THE BCS GAP EQUATION FOR SPIN-POLARIZED FERMIONS

ABRAHAM FREIJI, CHRISTIAN HAINZL, AND ROBERT SEIRINGER

ABSTRACT. We study the BCS gap equation for a Fermi gas with un-equal population of spin-up and spin-down states. For $\cosh(\delta_\mu/T) \leq 2$, with T the temperature and δ_μ the chemical potential difference, the question of existence of non-trivial solutions can be reduced to spectral properties of a linear operator, similar to the unpolarized case studied previously in [7, 8, 9]. For $\cosh(\delta_\mu/T) > 2$ the phase diagram is more complicated, however. We derive upper and lower bounds for the critical temperature, and study their behavior in the small coupling limit.

1. INTRODUCTION

Spin-polarized fermionic systems have attracted a lot of interest in recent years, and have been intensely studied both experimentally and theoretically. They are of relevance in various areas of physics, ranging from cold gases to nuclear stars.

The goal of our present paper is to study the BCS gap equation for such imbalanced systems. Experiments [17, 18] have demonstrated a significantly richer phase diagram than in the balanced, unpolarized case. The possibility of pairing in polarized systems was first pointed out in [16], and since then has been intensively studied in the literature, mostly under the assumption of a contact interaction among the fermions. We refer to [6] and [4] for recent reviews on this subject.

Our work is a continuation of recent papers [7, 8, 9, 10] where the BCS gap equation in the balanced case for systems with general pair interaction potential V was investigated. In particular, a criterion for potentials giving rise to superfluidity was derived. The form of the interaction in actual physical systems can be quite general, and hence it is important to keep V as general as possible. In the following we recall these results before we present the main results in the imbalanced case.

1.1. **Balanced fermionic systems.** Consider a gas consisting of neutral spin $\frac{1}{2}$ fermions. The kinetic energy of these particles is described by the non-relativistic Schrödinger operator, and their interaction by a pair potential

which we write for convenience as $2\lambda V$, with λ being a coupling parameter. According to Bardeen, Cooper and Schrieffer (BCS) [2] a superfluid state of the system is characterized by the existence of a non-trivial solution of the *gap equation*

$$\Delta(p) = -\frac{\lambda}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{V}(p-q) \frac{\Delta(q)}{E(q)} \tanh \frac{E(q)}{2T} \, dq \qquad (1.1)$$

at some temperature $T \geq 0$, with $E(p) = \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}$. Here, $\mu \in \mathbb{R}$ is the chemical potential and $\hat{V}(p) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} V(x) e^{-ipx} dx$ denotes the Fourier transform of V. The function $\Delta(p)$ is the order parameter which is closely related to the wavefunction of the *Cooper pairs*.

The detailed investigation of Eq. (1.1) in [7, 8, 9, 10, 11, 12, 13] is based on the observation in [8] that the existence of a non trivial solution of (1.1) is equivalent to the existence of a negative eigenvalue of the *linear* operator $K_T(-i\nabla) + \lambda V(x)$ on $L^2(\mathbb{R}^3)$, where

$$K_T(p) = \frac{|p^2 - \mu|}{\tanh \frac{|p^2 - \mu|}{2T}} \, .$$

The critical temperature $T_c(\lambda V)$ is thus defined by the equation

$$\inf \operatorname{spec} (K_{T_c} + \lambda V) = 0 \, . \qquad (1.2)$$

Since $K_T(p)$ is pointwise monotone in T, this defines T_c uniquely. For $T < T_c(\lambda V)$ the equation (1.1) has a non-trivial solution, while for $T \geq T_c(\lambda V)$ it has none.

For $\mu > 0$, $K_T(p)$ attains its minimum on the sphere $\Omega_\mu = \{p : p^2 = \mu\}$. In the small coupling limit $\lambda \to 0$, the study of T_c reduces to the study of an effective operator $\mathcal{V}_\mu : L^2(\Omega_\mu) \to L^2(\Omega_\mu)$, given by

$$(\mathcal{V}_\mu u)(p) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{\mu}} \int_{\Omega_\mu} \hat{V}(p-q) u(q) \, d\omega(q) \, , \qquad (1.3)$$

with $d\omega$ denoting the Lebesgue measure on the sphere. The lowest eigenvalue $e_\mu < 0$ of $\mathcal{V}_\mu$ determines the leading order of $T_c(\lambda V)$ as $\lambda \to 0$ [7]. In fact, in [7, 9] it is shown that

$$T_c = \mu \left(\frac{8e^{\gamma-2}}{\pi} + o(1) \right) e^{\pi/(2\sqrt{\mu}b_\mu)} \quad \text{as } \lambda \to 0 \, , \qquad (1.4)$$

where $\gamma \approx 0.577$ denotes Euler's constant, and $b_\mu = \pi \lambda e_\mu/(2\sqrt{\mu}) + O(\lambda^2)$, where the $O(\lambda^2)$ term is determined by another effective operator of $L^2(\Omega_\mu)$. The parameter b_μ can be interpreted as an effective scattering length (in second order Born approximation), which reduces to the usual scattering length of the interaction potential $2\lambda V$ in the low density limit, i.e., as $\mu \to 0$ [10].

2. Main results

We consider a gas of spin $\frac{1}{2}$ particles at temperature $T \geq 0$, interacting via a pair interaction potential $2\lambda V$. We assume the two spin states are unequally populated, but have equal masses. We introduce two separate chemical potentials, μ_+ and μ_-, for the spin up and spin down particles, respectively. We denote the average chemical potential by $\bar{\mu} = (\mu_+ + \mu_-)/2$, and half their difference by

$$\delta_\mu = \frac{\mu_+ - \mu_-}{2}, \qquad (2.1)$$

which we assume to be non-negative without loss of generality.

Let $\gamma_+(p)$ and $\gamma_-(p)$ denote the momentum distributions for the spin-up and spin-down fermions respectively. In addition, let α denote the Cooper pair wave-function. The BCS functional $\mathcal{F}_T$ is defined as

$$\mathcal{F}_T(\gamma_+, \gamma_-, \alpha) = \frac{1}{2} \int_{\mathbb{R}^3} (p^2 - \mu_+)\gamma_+(p)dp + \frac{1}{2} \int_{\mathbb{R}^3} (p^2 - \mu_-)\gamma_-(p)dp$$

$$+ \lambda \int_{\mathbb{R}^3} |\alpha(x)|^2 V(x)dx - \frac{T}{2}S(\gamma_+, \gamma_-, \alpha), \qquad (2.2)$$

where the entropy S is

$$S(\gamma_+, \gamma_-, \alpha) = -\int_{\mathbb{R}^3} \mathrm{Tr}_{\mathbb{C}^4}\left[\Gamma(p)\ln\Gamma(p)\right]dp, \qquad (2.3)$$

with

$$\Gamma(p) = \begin{pmatrix} \gamma_+(p) & 0 & 0 & \hat{\alpha}(p) \\ 0 & \gamma_-(p) & -\hat{\alpha}(p) & 0 \\ 0 & -\overline{\hat{\alpha}(p)} & 1 - \gamma_+(p) & 0 \\ \overline{\hat{\alpha}(p)} & 0 & 0 & 1 - \gamma_-(p) \end{pmatrix}. \qquad (2.4)$$

The functions $\gamma_\pm$ and α are constrained by demanding that $0 \leq \Gamma(p) \leq 1$ as an operator on $\mathbb{C}^4$, for all $p \in \mathbb{R}^3$.

The functional $\mathcal{F}_T$ can be heuristically derived [15, 8] by starting from a genuine many-body Hamiltonian and making three steps of simplifications. First, one considers only quasi-free (or generalized Hartree-Fock [1]) states. Second, one ignores the exchange and direct terms in the interaction energy. Finally, one assumes translation invariance and works out the energy per unit volume. The specific ansatz of the states in the form of Γ follows from assuming, in addition, that the total spin points in the direction of the applied field (quantified by δ_μ).

We will show below that on an appropriate domain the functional $\mathcal{F}_T$ attains its minimum and the corresponding Euler-Lagrange equation, expressed in terms of the *order parameter* $\Delta = -2\lambda\hat{V} * \hat{\alpha}$, takes the form

$$\Delta = -\lambda\hat{V} * \frac{\Delta}{2E}\left(\tanh\left(\frac{E + \delta_\mu}{2T}\right) + \tanh\left(\frac{E - \delta_\mu}{2T}\right)\right), \qquad (2.5)$$

where $E(p) = \sqrt{(p^2 - \bar{\mu})^2 + |\Delta(p)|^2}$. For convenience we define the convolution with a factor $(2\pi)^{-3/2}$ in front, i.e., $f*g(p) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} f(p-q)g(q)dq$.

In terms of

$$K_{T,\delta_\mu}^{\Delta(p)}(p) = \frac{2E(p)}{\tanh\left(\frac{E(p)+\delta_\mu}{2T}\right) + \tanh\left(\frac{E(p)-\delta_\mu}{2T}\right)}, \qquad (2.6)$$

the gap equation can conveniently be expressed as $\Delta = -\lambda \hat{V} * (\Delta/K_{T,\delta_\mu}^\Delta)$. Define also

$$\widetilde{K}_{T,\delta_\mu}(p) = \inf_{x>0} K_{T,\delta_\mu}^x(p).$$

The basis of our analysis is the following theorem.

THEOREM 1. *Let $V \in L^{3/2}(\mathbb{R}^3)$ be real-valued, $\bar{\mu} \in \mathbb{R}$, $T \geq 0$ and $\delta_\mu \geq 0$.*

(i) *If the linear operator $K_{T,\delta_\mu}^0(-i\nabla) + \lambda V(x)$ has a negative eigenvalue, then the functional $\mathcal{F}_T$ attains its minimum at some $\alpha \neq 0$.*

(ii) *If $\mathcal{F}_T$ attains its minimum at some $\alpha \neq 0$, then $\Delta(p) = -2\lambda \hat{V} * \hat{\alpha}(p)$ satisfied the BCS gap equation (2.5).*

(iii) *If the BCS equation has a non-trivial solution, then the linear operator $\widetilde{K}_{T,\delta_\mu}(-i\nabla) + \lambda V(x)$ has a negative eigenvalue.*

The proof of Theorem 1 is given in Section 3.

Remark 1. We shall show in Corollary 1 that for all pairs (δ_μ, T) with $\cosh(\delta_\mu/T) \leq 2$, the function K_{T,δ_μ}^Δ is pointwise monotone in Δ and hence

$$\widetilde{K}_{T,\delta_\mu}(p) = K_{T,\delta_\mu}^0(p) \quad \forall p.$$

This means that statements (i)$-$(iii) in Theorem 1 are actually *equivalent*, exactly as in the balanced case [8, Thm. 1]. Hence we have obtained a linear characterization for the existence of a non-trivial solution of the non-linear BCS gap equation. This does not hold if $\cosh(\delta_\mu/T) > 2$, however, as discussed below.

2.1. **Critical temperatures.** Since we do not have equivalence of the statements (i)$-$(iii) in Theorem 1 in case $\cosh(\delta_\mu/T) > 2$, we do not have a simple definition of the critical temperature as in (1.2) in this case. With the aid of the operators K_{T,δ_μ}^0 and $\widetilde{K}_{T,\delta\mu}$, we can define curves $T_{\delta_\mu}^i$ and $T_{\delta_\mu}^o$ in the (δ_μ, T) plane, given by

$$T_{\delta_\mu}^i(\lambda V) : \{(\delta_\mu, T) \,|\, \inf \operatorname{spec}(K_{T,\delta_\mu}^0 + \lambda V) = 0\}, \qquad (2.7)$$

$$T_{\delta_\mu}^o(\lambda V) : \{(\delta_\mu, T) \,|\, \inf \operatorname{spec}(\widetilde{K}_{T,\delta_\mu} + \lambda V) = 0\}. \qquad (2.8)$$

Below the curve $T_{\delta_\mu}^i$ one has $\inf \operatorname{spec}(K_{T,\delta_\mu}^0 + \lambda V) < 0$, and hence Theorem 1(i) implies that the functional $\mathcal{F}_T$ has a global minimum at some $\alpha \neq 0$. Above the curve $T_{\delta_\mu}^o$, however, Theorem 1(iii) implies that the gap equation does not have a non-trivial solution. Consequently the minimum of $\mathcal{F}_T$ is attained for $\alpha = 0$, the *normal* state. See Fig. 1.

In the region between $T_{\delta_\mu}^i$ and $T_{\delta_\mu}^o$, there can exist non-trivial solutions to the gap equation even if α vanishes identically for the minimizer of $\mathcal{F}_T$. In

fact it was shown in [5] by numerical calculations, using a contact interaction potential, that there is a parameter regime where the gap equation has a solution but the corresponding energy is higher than that of the normal state.

Using the methods developed in [7, 9] we are able to evaluate $T^{\mathrm{i}}_{\delta_\mu}(\lambda V)$ and $T^{\mathrm{o}}_{\delta_\mu}(\lambda V)$ in the small coupling limit $\lambda \to 0$, and obtain analogous formulas as in the balanced case in (1.4). With T_c denoting the critical temperature in the balanced case (at $\mu = \bar{\mu}$), we shall show that

$$T^{\#}_{\delta_\mu} = T_c e^{-\kappa^{\#}(\delta_\mu/T^{\#}_{\delta_\mu})} \tag{2.9}$$

as $\lambda \to 0$, with $\#$ standing for either i or o, and the $\kappa^{\#}$ are two explicit non-negative functions (defined in (4.9) and (4.29), respectively), with $\kappa^{\#}(0) = 0$, of course. We refer to Theorems 2 and 3 in Section 4 for the precise statements. The resulting curves are plotted in Figure 1.
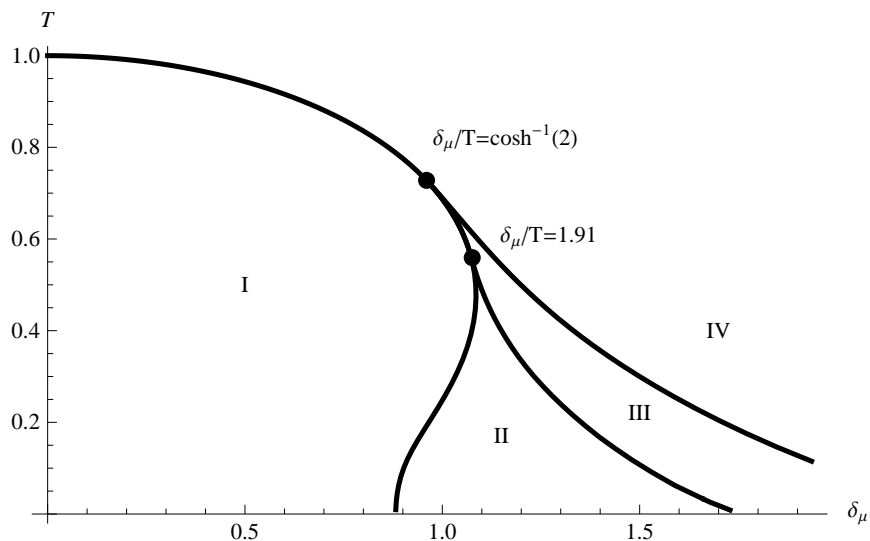

FIGURE 1. The curves $T^{\mathrm{i}}_{\delta_\mu}$, $T^{\mathrm{g}}_{\delta_\mu}$ and $T^{\mathrm{o}}_{\delta_\mu}$ (in increasing order), in units of T_c, the critical temperature in the balanced case. In region I the system is in a superfluid phase. In region IV, there are no non-trivial solutions to the BCS gap equation, and the system is the normal phase. In region III we prove, under additional assumptions on V, that the system is in a normal phase, even though non-trivial solutions to the gap equation might exist.

2.1.1. *A more detailed phase diagram.* Under additional assumptions on the interaction potential V we are able to derive a sharper upper bound on the

critical temperature, at least for small coupling. Let $T^{\mathrm{g}}_{\delta_\mu}$ be the curve defined by

$$T^{\mathrm{g}}_{\delta_\mu}(\lambda V) = \{(\delta_\mu, T) : \inf_{y>0} \inf \operatorname{spec}(K^y_{T,\delta_\mu} + \lambda V) = 0\}. \qquad (2.10)$$

Note the difference with $T^{\mathrm{o}}_{\delta_\mu}$ in (2.8), where the minimum is taken pointwise for every fixed p in $K^y_{T,\delta_\mu}(p)$. Recall also from Remark 1 that K^y_{T,δ_μ} is not monotone in y if $\cosh(\delta_\mu/T) > 2$, so the minimum of the spectrum of $K^y_{T,\delta_\mu} + \lambda V$ is not necessarily attained at $y = 0$.

In terms of α the BCS gap equation (2.5) can conveniently be written as

$$\left(K^\Delta_{T,\delta_\mu} + \lambda V\right)\alpha = 0. \qquad (2.11)$$

Let us *assume* that $K^\Delta_{T,\delta_\mu} + \lambda V$ has 0 as its lowest eigenvalue. For minimizers of the functional $\mathcal{F}_T$, this is the case, for instance, of $\hat{V}$ is negative, since in this case $\hat{\alpha}$ is negative and hence the ground state, by a Perron-Frobenius type argument.

In this case, we will show that Δ can be treated as a constant in the small coupling limit, with its value given by the one it attains on the Fermi surface. As a consequence, we shall show that if (δ_μ, T) lies above $T^{\mathrm{g}}_{\delta_\mu}$, the system is in a normal phase (in the sense that $\alpha = 0$ for the minimizer of $\mathcal{F}_T$), even though the BCS gap equation can have non-trivial solutions. Hence the sharper upper bound $T^{\mathrm{g}}_{\delta_\mu}$ on the critical temperature for superfluidity holds in place of $T^{\mathrm{o}}_{\delta_\mu}$; see Fig. 1. The precise formulation of our results is given in Theorem 4 in Section 4.

Numerical calculations in [5] (see, in particular, Fig. 2 there) predict that between the curves $T^{\mathrm{i}}_{\delta_\mu}$ and $T^{\mathrm{g}}_{\delta_\mu}$ the gap equation has at least two solutions. Moreover, there exists another dividing line inside this region separating the parameter regimes where the BCS functional is minimized by $\alpha = 0$ or $\alpha \neq 0$, i.e., the normal and the superfluid regime.

2.1.2. *A toy model in one dimension.* For the purpose of illustration let us now consider the BCS functional for a one-dimensional system, where the particles interact via a contact potential of the form $V(x) = -g\delta(x)$ with $g > 0$. In this case the gap equation takes a very simple form, and the order parameter Δ is constant. In fact, the BCS gap equation in this case is

$$\frac{1}{g} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{K^\Delta_{T,\delta_\mu}(p)} dp. \qquad (2.12)$$

The critical temperatures $T^{\mathrm{i}}_{\delta_\mu}$ and $T^{\mathrm{g}}_{\delta_\mu}$ are given implicitly via the equations

$$T^{\mathrm{i}}_{\delta_\mu} = \left\{(\delta_\mu, T) : \frac{1}{g} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{K^0_{T,\delta_\mu}(p)} dp\right\} \qquad (2.13)$$

$$T^{\mathrm{g}}_{\delta_\mu} = \left\{(\delta_\mu, T) : \frac{1}{g} = \max_\Delta \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{K^\Delta_{T,\delta_\mu}(p)} dp\right\}. \qquad (2.14)$$

The curve $T_{\delta_\mu}^{\mathrm{i}}$ encloses the region where the gap equation (2.12) has exactly one solution. Above $T_{\delta_\mu}^{\mathrm{o}}$ there are no solutions, while in-between (2.12) has exactly two solutions. Numerically this is easy to see, in fact. If one plots $\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{K_{T,\delta_\mu}^\Delta(p)}dp$ as a function of Δ, one observes that the graph crosses $1/g$ either once, twice, or not at all.

3. Preliminaries and proof of Theorem 1

We start by specifying the precise domain of definition of the BCS functions $\mathcal{F}_T$ in (2.2).

DEFINITION 1. Let $\mathcal{D}$ denote the set of functions $(\gamma_+, \gamma_-, \alpha)$, with $\gamma_+, \gamma_- \in L^1(\mathbb{R}^3, (1+p^2)dp)$, $0 \le \gamma_+(p), \gamma_-(p) \le 1$, and $\alpha \in H^1(\mathbb{R}^3, dx)$, satisfying $|\hat{\alpha}(p)|^2 \le \gamma_+(p)(1-\gamma_-(p))$ and $|\hat{\alpha}(p)|^2 \le \gamma_-(p)(1-\gamma_+(p))$.

For $(\gamma_+, \gamma_-, \alpha) \in \mathcal{D}$, the corresponding $\Gamma(p)$ in (2.4) satisfies $0 \le \Gamma \le 1$ as an operator on $\mathbb{C}^4$ for all $p \in \mathbb{R}^3$. Moreover, all the terms in $\mathcal{F}_T$ are well-defined under our assumptions on V.

Recall that $\bar{\mu} = (\mu_+ + \mu_-)/2$ denotes the average chemical potential and $\delta_\mu = (\mu_+ - \mu_-)/2$. It is easy to see that in the absence of a potential V, $\mathcal{F}_T$ is minimized on $\mathcal{D}$ by the *normal state* given by $\alpha = 0$, $\gamma_+^0(p) = [e^{\frac{1}{T}[(p^2-\bar{\mu})-\delta_\mu]} + 1]^{-1}$ and $\gamma_-^0(p) = [e^{\frac{1}{T}[(p^2-\bar{\mu})+\delta_\mu]} + 1]^{-1}$.

We start with the observation that $\mathcal{F}_T$ attains a minimum on $\mathcal{D}$.

PROPOSITION 1. *There exists a minimizer of $\mathcal{F}_T$ in $\mathcal{D}$.*

The proof is analogous to [8, Proposition 2], and we skip it for simplicity.

Let us now come to the proof of our main Theorem 1. We start with some preliminaries. Diagonalizing Γ allows to rewrite the entropy (2.3) in the form

$$S(\Gamma) = -\int_{\mathbb{R}^3}[(r+w)\ln(r+w) + (r-w)\ln(r-w)$$
$$+ (s+w)\ln(s+w) + (s-w)\ln(s-w)]dp, \quad (3.1)$$

where the functions s, r and w are given by

$$r = \frac{1}{2}(1+\gamma_+-\gamma_-), \ s = \frac{1}{2}(1-\gamma_++\gamma_-), \ w = \frac{1}{2}\sqrt{(1-\gamma_+-\gamma_-)^2 + 4|\hat{\alpha}|^2}.$$

Lemma 1. *Let $(\gamma_+, \gamma_-, \alpha) \in \mathcal{D}$ be a minimizer of $\mathcal{F}_T$ for $0 < T < \infty$. Then, for a.e. $p \in \mathbb{R}^3$,*

$$(\hat{V}*\hat{\alpha})(p) = -\frac{T}{4}\hat{\alpha}(p)\,(f_r(w) + f_s(w)) \tag{3.2}$$

$$\delta_\mu = \frac{\mu_+ - \mu_-}{2} = \frac{T}{2}\ln\left(\frac{r^2-w^2}{s^2-w^2}\right), \tag{3.3}$$

$$(p^2 - \bar{\mu}) = \frac{T}{4}(1-\gamma_+-\gamma_-)(f_r(w) + f_s(w)), \tag{3.4}$$

where $f_a(b) = \frac{1}{b}\ln\frac{a+b}{a-b}$.

Proof. Let $A_\epsilon := \{p \in \mathbb{R}^3 : (|\hat{\alpha}(p)|^2 - \gamma_+(1 - \gamma_-))(|\hat{\alpha}(p)|^2 - \gamma_-(1 - \gamma_+)) \geq \epsilon\}$ for some $\epsilon > 0$. Let $g \in H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ be such that the Fourier transform $\hat{g}$ of g is supported in A_ϵ. Then for small enough $|t|$, $(\gamma_+, \gamma_-, \alpha + tg) \in \mathcal{D}$. By the Lebesgue dominated convergence theorem, it follows that

$$\frac{d}{dt}\mathcal{F}_T(\gamma_+, \gamma_-, \alpha + tg)_{t=0} = 2 \Re \int \alpha(x)\overline{g(x)}V(x)dx$$
$$+ \frac{T}{2}\Re \int \hat{\alpha}(p)\overline{\hat{g}(p)}(f_r(w) + f_s(w))dp. \quad (3.5)$$

Here $\Re$ denotes the real part. Also, for $\hat{g}$ real, $(\gamma_+ + t\hat{g}, \gamma_-, \alpha) \in \mathcal{D}$ for small $|t|$, and we get

$$\frac{d}{dt}\mathcal{F}_T(\gamma_+ + t\hat{g}, \gamma_-, \alpha)_{t=0} = \frac{1}{2}\int (p^2 - \mu_+)\hat{g}(p)dp$$
$$+ \frac{T}{2}\int \left[\frac{(\gamma_+ + \gamma_- - 1)}{2}(f_r(w) + f_s(w)) + \ln\left(\frac{r^2 - w^2}{s^2 - w^2}\right)\right]dp \quad (3.6)$$

and, similarly,

$$\frac{d}{dt}\mathcal{F}_T(\gamma_+, \gamma_- + t\hat{g}, \alpha)_{t=0} = \frac{1}{2}\int (p^2 - \mu_-)\hat{g}(p)dp$$
$$+ \frac{T}{2}\int \left[\frac{(\gamma_+ + \gamma_- - 1)}{2}(f_r(w) + f_s(w)) - \ln\left(\frac{r^2 - w^2}{s^2 - w^2}\right)\right]dp. \quad (3.7)$$

Since $(\gamma_+, \gamma_-, \alpha)$ minimize $\mathcal{F}_T$ by assumption, the expressions in (3.5),(3.6) and (3.7) vanish. This implies that (3.2) holds, and also that

$$(p^2 - \mu_-) = \frac{T}{2}\left[\frac{(1 - \gamma_+ - \gamma_-)}{2}(f_r(w) + f_s(w)) + \ln\left(\frac{r^2 - w^2}{s^2 - w^2}\right)\right] \quad (3.8)$$

$$(p^2 - \mu_+) = \frac{T}{2}\left[\frac{(1 - \gamma_+ - \gamma_-)}{2}(f_r(w) + f_s(w)) - \ln\left(\frac{r^2 - w^2}{s^2 - w^2}\right)\right] \quad (3.9)$$

for a.e. $p \in A_\epsilon$. As in [8] we can argue that the measure of $B := \mathbb{R}^3 \backslash \cup_{\epsilon > 0} A_\epsilon$ is zero. Subtracting and adding Eqs. (3.8) and (3.9) implies (3.3) and (3.4). $\square$

Proof of Theorem 1. We consider the case where $T > 0$ in the proof, and leave the analogous case $T = 0$ to the reader. We proceed similarly to [8, Theorem 1]. To show (i), note that

$$K^0_{T,\delta_\mu}(p) = \frac{2(p^2 - \bar{\mu})}{\tanh\left(\frac{p^2 - \bar{\mu} + \delta_\mu}{2T}\right) + \tanh\left(\frac{p^2 - \bar{\mu} - \delta_\mu}{2T}\right)} = \frac{p^2 - \bar{\mu}}{1 - \gamma^0_+ - \gamma^0_-}, \quad (3.10)$$

where γ^0_+ and γ^0_- are the momentum distributions in the normal state defined in the beginning of this section. We shall show that if the normal state $\alpha = 0$ minimizes $\mathcal{F}_T$, then $\inf \text{spec}\,(K^0_{T,\delta_\mu} + \lambda V) \geq 0$.

Since it is true that for every $g \in C_0^\infty$, $\|(1 - \gamma_+^0 - \gamma_-^0)^2 + (tg)^2\|_\infty < 1$ for small enough $|t|$, it follows from the Lebesgue dominated convergence theorem that $\frac{d^2}{dt^2}\mathcal{F}_T(\gamma_+^0, \gamma_-^0, t\hat{g})$ exists for small t, and equals

$$\frac{d^2}{dt^2}\mathcal{F}_T(\gamma_+^0, \gamma_-^0, t\hat{g})\bigg|_{t=0}$$

$$= 2\lambda \int |\hat{g}(x)|^2 V(x)dx + T \int |g(p)|^2 \frac{\left[\ln\left(\frac{r^0+w^0}{r^0-w^0}\right) + \ln\left(\frac{s^0+w^0}{s^0-w^0}\right)\right]}{(1 - \gamma_+^0 - \gamma_-^0)}dp$$

at $t = 0$. Here $r^0 = \frac{1}{2}(1 + \gamma_+^0 - \gamma_-^0)$, $s^0 = \frac{1}{2}(1 - \gamma_+^0 + \gamma_-^0)$ and $w^0 = \frac{1}{2}(1 - \gamma_+^0 - \gamma_-^0)$. It is easy to see that

$$\ln\left(\frac{r^0+w^0}{r^0-w^0}\right) + \ln\left(\frac{s^0+w^0}{s^0-w^0}\right) = \frac{2(p^2 - \bar{\mu})}{T},$$

and hence

$$\frac{d^2}{dt^2}\mathcal{F}_T(\gamma_+^0, \gamma_-^0, t\hat{g})_{t=0} = 2\lambda \int |\hat{g}(x)|^2 V(x)dx + 2 \int K_{T,\delta_\mu}^0(p)|g(p)|^2 dp.$$

If the normal state minimizes $\mathcal{F}_T$ then clearly $\frac{d^2}{dt^2}\mathcal{F}_T(\gamma_+^0, \gamma_-^0, t\hat{g})_{t=0} \geq 0$ (keeping in mind that it is also stationary). This implies that $\langle \hat{g}, (K_{T,\delta_\mu}^0 + \lambda V)\hat{g}\rangle \geq 0$ for all $g \in C_0^\infty$, and hence proves the claim.

Next we shall show (ii). For a minimizer $(\gamma_+, \gamma_-, \alpha)$ of $\mathcal{F}_T$, we can define Δ via

$$\Delta(p) = 2\frac{p^2 - \bar{\mu}}{1 - \gamma_+ - \gamma_-}\hat{\alpha}(p). \tag{3.11}$$

Setting further $E(p) = \sqrt{(p^2 - \bar{\mu})^2 + |\Delta(p)|^2}$ and recalling that

$$2w = \sqrt{(1 - \gamma_+ - \gamma_-)^2 + 4|\hat{\alpha}|^2}$$

we obtain that

$$E(p) = 2w(p)\left|\frac{p^2 - \bar{\mu}}{1 - \gamma_+(p) - \gamma_-(p)}\right|.$$

If we further use (3.4) we get

$$E(p) = w\frac{T}{2}(f_r(w) + f_s(w)). \tag{3.12}$$

It then follows from the definition of $\Delta(p)$ and $E(p)$ that

$$\frac{\Delta(p)}{E(p)}w(p) = \hat{\alpha}(p). \tag{3.13}$$

Noting that

$$(f_r(w) + f_s(w)) = \frac{1}{\omega}\ln\frac{(r+w)(s+w)}{(r-w)(s-w)},$$

we obtain from (3.3) and the definition of $E(p)$ that

$$\frac{E(p) + \delta_\mu}{T} = \ln\left(\frac{r+w}{s-w}\right), \quad \frac{E(p) - \delta_\mu}{T} = \ln\left(\frac{s+w}{r-w}\right).$$

Since it is also true that

$$1 - \left(1 + \frac{r+w}{s-w}\right)^{-1} - \left(1 + \frac{s+w}{r-w}\right)^{-1} = 2w, \tag{3.14}$$

we get from (3.2) and (3.13) that

$$-\lambda V * \hat{\alpha} = \frac{\Delta(p)}{2}, \tag{3.15}$$

and hence

$$\Delta = -2\lambda V * \hat{\alpha} = -\lambda V * \frac{\Delta}{E}\left(1 - \frac{1}{1 + \frac{r+w}{s-w}} - \frac{1}{1 + \frac{s+w}{r-w}}\right).$$

From the above expressions, we observe that

$$e^{\frac{E(p)+\delta_\mu}{T}} = \frac{r+w}{s-w}, \quad e^{\frac{E(p)-\delta_\mu}{T}} = \frac{s+w}{r-w}.$$

Therefore we arrive at the BCS gap equation

$$\Delta = -\lambda V * \frac{\Delta}{E}\left(1 - \left(1 + e^{\frac{E(p)+\delta_\mu}{T}}\right)^{-1} - \left(1 + e^{\frac{E(p)-\delta_\mu}{T}}\right)^{-1}\right). \tag{3.16}$$

To see (iii), note that the gap equation can be written as

$$(K^\Delta_{T,\delta_\mu} + \lambda V)\alpha = 0,$$

with $\hat{\alpha}(p) = \Delta(p)/K^{\Delta(p)}_{T,\delta_\mu}(p)$. Consequently,

$$\langle \alpha, (\widetilde{K}_{T,\delta_\mu} + \lambda V)\alpha \rangle = \langle \alpha, (\widetilde{K}_{T,\delta_\mu} - K^\Delta_{T,\delta_\mu})\alpha \rangle \leq 0 \tag{3.17}$$

by the definition of $\widetilde{K}_{T,\delta_\mu}$. The inequality is, in fact, strict if Δ is not identically zero, since $\hat{\alpha}$ and Δ have the same support. This completes the proof of Theorem 1. $\qquad\square$

COROLLARY 1. *Assume that* $\cosh(\delta_\mu/T) \leq 2$. *Then the existence of a non-trivial solution of the BCS gap equation is* equivalent *to* $K^0_{T,\delta_\mu} + \lambda V$ *having a negative eigenvalue.*

Proof. The corollary follows directly from Theorem 1 and the fact that $K^\Delta_{T,\delta_\mu}(p)$ is pointwise monotone in Δ for all $\delta_\mu/T \in [0, \cosh^{-1}(2)]$. Hence $K^0_{T,\delta_\mu} = \widetilde{K}_{T,\delta_\mu}$ in this case. To see this one checks that the function $x \mapsto (\tanh(x+c)+\tanh(x-c))/x$ is monotone decreasing on $\mathbb{R}_+$ if $\cosh(2c) \leq 2$. $\qquad\square$

4. BOUNDS ON THE CRITICAL TEMPERATURE

As explained in Subsect. 2.1, the two linear operators in Theorem 1 determine upper and lower bounds $T^{\mathrm{o}}_{\delta_\mu}$ and $T^{\mathrm{i}}_{\delta_\mu}$ on the critical temperature. In this section we evaluate $T^{\mathrm{i}}_{\delta_\mu}(\lambda V)$ and $T^{\mathrm{o}}_{\delta_\mu}(\lambda V)$ in the limit of small λ. We will proceed similarly to [7, 9, 11]. Note that it follows from Corollary 1 that for $\cosh(\delta_\mu/T) \leq 2$ the two curves coincide, i.e., $T^{\mathrm{i}}_{\delta_\mu}(\lambda V) = T^{\mathrm{o}}_{\delta_\mu}(\lambda V)$. (Compare with Fig. 1.)

We start with $T^{\mathrm{i}}_{\delta_\mu}(\lambda V)$, defined in (2.7). The Birman-Schwinger principle implies that a pair $(\delta_\mu, T) \in T^{\mathrm{i}}_{\delta_\mu}(\lambda V)$ is characterized by the fact that the compact operator

$$B_{T,\delta_\mu} = \lambda |V|^{\frac{1}{2}} \frac{1}{K^0_{T,\delta_\mu}} V^{\frac{1}{2}} \tag{4.1}$$

has -1 as its lowest eigenvalue. Here we use the notation

$$V(x)^{1/2} = (\operatorname{sgn} V(x))|V(x)|^{1/2}.$$

That -1 has to be the lowest eigenvalue of B_{T,δ_μ} follows from the monotonicity of K^0_{T,δ_μ} in δ_μ, similarly to [7, Lemma 3.1]. In fact, since the spectrum of B_{T,δ_μ} goes to 0 for large δ_μ and is continuous in δ_μ, an eigenvalue of B_{T,δ_μ} smaller than -1 would correspond to an eigenvalue -1 for larger δ_μ, which is in contradiction to the monotonicity of $K^0_{T,\delta_\mu} + \lambda V$ in δ_μ and 0 being the lowest eigenvalue for the critical parameter.

For $\bar{\mu} > 0$, the operator B_{T,δ_μ} becomes singular as $(\delta_\mu, T) \to (0,0)$, and the key observation is that its singular part is represented by an operator $\mathcal{V}_{\bar{\mu}} : L^2(\Omega_{\bar{\mu}}) \to L^2(\Omega_{\bar{\mu}})$, given by

$$\left(\mathcal{V}_{\bar{\mu}} u\right)(p) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{\bar{\mu}}} \int_{\Omega_{\bar{\mu}}} \hat{V}(p - q) u(q) \, d\omega(q), \tag{4.2}$$

where $d\omega$ is the uniform Lebesgue measure on $\Omega_{\bar{\mu}}$. We note that the operator $\mathcal{V}_\mu$ has already appeared in the literature [3, 14]. If we assume that $V \in L^1(\mathbb{R}^3)$, then $\hat{V}(p)$ is a bounded continuous function, and hence $\mathcal{V}_{\bar{\mu}}$ is a Hilbert-Schmidt operator. In fact, $\mathcal{V}_{\bar{\mu}}$ is trace class, and its trace equals $\frac{\sqrt{\bar{\mu}}}{2\pi^2} \int_{\mathbb{R}^3} V(x) dx$. Let

$$e_{\bar{\mu}} = \inf \operatorname{spec} \mathcal{V}_{\bar{\mu}} \tag{4.3}$$

denote the infimum of the spectrum of $\mathcal{V}_{\bar{\mu}}$. Since $\mathcal{V}_{\bar{\mu}}$ is compact, we have $e_{\bar{\mu}} \leq 0$.

Let $\mathcal{F} : L^1(\mathbb{R}^3) \to L^2(\Omega_{\bar{\mu}})$ denote the bounded operator which maps $\psi \in L^1(\mathbb{R}^3)$ to the Fourier transform of ψ, restricted to the Fermi sphere $\Omega_{\bar{\mu}}$. For $V \in L^1(\mathbb{R}^3)$, multiplication by $|V|^{1/2}$ is a bounded operator from $L^2(\mathbb{R}^3)$ to $L^1(\mathbb{R}^3)$, and therefore $\mathcal{F}|V|^{1/2}$ is a bounded operator from $L^2(\mathbb{R}^3)$ to $L^2(\Omega_{\bar{\mu}})$.

Finally, as in [9], we define $\mathcal{W}_{\bar{\mu}}$ via its quadratic form, to be an operator on $L^2(\Omega_{\bar{\mu}})$ such that

$$\langle u|\mathcal{W}_{\bar{\mu}}|u\rangle = \int_0^\infty d|p| \left(\frac{|p|^2}{||p|^2 - \bar{\mu}|} \left[\int_{\mathbb{S}^2} d\Omega \left(|\hat{\varphi}(p)|^2 - |\hat{\varphi}(\sqrt{\bar{\mu}}p/|p|)|^2 \right) \right] \right.$$
$$\left. + \frac{1}{|p|^2} \int_{\mathbb{S}^2} d\Omega \, |\hat{\varphi}(\sqrt{\bar{\mu}}p/|p|)|^2 \right) . \qquad (4.4)$$

Here, $(|p|, \Omega) \in \mathbb{R}_+ \times \mathbb{S}^2$ denote spherical coordinates for $p \in \mathbb{R}^3$ and $\hat{\varphi}(p) = (2\pi)^{-3/2} \int_{\Omega_\mu} \hat{V}(p-q)u(q)d\omega(q)$. It was shown in [7] that (4.4) is well defined, despite the apparent singularity of the integral, and that $\mathcal{W}_{\bar{\mu}}$ is of Hilbert-Schmidt class.

For $\lambda > 0$, let

$$\mathcal{B}_{\bar{\mu}} = \lambda \frac{\pi}{2\sqrt{\bar{\mu}}} \mathcal{V}_{\bar{\mu}} - \lambda^2 \frac{\pi}{2\bar{\mu}} \mathcal{W}_{\bar{\mu}} , \qquad (4.5)$$

and let $\varrho(\lambda)$ denote its ground state energy,

$$\varrho(\lambda) = \inf \operatorname{spec} \mathcal{B}_{\bar{\mu}} . \qquad (4.6)$$

We note that if $e_{\bar{\mu}} < 0$, then $\varrho(\lambda) < 0$ for small λ. If the (normalized) eigenfunction $u \in L^2(\Omega_{\bar{\mu}})$ corresponding to $e_{\bar{\mu}}$ is unique, then

$$\varrho(\lambda) = \langle u|\mathcal{B}_{\bar{\mu}}|u\rangle + o(\lambda^3). \qquad (4.7)$$

In the degenerate case, this formula holds if one chooses u to be the eigenfunction of $\mathcal{V}_\mu$ that yields the largest value $\langle u|\mathcal{W}_\mu|u\rangle$ among all such (normalized) eigenfunctions.

4.1. **Evaluation of $T_{\delta_\mu}^{\mathrm{i}}$.** In this subsection we will derive the asymptotic behavior (2.9) of $T_{\delta_\mu}^{\mathrm{i}}(\lambda V)$ for small λ, which we present in the following theorem.

THEOREM 2. *Let $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ be real-valued and $\bar{\mu} > 0$. Assume that $e_{\bar{\mu}}$ defined in (4.3) is strictly negative, and let $\varrho(\lambda)$ be defined as in (4.6). Then any sequence of pairs $(\delta_\mu(\lambda), T(\lambda))$ on the curve $T_{\delta_\mu}^{i}(\lambda V)$ satisfies*

$$\lim_{\lambda \to 0} \left(\ln \frac{\bar{\mu}}{T} + \frac{\pi}{2\sqrt{\bar{\mu}}\varrho(\lambda)} - \kappa^{\mathrm{i}}(\delta_\mu/T) \right) = 2 - \gamma - \ln(8/\pi) \qquad (4.8)$$

where $\gamma \approx 0.5772$ is Euler's constant and

$$\kappa^{\mathrm{i}}(t) = \frac{1}{1 + e^t} \int_0^\infty \frac{1 - e^{-x}}{1 + e^{x+t}} \frac{dx}{x} + \frac{1}{1 + e^{-t}} \int_0^\infty \frac{1 - e^{-x}}{1 + e^{x-t}} \frac{dx}{x} - \ln \frac{\pi}{2} . \quad (4.9)$$

Note that $\kappa^{\mathrm{i}}(t) = \ln(t) + \gamma - \ln(\pi/2) + o(1)$ for large t, while $\kappa^{\mathrm{i}}(0) = 0$. We can rewrite Eq. (4.8) in the form

$$\kappa^{\mathrm{i}} \left(\frac{\delta_\mu}{T} \right) = \ln \left(\frac{8\bar{\mu}e^{\gamma - 2}e^{\frac{\pi}{2\sqrt{\bar{\mu}}\varrho(\lambda)}}}{\pi T} \right) + o(1).$$

If T_c denotes the critical temperature at $\delta_\mu = 0$, defined in Subsect. 1.1, this can also be written as

$$\kappa^{\mathrm{i}}\left(\frac{\delta_\mu}{T}\right) = \ln\left(\frac{T_c}{T}\right) + o(1)\,, \tag{4.10}$$

which agrees with (2.9) and is plotted in Figure 1. The region denoted by I defines the region where the system is guaranteed to be in a superfluid phase. The asymptotic behavior of κ^{i} implies that the curve hits the vertical axis at 1 and the horizontal axis at $\pi e^{-\gamma}/2 \approx 0.88$.

Proof. Let

$$m(T,\delta_\mu) = \frac{1}{4\pi\bar\mu} \int_{\mathbb{R}^3} \left(\frac{1}{K^0_{T,\delta_\mu}(p)} - \frac{1}{p^2}\right) dp\,, \tag{4.11}$$

and let $M(T,\delta_\mu)$ be the operator

$$M(T,\delta_\mu) = \frac{1}{K^0_{T,\delta_\mu}} - m(T,\delta_\mu)\mathcal{F}\mathcal{F}^*\,, \tag{4.12}$$

with $\mathcal{F}$ defined in the paragraph after Eq. (4.3). Following [7, Lemma 2] it is not difficult to see that $V^{1/2}M(T,\delta_\mu)|V|^{1/2}$ is bounded in Hilbert-Schmidt norm uniformly in (T,δ_μ).

Suppose ψ is an eigenstate of the operator $K^0_{T,\delta_\mu} + \lambda V$ corresponding to the lowest eigenvalue 0. By the Birman-Schwinger principle, $\phi = V^{\frac{1}{2}}\psi$ satisfies

$$-\phi = B_{T,\delta_\mu}\phi\,,$$

with B_{T,δ_μ} defined in (4.1). Moreover, -1 is the lowest eigenvalue of B_{T,δ_μ}, as argued above.

Since $V^{1/2}M(T,\delta_\mu)|V|^{1/2}$ is bounded uniformly in (T,δ_μ), the operator $1 + \lambda V^{1/2}M(T,\delta_\mu)|V|^{1/2}$ is invertible for λ small enough. Therefore

$$1 + \lambda|V|^{\frac{1}{2}}\frac{1}{K^0_{T,\delta_\mu}}V^{\frac{1}{2}} = 1 + \lambda|V|^{\frac{1}{2}}\left(m(T,\delta_\mu)\mathcal{F}\mathcal{F}^* + M(T,\delta_\mu)\right)V^{\frac{1}{2}} \tag{4.13}$$

$$= D\left(1 + \lambda m(T,\delta_\mu)D^{-1}V^{\frac{1}{2}}\mathcal{F}^*\mathcal{F}|V|^{\frac{1}{2}}\right),$$

where $D = 1 + \lambda V^{\frac{1}{2}}M(T,\delta_\mu)|V|^{\frac{1}{2}}$. The fact that -1 is an eigenvalue of (4.1) yields

$$D\left(1 + \lambda m(T,\delta_\mu)\frac{1}{1 + \lambda V^{\frac{1}{2}}M(T,\delta_\mu)|V|^{\frac{1}{2}}}V^{\frac{1}{2}}\mathcal{F}^*\mathcal{F}|V|^{\frac{1}{2}}\right)\phi = 0. \tag{4.14}$$

Since D is invertible, this is equivalent to the operator

$$\lambda m(T,\delta_\mu)\mathcal{F}|V|^{1/2}\frac{1}{1 + \lambda V^{1/2}M(T,\delta_\mu)|V|^{1/2}}V^{1/2}\mathcal{F}^* \tag{4.15}$$

having an eigenvalue -1 (using that AB and BA are isospectral if they are compact). Note that the latter operator is acting on $L^2(\Omega_{\bar\mu})$.

By expanding $(1 + \lambda V^{\frac{1}{2}} M(T, \delta_\mu) |V|^{\frac{1}{2}})^{-1}$ in a Neumann series we obtain the following implicit characterization of the corresponding temperature:

$$\lambda m(T, \delta_\mu) \text{inf spec} \left(\mathcal{F} \left(V - \lambda V M(T, \delta_\mu) V + o(\lambda^2) \right) \mathcal{F}^* \right) = -1. \qquad (4.16)$$

Note that $\mathcal{F} V \mathcal{F}^* = \sqrt{\bar{\mu}} \, \mathcal{V}_{\bar{\mu}}$, which was defined in (4.2). It follows that to leading order

$$\lim_{\lambda \to 0} \lambda m(T, \delta_\mu) = -\frac{1}{\text{inf spec} \mathcal{F} V \mathcal{F}^*} = -\frac{1}{\sqrt{\bar{\mu}} \, e_{\bar{\mu}}}. \qquad (4.17)$$

To derive the second order correction we employ $\mathcal{W}_{\bar{\mu}}$, which was defined in (4.4). It follows from (4.16), first order perturbation theory, the fact that $\mathcal{F} V \mathcal{F}^*$ is compact and that $\text{inf spec} \, \mathcal{F} V \mathcal{F}^* < 0$ by assumption, that

$$m(T, \delta_\mu) = \frac{-1}{\lambda \langle u | \mathcal{F} V \mathcal{F}^* | u \rangle - \lambda^2 \langle u | \mathcal{F} V M(T, \delta_\mu) V \mathcal{F}^* | u \rangle + O(\lambda^3)}, \qquad (4.18)$$

where u is the normalized eigenfunction corresponding to the lowest eigenvalue of $\mathcal{F} V \mathcal{F}^*$. If u is degenerate, we choose the u that minimizes the λ^2 term in the denominator of (4.18) among all such eigenfunctions. Eq. (4.18) represents an implicit equation for $T_{\delta_\mu}^i$. Since $\mathcal{F} V M(T, \delta_\mu) V \mathcal{F}^*$ is uniformly bounded and $T \to 0$ as $\lambda \to 0$, we have to evaluate the limit of $\langle u | \mathcal{F} V M(T, \delta_\mu) V \mathcal{F}^* | u \rangle$ as $(\delta_\mu, T) \to 0$. For this purpose, let $\varphi = V \mathcal{F}^* u$. Then

$$\langle u | \mathcal{F} V M(T, \delta_\mu) V \mathcal{F}^* | u \rangle$$
$$= \int_{\mathbb{R}^3} \frac{1}{K_{T,\delta_\mu}^0(p)} |\hat{\varphi}(p)|^2 \, dp - m(T, \delta_\mu) \int_{\Omega_{\bar{\mu}}} |\hat{\varphi}(p)|^2 \, d\omega(p) \qquad (4.19)$$
$$= \int_{\mathbb{R}^3} \left(\frac{1}{K_{T,\delta_\mu}^0(p)} \left[|\hat{\varphi}(p)|^2 - |\hat{\varphi}(\sqrt{\bar{\mu}} p/|p|)|^2 \right] + \frac{1}{p^2} |\hat{\varphi}(\sqrt{\bar{\mu}} p/|p|)|^2 \right) dp \, .$$

Recall the definition of $K_{T,\delta_\mu}^0(p)$ in (2.6). For a fixed δ_μ/T, it converges to $|p^2 - \bar{\mu}|$ as $T \to 0$. Using the fact that the spherical average of $|\hat{\varphi}(p)|^2$ is Lipschitz continuous (see [9]), we can interchange the limit and the radial integral over $|p|$, and hence obtain

$$\lim_{T \to 0} \langle u | \mathcal{F} V M(T, \delta_\mu) V \mathcal{F}^* | u \rangle = \langle u | \mathcal{W}_{\bar{\mu}} | u \rangle \, . \qquad (4.20)$$

We have thus shown that for $(\delta_\mu, T) \in T_{\delta_\mu}^i$

$$\lim_{\lambda \to 0} \left(m(T, \delta_\mu) + \frac{1}{\text{inf spec} \left(\lambda \sqrt{\bar{\mu}} \, V_{\bar{\mu}} - \lambda^2 \mathcal{W}_{\bar{\mu}} \right)} \right) = 0 \, . \qquad (4.21)$$

It remains to compute $m(T, \delta_\mu)$.

Lemma 2. *As* $(T, \delta_\mu) \to (0,0)$,

$$
\begin{aligned}
m(T, \delta_\mu) &= \frac{1}{4\pi\bar\mu} \int_{\mathbb{R}^3} \left(\frac{1}{K^0_{T,\delta_\mu}(p)} - \frac{1}{p^2} \right) dp \\
&= \frac{1}{\sqrt{\bar\mu}} \left(\ln \frac{\bar\mu}{T} + \gamma - 2 + \ln \frac{8}{\pi} - \kappa^{\mathrm{i}}(\delta_\mu/T) + o(1) \right). \quad (4.22)
\end{aligned}
$$

Proof. We split the integral into two parts according to whether $p^2 \le \bar\mu$ or $p^2 \ge \bar\mu$. By changing variables from $p^2 - \bar\mu$ to $-t$ and t, respectively, we see that $m(T, \delta_\mu)$ equals

$$
\begin{aligned}
&\frac{1}{2\bar\mu} \int_0^{\bar\mu} \left(\left(1 - \frac{1}{1 + e^{\frac{1}{T}(t+\delta_\mu)}} - \frac{1}{1 + e^{\frac{1}{T}(t-\delta_\mu)}} \right) \frac{\sqrt{\bar\mu - t}}{t} - \frac{1}{\sqrt{\bar\mu - t}} \right) dt \\
&+ \frac{1}{2\bar\mu} \int_0^\infty \left(\left(1 - \frac{1}{1 + e^{\frac{1}{T}(t+\delta_\mu)}} - \frac{1}{1 + e^{\frac{1}{T}(t-\delta_\mu)}} \right) \frac{\sqrt{\bar\mu + t}}{t} - \frac{1}{\sqrt{\bar\mu + t}} \right) dt.
\end{aligned}
$$
$$(4.23)$$

To simplify the notation, let us introduce the function

$$
\Upsilon_0(t) = \left(1 - \frac{1}{1 + e^{\frac{1}{T}(t+\delta_\mu)}} - \frac{1}{1 + e^{\frac{1}{T}(t-\delta_\mu)}} \right). \quad (4.24)
$$

We can rewrite (4.23) as

$$
\begin{aligned}
m(T, \delta_\mu) = &\frac{1}{2\bar\mu} \int_{\bar\mu}^\infty \left(\Upsilon_0(t) \frac{\sqrt{\bar\mu + t}}{t} - \frac{1}{\sqrt{\bar\mu + t}} \right) dt \\
&+ \frac{1}{2\bar\mu} \int_0^{\bar\mu} \Upsilon_0(t) \left(\frac{\sqrt{\bar\mu + t} + \sqrt{\bar\mu - t} - 2\sqrt{\bar\mu}}{t} \right) dt \\
&- \frac{1}{2\bar\mu} \int_0^{\bar\mu} \left(\frac{1}{\sqrt{\bar\mu + t}} + \frac{1}{\sqrt{\bar\mu - t}} \right) dt + \frac{1}{\sqrt{\bar\mu}} \int_0^{\bar\mu} \frac{\Upsilon_0(t)}{t} dt. \quad (4.25)
\end{aligned}
$$

Using the dominated convergence theorem, it is easy to see that

$$
\begin{aligned}
&\lim_{T, \delta_\mu \to 0} \int_{\bar\mu}^\infty \left(\Upsilon_0(t) \frac{\sqrt{\bar\mu + t}}{t} - \frac{1}{\sqrt{\bar\mu + t}} \right) dt \\
&= \int_{\bar\mu}^\infty \left(\frac{\sqrt{\bar\mu + t}}{t} - \frac{1}{\sqrt{\bar\mu + t}} \right) dt = 2\sqrt{\bar\mu} \ln \left(1 + \sqrt{2} \right).
\end{aligned}
$$

Moreover,

$$
\frac{1}{2\bar\mu} \int_0^{\bar\mu} \left(\frac{1}{\sqrt{\bar\mu + t}} + \frac{1}{\sqrt{\bar\mu - t}} \right) dt = \sqrt{\frac{2}{\bar\mu}}.
$$

Again by the dominated convergence theorem, the second integral becomes

$$
\frac{1}{2\bar\mu} \int_0^{\bar\mu} \frac{\sqrt{\bar\mu + t} + \sqrt{\bar\mu - t} - 2\sqrt{\bar\mu}}{t} dt = \frac{1}{\sqrt{\bar\mu}} \left(\ln 4 - \ln(1 + \sqrt{2}) + \sqrt{2} - 2 \right)
$$

in the limit $T, \delta_\mu \to 0$. With $c = \delta_\mu/T$, using the fact that $[1 + e^c]^{-1} + [1 + e^{-c}]^{-1} = 1$, we can rewrite the last term in (4.25) as

$$\frac{1}{\sqrt{\bar{\mu}}} \left[\int_0^{\frac{\bar{\mu}}{T}} \frac{1}{x} \left(\frac{1}{1 + e^c} - \frac{1}{1 + e^c e^x} \right) dx + \int_0^{\frac{\bar{\mu}}{T}} \frac{1}{x} \left(\frac{1}{1 + e^{-c}} - \frac{1}{1 + e^{-c} e^x} \right) dx \right].$$

Note that

$$\int_0^{\frac{\bar{\mu}}{T}} \frac{1}{x} \left(\frac{1}{1 + e^c} - \frac{1}{1 + e^c e^x} \right) dx = \frac{1}{1 + e^c} \int_0^{\frac{\bar{\mu}}{T}} \frac{1 - e^{-x}}{x} dx$$

$$- \frac{1}{1 + e^c} \int_0^{\frac{\bar{\mu}}{T}} \frac{1 - e^{-x}}{1 + e^c e^x} \frac{dx}{x}.$$

Using integration by parts,

$$\int_0^{\bar{\mu}/T} \frac{1 - e^{-x}}{x} dx = \ln \frac{\bar{\mu}}{T} \left(1 - e^{-\bar{\mu}/T} \right) - \int_0^{\bar{\mu}/T} \ln(x) e^{-x} \, dx.$$

Moreover, since

$$-\int_0^\infty e^{-x} \ln(x) dx = \gamma \tag{4.26}$$

(Euler's constant), we conclude that

$$\lim_{T, \delta_\mu \to 0} \left(\int_0^{\bar{\mu}/T} \frac{1 - e^{-x}}{x} dx - \ln \frac{\bar{\mu}}{T} \right) = \gamma.$$

The same argument applies with c replaced by $-c$, and hence

$$\frac{1}{\sqrt{\bar{\mu}}} \int_0^{\bar{\mu}} \frac{\Upsilon_0(t)}{t} dt = \frac{1}{\sqrt{\bar{\mu}}} \left(\gamma + \ln \frac{\bar{\mu}}{T} - \ln \frac{8}{\pi} - \kappa^{\mathrm{i}}(c) + o(1) \right).$$

Combining all the terms gives the statement of the lemma.   □

Theorem 2 follows from (4.21), (4.6), and (4.22).   □

4.2. **Evaluation of $T_{\delta_\mu}^{\mathrm{o}}$.** For $x \geq 0$ and $c \geq 0$, consider the function

$$f(x, c) = \frac{x}{\tanh \frac{x+c}{2} + \tanh \frac{x-c}{2}}.$$

For each c there is a $b(c)$ such that $x \mapsto f(x, c)$ attains its minimum at $b(c)$ and f is monotone increasing for all $x \geq b(c)$. Recall from Corollary 1 that $b(c) = 0$ for $c \leq \cosh^{-1}(2)$. We have

$$K_{T,\delta_\mu}^\Delta = T f \left(\sqrt{(p^2 - \bar{\mu})^2 + |\Delta|^2}/T, \delta_\mu/T \right).$$

Therefore, K_{T,δ_μ}^Δ is monotone in Δ whenever $\frac{|p^2 - \bar{\mu}|}{T} \geq b(\delta_\mu/T)$, and we have that

$$\widetilde{K}_{T,\delta_\mu} = \inf_\Delta K_{T,\delta_\mu}^\Delta = \begin{cases} K_{T,\delta_\mu}^0 & \text{for } \frac{|p^2 - \bar{\mu}|}{T} \geq b(\delta_\mu/T) \\ 2Tf(b(\delta_\mu/T), \delta_\mu/T) & \text{for } \frac{|p^2 - \bar{\mu}|}{T} < b(\delta_\mu/T). \end{cases}$$

$$\tag{4.27}$$

THEOREM 3. *Let $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ be real-valued and $\bar{\mu} > 0$. Assume that $e_{\bar{\mu}}$ defined in (4.3) is strictly negative, and let $\varrho(\lambda)$ be defined as in (4.6). Then any sequence of pairs $(\delta_\mu(\lambda), T(\lambda))$ on the curve $T^{\mathrm{o}}_{\delta_\mu}(\lambda V)$ satisfies*

$$\lim_{\lambda \to 0} \left(\ln \frac{\bar{\mu}}{T} + \frac{\pi}{2\sqrt{\bar{\mu}}\varrho(\lambda)} - \kappa^{\mathrm{o}}(\delta_\mu/T) \right) = 2 - \gamma - \ln \frac{8}{\pi}, \qquad (4.28)$$

where $\gamma \approx 0.5772$ is Euler's constant and

$$\kappa^{\mathrm{o}}(c) = \frac{1}{1+e^c} \int_b^\infty \frac{1-e^{-x}}{1+e^{x+c}} \frac{dx}{x} + \frac{1}{1+e^{-c}} \int_b^\infty \frac{1-e^{-x}}{1+e^{x-c}} \frac{dx}{x} \qquad (4.29)$$

$$+ \left(1 - e^{-b}\right) \ln b - \int_0^b \ln(x) e^{-x} \, dx - \frac{b}{2f(b,c)} - \ln \frac{\pi}{2},$$

with $b = b(c)$ defined above.

Recall that $b(c) = 0$ for $c \leq \cosh^{-1}(2)$, and hence $\kappa^{\mathrm{o}}(c) = \kappa^{\mathrm{i}}(c)$ in this case. For larger c they differ, however, as Figure 1 shows.

Proof. The proof works analogously to $T^{\mathrm{i}}_{\delta_\mu}$. The only difference is that $m(T, \delta_\mu)$ now has to be replaced by

$$\widetilde{m}(T, \delta_\mu) = \frac{1}{4\pi\mu} \int_{\mathbb{R}^3} \left(\frac{1}{\widetilde{K}_{T,\delta_\mu}(p)} - \frac{1}{p^2} \right) dp, \qquad (4.30)$$

and the operator $M(T, \delta_\mu)$ by

$$\widetilde{M}(T, \delta_\mu) = \frac{1}{\widetilde{K}_{T,\delta_\mu}(p)} - \widetilde{m}(T, \delta_\mu) \mathcal{F}\mathcal{F}^*. \qquad (4.31)$$

The result is that

$$\lim_{\lambda \to 0} \left(\widetilde{m}(T, \delta_\mu) + \frac{1}{\inf \operatorname{spec} \left(\lambda \sqrt{\bar{\mu}} \, \mathcal{V}_{\bar{\mu}} - \lambda^2 \mathcal{W}_{\bar{\mu}} \right)} \right) = 0 \qquad (4.32)$$

for $(\delta_\mu, T) \in T^{\mathrm{o}}_{\delta_\mu}$. What remains is to calculate $\widetilde{m}(T, \delta_\mu)$.

Lemma 3. *As $(T, \delta_\mu) \to (0, 0)$,*

$$\widetilde{m}(T, \delta_\mu) = \frac{1}{\sqrt{\bar{\mu}}} \left(\ln \frac{\bar{\mu}}{T} + \gamma - 2 + \ln \frac{8}{\pi} - \kappa^{\mathrm{o}}(\delta_\mu/T) + o(1) \right). \qquad (4.33)$$

Proof. For simplicity we will abbreviate $b\left(\delta_\mu/T\right)$ by b throughout the proof. First note that from (4.27) it follows that $\widetilde{m}(T, \delta_\mu)$ equals

$$\frac{1}{4\pi\bar{\mu}} \left[\int_{\frac{|p^2-\bar{\mu}|}{T} \geq b} \left(\frac{1}{K^0_{T,\delta_\mu}(p)} - \frac{1}{p^2} \right) dp + \int_{\frac{|p^2-\bar{\mu}|}{T} < b} \left(\frac{1}{f(b, \delta_\mu/T)} - \frac{1}{p^2} \right) dp \right]. \qquad (4.34)$$

For given $b > 0$,

$$\lim_{T, \delta_\mu \to 0} \frac{1}{4\pi\bar{\mu}} \int_{\frac{|p^2-\bar{\mu}|}{T} < b} \left(\frac{1}{2Tf(b, \delta_\mu/T)} - \frac{1}{p^2} \right) dp = \frac{b}{2\sqrt{\bar{\mu}}f(b, \delta_\mu/T)},$$

and the result obviously also holds for $b = 0$, where both sides are zero.

As in the proof of Lemma 2, we split the first integral in (4.34) into two parts according to $p^2 \leq \bar{\mu}$ and $p^2 \geq \bar{\mu}$, and change variables from $p^2 - \bar{\mu}$ to $-t$ and t, respectively. Introducing again the function Υ_0 in (4.24), the integral equals

$$
\frac{1}{2\bar{\mu}} \int_{\bar{\mu}}^{\infty} \left(\Upsilon_0(t) \frac{\sqrt{\bar{\mu}+t}}{t} - \frac{1}{\sqrt{\bar{\mu}+t}} \right) dt
$$
$$
+ \frac{1}{2\bar{\mu}} \int_{Tb}^{\bar{\mu}} \Upsilon_0(t) \left(\frac{\sqrt{\bar{\mu}+t} + \sqrt{\bar{\mu}-t} - 2\sqrt{\bar{\mu}}}{t} \right) dt
$$
$$
- \frac{1}{2\bar{\mu}} \int_{Tb}^{\bar{\mu}} \left(\frac{1}{\sqrt{\bar{\mu}+t}} + \frac{1}{\sqrt{\bar{\mu}-t}} \right) dt + \frac{1}{\sqrt{\bar{\mu}}} \int_{Tb}^{\bar{\mu}} \frac{\Upsilon_0(t)}{t} dt . \qquad (4.35)
$$

The first integral converges to $\bar{\mu}^{-1/2} \ln \left(1 + \sqrt{2} \right)$ as $T, \delta_\mu \to 0$. The second integral gives $\frac{1}{\sqrt{\bar{\mu}}} \left(-\ln(1 + \sqrt{2}) + \sqrt{2} + \ln 4 - 2 \right)$ and the third integral is $\sqrt{2/\bar{\mu}}$ in this limit.

To evaluate the last term in (4.35), we proceed as in Lemma 2, with the obvious modifications. The result is that

$$
\frac{1}{\sqrt{\bar{\mu}}} \int_{Tb}^{\bar{\mu}} \frac{\Upsilon_0(t)}{t} dt = \frac{1}{\sqrt{\bar{\mu}}} \left(\gamma + \ln \frac{\bar{\mu}}{T} - \ln \frac{8}{\pi} - \kappa^\circ(c) + o(1) \right) .
$$

Collecting all the terms proves the lemma. $\square$

This concludes the proof of Theorem 3. $\square$

4.3. Evaluation of $T^{\mathrm{g}}_{\delta_\mu}$ and a more detailed phase diagram.

Our final goal is derive a sharper upper bound on the critical temperature which separates the phases where the minimizer of the BCS functional has a vanishing or a non-vanishing α. We are able to do this under additional assumptions on the interaction potential V, which, in particular, imply that the BCS minimizer α is the ground state of the operator $K^\Delta_{T,\delta_\mu} + \lambda V$.

Recall the definition of $T^{\mathrm{g}}_{\delta_\mu}$ in (2.10).

THEOREM 4. *Let V be a radial function in $L^{3/2}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, with $\hat{V} \leq 0$ and $\hat{V}(0) < 0$. Assume that $e_\mu = \inf \operatorname{spec} \mathcal{V}_{\bar{\mu}} < 0$, and let $\varrho(\lambda)$ be defined as in (4.7). Then any sequence of pairs $(\delta_\mu(\lambda), T(\lambda))$ on the curve $T^{\mathrm{g}}_{\delta_\mu}(\lambda V)$ satisfies*

$$
\lim_{\lambda \to 0} \left(\ln \frac{\bar{\mu}}{T} + \frac{\pi}{2\sqrt{\bar{\mu}} \varrho(\lambda)} - \kappa^{\mathrm{g}}(\delta_\mu/T) \right) = 2 - \gamma - \ln(8/\pi) \qquad (4.36)
$$

where $\kappa^{\mathrm{g}}(t) = \inf_{d>0} \zeta(t, d)$, with

$$\zeta(t, d) = -\int_0^d e^{-x} \ln(x/d) \, dx + \int_d^\infty e^{-x} \ln\left(1 + \sqrt{1 - (d/x)^2}\right) dx \quad (4.37)$$

$$+ \frac{1}{1 + e^t} \int_d^\infty \frac{1 - e^{-x}}{1 + e^{x+t}} \frac{dx}{\sqrt{x^2 - d^2}}$$

$$+ \frac{1}{1 + e^{-t}} \int_d^\infty \frac{1 - e^{-x}}{1 + e^{x-t}} \frac{dx}{\sqrt{x^2 - d^2}} - \ln \pi.$$

Moreover, for any sequence $(\delta_\mu(\lambda), T(\lambda))$ that lies strictly above the curve $T^{\mathrm{g}}_{\delta_\mu}$, in the sense that

$$\lim_{\lambda \to 0} \left(\ln \frac{\bar{\mu}}{T} + \frac{\pi}{2\sqrt{\bar{\mu}}\varrho(\lambda)} - \kappa^{\mathrm{g}}(\delta_\mu/T) \right) < 2 - \gamma - \ln(8/\pi) \quad (4.38)$$

the BCS gap equation (2.5) cannot have a solution for small enough λ with the property that the operator $K^\Delta_{T,\delta_\mu} + \lambda V$ has 0 as lowest eigenvalue. In particular, the normal state minimizes $\mathcal{F}_T$ in this case.

We note that $\lim_{d \to 0} \xi(t, 0) = \kappa^{\mathrm{i}}(t)$. Moreover, $\kappa^{\mathrm{g}}(t) = \kappa^{\mathrm{i}}(t)$ for $t \leq \cosh^{-1}(2) \approx 1.32$. Numerically, one can check that this equality holds even on the larger interval $t \in [0, 1.91]$. (Compare with Fig. 1.)

The last statement of the theorem concerning the fact that the normal state minimizes $\mathcal{F}_T$ follows from a Perron-Frobenius argument, as explained in Subsect. 2.1.1. It implies that a non-vanishing and minimizing α is necessarily the ground state of $K^\Delta_{T,\delta_\mu} + \lambda V$, with eigenvalue zero.

Proof. By the previous arguments we obtain for $(\delta_\mu, T) \in T^{\mathrm{g}}_{\delta_\mu}$ the following behavior in the small coupling limit:

$$\lim_{\lambda \to 0} \left(\overline{m}(\delta_\mu, T) + \frac{1}{\inf \operatorname{spec}(\lambda\sqrt{\bar{\mu}}\,\mathcal{V}_{\bar{\mu}} - \lambda^2\mathcal{W}_{\bar{\mu}})} \right) = 0, \quad (4.39)$$

where

$$\overline{m}(\delta_\mu, T) = \frac{1}{4\pi\bar{\mu}} \max_y \int_{\mathbb{R}^3} \left(\frac{1}{K^y_{T,\delta_\mu}(p)} - \frac{1}{p^2} \right) dp.$$

For the first part of the theorem it remains to evaluate $\overline{m}(\delta_\mu, T)$, which is done in the following lemma.

Lemma 4. *In the limit $(T, \delta_\mu) \to (0, 0)$,*

$$\overline{m}(\delta_\mu, T) = \frac{1}{\sqrt{\bar{\mu}}} \left(\ln \frac{\bar{\mu}}{T} + \gamma - 2 + \ln \frac{8}{\pi} - \kappa^{\mathrm{g}}(\delta_\mu/T) + o(1) \right). \quad (4.40)$$

Proof. As in the proof of Lemma 2, we can rewrite $\overline{m}(\delta_\mu, T)$ as

$$\overline{m}(\delta_\mu, T) = \max_y I(\delta_\mu, T, y) \quad (4.41)$$

where

$$I(\delta_\mu, T, y) = \frac{1}{2\bar{\mu}} \int_{\bar{\mu}}^\infty \left(\Upsilon_y(t) \sqrt{\frac{\bar{\mu}+t}{t^2+y^2}} - \frac{1}{\sqrt{\bar{\mu}+t}} \right) dt \qquad (4.42)$$

$$+ \frac{1}{2\bar{\mu}} \int_0^{\bar{\mu}} \frac{\Upsilon_y(t)}{\sqrt{t^2+y^2}} \left(\sqrt{\bar{\mu}+t} + \sqrt{\bar{\mu}-t} - 2\sqrt{\bar{\mu}} \right) dt$$

$$- \frac{1}{2\bar{\mu}} \int_0^{\bar{\mu}} \left(\frac{1}{\sqrt{\bar{\mu}+t}} + \frac{1}{\sqrt{\bar{\mu}-t}} \right) dt + \frac{1}{\sqrt{\bar{\mu}}} \int_0^{\bar{\mu}} \frac{\Upsilon_y(t)}{\sqrt{t^2+y^2}} dt$$

and

$$\Upsilon_y(t) = \left(1 - \frac{1}{1+e^{\frac{\sqrt{t^2+y^2}+\delta_\mu}{T}}} - \frac{1}{1+e^{\frac{\sqrt{t^2+y^2}-\delta_\mu}{T}}} \right).$$

It is clear that the maximum is attained for a y that goes to zero as $(T, \delta_\mu) \to (0, 0)$. Using the Lebesgue dominated convergence theorem, one observes that in the limit $T, y, \delta_\mu \to 0$ the first integral converges to $\bar{\mu}^{-1/2} \ln\left(1+\sqrt{2}\right)$, the second becomes $\bar{\mu}^{-1/2}\left(-\ln(\sqrt{2}-1) + \sqrt{2} + \ln 4 - 2\right)$, and the third $\sqrt{2/\bar{\mu}}$.

It remains to compute the last integral in (4.42). Changing variables to $x = \sqrt{t^2+y^2}/T$, we can rewrite it as

$$\int_d^{\sqrt{d^2+(\bar{\mu}/T)^2}} \frac{1 - \frac{1}{1+e^{x+c}} - \frac{1}{1+e^{x-c}}}{\sqrt{x^2-d^2}} \, dx = I + II := \qquad (4.43)$$

$$= \int_d^{\sqrt{d^2+(\bar{\mu}/T)^2}} \frac{\frac{1}{1+e^c} - \frac{1}{1+e^{x+c}}}{\sqrt{x^2-d^2}} \, dx + \int_d^{\sqrt{d^2+(\bar{\mu}/T)^2}} \frac{\frac{1}{1+e^{-c}} - \frac{1}{1+e^{x-c}}}{\sqrt{x^2-d^2}} \, dx,$$

where $c = \frac{\delta_\mu}{T}$ and $d = \frac{y}{T}$. Now I in (4.43) equals

$$I = \frac{1}{1+e^c} \int_d^{\sqrt{d^2+(\bar{\mu}/T)^2}} \left[\frac{1-e^{-x}}{\sqrt{x^2-d^2}} - \frac{1}{\sqrt{x^2-d^2}} \left(\frac{1-e^{-x}}{1+e^{x+c}} \right) \right] dx, \quad (4.44)$$

and similarly for II, replacing c by $-c$. For the second term in the integrand, we can simply replace the upper integration boundary by ∞ as $T \to 0$. To evaluate the integral of the first term, we integrate by parts and obtain

$$\left(1 - e^{-\sqrt{d^2+(\bar{\mu}/T)^2}} \right) \ln\left(\frac{\bar{\mu}}{T} + \sqrt{d^2+(\bar{\mu}/T)^2} \right) - \left(1 - e^{-d} \right) \ln d \qquad (4.45)$$

$$- \int_d^{\sqrt{d^2+(\bar{\mu}/T)^2}} e^{-x} \ln\left(x + \sqrt{x^2-d^2} \right) dx.$$

The latter integral converges as $T \to 0$. Proceeding in the same way with II, we have thus shown that

$$\int_0^{\bar{\mu}} \frac{\Upsilon_y(t)}{\sqrt{t^2 + y^2}} dt \tag{4.46}$$

$$= \ln\left(\frac{\bar{\mu}}{T}\right) + \ln 2 - \left(1 - e^{-d}\right)\ln d - \int_d^\infty e^{-x} \ln\left(x + \sqrt{x^2 - d^2}\right) dx + o(1)$$

$$- \frac{1}{1 + e^c} \int_d^\infty \frac{1 - e^{-x}}{1 + e^{x+c}} \frac{dx}{\sqrt{x^2 - d^2}} - \frac{1}{1 + e^{-c}} \int_d^\infty \frac{1 - e^{-x}}{1 + e^{x-c}} \frac{dx}{\sqrt{x^2 - d^2}} \, .$$

Combining all the terms, keeping in mind (4.26), we arrive at (4.40). $\square$

We now turn to the proof of the second part of Theorem 4. The order parameter $\Delta(p)$ is assumed to satisfy the BCS gap equation. Under the assumption that α is the ground state of $K^\Delta_{T,\delta_\mu} + \lambda V$ we can use the Birman-Schwinger principle to show, as in [9, Lemma 4], that

$$\Delta(p) = -g(\lambda)\left(\int_{\Omega_{\bar{\mu}}} \hat{V}(p - q) d\omega(q) + \lambda \nu_\lambda(p)\right), \tag{4.47}$$

with $\|\nu_\lambda\|_\infty \leq C$ uniformly in λ, and $g(\lambda)$ a normalization constant determined by the gap equation. Eq. (4.47) can be derived using standard perturbation theory applied to the operator

$$\lambda m(T, \delta_\mu, \Delta) \mathcal{F}|V|^{1/2} \frac{1}{1 + \lambda V^{1/2} M(T, \delta_\mu, \Delta)|V|^{1/2}} V^{1/2} \mathcal{F}^* \tag{4.48}$$

where

$$m(T, \delta_\mu, \Delta) = \frac{1}{4\pi\bar{\mu}} \int_{\mathbb{R}^3} \left(\frac{1}{K^\Delta_{T,\delta_\mu}(p)} - \frac{1}{p^2}\right) dp \, ,$$

and M is defined as in (4.12), with K^0_{T,δ_μ} replaced by K^Δ_{T,δ_μ} and $m(T, \delta_\mu)$ by $m(T, \delta_\mu, \Delta)$. The eigenvector corresponding to the lowest eigenvalue -1 of this operator is, to leading order, given by the lowest eigenvector of $\mathcal{V}_{\bar{\mu}}$, which is the constant function $u(p) = \frac{1}{\sqrt{4\pi\bar{\mu}}}$. In fact, the radial symmetry of V implies that the eigenstates of $\mathcal{V}_{\bar{\mu}}$ are the spherical harmonics, and the fact that $\hat{V} \leq 0$ forces the ground state of $\mathcal{V}_{\bar{\mu}}$ to have a fixed sign, hence to be constant. The lowest eigenvector of (4.48) is, therefore, $\phi = u + \lambda \xi_\lambda$, with ξ_λ uniformly bounded in $L^2(\Omega_{\bar{\mu}})$ norm. With the aid of the Birman-Schwinger principle, α can be recovered from ϕ via $\alpha = cV^{1/2}\mathcal{F}^*\phi$ for some normalization constant c. Hence we have

$$\Delta(p) = 2K^\Delta_{T,\delta_\mu}(p)\hat{\alpha}(p) = -2\lambda\widehat{V\alpha}(p) = -2\lambda c\widehat{|V|^{1/2}\phi}(p) \, ,$$

which implies (4.47).

Using the Lipschitz continuity of the main term in (4.47), we can argue as in [9, Thm. 2] that for small λ the function $m(\delta_\mu, T, \Delta)$ is determined by

the value of $\Delta(p)$ at the Fermi surface $p^2 = \bar{\mu}$, i.e.,

$$m\left(\delta_\mu, T, \Delta\right) = m\left(\delta_\mu, T, \Delta(\sqrt{\bar{\mu}})\right) + o(1)\,.$$

This implies further that

$$\lim_{\lambda \to 0}\left(m\left(\delta_\mu, T, \Delta(\sqrt{\bar{\mu}})\right) + \frac{1}{\inf \operatorname{spec}\left(\lambda\sqrt{\bar{\mu}}\,\mathcal{V}_{\bar{\mu}} - \lambda^2 \mathcal{W}_{\bar{\mu}}\right)}\right) = 0\,. \qquad (4.49)$$

Since, by definition,

$$m\left(\delta_\mu, T, \Delta(\sqrt{\bar{\mu}})\right) \leq \overline{m}(\delta_\mu, T)\,,$$

it follows from (4.39) that (4.49) cannot be satisfied for (δ_μ, T) outside of $T_{\delta_\mu}^{\mathrm{g}}$, i.e., for such (δ_μ, T) such that

$$\lim_{\lambda \to 0}\left(\overline{m}\left(\delta_\mu, T\right) + \frac{1}{\inf \operatorname{spec}\left(\lambda\sqrt{\bar{\mu}}\,\mathcal{V}_{\bar{\mu}} - \lambda^2 \mathcal{W}_{\bar{\mu}}\right)}\right) < 0\,. \qquad (4.50)$$

This completes the proof of the theorem. $\qquad \square$

REFERENCES

[1] V. Bach, E.H. Lieb, J.P. Solovej, *Generalized Hartree-Fock theory and the Hubbard model*, J. Stat. Phys. **76**, 3–89 (1994).

[2] J. Bardeen, L. Cooper, J. Schrieffer, *Theory of Superconductivity*, Phys. Rev. **108**, 1175–1204 (1957).

[3] M.S. Birman, D.R. Yafaev, *Spectral properties of the scattering matrix*, St. Petersburg Math. J. **4**, 1055–1079 (1993).

[4] I. Bloch, J. Dalibard, W. Zwerger, *Many-Body Physics with Ultracold Gases*, Rev. Mod. Phys. **80**, 885 (2008).

[5] H. Caldas, A. L. Mota, *Temperature effects in a Fermi Gas with Population Imbalance*, J. Stat. Mech.: Theory and Experiment, P08013 (2008).

[6] Q. Chen, J. Stajic, S. Tan, K. Levin, *BCS–BEC crossover: From high temperature superconductors to ultracold superfluids*, Phys. Rep. **412**, 1–88 (2005).

[7] R.L. Frank, C. Hainzl, S. Naboko, R. Seiringer, *The critical temperature for the BCS equation at weak coupling*, J. Geom. Anal. **17**, 559–568 (2007).

[8] C. Hainzl, E. Hamza, R. Seiringer, J.P. Solovej, *The BCS functional for general pair interactions*, Comm. Math. Phys. **281**, 349–367 (2008).

[9] C. Hainzl, R. Seiringer, *Critical temperature and energy gap for the BCS equation*, Phys. Rev. B **77**, 184517-1–10 (2008).

[10] C. Hainzl, R. Seiringer, *The BCS Critical Temperature for Potentials with Negative Scattering Length*, Lett. Math. Phys. **84**, 99–107 (2008).

[11] C. Hainzl, R. Seiringer, *Spectral properties of the BCS gap equation of superfluidity*, in: Mathematical Results in Quantum Mechanics, proceedings of QMath10, Moeciu, Romania, September 10–15, 2007, World Scientific (2009).

[12] C. Hainzl, R. Seiringer, *A linear criterion for solutions of non-linear equations, with application to the BCS gap equation*, in: Spectral and Scattering Theory for Quantum Magnetic Systems, P. Briet et al, eds., CIRM, Marseilles, July 711, 2008, Amer. Math. Soc. (2009).

[13] C. Hainzl, R. Seiringer, *Asymptotic behavior of eigenvalues for Schrödinger type operators with degenerate kinetic energy*, Math. Nachr. **283**, 489–499 (2010).

[14] A. Laptev, O. Safronov, T. Weidl, *Bound state asymptotics for elliptic operators with strongly degenerate symbols*, in: Nonlinear Problems in Mathematical Physics and Related Topics I, pp. 233–246, Int. Math. Ser. (N.Y.), Kluwer/Plenum, New York (2002).

[15] A.J. Leggett, *Diatomic Molecules and Cooper Pairs*, in: Modern trends in the theory of condensed matter, A. Pekalski, R. Przystawa, eds., Springer (1980).

[16] G. Sarma, *On the influence of a uniform exchange field acting on the spins of the conduction electrons in a superconductor*, J. Phys. Chem. Solids **24**, 1029–1032 (1963).

[17] M.W. Zwierlein, C.H. Schunck, A. Schirotzek, W. Ketterle, *Fermionic Superfluidity with Imbalanced Spin Populations*, Science **311**, 492–496 (2006).

[18] M.W. Zwierlein, C.H. Schunck, A. Schirotzek, W. Ketterle, *Direct observation of the superfluid phase transition in ultracold Fermi gases*, Nature **442**, 54–58 (2006).

(Abraham Freiji), DEPARTMENT OF NEUROLOGY, UAB, SC 350, 1530 3RD AVE S, BIRMINGHAM, AL 35294, USA

E-mail address: `afreiji@uab.edu`

(Christian Hainzl) MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, 72076 TÜBINGEN, GERMANY

E-mail address: `christian.hainzl@uni-tuebingen.de`

(Robert Seiringer), DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, 805 SHERBROOKE STREET WEST, MONTREAL, QC, H3A 2K6, CANADA

E-mail address: `rseiring@math.mcgill.ca`