Harmonic analysis and the Riemann-Roch theorem.

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1. Let D be a smooth projective curve over a finite field k. It is known (see, e.g., [6, §3]) that the Poisson summation formula applied to the discrete subgroup $k(D)$ of the adelic space $\mathbb{A}_D$ implies the Riemann-Roch theorem on the curve D. In this note we will show how the two-dimensional Poisson formulas (see [3, §5.9] and [4, §13]) imply the Riemann-Roch theorem (without the Noether formula) on a projective smooth algebraic surface X over k.

First, we need some general proposition. Let $E=(I,F,V)$ be a C_2-space over the field k (see [2]). Recall that for any $i,j \in I$ we have constructed in [3, §5.2] a one-dimensional $\mathbb{C}$-vector space of virtual measures $\mu(F(i) \mid F(j)) = \mu(F(i)/F(l))^* \otimes_{\mathbb{C}} \mu(F(l)/F(j))$, where $l \in I$ such that $l \le i$, $l \le j$, and $\mu(H)$ is the space of $\mathbb{C}$-valued Haar measures on a C_1-space H. The space $\mu(F(i) \mid F(j))$ does not depend on the choice of $l \in I$ up to a canonical isomorphism.

Let $0 \to A \to E \to B \to 0$ be an admissible triple of C_2-spaces over k. Let $A=(J,G,W)$ as a C_2-space, and $W = F(j)$ for some $j \in I$. Then A is a cC_2-space and B is a dC_2-space (see [3, §5.1]). Let $o \in I$ and $\mu \in \mu(W/F(o)\cap W)$, $\nu \in \nu(F(o)/F(o)\cap W)^*$. Then in [3, form. (164)] we have constructed the characteristic element $\delta_{A,\mu\otimes\nu} \in \mathcal{D}'_{F(o)}(E)$. We note that $\mu\otimes\nu \in \mu(F(o) \mid W)$. Therefore we can replace $\mu\otimes\nu$ by $\eta \in \mu(F(o)\mid W)$ and write $\delta_{A,\eta} \in \mathcal{D}'_{F(o)}(E)$ instead of the previous notation.

Let $0 \to L \to E \to M \to 0$ be an admissible triple of C_2-spaces over k such that L is a cfC_2-space and M is a dfC_2-space (see [3, §5.1]). In [3, form. (169)] we have constructed the characteristic element $\delta_L \in \mathcal{D}_{F(o)}(E)$. We note that for any $i,j \in I$ the space L defines a non-zero element $\mu_{L,F(i),F(j)} \in \mu(F(i) \mid F(j))$ in the following way. Let $L=(K,T,U)$ as a C_2-space. Choose some $l \in I$ such that $l \le i$, $l \le j$. Then $\mu_{L,F(i),F(j)} = \mu^{-1}_{L,F(l),F(i)} \otimes \mu_{L,F(l),F(j)}$, where for any $m \le n \in I$ we define $\mu_{L,m,n} \in \mu(F(n)/F(m))$ as $\mu_{L,m,n}(U\cap F(n)/U \cap F(m)) = 1$. The element $\mu_{L,F(i),F(j)}$ does not depend on the choice of $l \in I$.

There is a natural pairing $< \cdot,\cdot >: \mathcal{D}_{F(o)}(E) \times D'_{F(o)}(E) \to \mathbb{C}$. From the above definitions it is easy to prove the following proposition.

Proposition 1

$$< \delta_L, \delta_{A,\eta} >= \frac{\eta}{\mu_{L,F(o),W}}.$$

*Both authors are supported by RFBR (grant no. 11-01-00145-a) and by a program of President of RF for supporting of Leading Scientific Schools (grant no. NSh-4713.2010.1).

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2. Let X be a smooth projective algebraic surface over a finite field k. Let $\mid k \mid = q$. For any quasicoherent sheaf $\mathcal{F}$ on X there is an adelic complex $\mathcal{A}_X(\mathcal{F})$ such that $H^*(\mathcal{A}_X(\mathcal{F})) = H^*(X, \mathcal{F})$. Let $C \in \mathrm{Div}(X)$. For the sheaf $\mathcal{O}_X(C)$ on X we will write this complex in the following way:

$$\mathbb{A}_{0,C} \oplus \mathbb{A}_{1,C} \oplus \mathbb{A}_{2,C} \longrightarrow \mathbb{A}_{01,C} \oplus \mathbb{A}_{02,C} \oplus \mathbb{A}_{12,C} \longrightarrow \mathbb{A}_{012,C},$$

where $\mathbb{A}_{*,\,C} = \mathbb{A}_{X,*}(\mathcal{O}_X(C))$ (see the corresponding notations and definitions in [4, §14.1]), and we have omitted indication on X in the notations of subgroups of the adelic complex, because we will work only with one algebraic surface X during this note. We note that that all the groups $\mathbb{A}_{*,C}$ are subgroups of the group $\mathbb{A}_{012,C}$. Besides, the following groups does not depend on $C \in \mathrm{Div}(X)$:

$$\mathbb{A}_{0,C} = \mathbb{A}_0, \quad \mathbb{A}_{01,C} = \mathbb{A}_{01}, \quad \mathbb{A}_{02,C} = \mathbb{A}_{02}, \quad \mathbb{A}_{012,C} = \mathbb{A}_{012} = \mathbb{A}.$$

Moreover, $\mathbb{A} \subset \prod_{x \in D} K_{x,D}$, where $x \in D$ runs over all pairs with irreducible curve D on X and x is a point on D. The ring $K_{x,D}$ is a finite product of two-dimensional local fields with the last residue field $k(x)$.

We fix a rational differential form $\omega \in \Omega^2_{k(X)/k}$. Let $(\omega) \in \mathrm{Div}(X)$ be the corresponding divisor. The following pairing (which depends on ω) is well-defined, symmetric and non-degenerate:

$$\mathbb{A} \times \mathbb{A} \longrightarrow k \quad : \quad \{f_{x,D}\} \times \{g_{x,D}\} \mapsto \sum_{x \in D} \mathrm{Tr}_{k(x)/k} \circ \mathrm{res}_{x,D}(f_{x,D}\, g_{x,D}\, \omega), \qquad (1)$$

where $\mathrm{res}_{x,D}$ is the two-dimensional residue. For any k-subspace $V \subset \mathbb{A}$ we will denote by $V^\perp$ the annihilator of V in $\mathbb{A}$ with respect to the pairing (1). Using the reciprocity laws for the residues of differential forms on X (the reciprocity laws "around a point" and the reciprocity laws "along a curve") one can prove the following proposition.

Proposition 2 *We have the following properties.*

$$\mathbb{A}_0^\perp = \mathbb{A}_{01} + \mathbb{A}_{02}, \quad \mathbb{A}_{1,C}^\perp = \mathbb{A}_{01} + \mathbb{A}_{12,(\omega)-C}, \quad \mathbb{A}_{2,C}^\perp = \mathbb{A}_{02} + \mathbb{A}_{12,(\omega)-C}$$
$$\mathbb{A}_{01}^\perp = \mathbb{A}_{01}, \quad \mathbb{A}_{02}^\perp = \mathbb{A}_{02}, \quad \mathbb{A}_{12,C}^\perp = \mathbb{A}_{12,(\omega)-C}.$$

We note that $\mathbb{A} = \lim\limits_{\substack{\longrightarrow \\ C \in \mathrm{Div}(X)}} \mathbb{A}_{12,C}$, and $\mathbb{A}_{12,C} = \lim\limits_{\substack{\longleftarrow \\ C' \le C}} \mathbb{A}_{12,C}/\mathbb{A}_{12,C'}$. For any $C' \le C$ the k-space $\mathbb{A}_{12,C}/\mathbb{A}_{12,C'}$ has the natural structure of a complete C_1-space over the field k. Hence we obtain that the k-space $\mathbb{A}$ has the following structure of a complete C_2-space over k: $(\mathrm{Div}(X), F, \mathbb{A})$, where $F(C) = \mathbb{A}_{12,C}$ for $C \in \mathrm{Div}(X)$. For simplicity we will use the same notation $\mathbb{A}$ for this C_2-space, i.e. we will omit the partially ordered set $\mathrm{Div}(X)$ and the function F. The subspaces $\mathbb{A}_{*,C}$ of $\mathbb{A}$ (and the factor-spaces by these subspaces) have induced structures of C_2-spaces, which we will also denote by the same notations $\mathbb{A}_{*,C}$ (by notations for factor-spaces).

From proposition 2 it follows that the C_2-dual space (see [3, §5.1]) $\check{\mathbb{A}}$ coincides with the C_2-space $\mathbb{A}$ itself:

$$\check{\mathbb{A}} = \lim_{\substack{\longleftarrow \\ C \in \mathrm{Div}(X)}} \lim_{\substack{\longrightarrow \\ C' \le C}} \mathbb{A}_{12,C'}^\perp/\mathbb{A}_{12,C}^\perp = \lim_{\substack{\longrightarrow \\ C' \in \mathrm{Div}(X)}} \lim_{\substack{\longleftarrow \\ C \ge C'}} \mathbb{A}_{12,(\omega)-C'}/\mathbb{A}_{12,(\omega)-C} = \mathbb{A}.$$

3. For any $E \in \mathrm{Div}(X)$ we denote $h^i(E) = \dim_k H^i(X, \mathcal{O}_X(E))$, where $0 \le i \le 2$. We fix any $H, C \in \mathrm{Div}(X)$. We consider the following admissible triple of complete C_2-spaces over k:

$$0 \longrightarrow \mathbb{A}_0 \longrightarrow \mathbb{A}_{01} \longrightarrow \mathbb{A}_{01}/\mathbb{A}_0 \longrightarrow 0. \qquad (2)$$

The space $\mathbb{A}_0$ is a cfC_2-space, and the space $\mathbb{A}_{01}/\mathbb{A}_0$ is a dfC_2-space. Therefore there is the characteristic element $\delta_{\mathbb{A}_0} \in \mathcal{D}_{\mathbb{A}_{1,H}}(\mathbb{A}_{01})$.

Now we consider the following admissible triple of complete C_2-spaces over k:

$$0 \longrightarrow \mathbb{A}_{1,C} \longrightarrow \mathbb{A}_{01} \longrightarrow \mathbb{A}_{01}/\mathbb{A}_{1,C} \longrightarrow 0. \qquad (3)$$

We note that the space $\mathbb{A}_{01}$ is a dfC_2-space. Therefore for any $H', C' \in \mathrm{Div}(X)$ there is a natural element $\delta_{H',C'} \in \mu(\mathbb{A}_{1,H'} \mid \mathbb{A}_{1,C'})$ which is uniquely defined by the following two conditions: 1) $\delta_{H',M'} \otimes \delta_{M',C'} = \delta_{H',C'}$ for any $H', M', C' \in \mathrm{Div}(X)$, and 2) if $H' \le C'$ then $\delta_{H',C'} \in \mu(\mathbb{A}_{1,C'}/\mathbb{A}_{1,H'})$ is defined as $\delta_{H',C'}((0)) = 1$, where (0) is the zero subspace in the discrete C_1-space $\mathbb{A}_{1,C'}/\mathbb{A}_{1,H'}$. Besides, the space $\mathbb{A}_{1,C}$ is a cC_2-space, and the space $\mathbb{A}_{01}/\mathbb{A}_{1,C}$ is a dC_2-space. Hence there is the characteristic element $\delta_{\mathbb{A}_{1,C}, \delta_{H,C}} \in \mathcal{D}'_{\mathbb{A}_{1,H}}(\mathbb{A}_{01})$.

Lemma 1 *We have the following equality:*

$$< \delta_{\mathbb{A}_0} \, , \, \delta_{\mathbb{A}_{1,C}, \delta_{H,C}} >= q^{h^0(C) - h^0(H)}.$$

Proof We will use proposition 1. From this proposition it follows that it is enough to consider $H \le C$. In this case, by this proposition again, we have $< \delta_{\mathbb{A}_0}, \delta_{\mathbb{A}_{1,C}, \delta_{H,C}} >= q^{\dim_k V}$, where the k-vector space $V = (\mathbb{A}_0 \cap \mathbb{A}_{1,C})/(\mathbb{A}_0 \cap \mathbb{A}_{1,H})$. Now we use $\mathbb{A}_0 \cap \mathbb{A}_{1,E} = H^0(X, \mathcal{O}_X(E))$ for any $E \in \mathrm{Div}(X)$. The lemma is proved.

Now we fix any $P, Q \in \mathrm{Div}(X)$. We consider the following admissible triple of complete C_2-spaces over k:

$$0 \longrightarrow \mathbb{A}_{02}/\mathbb{A}_0 \longrightarrow \mathbb{A}/\mathbb{A}_{01} \longrightarrow \mathbb{A}/(\mathbb{A}_{02} + \mathbb{A}_{01}) \longrightarrow 0, \qquad (4)$$

where we use that $\mathbb{A}_0 = \mathbb{A}_{01} \cap \mathbb{A}_{02}$. The space $\mathbb{A}_{02}/\mathbb{A}_0$ is a cfC_2-space, and the space $\mathbb{A}/(\mathbb{A}_{01} + \mathbb{A}_{02})$ is a dfC_2-space. Therefore there is the characteristic element $\delta_{\mathbb{A}_{02}/\mathbb{A}_0} \in \mathcal{D}_{\mathbb{A}_{12,P}/\mathbb{A}_{1,P}}(\mathbb{A}/\mathbb{A}_{01})$.

Now we consider the following admissible triple of complete C_2-spaces over k:

$$0 \longrightarrow \mathbb{A}_{12,Q}/\mathbb{A}_{1,Q} \longrightarrow \mathbb{A}/\mathbb{A}_{01} \longrightarrow \mathbb{A}/(\mathbb{A}_{12,Q} + \mathbb{A}_{01}) \longrightarrow 0, \qquad (5)$$

where we use that $\mathbb{A}_{1,Q} = \mathbb{A}_{01} \cap \mathbb{A}_{12,Q}$. We note that the space $\mathbb{A}/\mathbb{A}_{01}$ is a cfC_2-space. Therefore for any $P', Q' \in \mathrm{Div}(X)$ there is the following natural element $1_{P',Q'} \in \mu(\mathbb{A}_{12,P'}/\mathbb{A}_{1,P'} \mid \mathbb{A}_{1,Q'}/\mathbb{A}_{1,Q'})$ which is uniquely defined by the following two conditions: 1) $1_{P',R'} \otimes 1_{R',Q'} = 1_{P',Q'}$ for any $P', R', Q' \in \mathrm{Div}(X)$, and 2) if $P' \le Q'$ then $1_{P',Q'} \in \mu((\mathbb{A}_{12,Q'}/\mathbb{A}_{1,Q'})/(\mathbb{A}_{12,P'}/\mathbb{A}_{1,P'}))$ is defined as $1_{P',Q'}((\mathbb{A}_{12,Q'}/\mathbb{A}_{1,Q'})/(\mathbb{A}_{12,P'}/\mathbb{A}_{1,P'})) = 1$, since $(\mathbb{A}_{12,Q'}/\mathbb{A}_{1,Q'})/(\mathbb{A}_{12,P'}/\mathbb{A}_{1,P'})$ is a compact C_1-space. Besides, the space $\mathbb{A}_{12,Q}/\mathbb{A}_{1,Q}$ is a cC_2-space, and the space $\mathbb{A}/(\mathbb{A}_{12,Q} + \mathbb{A}_{01})$ is a dC_2-space. Hence there is the characteristic element $\delta_{\mathbb{A}_{12,Q}/\mathbb{A}_{1,Q}, 1_{P,Q}} \in \mathcal{D}'_{\mathbb{A}_{12,P}/\mathbb{A}_{1,P}}(\mathbb{A}/\mathbb{A}_{01})$.

Lemma 2 *We have the following equality:*

$$< \delta_{\mathbb{A}_{02}/\mathbb{A}_0} \, , \, \delta_{\mathbb{A}_{12,Q}/\mathbb{A}_{1,Q}, 1_{P,Q}} >= q^{h^2(Q)-h^2(P)} .$$

Proof We will use proposition 1. By this proposition, it is enough to consider $P \ge Q$. In this case, by this proposition again, we have $< \delta_{\mathbb{A}_{02}/\mathbb{A}_0} \, , \, \delta_{\mathbb{A}_{12,Q}/\mathbb{A}_{1,Q}, 1_{P,Q}} >= q^{\dim_k W}$, where the k-vector space $W = (\mathbb{A}_{01} + \mathbb{A}_{02} + \mathbb{A}_{12,P})/(\mathbb{A}_{01} + \mathbb{A}_{02} + \mathbb{A}_{12,Q})$. Now we use that from the adelic complex $\mathcal{A}_X(\mathcal{O}_X(E))$ we have $\mathbb{A}/(\mathbb{A}_{01} + \mathbb{A}_{02} + \mathbb{A}_{12,E}) = H^2(X, \mathcal{O}_X(E))$ for any $E \in \mathrm{Div}(X)$. The lemma is proved.

Now we suppose that $Q = (\omega) - C$ and $P = (\omega) - H$. From proposition 2 it follows that triple (4) is a C_2-dual sequence to triple (2), and triple (5) is a C_2-dual sequence to triple (3). We have also the two-dimensional Fourier transforms $\mathbf{F} : \mathcal{D}_{\mathbb{A}_{1,H}}(\mathbb{A}_{01}) \to \mathcal{D}_{\mathbb{A}_{12,P}/\mathbb{A}_{1,P}}(\mathbb{A}/\mathbb{A}_{01})$ and $\mathbf{F} : \mathcal{D}'_{\mathbb{A}_{1,H}}(\mathbb{A}_{01}) \to \mathcal{D}'_{\mathbb{A}_{12,P}/\mathbb{A}_{1,P}}(\mathbb{A}/\mathbb{A}_{01})$ (see [3, §5.4.2] and [4, §8.2]), which we denote by the same letter, although they act from various spaces. Now by the two-dimensional Poisson formula II (see [3, th. 3]) we have $\mathbf{F}(\delta_{\mathbb{A}_0}) = \delta_{\mathbb{A}_{02}/\mathbb{A}_0}$. By the two-dimensional Poisson formula I (see [3, th. 2]) we have $\mathbf{F}(\delta_{\mathbb{A}_{1,C}, \delta_{H,C}}) = \delta_{\mathbb{A}_{12,Q}/\mathbb{A}_{1,Q}, 1_{P,Q}}$. (We used that according to [3, form. (103)] we have $\mu(\mathbb{A}_{1,H} \mid \mathbb{A}_{1,C}) = \mu(\mathbb{A}_{12,P}/\mathbb{A}_{1,P} \mid \mathbb{A}_{1,Q}/\mathbb{A}_{1,Q})$, and $\delta_{H,C} \mapsto 1_{P,Q}$ under this isomorphism.) Now since $\mathbf{F} \circ \mathbf{F}(g) = g$ for $g = \delta_{\mathbb{A}_0}$ or $g = \delta_{\mathbb{A}_{1,C}, \delta_{H,C}}$, and the maps $\mathbf{F}$ are conjugate with respect to each other (see [3, prop. 24]), we have that $< \delta_{\mathbb{A}_0}, \delta_{\mathbb{A}_{1,C}, \delta_{H,C}} >=< \mathbf{F}(\delta_{\mathbb{A}_0}), \mathbf{F}(\delta_{\mathbb{A}_{1,C}, \delta_{H,C}}) >$. Hence and from lemmas 1-2 we obtain for any $H, C \in \mathrm{Div}(X)$ the following equality:

$$h^0(C) - h^0(H) = h^2((\omega) - C) - h^2((\omega) - H). \tag{6}$$

4. For any $E \in \mathrm{Div}(X)$ we denote the Euler characteristic $\chi(E) = h^0(E) - h^1(E) + h^2(E)$. We fix any $R, S \in \mathrm{Div}(X)$. We consider the following admissible triple of complete C_2-spaces over k:

$$0 \longrightarrow \mathbb{A}_{02} \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}/\mathbb{A}_{02} \longrightarrow 0. \tag{7}$$

The space $\mathbb{A}_{02}$ is a cfC_2-space, and the space $\mathbb{A}/\mathbb{A}_{02}$ is a dfC_2-space. Therefore there is the characteristic element $\delta_{\mathbb{A}_{02}} \in \mathcal{D}_{\mathbb{A}_{12,R}}(\mathbb{A})$.

Now we consider the following admissible triple of complete C_2-spaces over k:

$$0 \longrightarrow \mathbb{A}_{12,S} \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}/\mathbb{A}_{12,S} \longrightarrow 0. \tag{8}$$

The subspace $\mathbb{A}_{01}$ uniquely defines an element $\nu_{R',S'} \in \mu(\mathbb{A}_{12,R'} \mid \mathbb{A}_{12,S'})$ for any $R', S' \in \mathrm{Div}(X)$ in the following way. If $R' \le S'$, then we consider the following admissible triple of C_1-spaces:

$$0 \longrightarrow \mathbb{A}_{1,S'}/\mathbb{A}_{1,R'} \longrightarrow \mathbb{A}_{12,S'}/\mathbb{A}_{12,R'} \longrightarrow \mathbb{A}_{12,S'}/\mathbb{A}_{1,S'} + \mathbb{A}_{12,R'} \longrightarrow 0,$$

where $\mathbb{A}_{1,S'}/\mathbb{A}_{1,R'}$ is a discrete C_1-space, and $\mathbb{A}_{12,S'}/\mathbb{A}_{1,S'} + \mathbb{A}_{12,R'}$ is a compact C_1-space. Now $\nu_{R',S'} \in \mu(\mathbb{A}_{12,S'}/\mathbb{A}_{12,R'})$ is equal to $\delta_0 \otimes 1$, where $\delta_0((0)) = 1$, $\delta_0 \in \mu(\mathbb{A}_{1,S'}/\mathbb{A}_{1,R'})$, and $1(\mathbb{A}_{12,S'}/\mathbb{A}_{1,S'} + \mathbb{A}_{12,R'}) = 1$, $1 \in \mu(\mathbb{A}_{12,S'}/\mathbb{A}_{1,S'} + \mathbb{A}_{12,R'})$. For arbitrary R', S' the element $\nu_{R',S'}$ is defined by the following rule: $\nu_{R',S'} = \nu_{R',T'} \otimes \nu_{T',S'}$, where $T' \in \mathrm{Div}(X)$ is any. The space $\mathbb{A}_{12,S}$ is a cC_2-space, and the space $\mathbb{A}/\mathbb{A}_{12,S}$ is a dC_2-space. Hence there is the characteristic element $\delta_{\mathbb{A}_{12,S}, \nu_{R,S}} \in \mathcal{D}'_{\mathbb{A}_{12,R}}(\mathbb{A})$.

Lemma 3 *We have the following equality:*

$$< \delta_{\mathbb{A}_{02}} \, , \, \delta_{\mathbb{A}_{12,S}, \nu_{R,S}} >= q^{\chi(S)-\chi(R)}.$$

Proof We will use proposition 1. From this proposition it follows that it is enough to consider $R \le S$. In this case, by this proposition again, we have $< \delta_{\mathbb{A}_{02}}, \delta_{\mathbb{A}_{12,S}, \nu_{R,S}} >= q^a$, where a is equal to the Euler characteristic of the following complex, which has the finite-dimensional over k cohomology groups:

$$\mathbb{A}_{1,S}/\mathbb{A}_{1,R} \, \oplus \, \mathbb{A}_{2,S}/\mathbb{A}_{2,R} \longrightarrow \mathbb{A}_{12,S}/\mathbb{A}_{12,R}. \tag{9}$$

Complex (9) is the factor-complex of the adelic complex $\mathcal{A}_X(\mathcal{O}_X(S))$ by the adelic complex $\mathcal{A}_X(\mathcal{O}_X(R))$. Therefore the Euler characteristic of complex (9) is the difference of the Euler characteristics of corresponding adelic complexes. The lemma is proved.

From proposition 2 it follows that triple (7) itself is a C_2-dual sequence to triple (7), and triple (8) is a C_2-dual sequence to triple (8) when $S \mapsto (\omega) - S$. We have also the two-dimensional Fourier transforms $\mathbf{F} : \mathcal{D}_{\mathbb{A}_{12,R}}(\mathbb{A}) \to \mathcal{D}_{\mathbb{A}_{12,(\omega)-R}}(\mathbb{A})$ and $\mathbf{F} : \mathcal{D}'_{\mathbb{A}_{12,R}}(\mathbb{A}) \to \mathcal{D}'_{\mathbb{A}_{12,(\omega)-R}}(\mathbb{A})$. By the two-dimensional Poisson formulas (see [3, th. 2-th. 3]) we have $\mathbf{F}(\delta_{\mathbb{A}_{02}}) = \delta_{\mathbb{A}_{02}}$ and $\mathbf{F}(\delta_{\mathbb{A}_{12,S}, \nu_{R,S}}) = \delta_{\mathbb{A}_{12,(\omega)-S}, \nu_{(\omega)-R, (\omega)-S}}$. (We used that from proposition 2 it follows that $\nu_{R,S} \mapsto \nu_{(\omega)-R, (\omega)-S}$ under the natural isomorphism $\mu(\mathbb{A}_{12,R} \mid \mathbb{A}_{12,S}) = \mu(\mathbb{A}_{12,(\omega)-R} \mid \mathbb{A}_{12,(\omega)-S})$.) From [3, prop. 24] we have $< \delta_{\mathbb{A}_{02}} \, , \, \delta_{\mathbb{A}_{12,S}, \nu_{R,S}} >=< \mathbf{F}(\delta_{\mathbb{A}_{02}}) \, , \, \mathbf{F}(\delta_{\mathbb{A}_{12,S}, \nu_{R,S}}) >$. Hence and from lemma 3 we have that $\chi(S) - \chi(R) = \chi((\omega) - S) - \chi((\omega) - R)$. If we put $R = (\omega) - S$, then for any $S \in \mathrm{Div}(X)$ we obtain from the previous formula the following equality:

$$\chi(S) = \chi((\omega) - S). \tag{10}$$

5. In section **1** we introduced the element $\mu_{L,F(i),F(j)} \in \mu(F(i), F(j)))$ for the admissible monomorphism of C_2-spaces $L \to E$. When $L = \mathbb{A}_{02}$, $E = \mathbb{A}$, $F(i) = \mathbb{A}_{12,R}$, $F(j) = \mathbb{A}_{12,S}$ for $R, S \in \mathrm{Div}(X)$ we will denote this element by $\mu_{R,S}$. From the proof of lemma 3 it follows that

$$q^{\chi(S)-\chi(R)} = \frac{\nu_{R,S}}{\mu_{R,S}}. \tag{11}$$

For any $g \in \mathbb{A}^*$ and any $R, S \in \mathrm{Div}(X)$ we have a natural action: $g^* : \mu(\mathbb{A}_{12,R} \mid \mathbb{A}_{12,S}) \to \mu(g\mathbb{A}_{12,R} \mid g\mathbb{A}_{12,S})$. Hence we obtain a central extension (see also [3, §5.5.3]):

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \widehat{\mathbb{A}^*} \xrightarrow{\pi} \mathbb{A}^* \longrightarrow 1,$$

where $\mathbb{A}^* = \{(g, \phi) \; : \; g \in \mathbb{A}^*, \phi \in \mu(\mathbb{A}_{12,0} \mid g\mathbb{A}_{12,0}), \phi \ne 0\}$, and $(g_1, \phi_1)(g_2, \phi_2) = (g_1 g_2, \phi_1 \otimes g_1^*(\phi_2))$. (Here $\mathbb{A}_{12,0}$ is the group connected with the zero divisor on X.) For any $g_1, g_2 \in \mathbb{A}^*$ we denote $\langle g_1, g_2 \rangle = [\widehat{g_1}, \widehat{g_2}] \in \mathbb{C}^*$, where $\widehat{g_i} \in \widehat{\mathbb{A}^*}$ are any such that $\pi(\widehat{g_i}) = g_i$. The element $\langle g_1, g_2 \rangle$ does not depend on the choice of appropriate elements $\widehat{g_i}$. From [1] it follows the following equality:

$$\langle g_1, g_2 \rangle = \prod_{x \in D} q^{-[k(x):k] \, ({g_1}_{x,D}, {g_2}_{x,D})_{x,D}}, \tag{12}$$

where $(\cdot,\cdot)_{x,D}$ is the composition of the maps: $K^*_{x,D} \times K^*_{x,D} \to K_2(K_{x,D}) \stackrel{\partial_2}{\to} \overline{K}^*_{x,D} \stackrel{\partial_1}{\to} \mathbb{Z}$.

For any $E \in \mathrm{Div}(X)$ we choose an element $j_{1,E} \in \mathbb{A}^*_{01}$ such that $\mathbb{A}_{1,E} = j_{1,E}\mathbb{A}_{1,0}$, and an element $j_{2,E} \in \mathbb{A}^*_{02}$ such that $\mathbb{A}_{2,E} = j_{2,E}\mathbb{A}_{2,0}$, where we take the product inside the ring $\mathbb{A}$. Now from [5, §2.2] and from (12) it follows the following formula for any $C, H \in \mathrm{Div}(X)$ ((C,H) means the intersection index of divisors C and H on X):

$$\langle j_{2,C}, j_{1,H} \rangle = q^{-(C,H)}. \tag{13}$$

Since we can take $j_{1,E_1+E_2} = j_{1,E_1} j_{2,E_2}$ and $j_{2,E_1+E_2} = j_{2,E_1} j_{2,E_2}$, we obtain $j_{1,E_1}\mathbb{A}_{1,E_2} = \mathbb{A}_{1,E_1+E_2}$ and $j_{2,E_1}\mathbb{A}_{2,E_2} = \mathbb{A}_{2,E_1+E_2}$ for any $E_1, E_2 \in \mathrm{Div}(X)$. Hence we have $j^*_{1,E}(\nu_{R,S}) = \nu_{R+E,S+E}$ and $j^*_{2,E}(\mu_{R,S}) = \mu_{R+E,S+E}$ for any $R, S, E \in \mathrm{Div}(X)$. For any $C \in \mathrm{Div}(X)$ we choose $\widehat{j_{2,C}} = (j_{2,C}, \nu_{0,C}) \in \widehat{\mathbb{A}}^*$ and $\widehat{j_{1,(\omega)-C}} = (j_{1,(\omega)-C}, \mu_{0,(\omega)-C}) \in \widehat{\mathbb{A}}^*$. We have

$$\langle j_{2,C}, j_{1,(\omega)-C} \rangle = \frac{\widehat{j_{2,C}}\ \widehat{j_{1,(\omega)-C}}}{\widehat{j_{1,(\omega)-C}}\ \widehat{j_{2,C}}} = \frac{\nu_{0,C} \otimes j^*_{2,C}(\mu_{0,(\omega)-C})}{\mu_{0,(\omega)-C} \otimes j^*_{1,(\omega)-C}(\nu_{0,C})} = \frac{\nu_{0,C} \otimes \mu_{C,(\omega)}}{\mu_{0,(\omega)-C} \otimes \nu_{(\omega)-C,(\omega)}} =$$

$$= \frac{\nu_{0,C} \otimes \mu_{C,(\omega)-C} \otimes \mu_{(\omega)-C,(\omega)}}{\mu_{0,C} \otimes \mu_{C,(\omega)-C} \otimes \nu_{(\omega)-C,(\omega)}} = \frac{\nu_{0,C}}{\mu_{0,C}}\ \frac{\mu_{(\omega)-C,(\omega)}}{\nu_{(\omega)-C,(\omega)}}. \tag{14}$$

From (11) and (10) we obtain $\frac{\nu_{0,C}}{\mu_{0,C}} = \frac{\mu_{(\omega)-C,(\omega)}}{\nu_{(\omega)-C,(\omega)}} = q^{\chi(C)-\chi(0)}$. Therefore from (14) and (13) we have $2(\chi(C) - \chi(0)) = -(C, (\omega) - C)$ for any $C \in \mathrm{Div}(X)$. From the last equality and formula (6) we obtain the Riemann-Roch theorem in the following form.

Theorem 1 *For any $C \in \mathrm{Div}(X)$ and $\omega \in \Omega^2_{k(X)}$ we have the following equality*

$$h^0(C) - h^1(C) + h^0((\omega) - C) = h^0(0) - h^1(0) + h^0((\omega)) - \frac{1}{2}(C, (\omega) - C).$$

References

[1] Osipov D.V., *Central extensions and reciprocity laws on algebraic surfaces*, (Russian) Mat. Sb. 196:10 (2005), 111-136; translation in Sb. Math. 196:10 (2005), 1503-1527.

[2] Osipov D., *Adeles on n-dimensional schemes and categories C_n*, International Journal of Mathematics, vol. 18, no. 3 (2007), 269-279.

[3] Osipov D.V., Parshin A.N., *Harmonic analisys on local fields and adelic spaces. I*, Izvestiya RAN: Ser. Mat., 2008, 72:5, pp. 77-140; english translation in Izvestiya: Mathematics, 2008, 72:5, pp. 915-976.

[4] Osipov D.V., Parshin A.N., *Harmonic analisys on local fields and adelic spaces. II*, to appear in Izvestiya RAN: Ser. Mat., 2011, 72:4, e-print arXiv:0912.1577.

[5] Parshin A.N., *Chern classes, adeles and L-functions*, Journal für die reine and angewandte Mathematik, Band 341, 1983, pp. 174-192.

[6] Parshin A. N., *Higher-dimensional local fields and L-functions*, Invitation to higher local fields (Münster, 1999), 199–213 (electronic), Geom. Topol. Monogr., 3, Geom. Topol. Publ., Coventry, 2000.