R^f \wedge \beta,$$

where β is almost smooth, and by (1.5) and (9.2),

$$(9.3) \quad |\phi'| \leq C|f|^\mu.$$

Now take a smooth modification $\pi : \tilde{X} \rightarrow X$ such that $\beta = \pi_* \tilde{\beta}$, where $\tilde{\beta}$ is smooth, and $f = f^0 f'$, where f^0 is a section of a line bundle and f' is nonvanishing. Then

$\phi R^f \wedge \omega$ is the push-forward under π of a finite sum of currents like

$$(\pi^* \phi') \bar{\partial} \frac{1}{(f^0)^\mu} \wedge \text{smooth},$$

cf., (8.4), (8.5), and in view of (9.3) they must vanish. Thus $\phi R^f \wedge \omega = 0$ and (ii) is proved as (i). \square

If $X = \mathbb{C}^n$ and $m \leq n$, then, cf., (6.2), there are no cohomological obstructions at all, and so we get the estimate (1.7).

Remark 9.2. If

$$(9.4) \quad \text{codim}(Z^f \cap X^\ell) \geq \mu + \ell + 1, \quad \ell \geq 0$$

where X^ℓ is as in Section 3.6 (thus either $X_{\text{sing}} \cap Z^f = \emptyset$ or $m < n$), then Theorem 1.1 (i) holds with $\mu_0 = 0$. To see this, take $\rho \geq \deg \Phi + \mu d^{c_\infty} \deg X$ in the proof of Theorem 1.1. Then $\phi R^f = 0$ on X_{reg} , and thus $\phi R^f \wedge \omega = 0$ there. From (9.4) and the dimension principle it follows that $\phi R^f \wedge \omega_0 = 0$.

To see this, take $\rho \geq \deg \Phi + \mu d^{c_\infty} \deg X$ in the proof of Theorem 1.1. Then $\phi R^f = 0$ on X_{reg} , and thus $\phi R^f \wedge \omega$ has support on $Z^f \cap X^0$. Since $\phi R^f \wedge \omega_0$ has bidegree at most (n, μ) and $\text{codim}(Z^f \cap X^0) \geq \mu + 1$ by (9.4), it follows from the dimension principle that $\phi R^f \wedge \omega_0 = 0$.

Thus $\phi R^f \wedge \omega_1 = \phi R^f \wedge \alpha^1 \omega_0$ vanishes outside X^1 , so again by (9.4) and the dimension principle we find that $\phi R^f \wedge \omega_1$ vanishes identically. By induction, $\phi R^f \wedge \omega = 0$. \square

Example 9.3. In light of Example 2.3 of Kollár in [24] one can see that the power c_∞ in Theorem 1.1 cannot be improved: Let $X = \mathbb{P}^n$ and let m be an integer with $2 \leq m \leq n$. Consider the m polynomials

$$z_1^d, z_1 z_m^{d-1} - z_2^d, \dots, z_{m-2} z_m^{d-1} - z_{m-1}^d, z_{m-1} z_m^{d-1} - 1,$$

in \mathbb{C}^n . The associated projective variety $\{z_0 = z_1 = \dots = z_{m-1} = 0\} \subset X_\infty$ has codimension m , and hence $c_\infty = m$, cf., (1.3). It follows from Theorem 1.1 that we have a representation (1.1) with $\Phi = 1$ and $\deg F_j Q_j \leq m d^m$ (if d is not too small). However, if Q_j are any polynomials so that (1.1) holds with $\Phi = 1$, then by considering the curve

$$t \mapsto (t^{d^{m-1}-1}, t^{d^{m-2}-1}, \dots, t^{d-1}, 1/t, 0, \dots, 0),$$

one can conclude that Q_1 must have degree at least $d^m - d$ so that $\deg F_1 Q_1 \geq d^m$. \square

In [3] is used a slight generalization of the Koszul complex to deal with a positive power \mathcal{J}_f^ℓ of \mathcal{J}_f , cf. [15, p. 439]. The first mapping in the complex is the natural mapping $E^{\otimes \ell} \rightarrow \mathbb{C}$ induced by the f_j . The associated residue current is the push-forward of currents like

$$\bar{\partial} \frac{1}{(f^0)^k} \wedge \text{smooth}$$

for $\ell \leq k \leq \mu + \ell - 1$. By an analogous proof we get the following generalization of Theorem 1.1.

Theorem 9.4. *With the notation in Theorem 1.1, if*

$$|\Phi| \leq C |F|^{\mu + \mu_0 + \ell - 1}$$

locally on V , then $\Phi \in (F_j)^\ell$ and there are polynomials Q_I such that

$$\Phi = \sum_{I_1 + \dots + I_m = \ell} F_1^{I_1} \dots F_m^{I_m} Q_I$$

and

$$\begin{aligned} \deg(F_1^{I_1} \dots F_m^{I_m} Q_I) \leq \\ \max(\deg \Phi + (\mu + \mu_0 + \ell - 1)d^{c_\infty} \deg X, d(\min(m, n+1) + \ell - 1) + \nu, d\ell + \kappa). \end{aligned}$$

There is also an analogous generalization of part (ii) of Theorem 1.1.

Proof of Theorem 2.1. Choose $\rho = \max(\deg \Phi + \mu d^{c_\infty} \deg X, d \min(m, n+1) + \nu, d + \kappa)$. As in the proof of Theorem 1.1, let ϕ be the ρ -homogenization of Φ and let $R^f \wedge \omega$ be the residue current associated with the Koszul complex of the homogenizations f_j of F_j , and make the decomposition

$$(9.5) \quad R^f \wedge \omega = \mathbf{1}_V R^f \wedge \omega + \mathbf{1}_{X_\infty} R^f \wedge \omega.$$

Since $\text{codim } Z^f \cap V \geq m$, the duality principle for a complete intersection implies that $\phi R^f = 0$ in $V_{\text{reg}} = V \setminus X^0$, and so $\mathbf{1}_V \phi R^f \wedge \omega_0$ has support on $Z^f \cap X^0$.

Consider now the normalization of the blow-up $\nu: X_+ \rightarrow X$, and let $R^+ := \sum R_k^+$ be as in the proof of Lemma 8.1. Let W_j be the irreducible components of $W = \nu^{-1} Z^f$ that are contained in $\nu^{-1} X_\infty$ and let W' be their union. We claim that

$$(9.6) \quad \mathbf{1}_{X_\infty} R^f = \nu_*(\mathbf{1}_{W'} R^+).$$

In fact, by (3.2),

$$(9.7) \quad \mathbf{1}_{X_\infty} R^f = \nu_*(\mathbf{1}_{\nu^{-1} X_\infty} R^+) = \nu_*(\mathbf{1}_{\nu^{-1} X_\infty} (\mathbf{1}_{W'} + \mathbf{1}_{W \setminus W'}) R^+).$$

By (3.3), $\mathbf{1}_{\nu^{-1} X_\infty} \mathbf{1}_{W'} R^+ = \mathbf{1}_{W'} R^+$. Moreover,

$$\mathbf{1}_{\nu^{-1} X_\infty} \mathbf{1}_{W \setminus W'} \wedge \bar{\partial} \frac{1}{(f^0)^k} = 0$$

by the dimension principle since $\nu^{-1} X_\infty \cap W \setminus W'$ has codimension at least 2 in X_+ . In view of (8.4) we conclude that $\mathbf{1}_{\nu^{-1} X_\infty} \mathbf{1}_{W \setminus W'} R^+ = 0$, and so (9.6) follows from (9.7).

Claim. $\mathbf{1}_{X_\infty} \phi R^f$ vanishes outside X^0 .

In view of (9.6), the dimension principle, and (8.4) it is enough to show that $\nu^* \phi \mathbf{1}_{W'} R^+$ vanishes in a neighborhood of each point x on W' where W' and W are regular. Consider now such a regular point x on say W_j . In a neighborhood of x we have that $f^0 = s^{r_j} v$, where s is a coordinate function and v is nonvanishing and r_j is as in Section 8. By the choice of ρ , $\nu^* \phi$ vanishes to order (at least) $d^{c_\infty} \mu \deg X$ on W' and thus in view of (9.2) it vanishes to order μr_j on W_j and hence it has a factor $s^{\mu r_j}$. Now in a neighborhood of x we have

$$\bar{\partial} \frac{1}{(f^0)^k} = \bar{\partial} \frac{1}{s^{kr_j}} \cdot \text{smooth}$$

and it follows that it is annihilated by $\nu^* \phi$ for $k \leq \mu$. Thus the claim follows in light of (8.4).

Summing up so far we have found, in view of (9.5), that $\phi R^f \wedge \omega_0$ has support on $Z^f \cap X^0$. By (2.2) and the dimension principle we conclude that it vanishes identically, since the bidegree of R^f is at most $(0, \mu)$. Thus $\phi R^f \wedge \omega_1 = \phi R^f \wedge \alpha^1 \omega_0$

vanishes outside X^1 , and by (2.2) and the dimension principle, it vanishes identically. By induction, it follows that $\phi R^f \wedge \omega_\ell = 0$ for each ℓ . We conclude that $\phi R^f \wedge \omega = 0$. Now the theorem follows from Theorem 5.3 as in previous proofs. \square

10. PROOFS OF THEOREM 1.4 AND VARIATIONS

We first look at the case when X is smooth.

Theorem 10.1. *Let X be a smooth projective variety, let $L \rightarrow X$ be an ample Hermitian line bundle, and let $A \rightarrow X$ is a line bundle that is either ample or big and nef. Moreover, let f_1, \dots, f_m be global holomorphic sections of L , and let ϕ be a section of*

$$L^{\otimes s} \otimes K_X \otimes A,$$

where $s \geq \min(m, n + 1)$. If

$$(10.1) \quad |\phi| \leq C|f|^\mu$$

on X , then there are holomorphic sections q_j of $L^{\otimes(s-1)} \otimes K_X \otimes A$ such that

$$(10.2) \quad f_1 q_1 + \dots + f_m q_m = \phi.$$

Let \mathcal{J}_f be the ideal sheaf generated by f_j and assume that the associated distinguished varieties Z_k have multiplicities r_k , cf., Section 8. If ϕ vanishes to (at least) order $r_k \mu$ at a generic point on Z_k , for each k , then, cf., e.g., the proof of Lemma 10.5.2 in [26], (10.1) holds, and thus we have

Corollary 10.2. *If ϕ vanishes to order $r_k \mu$ at a generic point on Z_k , for each k , then we have a representation (10.2).*

This corollary is precisely part (iii) of the main theorem in [15, p. 430], except for that we have μr_k rather than $(n + 1)r_k$, cf., the discussion in Remark 1.2. Recall from Section 8 that one can estimate the multiplicities r_k ; for instance $r_k \leq \deg_L X$, see (8.2).

One can also have a mixed hypothesis, and for instance assume that (10.1) holds outside a hypersurface H and that ϕ vanishes to order μr_k on each distinguished variety Z_k contained in H ; this would lead to an ‘‘abstract’’ Hickel theorem.

Proof of Theorem 10.1. Let E_\bullet, δ_f be the Koszul complex generated by f_1, \dots, f_m , as in Example 3.1, tensorized with $L^{\otimes s} \otimes A \otimes K_X$ and let R^f be the associated residue current on X . From the hypothesis (10.1) and Lemma 8.1 we conclude that $R^f \phi = 0$. The bundle E_k is a direct sum of line bundles $L^{\otimes(s-k)} \otimes A \otimes K_X$ and so all the relevant cohomology groups (5.1) vanish by Kodaira’s vanishing theorem, or, at the top degree, by the Kawamata-Viehweg vanishing theorem if A is nef and big. Thus Theorem 10.1 follows from Proposition 5.1. \square

Proof of Theorem 1.4. Let E_\bullet^f, δ_f be the Koszul complex generated by f_1, \dots, f_m tensorized with $L^{\otimes s}$. The choice of s guarantees that (5.1) is satisfied and thus by the same arguments as in the proof of Theorem 5.3 (ii) we get the desired holomorphic solution to (1.9) as soon as we have a smooth solution to

$$(10.3) \quad \nabla_f W = \phi$$

on X . Hence to prove the theorem it suffices to show that there is a μ_0 such that we can find a smooth solution to (10.3) for each global section ϕ of $L^{\otimes s}$ that satisfies (1.8). The strategy will be to follow and further elaborate the proof of Theorem 5.3 (i). Note that we cannot apply Theorem 5.3 (i) directly since a priori L and the sections f_j are only defined on X .

We first claim that there is an embedding $i: X \rightarrow Y$ into a smooth projective manifold Y and a line bundle $\mathbf{L} \rightarrow Y$ such that $L = \mathbf{L}|_X$, i.e., $L = i^*\mathbf{L}$. In fact, if M is large enough, there are embeddings $i_j: X \rightarrow \mathbb{P}^{N_j}$, $j = 1, 2$, such that $\mathcal{O}(1)_{\mathbb{P}^{N_1}}|_X = L^M$ and $\mathcal{O}(1)_{\mathbb{P}^{N_2}}|_X = L^{M+1}$. If $\pi_j: \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \rightarrow \mathbb{P}^{N_j}$, then $\mathbf{L} := \pi_2^*\mathcal{O}(1)_{\mathbb{P}^{N_2}} \otimes \pi_1^*\mathcal{O}(-1)_{\mathbb{P}^{N_1}}$ is a line bundle over $Y := \mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$ and its restriction to $X \simeq \Delta_{X \times X} \subset Y$ is precisely L . This argument was recently communicated to us by R. Lazarsfeld. Let $\mathcal{O}(E_\bullet^g), g$ be an Hermitian resolution of $\mathcal{O}^Y/\mathcal{I}_X$ in Y .

In general we cannot assume that the f_j extends holomorphically to Y or even a neighborhood of X in Y . However, let E_\bullet^h, h be a complex that is isomorphic to but disjoint from E_\bullet^g, g . Then, in view of Lemma 5.6, we can choose smooth ∇_h -closed extensions $\tilde{f}_j \in \oplus_i \mathcal{E}_{0,i}(E_i^h \otimes \mathbf{L})$ of f_j to Y , as defined in Section 5. Let E^1, \dots, E^m be trivial line bundles as in Example 3.1, with basis elements e_1, \dots, e_m , respectively, and let \tilde{f} be the section $\tilde{f} := f_j e_j^*$ of $E_\bullet^h \otimes E^*$, where $E := \bigoplus_{j=1}^m \mathbf{L}^{-1} \otimes E^j$ and e_j^* are the dual basis elements. Note that each \tilde{f}_j has even degree so that \tilde{f} has odd degree.

Inspired by Example 3.1 we want to construct a Koszul complex of \tilde{f} as an extension of E_\bullet^f, δ_f and an associated residue current. In order to do this we will need to take products of sections of E_\bullet^h . We therefore introduce $E_\bullet^H := \bigcup_{k \geq 1} (E_\bullet^h)^{\otimes k}$. Since E_0^h is the trivial line bundle, $(E_\bullet^h)^{\otimes k}$ is a natural subcomplex of $(E_\bullet^h)^{\otimes (k+1)}$ and thus the definition makes sense. In fact, our objects will all take values in $(E_\bullet^h)^{\otimes n}$. Next let $\delta_{\tilde{f}}: E_\bullet^H \otimes \Lambda^k E \rightarrow E_\bullet^H \otimes \Lambda^{k-1} E$ be contraction with \tilde{f} , i.e., for a section $\xi = \sum_{I=\{i_1, \dots, i_k\}} \xi_I \wedge e_I$, with $\xi_I \in E_\bullet^H$ and $e_I = e_{i_1} \wedge \dots \wedge e_{i_k}$, of $E_\bullet^H \otimes \Lambda^k E$, $\delta_{\tilde{f}} \xi = \sum_I (-1)^{\deg \xi_I} \xi_I \sum_j (-1)^{j-1} \tilde{f}_{i_j} \wedge e_{I \setminus i_j}$.

As long as we restrict to X we can write $\tilde{f} = f - f'$, where $f := \sum f_j e_j^*$ and f' has positive form degree. Let $\delta_{f'}$ be defined analogously to $\delta_{\tilde{f}}$ and let σ be the section of $E^{\tilde{f}}$ over $X \setminus Z$ of pointwise minimal norm such that $\delta_{\tilde{f}} \sigma = 1$ there, cf. Example 3.1. Then

$$\delta_{\tilde{f}} \sigma = \delta_f \sigma - \delta_{f'} \sigma = 1 - \delta_{f'} \sigma$$

on $X \setminus Z$. Notice that $\delta_{f'} \sigma$ has even degree, and form bidegree at least $(0, 1)$, so that

$$\frac{1}{1 - \delta_{f'} \sigma} = 1 + \delta_{f'} \sigma + (\delta_{f'} \sigma)^2 + \dots + (\delta_{f'} \sigma)^n$$

is a form on $X \setminus Z$ with values in $E_\bullet^H \otimes \Lambda^\bullet E$. Let $\tilde{\sigma} := \sigma / (1 - \delta_{f'} \sigma)$ on $X \setminus Z$; then $\delta_{\tilde{f}} \tilde{\sigma} = 1$ on $X \setminus Z$. Next, let

$$\tilde{u} = \frac{\tilde{\sigma}}{(\delta_{\tilde{f}} + \nabla_h) \tilde{\sigma}} = \sum_{k \geq 1} \tilde{\sigma} \wedge (-\nabla_h \tilde{\sigma})^{k-1},$$

cf. Example 3.1. Note that $\delta_{\tilde{f}}$ anticommutes with (the extension to $E_\bullet^H \otimes \Lambda^\bullet E$ of) ∇_h , i.e., $\delta_{\tilde{f}} \otimes \nabla_h = -\nabla_h \otimes \delta_{\tilde{f}}$. It follows that

$$(\delta_{\tilde{f}} + \nabla_h) \tilde{u} = 1$$

on $X \setminus Z$, cf. Section 3.2.

Let R^g be the residue current associated with the resolution $\mathcal{O}(E_\bullet^g), g$. Recall from Section 3.6 that if α is a sufficiently smooth form on X , then $\alpha \wedge R^g$ is a well-defined; in particular, $\chi(|f|^2/\epsilon)\tilde{u} \wedge R^g$ is a well-defined current in Y with values in $E_\bullet^H \otimes \Lambda^\bullet E \otimes E_\bullet^g$. Letting

$$\nabla = g + \delta_{\tilde{f}} + \nabla_h = g + \delta_{\tilde{f}} + h - \bar{\partial},$$

note that

$$\nabla(\chi(|f|^2/\epsilon)\tilde{u} \wedge R^g + U^g) = I - \tilde{R}^\epsilon \wedge R^g,$$

where $\tilde{R}^\epsilon = I - \chi(|f|^2/\epsilon)I + \bar{\partial}\chi(|f|^2/\epsilon) \wedge \tilde{u}$.

We claim that $\chi(|f|^2/\epsilon)\tilde{u} \wedge R^g$ has a limit when $\epsilon \rightarrow 0$. To see this, recall from Section 3.6, using the notation from that section, that $\chi(|f|^2/\epsilon)\tilde{u} \wedge R^g \wedge \gamma = i_*(\chi(|f|^2/\epsilon)\tilde{u} \wedge \omega)$. Next, notice that

$$(10.4) \quad \tilde{\sigma} \wedge (-\nabla_h \tilde{\sigma})^{k-1} = \sigma \wedge (\bar{\partial}\sigma)^{k-1} \wedge \sum_{j=0}^n c_j^k (\delta_{f'} \sigma)^j,$$

for some numbers c_j^k , since $\sigma \wedge \sigma = 0$. Let $\pi : \tilde{X} \rightarrow X$ be a smooth modification such that $\pi^*\omega$ is semimeromorphic and $\pi^*\sigma$ is of the form σ'/f^0 , cf. Section 8. Then $\pi^*\tilde{u}$ is a finite sum of terms $\alpha_k/(f^0)^k$, where α_k are smooth, and hence, by [12], $\lim_{\epsilon \rightarrow 0} \pi^*(\chi(|f|^2/\epsilon)\tilde{u} \wedge \omega)$ exists. Since γ is non-vanishing it follows that the limit of $\chi(|f|^2/\epsilon)\tilde{u} \wedge R^g$ exists.

Let

$$\tilde{U} \wedge R^g = \lim_{\epsilon \rightarrow 0} \chi(|f|^2/\epsilon)\tilde{u} \wedge R^g, \quad \tilde{R} \wedge R^g = \lim_{\epsilon \rightarrow 0} \tilde{R}^\epsilon \wedge R^g.$$

Then

$$\nabla(\tilde{U} \wedge R^g + U^g) = I - \tilde{R} \wedge R^g,$$

and if Φ is a smooth ∇_g -closed extension of ϕ as in Lemma 5.6 (regarded as a section of $E_\bullet^H \otimes \Lambda^\bullet E \otimes E_\bullet^g$), it follows that

$$(10.5) \quad \nabla((\tilde{U} \wedge R^g + U^g) \wedge \Phi) = \Phi$$

in Y as soon as

$$(10.6) \quad \tilde{R} \wedge R^g \phi = 0,$$

since, as was noted in the proof of Lemma 5.6, $R^g \wedge \Phi = R^g \phi$.

We claim that there is a μ_0 , only depending on X , such that (10.6) holds as soon as ϕ satisfies (1.8). Note that (10.6) is equivalent to that $\tilde{R} \wedge R^g \phi \wedge \gamma = \lim_{\epsilon \rightarrow 0} i_*(\tilde{R}^\epsilon \phi \wedge \omega)$ vanishes. Let $\tau : \tilde{X} \rightarrow X$ be a smooth modification as in Section 4, so that $\tau^*\omega = \frac{\text{smooth}}{s^{\alpha+1}}$ as in the proof of Lemma 8.1. Following that proof, the action of $\tilde{R}^\epsilon \phi \wedge \omega$ on a test form is a sum of integrals like (suppressing τ^* for simplicity)

$$(10.7) \quad \int_{\tilde{X}} \frac{ds_1 \wedge \dots \wedge ds_n}{s_1 \cdots s_n} \wedge \partial_s^\alpha (\tilde{R}^\epsilon \phi \wedge \xi),$$

where ξ is smooth. As in the proof of Lemma 8.1 it is enough to consider components of \tilde{X} where f does not vanish identically.

As in the proof of Lemma 8.1, $\partial_s^\alpha \tilde{R}^\epsilon$ has a certain homogeneity that increases with ℓ . Indeed, in view of (10.4), \tilde{R}^ϵ is a finite sum of terms like

$$\bar{\partial}\chi(|f|^2/\epsilon) \wedge \sigma \wedge (\bar{\partial}\sigma)^{k-1} \wedge (f' \cdot \sigma)^j,$$

where $k + j \leq n$ for degree reasons; recall that f' has form degree at least $(0, 1)$. Now, if we take a smooth modification $\pi : \tilde{X} \rightarrow \tilde{X}$ such that π^*f is principal, then $\pi^*\sigma = \text{smooth}/f^0$, $\pi^*\left(\frac{\partial}{\partial s_j}\sigma\right) = \text{smooth}/(f^0)^2$ and combining with the proof of Lemma 8.1 it follows that $\pi^*(\partial_s^\ell \tilde{R}^\epsilon)$ is like $1/(f^0)^{n+|\ell|+1}$ with support where $|f|^2 \sim \epsilon$. Choose $\mu_0 \geq n + |\alpha| + 1$. Then $|\partial_s^\ell \phi| \leq C|f|^{n+|\alpha|-|\ell|+1}$, cf. the proof of Lemma 8.1. Now by dominated convergence (10.7) tends to zero when $\epsilon \rightarrow 0$, and since the choice of μ_0 only depends on α and n the claim follows.

To sum up so far, if ϕ satisfies (1.8), we have a current solution to $\nabla\Psi = \Phi$. By a slight modification of Proposition 3.4 we also have a smooth solution. To see this, let $E_\bullet^F = \Lambda^\bullet E \otimes E_\bullet^g$ and let \mathcal{M}_\bullet and $\mathcal{M}_\bullet^\mathcal{E}$ be defined as in Section 3.3, but for the complex E_\bullet^H instead of E_\bullet^f . Then we have the double complex

$$\mathcal{B}_{\ell,k} = \bigoplus_j \mathcal{C}_{0,j}(E_{j+k}^H \otimes E_\ell^F) =: \mathcal{M}_k(E_\ell^F)$$

with mappings $\nabla_h : \mathcal{B}_{\ell,k} \rightarrow \mathcal{B}_{\ell,k-1}$ and $F := g + \delta_{\tilde{f}} : \mathcal{B}_{\ell,k} \rightarrow \mathcal{B}_{\ell-1,k}$; indeed note that $\nabla_h \circ F = -F \circ \nabla_h$. If $\mathcal{B}_j := \bigoplus_{\ell+k=j} \mathcal{B}_{\ell,k}$ we get the associated total complex

$$\dots \xrightarrow{\nabla} \mathcal{B}_j \xrightarrow{\nabla} \mathcal{B}_{j-1} \xrightarrow{\nabla} \dots$$

Let $\mathcal{B}_{\ell,k}^\mathcal{E} = \bigoplus_j \mathcal{E}_{0,j}(E_{j+k}^H \otimes E_\ell^F) =: \mathcal{M}_k^\mathcal{E}(E_\ell^F)$ with total complex $\mathcal{B}_\bullet^\mathcal{E}$. Moreover, let $\mathcal{B}_\bullet(X)$ and $\mathcal{B}_\bullet^\mathcal{E}(X)$ be the associated complexes of global sections. Note that we have natural mappings

$$(10.8) \quad H^j(\mathcal{B}_\bullet^\mathcal{E}(X)) \rightarrow H^j(\mathcal{B}_\bullet(X)), \quad j \in \mathbb{Z}.$$

Proposition 3.4 implies that the natural mappings $H^k(\mathcal{M}_\bullet^\mathcal{E}(X, E_\ell^F)) \rightarrow H^k(\mathcal{M}_\bullet(X, E_\ell^F))$ are isomorphisms. Now, by repeating the proof of Proposition 3.4 with \mathcal{M}_\bullet , $\mathcal{M}_\bullet^\mathcal{E}$, $\mathcal{C}_{0,\bullet}$, and $\mathcal{E}_{0,\bullet}$ replaced by \mathcal{B}_\bullet , $\mathcal{B}_\bullet^\mathcal{E}$, \mathcal{M}_\bullet , and $\mathcal{M}_\bullet^\mathcal{E}$, respectively, using that the double complex $\mathcal{B}_{\ell,k}$ is bounded in the ℓ -direction, we can therefore prove that the mappings (10.8) are in fact isomorphisms, and so the current solution (10.5) gives a smooth solution to $\nabla\Psi = \Phi$.

Let lower indices (i, j, k) denote components in $E_i^H \otimes \Lambda^j E \otimes E_k^g$. Then $\Phi = \Phi_{0,0,0} + \Phi_{0,0,1} + \dots + \Phi_{0,0,n}$, where $\Phi_{0,0,k}$ has form bidegree $(0, k)$. Notice that we have the decomposition $\tilde{f} = f_0 - f'$ in Y , where f_0 denotes the 0-component of \tilde{f} and hence is a smooth extension of f to Y . It follows that

$$(10.9) \quad h\Psi_{1,0,0} + \delta_{f_0}\Psi_{0,1,0} + g\Psi_{0,0,1} = \Phi_0, \\ h\Psi_{1,j,0} + \delta_{f_0}\Psi_{0,j+1,0} + g\Psi_{0,j,1} - \bar{\partial}\Psi_{0,j,0} = 0, \quad j \geq 1,$$

Indeed, note that $\delta_{f'}\Psi_{i,j,k}$ has positive degree in E_\bullet^H for all non-vanishing $\Psi_{i,j,k}$. Since Ψ is smooth, we can define the smooth forms $W_j := i^*\Psi_{0,j,0}$ on X . Since $g\Psi_{0,j,1} = g^1\Psi_{0,j,1}$ and $h\Psi_{1,j,0} = h^1\Psi_{1,j,0}$ are in \mathcal{J}_X , (10.9) implies

$$\delta_f W_1 = \phi, \quad \delta_f W_{j+1} - \bar{\partial}W_j = 0, \quad j \geq 1.$$

Thus we have shown that if $\mu_0 \geq n + |\alpha| + 1$ and ϕ satisfies (1.8), then we get a smooth solution to $\nabla_f W = \phi$; this concludes the proof. \square

Remark 10.3. If E_\bullet^h, h is a Koszul complex, then we just simply take $E_\bullet^H = E_\bullet^h$, since the desired "product" already exists within E_\bullet^h . \square

In analogy with Theorem 9.4 we also have the following generalizations of Theorems 10.1 and 1.4.

Theorem 10.4. *With the notation in Theorem 10.1, if ϕ is a section of $L^{\otimes s} \otimes K_X \otimes A$, where $s \geq \min(m, n+1) + \ell - 1$, and*

$$|\phi| \leq C|f|^{\mu+\ell-1},$$

there are holomorphic sections q_I , $|I| = \ell$, of $L^{\otimes(s-\ell)} \otimes K_X \otimes A$, such that

$$(10.10) \quad \phi = \sum_{I_1+\dots+I_m=\ell} f_1^{I_1} \cdots f_m^{I_m} q_I.$$

With the notation in Theorem 1.4, if ϕ is a section of $L^{\otimes s}$ with $s \geq \nu_L + \min(m, n+1) + \ell - 1$ such that

$$|\phi| \leq C|f|^{\mu_0+\mu+\ell-1},$$

then there are holomorphic sections q_I of $L^{\otimes(s-\ell)}$ such that (10.10) holds.

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