Confluent A-hypergeometric functions and rapid decay homology cycles *

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Abstract

We study confluent A-hypergeometric functions introduced by Adolphson [1]. In particular, we give their integral representations by using rapid decay homology cycles of Hien [10] and [11]. The method of toric compactifications introduced in [21] and [25] will be used to prove our main theorem.

1 Introduction

The theory of A-hypergeometric systems introduced by Gelfand-Kapranov-Zelevinsky [7] is a vast generalization of that of classical hypergeometric differential equations. As in the case of hypergeometric equations, the holomorphic solutions to their A-hypergeometric systems (i.e. the A-hypergeometric functions) admit power series expansions ([7]) and integral representations ([8]). Moreover this theory has deep connections with other fields of mathematics, such as toric varieties, projective duality, period integrals, mirror symmetry and combinatorics. Also from the viewpoint of $\mathcal{D}$-module theory (see [15] and [16] etc.), A-hypergeometric $\mathcal{D}$-modules are very elegantly constructed in [8]. For the recent development of this subject see [32] and [33] etc. In [3], [8], [13] and [36] etc. the monodromy of their A-hypergeometric functions was studied. In [1] Adolphson generalized the hypergeometric systems of Gelfand-Kapranov-Zelevinsky [7] to the confluent (i.e. irregular) case and proved many important results. However the construction of the confluent A-hypergeometric $\mathcal{D}$-modules is not functorial as in [7] and [8]. This leads us to some difficulties in obtaining the integral representations of their holomorphic solutions. In this paper, we construct Adolphson's confluent A-hypergeometric $\mathcal{D}$-modules functorially as in [8]. Note that recently the same problem was solved more completely in Saito [31] and Schulze-Walther [34], [35] by using commutative algebras. However our approach is based on sheaf-theoretical methods and totally different from theirs. In this paper we also go a little bit further and give an integral representation of A-hypergeometric functions

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by using the rapid decay homology cycles introduced in Hien [10] and [11]. Our integral representation

$$u(z) = \int_{\gamma_z} \exp(\sum_{j=1}^{N} z_j x^{a(j)}) x_1^{c_1-1} \cdots x_n^{c_n-1} dx_1 \wedge \cdots \wedge dx_n \tag{1.1}$$

coincides with the one in Adolphson [1, Equation (2.6)], where $\gamma = \{\gamma_z\}$ is a family of real n-dimensional topological cycles γ_z in the algebraic torus $T = (\mathbb{C}^*)_x^n$ on which the function $\exp(\sum_{j=1}^{N} z_j x^{a(j)}) x_1^{c_1-1} \cdots x_n^{c_n-1}$ rapidly decays at infinity. See Sections 3 and 4 for the details. In other words, we could give a geometric meaning to Adolphson's formula [1, Equation (2.6)] by using rapid decay homology cycles. Note that this integral representation can be considered as a natural generalization of those for the classical Bessel and Airy functions etc. Recall that in the case of hypergeometric functions associated to hyperplane arrangements the same problem was precisely studied by Kimura-Haraoka-Takano [19] etc. We hope that our geometric construction would be useful in the explicit study of Adolphson's confluent A-hypergeometric functions. In the course of the proof, we will use the method of toric compactifications introduced in [21] and [25]. Moreover we will introduce Proposition 3.3 which enables us to calculate Hien's rapid decay homologies in [10], [11] by using (usual) relative twisted homologies. By Proposition 3.3 and Lemmas 3.4 and 3.5 we can calculate the rapid decay homologies very explicitly in many cases.

2 Adolphson's results

First of all, we recall the definition of the confluent A-hypergeometric systems introduced by Adolphson [1] and their important properties. In this paper, we essentially follow the terminology of [15] and [16] etc. Let $A = \{a(1), a(2), \dots, a(N)\} \subset \mathbb{Z}^n$ be a finite subset of the lattice $\mathbb{Z}^n$. Assume that A generates $\mathbb{Z}^n$ as in [7] and [8]. Following [1] we denote by Δ the convex hull of $A \cup \{0\}$ in $\mathbb{R}^n$. By definition Δ is an n-dimensional polytope. Let $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ be a parameter vector. Moreover consider the $n \times N$ integer matrix

$$A := \begin{pmatrix} {}^t a(1) & {}^t a(2) & \cdots & {}^t a(N) \end{pmatrix} = (a_{i,j}) \in M(n, N, \mathbb{Z}) \tag{2.1}$$

whose j-th column is ${}^t a(j)$. Then Adolphson's confluent A-hypergeometric system on $X = \mathbb{C}^A = \mathbb{C}_z^N$ associated with the parameter vector $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ is

$$\left(\sum_{j=1}^{N} a_{i,j} z_j \frac{\partial}{\partial z_j} + c_i \right) u(z) = 0 \quad (1 \leq i \leq n), \tag{2.2}$$

$$\left\{ \prod_{\mu_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-\mu_j} \right\} u(z) = 0 \quad (\mu \in \mathrm{Ker} A \cap \mathbb{Z}^N). \tag{2.3}$$

Remark 2.1. *In [1] Adolphson does not assume that A generates $\mathbb{Z}^n$. However we need this condition to obtain a geometric construction of his confluent A-hypergeometric systems. Even when A does not generate $\mathbb{Z}^n$, by a suitable linear coordinate change of $\mathbb{R}^n$ we can get an equivalent system for $A' \subset \mathbb{Z}^n$ and $c' \in \mathbb{C}^n$ such that A' generates $\mathbb{Z}^n$. Namely our condition is not restrictive at all.*

Let $D(X)$ be the Weyl algebra over X and consider the differential operators

$$Z_{i,c} := \sum_{j=1}^{N} a_{ij} z_j \frac{\partial}{\partial z_j} + c_i \quad (1 \leq i \leq n), \tag{2.4}$$

$$\Box_\mu := \prod_{\mu_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-\mu_j} \qquad (\mu \in \mathrm{Ker} A \cap \mathbb{Z}^N) \tag{2.5}$$

in it. Then the above system is naturally identified with the left $D(X)$-module

$$M_{A,c} = D(X) / \left(\sum_{1 \leq i \leq n} D(X) Z_{i,c} + \sum_{\mu \in \mathrm{Ker} A \cap \mathbb{Z}^N} D(X) \Box_\mu \right). \tag{2.6}$$

Let $\mathcal{D}_X$ be the sheaf of differential operators over the "algebraic variety" X and define a coherent $\mathcal{D}_X$-module by

$$\mathcal{M}_{A,c} = \mathcal{D}_X / \left(\sum_{1 \leq i \leq n} \mathcal{D}_X Z_{i,c} + \sum_{\mu \in \mathrm{Ker} A \cap \mathbb{Z}^N} \mathcal{D}_X \Box_\mu \right). \tag{2.7}$$

Then $\mathcal{M}_{A,c}$ is the localization of the left $D(X)$-module $M_{A,c}$ (see [15, Proposition 1.4.4 (ii)] etc.). Adolphson [1] proved that $\mathcal{M}_{A,c}$ is holonomic. In fact, he proved the following more precise result.

Definition 2.2. *(Adolphson [1, page 274], see also [28] etc.) For $z \in X = \mathbb{C}^A$ we say that the Laurent polynomial $h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)}$ is non-degenerate if for any face Γ of Δ not containing the origin $0 \in \mathbb{R}^n$ we have*

$$\left\{ x \in T = (\mathbb{C}^*)^n \mid h_z^\Gamma(x) = \frac{\partial h_z^\Gamma}{\partial x_1}(x) = \cdots\cdots = \frac{\partial h_z^\Gamma}{\partial x_n}(x) = 0 \right\} = \emptyset, \tag{2.8}$$

where we set $h_z^\Gamma(x) = \sum_{j : a(j) \in \Gamma} z_j x^{a(j)}$.

Let $\Omega \subset X$ be the Zariski open subset of X consisting of $z \in X = \mathbb{C}^A$ such that the Laurent polynomial $h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)}$ is non-degenerate. Then Adolphson's result in [1, Lemma 3.3] asserts that the holonomic $\mathcal{D}_X$-module $\mathcal{M}_{A,c}$ is an integrable connection on Ω (i.e. the characteristic variety of $\mathcal{M}_{A,c}$ is contained in the zero section of the cotangent bundle $T^*\Omega$). Now let X^{an} (resp. Ω^{an}) be the underlying complex analytic manifold of X (resp. Ω) and consider the holomorphic solution complex $\mathrm{Sol}_X(\mathcal{M}_{A,c}) \in \mathbf{D}^{\mathrm{b}}(X^{\mathrm{an}})$ of $\mathcal{M}_{A,c}$ defined by

$$\mathrm{Sol}_X(\mathcal{M}_{A,c}) = R\mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{M}_{A,c})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \tag{2.9}$$

(see [15] etc. for the details). Then by the above Adolphson's result, $\mathrm{Sol}_X(\mathcal{M}_{A,c})$ is a local system on Ω^{an}. Moreover he proved the following remarkable result.

Theorem 2.3. *(Adolphson [1, Corollary 5.20]) Assume that the parameter vector $c \in \mathbb{C}^n$ is semi-nonresonant (see [1, page 284]). Then the rank of the local system $H^0\mathrm{Sol}_X(\mathcal{M}_{A,c})$ on Ω^{an} is equal to the normalized n-dimensional volume $\mathrm{Vol}_{\mathbb{Z}}(\Delta) \in \mathbb{Z}$ of Δ with respect to the lattice $\mathbb{Z}^n$.*

This is a generalization of the famous result of Gelfand-Kapranov-Zelevinsky in [7] to the confluent case. The sections of the local system $H^0\mathrm{Sol}_X(\mathcal{M}_{A,c})|_{\Omega^{\mathrm{an}}}$ are called A-hypergeometric functions (associated to the parameter $c \in \mathbb{C}^n$).

3 Hien's rapid decay homologies

In this section, we review Hien's theory of rapid decay homologies invented in [10] and [11]. Let U be a smooth quasi-projective variety of dimension n and $(\mathcal{E}, \nabla)$ an integrable connection on it. We consider $(\mathcal{E}, \nabla)$ as a left $\mathcal{D}_U$-module and set

$$\mathrm{DR}_U(\mathcal{E}) = \Omega_{U^{\mathrm{an}}} \otimes^L_{\mathcal{D}_{U^{\mathrm{an}}}} \mathcal{E}^{\mathrm{an}} \simeq \Omega^{\cdot}_{U^{\mathrm{an}}} \otimes_{\mathcal{O}_{U^{\mathrm{an}}}} \mathcal{E}^{\mathrm{an}}[n]. \tag{3.1}$$

Assume that $i : U \hookrightarrow Z$ is a smooth projective compactification of U such that $D = Z \setminus U$ is a normal crossing divisor and the extension $i_*\mathcal{E}$ of $\mathcal{E}$ to Z admits a good lattice in the sense of Sabbah [30] and Mochizuki [26]. Such a compactification always exists by the fundamental theorem established by Mochizuki [26]. Now let $\pi : \tilde{Z} \longrightarrow Z^{\mathrm{an}}$ be the real oriented blow-up of Z^{an} in [10], [11] and set $\tilde{D} = \pi^{-1}(D^{\mathrm{an}})$. Recall that π induces an isomorphism $\tilde{Z} \setminus \tilde{D} \xrightarrow{\sim} Z^{\mathrm{an}} \setminus D^{\mathrm{an}}$. More precisely, for each point $q \in D^{\mathrm{an}}$ by taking a local coordinate $(x_1, \ldots, x_n)$ on a neighborhood of q such that $q = (0, \ldots, 0)$ and $D^{\mathrm{an}} = \{x_1 \cdots x_k = 0\}$ the morphism π is explicitly given by

$$([0, \varepsilon) \times S^1)^k \times B(0; \varepsilon)^{n-k} \longrightarrow B(0; \varepsilon)^k \times B(0; \varepsilon)^{n-k} \tag{3.2}$$

$$(\{(r_i, e^{i\theta_i})\}_{i=1}^k, x_{k+1}, \ldots, x_n) \longmapsto (\{r_i e^{i\theta_i}\}_{i=1}^k, x_{k+1}, \ldots, x_n), \tag{3.3}$$

where we set $B(0; \varepsilon) = \{x \in \mathbb{C} \mid |x| < \varepsilon\}$ for $0 < \varepsilon$. For $p \geq 0$ and a subset $B \subset \tilde{Z}$ denote by $S_p(B)$ the $\mathbb{C}$-vector space generated by the piecewise smooth maps $c : \Delta^p \longrightarrow B$ from the p-dimensional simplex Δ^p. We denote by $\mathcal{C}^{-p}_{\tilde{Z},\tilde{D}}$ the sheaf on $\tilde{Z}$ associated to the presheaf

$$V \longmapsto S_p(\tilde{Z}, (\tilde{Z} \setminus V) \cup \tilde{D}) = S_p(\tilde{Z}) / S_p((\tilde{Z} \setminus V) \cup \tilde{D}). \tag{3.4}$$

Namely $\mathcal{C}^{-p}_{\tilde{Z},\tilde{D}}$ is the sheaf of the relative p-chains on the pair $(\tilde{Z}, \tilde{D})$. Now let $\mathcal{L} := H^{-n}\mathrm{DR}_U(\mathcal{E}) = \mathrm{Ker}\{\nabla^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \longrightarrow \Omega^1_{U^{\mathrm{an}}} \otimes_{\mathcal{O}_{U^{\mathrm{an}}}} \mathcal{E}^{\mathrm{an}}\}$ be the sheaf of horizontal sections of the analytic connection $(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$ and $\iota : U^{\mathrm{an}} \hookrightarrow \tilde{Z}$ the inclusion. Then $\iota_*\mathcal{L}$ is a local system on $\tilde{Z}$. We define the sheaf $\mathcal{C}^{-p}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L})$ of the relative twisted p-chains on the pair $(\tilde{Z}, \tilde{D})$ with coefficients in $\iota_*\mathcal{L}$ by $\mathcal{C}^{-p}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L}) = \mathcal{C}^{-p}_{\tilde{Z},\tilde{D}} \otimes_{\mathbb{C}_{\tilde{Z}}} \iota_*\mathcal{L}$.

Definition 3.1. *(Hien [10] and [11]) A section $\gamma = c \otimes s \in \Gamma(V; \mathcal{C}^{-p}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L}))$ is called a rapid decay chain if for any point $q \in c(\Delta^p) \cap \tilde{D} \cap V$ the following condition holds:*

In a local coordinate $(x_1, \ldots, x_n)$ on a neighborhood of q such that $q = (0, \ldots, 0)$ and $D^{\mathrm{an}} = \{x_1 \cdots x_k = 0\}$ by taking a local trivialization $(i_*\mathcal{E})^{\mathrm{an}} \simeq \oplus_{i=1}^r \mathcal{O}_{Z^{\mathrm{an}}}(*D^{\mathrm{an}})e_i$ for a free basis $e_1, \ldots, e_r$ and setting $s = \sum_{i=1}^r f_i \cdot \iota_* i^{-1} e_i$ ($f_i \in \iota_*\mathcal{O}_{Z^{\mathrm{an}}}$), for any $1 \leq i \leq r$ and $N = (N_1, \ldots, N_k) \in \mathbb{N}^k$ there exists $C_N > 0$ such that

$$|f_i(x)| \leq C_N |x_1|^{N_1} \cdots |x_k|^{N_k} \tag{3.5}$$

for any $x \in (c(\Delta^p) \setminus \tilde{D}) \cap V$ with small $|x_1|, \ldots, |x_k|$.

In particular, if $c(\Delta^p) \cap \tilde{D} \cap V = \emptyset$ we do not impose any condition on $s \in \iota_*\mathcal{L}$.

Note that this definition does not depend on the local coordinate $(x_1, \ldots, x_n)$ nor the local trivialization $(i_*\mathcal{E})^{\mathrm{an}} \simeq \oplus_{i=1}^r \mathcal{O}_{Z^{\mathrm{an}}}(*D^{\mathrm{an}})e_i$. We denote by $\mathcal{C}^{\mathrm{rd},-p}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L})$ the subsheaf of

$\mathcal{C}^{-p}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L})$ consisting of rapid decay chains. According to Hien [10] and [11], $\mathcal{C}^{\mathrm{rd},-p}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L})$ is a fine sheaf. Then we otain a complex of fine sheaves on $\tilde{Z}$:

$$\mathcal{C}^{\mathrm{rd},-\cdot}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L}) = \left[\cdots \longrightarrow \mathcal{C}^{\mathrm{rd},-(p+1)}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L}) \longrightarrow \mathcal{C}^{\mathrm{rd},-p}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L}) \longrightarrow \mathcal{C}^{\mathrm{rd},-(p-1)}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L}) \longrightarrow \cdots \right]. \tag{3.6}$$

Definition 3.2. *(Hien [10] and [11]) For $p \in \mathbb{Z}$ we set*

$$H^{\mathrm{rd}}_p(U;\mathcal{E}) := H^{-p}\Gamma(\tilde{Z};\mathcal{C}^{\mathrm{rd},-\cdot}_{\tilde{Z},\tilde{D}}(\iota_*\mathcal{L})) \tag{3.7}$$

and call it the p-th rapid decay homology group associated to the integrable connection $\mathcal{E}$.

If $\mathcal{E} \simeq \mathcal{O}_U$ and its analytification $\mathcal{E}^{\mathrm{an}} \simeq \mathcal{O}_{U^{\mathrm{an}}}$ induces an isomorphism $\mathcal{L} \simeq \mathbb{C}_{U^{\mathrm{an}}} g$ ($\subset \mathcal{O}_{U^{\mathrm{an}}}$) for a possibly multi-valued holomorphic function $g : U^{\mathrm{an}} \longrightarrow \mathbb{C}$, we call $H^{\mathrm{rd}}_p(U;\mathcal{E})$ the p-th rapid decay homology group associated to the function g. In the special case where $g(x) = \exp(h(x))g_0(x)$ for a meromorphic function h on Z^{an} with poles in D^{an} and a possibly multi-valued holomorphic function g_0 on U^{an}, we shall give a purely topological interpretation of $H^{\mathrm{rd}}_p(U;\mathcal{E})$. By the admissibility of $\mathcal{E}$ the meromorphic function h has no point of indeterminacy on the whole Z^{an} (see [12, the paragraph just below Definition 2.1] etc.). By $\iota : U^{\mathrm{an}} \hookrightarrow \tilde{Z}$ we consider U^{an} as an open subset of $\tilde{Z}$ and set

$$P = \tilde{D} \cap \overline{\{x \in U^{\mathrm{an}} \mid \mathrm{Re}h(x) \geq 0\}}. \tag{3.8}$$

Let $D = D_1 \cup \cdots \cup D_m$ be the irreducible decomposition of D. For $1 \leq i \leq m$ let $b_i \in \mathbb{Z}$ be the order of the meromorphic function h along D_i. If $b_i \geq 0$ we say that the irreducible component D_i is irrelevant. Denote by D' the union of the irrelevant components of D. Then we set $Q = \tilde{D} \setminus \{P \cup \pi^{-1}(D')^{\mathrm{an}}\}$. Note that Q is an open subset of $\tilde{D}$ (i.e. the set of the rapid decay directions of the function g in $\tilde{D}$). By dividing U^{an} into small sectors and using a homotopy argument on each of them, we can easily prove the following useful proposition.

Proposition 3.3. *In the situation as above, we have an isomorphism*

$$H^{\mathrm{rd}}_p(U;\mathcal{E}) \simeq H_p(U^{\mathrm{an}} \cup Q, Q; \iota_*(\mathbb{C}_{U^{\mathrm{an}}} g_0)) \tag{3.9}$$

for any $p \in \mathbb{Z}$, where the right hand side is the p-th relative twisted homology group of the pair $(U^{\mathrm{an}} \cup Q, Q)$ with coefficients in the rank-one local system $\iota_*(\mathbb{C}_{U^{\mathrm{an}}} g_0)$ on $\tilde{Z}$.

The following lemma will be used in Section 4.

Lemma 3.4. *In the situation as above, for $q \in D^{\mathrm{an}}$ let $k \geq 0$ be the number of the relevant irreducible components of D^{an} passing through q. Assume that $k \geq 2$. Then for a small open neighborhood V of q in Z^{an} we have*

$$\sum_{p \in \mathbb{Z}} (-1)^p \mathrm{dim} H_p((V \cap U^{\mathrm{an}}) \cup (\pi^{-1}(V) \cap Q), (\pi^{-1}(V) \cap Q); \iota_*(\mathbb{C}_{U^{\mathrm{an}}} g_0)) = 0. \tag{3.10}$$

In the sequel, we consider the more special case where $U = \mathbb{C}_x^*$ and $\mathcal{E}$ is an integrable connection on U such that $\mathcal{L} = H^{-1}\mathrm{DR}_U(\mathcal{E}) \simeq \mathbb{C}_{U^{\mathrm{an}}} \exp(h(x))g_0(x)$ for a Laurent polynomial $h(x) = \sum_{i\in\mathbb{Z}} a_i x^i$ and a possibly multi-valued holomorphic function g_0 on U^{an}. Then we can take the projective line $\mathbb{P}$ to be the compactification Z of $U = \mathbb{C}_x^*$. In this case, we have $D = Z \setminus U = D_1 \cup D_2$, where we set $D_1 = \{0\}$ and $D_2 = \{\infty\}$. For the real oriented blow-up $\pi : \tilde{Z} \longrightarrow Z^{\mathrm{an}}$ of Z^{an} the subset $\tilde{D} = \pi^{-1}(D^{\mathrm{an}})$ of $\tilde{Z}$ is a union of two circles $\tilde{D}_i := \pi^{-1}(D_i^{\mathrm{an}}) \simeq S^1$ $(i = 1, 2)$. Moreover the open subset $Q \subset \tilde{D}$ is a union of open intervals in $\tilde{D}_1 \cup \tilde{D}_2 \simeq S^1 \cup S^1$. Let $NP(h) \subset \mathbb{R}$ be the Newton polytope of h i.e. the convex hull of the set $\{i \in \mathbb{Z} \mid a_i \neq 0\}$ in $\mathbb{R}$. Finally denote by $\Delta \subset \mathbb{R}$ the convex hull of $NP(h) \cup \{0\}$ in $\mathbb{R}$. Then by using Proposition 3.3 we can easily prove the following result.

Lemma 3.5. *In the situation as above, we have*
(i) The dimension of the rapid decay homology group $H_p^{\mathrm{rd}}(U; \mathcal{E})$ is $\mathrm{Vol}_{\mathbb{Z}}(\Delta)$ if $p = 1$ and zero otherwise.
(ii) Assume that $\Delta = [-m, 0]$ (resp. $\Delta = [0, m]$) for some $m > 0$. Then $Q \subset \tilde{D}$ is a union of open intervals $Q_1, Q_2, \dots, Q_m$ in $\tilde{D}_1 \simeq S^1$ (resp. in $\tilde{D}_2 \simeq S^1$) and the first rapid decay homology group $H_1^{\mathrm{rd}}(U; \mathcal{E})$ has a basis formed by the m cycles

$$[\gamma_i] \in H_1^{\mathrm{rd}}(U; \mathcal{E}) \qquad (i = 1, 2, \dots, m), \tag{3.11}$$

where γ_i is a 1-dimensional smooth chain in $U^{\mathrm{an}} \cup Q$ starting from a point in Q_i and going directly to that in Q_{i+1} (here we set $Q_{m+1} = Q_1$).
(iii) Assume that $\Delta = [-m_1, m_2]$ for some $m_1, m_2 > 0$. Then $Q \subset \tilde{D}$ is a union of open intervals $Q_1, Q_2, \dots, Q_{m_1}$ in $\tilde{D}_1 \simeq S^1$ and the ones $Q'_1, Q'_2, \dots, Q'_{m_2}$ in $\tilde{D}_2 \simeq S^1$. If moreover the function g_0 has a non-trivial monodromy around the origin, then the first rapid decay homology group $H_1^{\mathrm{rd}}(U; \mathcal{E})$ has a basis formed by the $m_1 + m_2$ cycles

$$[\gamma_i] \in H_1^{\mathrm{rd}}(U; \mathcal{E}) \qquad (i = 1, 2, \dots, m_1) \tag{3.12}$$

and

$$[\gamma'_i] \in H_1^{\mathrm{rd}}(U; \mathcal{E}) \qquad (i = 1, 2, \dots, m_2), \tag{3.13}$$

where γ_i (resp. γ'_i) is a 1-dimensional smooth chain in $U^{\mathrm{an}} \cup Q$ starting from a point in Q_i (resp. Q'_i) and going directly to that in Q_{i+1} (resp. Q'_{i+1}).

4 A geometric construction of integral representations

In this section we give a geometric construction of Adolphson's confluent A-hypergeometric $\mathcal{D}$-module $\mathcal{M}_{A,c}$ and apply it to obtain the integral representations of A-hypergeometric functions. Let $Y = (\mathbb{C}^A)^* = \mathbb{C}_\zeta^N$ be the dual vector space of $X = \mathbb{C}^A = \mathbb{C}_z^N$, where ζ is the dual coordinate of z. As in [8], to $A \subset \mathbb{Z}^n$ we associate a morphism

$$j : T = (\mathbb{C}^*)_x^n \longrightarrow Y = (\mathbb{C}^A)^* = \mathbb{C}_\zeta^N \tag{4.1}$$

defined by $x \longmapsto (x^{a(1)}, x^{a(2)}, \dots, x^{a(N)})$. Since we assume here that A generates $\mathbb{Z}^n$, j is an embedding. Let $I \subset \mathbb{C}[\zeta_1, \dots, \zeta_N]$ be the defining ideal of the closure $\overline{j(T)}$ of $j(T) \subset Y$ in

Y. Moreover denote by $D(Y)$ the Weyl algebra over Y. Then we have a ring isomorphism

$$\wedge : D(X) \xrightarrow{\sim} D(Y) \tag{4.2}$$

defined by

$$(\frac{\partial}{\partial z_j})^\wedge = \zeta_j, \qquad (z_j)^\wedge = -\frac{\partial}{\partial \zeta_j} \qquad (j = 1, 2, \ldots, N). \tag{4.3}$$

We call $\wedge$ the Fourier transform (see Malgrange [22] etc. for the details). Via this $\wedge$, the Adolphson's system $M_{A,c}$ is transformed to the one

$$(Z_{i,c})^\wedge v(\zeta) = 0 \quad (1 \le i \le n), \tag{4.4}$$

$$f(\zeta)v(\zeta) = 0 \quad (f \in I) \tag{4.5}$$

on $Y = (\mathbb{C}^A)^*$. Note that this system has no holomorphic solution. Let

$$N_{A,c} = M_{A,c}^\wedge = D(Y) / \left(\sum_{1 \le i \le n} D(Y)(Z_{i,c})^\wedge + \sum_{f \in I} D(Y) f \right) \tag{4.6}$$

be the corresponding left $D(Y)$-module and $\mathcal{N}_{A,c}$ the coherent $\mathcal{D}_Y$-module associated to it. By a theorem of Hotta [14] $\mathcal{N}_{A,c}$ is regular holonomic. Now on $T = (\mathbb{C}^*)^n_x$ we define a holonomic $\mathcal{D}_T$-module $\mathcal{R}_c$ by

$$\mathcal{R}_c = \mathcal{D}_T / \sum_{1 \le i \le n} \mathcal{D}_T \left\{ x_i \frac{\partial}{\partial x_i} + (1 - c_i) \right\}. \tag{4.7}$$

This is an integrable connection on T and we have

$$\mathrm{DR}_T(\mathcal{R}_c) \simeq (\mathbb{C}_{T^{\mathrm{an}}} x_1^{-c_1+1} \cdots x_n^{-c_n+1})[n]. \tag{4.8}$$

Let $v = [1] \in \mathcal{N}_{A,c}$ and $w_0 = [1] \in \mathcal{R}_c$ be the canonical generators. Recall that the transfer bimodule $\mathcal{D}_{T \longrightarrow Y}$ has the canonical section $1_{T \longrightarrow Y}$. We define a section $1_{Y \longleftarrow T}$ of $\mathcal{D}_{Y \longleftarrow T} = \Omega_T \otimes_{\mathcal{O}_T} \mathcal{D}_{T \longrightarrow Y} \otimes_{j^{-1}\mathcal{O}_Y} j^{-1}\Omega_Y^{\otimes -1}$ by

$$1_{Y \longleftarrow T} = (dx_1 \wedge \cdots \wedge dx_n) \otimes 1_{T \longrightarrow Y} \otimes j^{-1}(d\zeta_1 \wedge \cdots \wedge d\zeta_N)^{\otimes -1}. \tag{4.9}$$

Note that this definition of $1_{Y \longleftarrow T}$ depends on the coordinates of Y and T. Then we obtain a section w of the holonomic $\mathcal{D}_Y$-module

$$\mathcal{S}_{A,c} := \int_j \mathcal{R}_c = j_*(\mathcal{D}_{Y \longleftarrow T} \otimes_{\mathcal{D}_T} \mathcal{R}_c) \tag{4.10}$$

defined by $w = j_*(1_{Y \longleftarrow T} \otimes w_0)$. We can easily check that this section $w \in \mathcal{S}_{A,c}$ satisfies the system (4.4). Hence as in [8, page 268-269], we obtain a morphism

$$\Psi : \mathcal{N}_{A,c} \longrightarrow \mathcal{S}_{A,c} = \int_j \mathcal{R}_c \tag{4.11}$$

of left $\mathcal{D}_Y$-modules which sends the canonical generator $v = [1] \in \mathcal{N}_{A,c}$ to $w \in \mathcal{S}_{A,c}$.

Definition 4.1. *(Gelfand-Kapranov-Zelevinsky [8, page 262]) For a face Γ of Δ containing the origin $0 \in \mathbb{R}^n$ we denote by $\mathrm{Lin}(\Gamma) \subset \mathbb{C}^n$ the $\mathbb{C}$-linear span of Γ. We say that the parameter vector $c \in \mathbb{C}^n$ is nonresonant (with respect to A) if for any face Γ of Δ of codimension 1 such that $0 \in \Gamma$ we have $c \notin \{\mathbb{Z}^n + \mathrm{Lin}(\Gamma)\}$.*

Recall that if $c \in \mathbb{C}^n$ is nonresonant then it is semi-nonresonant in the sense of [1, page 284]. The following result was proved by Saito [31] and Schulze-Walther [34], [35] by using commutative algebras. Here we give a geometric proof to it.

Lemma 4.2. *Assume that the parameter vector $c \in \mathbb{C}^n$ is nonresonant. Then the holonomic $\mathcal{D}_Y$-module $\mathcal{S}_{A,c}$ is irreducible.*

Proof. Note that $\mathrm{DR}_T(\mathcal{R}_c) \simeq (\mathbb{C}_{T^{\mathrm{an}}} x_1^{-c_1+1} \cdots x_n^{-c_n+1})[n]$ is an irreducible perverse sheaf on T^{an}. Then also its minimal extension by the locally closed embedding j is irreducible (see [15, Corollary 8.2.10] etc.). As in [8, Theorem 3.5 and Propositions 3.2 and 4.4] it suffices to show that the canonical morphism

$$j_!(\mathbb{C}_{T^{\mathrm{an}}} x_1^{c_1-1} \cdots x_n^{c_n-1}) \longrightarrow Rj_*(\mathbb{C}_{T^{\mathrm{an}}} x_1^{c_1-1} \cdots x_n^{c_n-1}) \tag{4.12}$$

is a quasi-isomorphism. For this, we have only to prove the vanishing $Rj_*(\mathbb{C}_{T^{\mathrm{an}}} x_1^{c_1-1} \cdots x_n^{c_n-1})_q \simeq 0$ for any $q \in \overline{j(T)} \setminus j(T)$. Note that by the nonresonance of $c \in \mathbb{C}^n$ for any $p \in \mathbb{Z}$ and the local system $\mathcal{L} := \mathbb{C}_{T^{\mathrm{an}}} x_1^{c_1-1} \cdots x_n^{c_n-1}$ on T we have $H^p(T^{\mathrm{an}}; \mathcal{L}) = 0$. Let $S(A) \subset \mathbb{Z}^n$ (resp. $K(A) \subset \mathbb{R}^n$) be the semigroup (resp. the convex cone) generated by A. Recall that there exists a natural bijection between the faces of $K(A)$ and the T-orbits in $\overline{j(T)}$. First consider the case where $0 \in \mathbb{R}^n$ is an apex of $K(A)$ and $q = 0 \in Y = \mathbb{C}^N_\zeta$. If $0 \in A$ i.e. $0 = a(j)$ for some $1 \leq j \leq N$ we have $\overline{j(T)} \subset \{\zeta_j = 0\} \simeq \mathbb{C}^{N-1}$. Hence we may assume that $0 \notin A$ from the first. In this case, $\{0\} \subset \overline{j(T)}$ is the unique 0-dimensional T-orbit in $\overline{j(T)}$ which corresponds to $\{0\} \prec K(A)$. From now on, we will prove that $Rj_*(\mathcal{L})_0 \simeq 0$. By our assumption there exists a linear function $l : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $l(\mathbb{Z}^n) \subset \mathbb{Z}$ and $K(A) \setminus \{0\} \subset \{l > 0\}$. We define a real-valued function $\varphi : Y = \mathbb{C}^N_\zeta \longrightarrow \mathbb{R}$ by

$$\varphi(\zeta) = |\zeta_1|^{\frac{C}{l(a(1))}} + \cdots + |\zeta_N|^{\frac{C}{l(a(N))}}, \tag{4.13}$$

where we take $C \in \mathbb{Z}_{>0}$ large enough so that φ and its level sets $\varphi^{-1}(b)$ $(b > 0)$ are smooth. Let $(l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n$ be the coefficients of the linear function l. Define an action of the multiplicative group $\mathbb{R}_{>0}$ on T by

$$r \cdot (x_1, \ldots, x_n) = (r^{l_1} x_1, \ldots, r^{l_n} x_n) \tag{4.14}$$

for $r \in \mathbb{R}_{>0}$. Then we have

$$j(r \cdot x) = (r^{l(a(1))} x^{a(1)}, \ldots, r^{l(a(N))} x^{a(N)}) \tag{4.15}$$

and hence

$$\varphi(j(r \cdot x)) = r^C \varphi(j(x)). \tag{4.16}$$

Therefore by the action of $\mathbb{R}_{>0}$ on $Y = \mathbb{C}^N_\zeta$ defined by

$$r \cdot (\zeta_1, \ldots, \zeta_N) = (r^{l(a(1))} \zeta_1, \ldots, r^{l(a(N))} \zeta_N), \tag{4.17}$$

a level set $\varphi^{-1}(b)$ $(b>0)$ of φ is sent to the one $\varphi^{-1}(r^C b)$. Moreover this action preserves the T-orbits in $\overline{j(T)}$. Let $O \subset \overline{j(T)}$ be such a T-orbit. Then all the level sets $\varphi^{-1}(b)$ $(b>0)$ of φ are transversal to O, or all are not. But the latter case cannot occur by the Sard theorem. Then we obtain an isomorphism

$$H^pRj_*(\mathcal{L})_0 \simeq H^p(\mathbb{C}^N; Rj_*(\mathcal{L})) \simeq H^p(T^{\mathrm{an}}; \mathcal{L}) \simeq 0 \tag{4.18}$$

for any $p \in \mathbb{Z}$. Next consider the remaining case where $q \in O$ for a T-orbit O in $\overline{j(T)}$ such that $\dim O \geq 1$. Then in a neighborhood of q, the variety $\overline{j(T)}$ is a product $W \times O$ for an affine toric variety $W \subset \mathbb{C}^{N'}$ and $j(T) = (T_1 \sqcup \cdots \sqcup T_k) \times O$ for some tori $T_i \simeq (\mathbb{C}^*)^{n-\dim O}$. See [9, Chapter 5, Theorem 3.1] and the proof of [23, Theorem 4.9] etc. for the details. Moreover for the semigroup $S(A_O) \subset \mathbb{Z}^{n-\dim O}$ generated by a finite subset $A_O \subset \mathbb{Z}^{n-\dim O}$ we have $\overline{T_i} \simeq \mathrm{Spec}(\mathbb{C}[S(A_O)]) \subset W$ $(i=1,2,\dots,k)$. These varieties $\overline{T_i}$ are the irreducible components of W. For the explicit construction of $\overline{T_i}$ see the proof of [23, Theorem 4.9]. By this construction $0 \in \mathbb{R}^{n-\dim O}$ is an appex of the convex cone $K(A_O) \subset \mathbb{R}^{n-\dim O}$ generated by A_O. Let $p_2 : W \times O \longrightarrow O$ and $q_2 : T_i \times O \longrightarrow O$ be the second projections. Then it follows from the nonresonance of $c \in \mathbb{C}^n$ the restriction of $\mathcal{L}$ to $q_2^{-1}p_2(q) \simeq T_i$ is a non-constant local system. So we can apply our previous arguments and prove $Rj_*(\mathcal{L})_q \simeq 0$ in this case, too. This completes the proof. $\square$

By Lemma 4.2, if $c \in \mathbb{C}^n$ is nonresonant the non-trivial morphism Ψ should be surjective. According to Schulze-Walther [34, Corollary 3.8] the morphism Ψ is also an isomorphism in this case. Let $\vee : D(Y) \xrightarrow{\sim} D(X)$ be the inverse of the Fourier transform $\wedge$. Then we have an isomorphism $N_{A,c}^{\vee} \simeq M_{A,c}$ of left $D(X)$-modules. The corresponding coherent $\mathcal{D}_X$-module $\mathcal{N}_{A,c}^{\vee} \simeq \mathcal{M}_{A,c}$ can be more geometrically constructed as follows. Let $\sigma = < \cdot, \cdot >: X \times Y \longrightarrow \mathbb{C}$ be the canonical pairing defined by $< z, \zeta > = \sum_{j=1}^N z_j \zeta_j$ and $p_1 : X \times Y \longrightarrow X$ (resp. $p_2 : X \times Y \longrightarrow Y$) the first (resp. second) projection. Then we have the following theorem due to Katz-Laumon [17].

Theorem 4.3. *(Katz-Laumon [17]) In the situation as above, we have an isomorphism*

$$\mathcal{N}_{A,c}^{\vee} \simeq \int_{p_1} \left\{ (p_2^* \mathcal{N}_{A,c}) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{O}_{X\times Y} e^{\sigma} \right\}, \tag{4.19}$$

where $\mathcal{O}_{X\times Y}e^{\sigma}$ is the integrable connection associated to $e^{\sigma} : X \times Y \longrightarrow \mathbb{C}$ (see [22] etc.).

In the same way, we have

$$\mathcal{S}_{A,c}^{\vee} = \int_{p_1} \left\{ (p_2^* \mathcal{S}_{A,c}) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{O}_{X\times Y} e^{\sigma} \right\}. \tag{4.20}$$

From now on, we assume that $c \in \mathbb{C}^n$ is nonresonant. Then by Lemma 4.2 we obtain surjective morphisms $N_{A,c} \longrightarrow \mathcal{S}_{A,c}(Y)$ and $\mathcal{M}_{A,c}(X) \simeq N_{A,c}^{\vee} \longrightarrow \mathcal{S}_{A,c}^{\vee}(X)$. Hence we obtain a surjective morphism

$$\mathcal{M}_{A,c} \simeq \mathcal{N}_{A,c}^{\vee} \longrightarrow \mathcal{S}_{A,c}^{\vee} \tag{4.21}$$

of left $\mathcal{D}_X$-modules. Let $e^{\tau} : X \times T \longrightarrow \mathbb{C}$ be the function defined by $e^{\tau}(z,x) = \exp(\sum_{j=1}^N z_j x^{a(j)})$ and $q_1 : X \times T \longrightarrow X$ (resp. $q_2 : X \times T \longrightarrow T$) the first (resp.

second) projection. Then by the base change theorem [15, Theorem 1.7.3], we have the isomorphism

$$\mathcal{S}_{A,c}^{\vee} \simeq \int_{q_1} \left\{ (q_2^* \mathcal{R}_c) \otimes_{\mathcal{O}_{X\times T}} \mathcal{O}_{X\times T} e^{\tau} \right\}. \tag{4.22}$$

Namely $\mathcal{S}_{A,c}^{\vee}$ is the direct image of the integrable connection

$$\mathcal{K} = (q_2^* \mathcal{R}_c) \otimes_{\mathcal{O}_{X\times T}} \mathcal{O}_{X\times T} e^{\tau} \tag{4.23}$$

on $X \times T$ by q_1. Define a function $g : X \times T \longrightarrow \mathbb{C}$ by

$$g(z,x) = \exp(\sum_{j=1}^{N} z_j x^{a(j)}) x_1^{c_1 - 1} \cdots x_n^{c_n - 1}. \tag{4.24}$$

Then by the results of Hien-Roucairol [12] the holomorphic solution complex

$$\mathrm{Sol}_X(\mathcal{S}_{A,c}^{\vee}) = R\mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{S}_{A,c}^{\vee})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \tag{4.25}$$

of $\mathcal{S}_{A,c}^{\vee}$ is expressed by the rapid decay homology groups associated the function g. Indeed, for $z \in \Omega$ let $\mathcal{K}_z$ (resp. $g_z : T \longrightarrow \mathbb{C}$) be the restriction of the connection $\mathcal{K}$ (resp. the function g) to $U_z := q_1^{-1}(z) \simeq T \subset \Omega \times T$. Namely we set

$$g_z(x) = \exp(\sum_{j=1}^{N} z_j x^{a(j)}) x_1^{c_1 - 1} \cdots x_n^{c_n - 1}. \tag{4.26}$$

for $z \in U_z \simeq T$. Then for the dual connection $\mathcal{K}_z^*$ of $\mathcal{K}_z$ we have

$$H^{-n}\mathrm{DR}_T(\mathcal{K}_z^*) \simeq \mathbb{C}_{U_z^{\mathrm{an}}} g_z. \tag{4.27}$$

Moreover for any $p \in \mathbb{Z}$, the rapid decay homology groups

$$H_p^{\mathrm{rd}}(U_z; \mathcal{K}_z^*) \qquad (z \in \Omega^{\mathrm{an}}) \tag{4.28}$$

associated to the integrable connections $\mathcal{K}_z^*$ (or to the functions $g_z : T \longrightarrow \mathbb{C}$) are isomorphic to each other and define a local system $\mathcal{H}_p^{\mathrm{rd}}$ on Ω^{an}. The following result is essentially due to Hien-Roucairol [12].

Theorem 4.4. *(Hien-Roucairol [12]) In the situation as above, for any $p \in \mathbb{Z}$ we have an isomorphism*

$$\mathcal{H}_{n+p}^{\mathrm{rd}} \simeq H^p \mathrm{Sol}_X(\int_{q_1} \mathcal{K}) \simeq H^p \mathrm{Sol}_X(\mathcal{S}_{A,c}^{\vee}) \tag{4.29}$$

of local systems on Ω^{an}.

In [1, Section 3] Adolphson proved that $\mathcal{M}_{A,c}$ is an integrable connection on Ω. Then by the surjective morphism $\mathcal{M}_{A,c} \simeq \mathcal{N}_{A,c}^{\vee} \longrightarrow \mathcal{S}_{A,c}^{\vee}$ we find that $\mathcal{S}_{A,c}^{\vee}$ is also an integrable connection on Ω. This in particular implies that for any $p \neq 0$ we have $H^p\mathrm{Sol}_X(\mathcal{S}_{A,c}^{\vee}) \simeq 0$. Hence we get

$$H_{n+p}^{\mathrm{rd}}(U_z; \mathcal{K}_z^*) \simeq 0 \qquad (p \neq 0, \quad z \in \Omega^{\mathrm{an}}). \tag{4.30}$$

It follows also from the surjection $\mathcal{M}_{A,c} \longrightarrow \mathcal{S}_{A,c}^{\vee}$ that we have an injection

$$\begin{aligned}\Phi : \mathcal{H}_n^{\mathrm{rd}} &\simeq \mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{S}_{A,c}^{\vee})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) &(4.31)\\ &\hookrightarrow \mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{M}_{A,c})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}). &(4.32)\end{aligned}$$

By using the generator

$$u = [1] \in \mathcal{M}_{A,c} = \mathcal{D}_X / \left(\sum_{1 \leq i \leq n} \mathcal{D}_X Z_{i,c} + \sum_{\mu \in \mathrm{Ker} A \cap \mathbb{Z}^N} \mathcal{D}_X \square_\mu \right) \tag{4.33}$$

of $\mathcal{M}_{A,c}$ we regard $\mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{M}_{A,c})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ as a subsheaf of $\mathcal{O}_{X^{\mathrm{an}}}$. Then we have the following result.

Theorem 4.5. *Assume that the parameter vector $c \in \mathbb{C}^n$ is nonresonant. Then the morphism Φ induces an isomorphism*

$$\mathcal{H}_n^{\mathrm{rd}} \simeq \mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{M}_{A,c})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \tag{4.34}$$

of local systems on Ω^{an}. Moreover this isomorphism is given by the integral

$$\gamma \longmapsto \left\{ \Omega^{\mathrm{an}} \ni z \longmapsto \int_{\gamma_z} \exp(\sum_{j=1}^{N} z_j x^{a(j)}) x_1^{c_1 - 1} \cdots x_n^{c_n - 1} dx_1 \wedge \cdots \wedge dx_n \right\}, \tag{4.35}$$

where for a family γ of rapid decay cycles and $z \in \Omega^{\mathrm{an}}$ we denote by γ_z the restriction $\gamma \cap U_z$ of γ to $U_z = q_1^{-1}(z) \simeq T$.

Note that this integral representation of the confluent A-hypergeometric functions $\mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{M}_{A,c})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ coincides with the one in Adolphson [1, Equation (2.6)].

Proof. Recall that the sheaf $\mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{M}_{A,c})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ is a local system on Ω^{an}. Moreover by [1, Corollary 5.20] its rank is $\mathrm{Vol}_{\mathbb{Z}}(\Delta)$. So it suffices to show that for any $z \in \Omega^{\mathrm{an}}$ the dimension of the n-th rapid decay homology group $H_n^{\mathrm{rd}}(U_z; \mathcal{K}_z^*)$ is also $\mathrm{Vol}_{\mathbb{Z}}(\Delta)$. Let

$$\mathrm{Eu}^{\mathrm{rd}}(U_z; \mathcal{K}_z^*) := \sum_{p \in \mathbb{Z}} (-1)^p \mathrm{dim} H_p^{\mathrm{rd}}(U_z; \mathcal{K}_z^*) \tag{4.36}$$

be the rapid decay Euler characteristic. Then by (4.30) we have only to prove the equality

$$\mathrm{Eu}^{\mathrm{rd}}(U_z; \mathcal{K}_z^*) = (-1)^n \mathrm{Vol}_{\mathbb{Z}}(\Delta). \tag{4.37}$$

Let Σ_0 be the dual fan of Δ in $\mathbb{R}^n$ and Σ its smooth subdivision. Denote by Z_Σ the smooth toric variety associated to the fan Σ. Then Z_Σ is a smooth compactification of $U_z \simeq T$ such that $Z_\Sigma \setminus U_z$ is a normal crossing divisor. By using the non-degeneracy of the Laurent polynomial $h_z(x) = \sum_{j=1}^N z_j x^{a(j)}$, as in [25, Section 3] we can construct a blow-up $Z := \tilde{Z_\Sigma}$ of Z_Σ such that the meromorphic extension of h_z to it has no point of indeterminacy. Hence we can use this smooth compactification Z of $U_z \simeq T$ and the normal crossing divisor $D := Z \setminus U_z$ in it to define the rapid decay homology groups associated to the function $g_z(x) = \exp(h_z(x)) x_1^{c_1 - 1} \cdots x_n^{c_n - 1}$ (see [12, the paragraph just below Definition 2.1] etc.). Let $\pi : \tilde{Z} \longrightarrow Z^{\mathrm{an}}$ be the real oriented blow-up of Z^{an} along

D^{an} and set $\tilde{D} = \pi^{-1}(D^{\mathrm{an}})$. Then, as in Section 3 we define the rapid decay homology groups $H_p^{\mathrm{rd}}(U_z; \mathcal{K}_z^*)$ by using $\pi : \tilde{Z} \longrightarrow Z^{\mathrm{an}}$, $\tilde{D}$ and g_z etc. By Proposition 3.3, Lemmas 3.4 and 3.5, Mayer-Vietoris exact sequences for relative twisted homology groups and the geometry of $\tilde{D} \subset \tilde{Z}$, we can easily calculate the rapid decay Euler characteristic $\mathrm{Eu}^{\mathrm{rd}}(U_z; \mathcal{K}_z^*)$ and prove the equality (4.37). This completes the proof of the isomorphism (4.34). Let us prove the remaining assertion. Denote the distinguished section $(q_2^* w_0) \otimes e^{\tau}$ of the integrable connection $\mathcal{K} = (q_2^* \mathcal{R}_c) \otimes_{\mathcal{O}_{X \times T}} \mathcal{O}_{X \times T} e^{\tau}$ by t. Let $\Omega^{\cdot}_{X \times T/X} \otimes_{\mathcal{O}_{X \times T}} \mathcal{K}$ be the relative algebraic de Rham complex of $\mathcal{K}$ associated to the morphism $q_1 : X \times T \longrightarrow X$. Then we have an isomorphism

$$\mathcal{S}_{A,c}^{\vee} \simeq \int_{q_1} \mathcal{K} \simeq H^n \left\{ (q_1)_* (\Omega^{\cdot}_{X \times T/X} \otimes_{\mathcal{O}_{X \times T}} \mathcal{K}) \right\}. \tag{4.38}$$

For a relative n-form $\omega \in (q_1)_* \Omega^n_{X \times T/X}$ denote by $\mathrm{cl}(\omega \otimes t)$ the section of $\mathcal{S}_{A,c}^{\vee}$ which corresponds to the cohomology class $[(q_1)_*(\omega \otimes t)] \in H^n \{ (q_1)_* (\Omega^{\cdot}_{X \times T/X} \otimes_{\mathcal{O}_{X \times T}} \mathcal{K}) \}$ by the above isomorphism. According to the result of [12], by the isomorphism

$$\mathcal{H}_n^{\mathrm{rd}} \simeq \mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{S}_{A,c}^{\vee})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \tag{4.39}$$

of local systems on Ω^{an}, a family of rapid decay cycles $\gamma \in \mathcal{H}_n^{\mathrm{rd}}$ is sent to the section

$$\left[(\mathcal{S}_{A,c}^{\vee})^{\mathrm{an}} \ni f \otimes \mathrm{cl}(\omega \otimes t) \longmapsto \left\{ \Omega^{\mathrm{an}} \ni z \longmapsto f(z) \int_{\gamma_z} \exp(\sum_{j=1}^{N} z_j x^{a(j)}) x_1^{c_1 - 1} \cdots x_n^{c_n - 1} \omega \right\} \right] \tag{4.40}$$

($f \in \mathcal{O}_{X^{\mathrm{an}}}$) of $\mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{S}_{A,c}^{\vee})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$. Then the remaining assertion follows from the lemma below. This completes the proof. □

Lemma 4.6. *By the morphism*

$$\mathcal{M}_{A,c} \longrightarrow \mathcal{S}_{A,c}^{\vee} \simeq H^n \left\{ (q_1)_* (\Omega^{\cdot}_{X \times T/X} \otimes_{\mathcal{O}_{X \times T}} \mathcal{K}) \right\} \tag{4.41}$$

the canonical section $u = [1] \in \mathcal{M}_{A,c}$ is sent to the cohomology class $\mathrm{cl}((dx_1 \wedge \cdots \wedge dx_n) \otimes t)$.

Proof. First note that the morphism $\Psi^{\vee}(X) : \mathcal{M}_{A,c}(X) \simeq M_{A,c} \simeq N_{A,c}^{\vee} \longrightarrow \mathcal{S}_{A,c}^{\vee}(X) \simeq \mathcal{S}_{A,c}(Y)$ sends the canonical generator $u = [1] \in \mathcal{M}_{A,c}(X)$ to $w = j_*(1_{Y \longleftarrow T} \otimes w_0) \in \mathcal{S}_{A,c}(Y)$. On the other hand, by (4.20) we have an isomorphism

$$\mathcal{S}_{A,c}^{\vee} \simeq H^N \left[(p_1)_* \left\{ \Omega^{\cdot}_{X \times Y/X} \otimes_{\mathcal{O}_{X \times Y}} (p_2^* \mathcal{S}_{A,c}) \otimes_{\mathcal{O}_{X \times Y}} \mathcal{O}_{X \times Y} e^{\sigma} \right\} \right]. \tag{4.42}$$

Then by Malgrange's simple proof [22, page 135] of Theorem 4.3, via this isomorphism the section $w \in \mathcal{S}_{A,c}^{\vee}(X) \simeq \mathcal{S}_{A,c}(Y)$ corresponds to the cohomology class

$$[(p_1)_* \left\{ (d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes (p_2^* w) \otimes e^{\sigma} \right\}]. \tag{4.43}$$

Let $\tilde{j} : X \times T \hookrightarrow X \times Y$ be the embedding induced by j. By the isomorphism

$$\mathcal{S}_{A,c}^{\vee} \simeq H^N \left[(p_1)_* \left\{ \Omega^{\cdot}_{X \times Y/X} \otimes_{\mathcal{O}_{X \times Y}} \tilde{j}_* (\mathcal{D}_{X \times Y \longleftarrow X \times T} \otimes_{X \times T} \mathcal{K}) \right\} \right] \tag{4.44}$$

the above cohomology class corresponds to the one

$$\rho := \left[(p_1)_* \left\{ (d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes \tilde{j}_* (1_{X \times Y \longleftarrow X \times T} \otimes t) \right\} \right], \tag{4.45}$$

where the section $1_{X\times Y\longleftarrow X\times T}\in\mathcal{D}_{X\times Y\longleftarrow X\times T}$ is defined similarly to $1_{Y\longleftarrow T}\in\mathcal{D}_{Y\longleftarrow T}$. Then it suffices to show that via the isomorphism

$$\mathcal{S}^{\vee}_{A,c}\simeq\int_{p_1}\int_{\tilde{j}}\mathcal{K}\simeq\int_{q_1}\mathcal{K} \tag{4.46}$$

the cohomology class ρ is sent to the one $\mathrm{cl}((dx_1\wedge\cdots\wedge dx_n)\otimes t)=[(q_1)_*\{(dx_1\wedge\cdots\wedge dx_n)\otimes t\}]$ in

$$\int_{q_1}\mathcal{K}\simeq H^n\left\{(q_1)_*(\Omega^{\cdot}_{X\times T/X}\otimes_{\mathcal{O}_{X\times T}}\mathcal{K})\right\}. \tag{4.47}$$

Since X and $X\times Y$ are affine, we have only to prove that via the isomorphism

$$H^N\Gamma(X\times Y;\Omega^{\cdot}_{X\times Y/X}\otimes_{\mathcal{O}_{X\times Y}}\int_{\tilde{j}}\mathcal{K}) \tag{4.48}$$

$$\simeq\quad H^n\Gamma(X\times T;\Omega^{\cdot}_{X\times T/X}\otimes_{\mathcal{O}_{X\times T}}\mathcal{K}) \tag{4.49}$$

the cohomology class

$$\left[(d\zeta_1\wedge\cdots\wedge d\zeta_N)\otimes\tilde{j}_*(1_{X\times Y\longleftarrow X\times T}\otimes t)\right] \tag{4.50}$$

is sent to the one $[(dx_1\wedge\cdots\wedge dx_n)\otimes t]$. With the help of the fact that $X\times T$ is affine and $Y=\mathbb{C}^N$, we can easily prove it by an explicit calculation. This completes the proof. $\square$

As a corollary of Theorem 4.5, we recover the following Saito and Schulze-Walther's geometric (functorial) construction of Adolphson's confluent A-hypergeometric $\mathcal{D}$-module $\mathcal{M}_{A,c}$ on $\Omega\subset X=\mathbb{C}^A$.

Corollary 4.7. *(Saito [31] and Schulze-Walther [34], [35]) Assume that the parameter vector $c\in\mathbb{C}^n$ is nonresonant. Then we have an isomorphism $\mathcal{M}_{A,c}\overset{\sim}{\longrightarrow}\mathcal{S}^{\vee}_{A,c}$ of integrable connections on Ω. In particular, $\mathcal{M}_{A,c}$ is an irreducible connection there.*

This result was first obtained in Saito [31] and Schulze-Walther [34], [35] by using totally different methods. In fact, they proved moreover that we have an isomorphism $\mathcal{M}_{A,c}\overset{\sim}{\longrightarrow}\mathcal{S}^{\vee}_{A,c}$ on the whole X.

Remark 4.8. *Since $\mathcal{N}_{A,c}$ is regular holonomic by a theorem of Hotta [14], it is also regular at infinity in the sense of Daia [4]. Then by using the Fourier-Sato transforms (see [16] and [22] etc.), we can apply the main theorem of Daia [4] to get a topological construction of the sheaf of the confluent A-hypergeometric functions $\mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}((\mathcal{M}_{A,c})^{\mathrm{an}},\mathcal{O}_{X^{\mathrm{an}}})$. This construction is valid even when the parameter $c\in\mathbb{C}^n$ is not nonresonant.*

Example 4.9. *Assume that $n=1$ and $T=\mathbb{C}^*_x$.*
(i) If $A=\{1,-1\}\subset\mathbb{Z}$ our integral representation of the A-hypergeometric functions $u(z_1,z_2)$ on $\mathbb{C}^2_z$ is

$$u(z_1,z_2)=\int_{\gamma_z}\exp(z_1x+\frac{z_2}{x})x^{c-1}dx. \tag{4.51}$$

By the restriction of $u(z_1,z_2)$ by the injective map $\mathbb{C}_t\hookrightarrow\mathbb{C}^2_z$, $t\longmapsto(\frac{t}{2},-\frac{t}{2})$ we obtain the classical Bessel function

$$v(t)=\frac{1}{2\pi i}\int_{\gamma_{(\frac{t}{2},-\frac{t}{2})}}\exp(\frac{tx}{2}-\frac{t}{2x})x^{-\nu-1}dx \tag{4.52}$$

for the parameter $\nu = -c$.
(ii) If $A = \{3, 1\} \subset \mathbb{Z}$ *our integral representation of the A-hypergeometric functions* $u(z_1, z_2)$ *on* $\mathbb{C}_z^2$ *is*

$$u(z_1, z_2) = \int_{\gamma_z} \exp(z_1 x^3 + z_2 x) x^{c-1} dx. \tag{4.53}$$

By the restriction of $u(z_1, z_2)$ *by the injective map* $\mathbb{C}_t \hookrightarrow \mathbb{C}_z^2$, $t \longmapsto (\frac{1}{3}, -t)$ *we obtain the classical Airy function*

$$v(t) = \frac{1}{2\pi i} \int_{\gamma_{(\frac{1}{3}, -t)}} \exp(\frac{x^3}{3} - tx) dx \tag{4.54}$$

for $c = 1$.

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