A LOWER BOUND ON BLOWUP RATES FOR THE 3D INCOMPRESSIBLE EULER EQUATION AND A SINGLE EXPONENTIAL BEALE-KATO-MAJDA ESTIMATE

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Abstract. We prove a Beale-Kato-Majda criterion for the loss of regularity for solutions of the incompressible Euler equations in $H^s(\mathbb{R}^3)$, for $s > \frac{5}{2}$. Instead of double exponential estimates of Beale-Kato-Majda type, we obtain a single exponential bound on $\|u(t)\|_{H^s}$ involving the length parameter introduced by P. Constantin in [3]. In particular, we derive lower bounds on the blowup rate of such solutions.

1. Introduction

In this paper, we revisit the Beale-Kato-Majda criterion for the breakdown of smooth solutions to the $3D$ Euler equations.

More precisely, we consider the incompressible Euler equations

$$\partial_t u + (u \cdot \nabla) u + \nabla p \,=\, 0 \tag{1.1}$$

$$\nabla \cdot u \,=\, 0 \tag{1.2}$$

$$u(x,0) \,=\, u_0 \tag{1.3}$$

for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1 \le i \le 3} \in \mathbb{R}^3$ and pressure $p = p(x,t) \in \mathbb{R}$, for position $x \in \mathbb{R}^3$ and time $t \in [0,\infty)$.

Existence and uniqueness of local in time solutions to (1.1) – (1.3) in the space

$$C([0,T], H^s) \cap C^1([0,T]; H^{s-1}), \tag{1.4}$$

has long been known for $s > \frac{5}{2}$, see for instance [7]. However, it is an open problem to determine whether such solutions can lose their regularity in finite time. An important result that addresses the question of a possible loss of regularity of solutions to Euler equations (1.1) – (1.3) is the criterion formulated by Beale-Kato-Majda [1] in terms of the L^∞ norm of the vorticity $\omega = \nabla \wedge u$. More precisely, Beale-Kato-Majda in [1] proved the following theorem:

Theorem 1.1. *Let u be a solution to* (1.1) – (1.3) *in the class* (1.4) *for $s \geq 3$ integer. Suppose that there exists a time T^* such that the solution cannot be continued in the class* (1.4) *to $T = T^*$. If T^* is the first such time, then*

$$\int_0^{T^*} \|\omega(\cdot,t)\|_{L^\infty}\, dt = \infty. \tag{1.5}$$

The theorem is proved with a contradiction argument. Under the assumption

$$\int_0^{T^*} \|\omega(\cdot,t)\|_{L^\infty}\,dt < \infty\,,$$

the authors of [1] show that $\|u(\cdot,t)\|_{H^s} \le C_0$, for all $t < T^*$ contradicting the hypothesis that T^* is the first time such that the solution cannot be continued to $T = T^*$. In particular, Beale-Kato-Majda obtain a double exponential bound for $\|u(\cdot,t)\|_{H^s}$, which follows from the following estimates:

Step 1 An energy-type bound on $\|u\|_{H^s}$ in terms of $\|Du\|_{L^\infty}$, where $Du = [\partial_i u_j]_{ij}$ is a 3×3-matrix valued function. More specifically, one applies the operator D^α to equations (1.1)-(1.2), where α is an integer-valued multi-index with $|\alpha| \le s$ and uses a certain commutator estimate to derive

$$\frac{d}{dt}\|u(\cdot,t\|_{H^s}^2 \le 2C\|Du\|_{L^\infty}\|u(\cdot,t)\|_{H^s}^2, \tag{1.6}$$

which via Gronwall's inequality gives the bound:

$$\|u(\cdot,t)\|_{H^s} \;\le\; \|u_0\|_{H^s}\,\exp\left(C\int_0^t \|Du(\cdot,\tau)\|_{L^\infty}\,d\tau\right). \tag{1.7}$$

Step 2 An estimate on $\|Du(\cdot,t)\|_{L^\infty}$ based on the quantities $\|\omega(\cdot,t)\|_{L^\infty}$, $\|\omega(\cdot,t)\|_{L^2}$, and $\log^+\|u(\cdot,t)\|_{H^3}$, given by

$$\|Du(\cdot,t)\|_{L^\infty} \;\le\; C\left\{1 + \left(1 + \log^+\|u(\cdot,t)\|_{H^3}\right)\|\omega(\cdot,t)\|_{L^\infty} + \|\omega(\cdot,t)\|_{L^2}\right\}, \tag{1.8}$$

where C is a universal constant.

Step 3 The bound on $\|\omega(\cdot,t)\|_{L^2}$ in terms of $\|\omega(\cdot,t)\|_{L^\infty}$ given by

$$\frac{d}{dt}\|\omega(\cdot,t)\|_{L^2}^2 \;\le\; 2\,\widetilde{C}\,\|\omega(\cdot,t)\|_{L^\infty}\,\|\omega(\cdot,t)\|_{L^2}^2\,,$$

which follows from taking the $L^2(\mathbb{R}^3)$-inner product of ω with the equation for vorticity. Then, Gronwall's inequality yields

$$\|\omega(\cdot,t)\|_{L^2} \le \|\omega(\cdot,0)\|_{L^2}\,\exp\left(\widetilde{C}\int_0^t \|\omega(\,\cdot\,,\tau)\|_{L^\infty}\,d\tau\right). \tag{1.9}$$

Consequently, one obtains the double exponential bound

$$\|u(\cdot,t)\|_{H^s} \;\le\; \|u_0\|_{H^s}\,\exp\left(\exp\left(\widetilde{C}\int_0^t \|\omega(\,\cdot\,,\tau)\|_{L^\infty}\,d\tau\right)\right). \tag{1.10}$$

from combining (1.7), (1.8) and (1.9).

It is an open question whether (1.10) is sharp[1]. While we do not attempt to answer that question itself in this paper, we obtain a single exponential bound on the H^s-norm of solution to Euler equations (1.1) - (1.3) in terms of the quantity

$$\ell_\delta(t) \;=\; \min\left\{L\,,\;\left(\frac{\|\omega(t)\|_{C^\delta}}{\|u_0\|_{L^2}}\right)^{-\frac{2}{2\delta+5}}\right\}, \tag{1.11}$$

[1] Single exponential bounds have been obtained in other solution spaces than those displayed above, see for instance [8] for such a result in BMO.

where

$$\|\omega\|_{C^\delta} \;=\; \sup_{|x-y|<L} \frac{|\omega(x)-\omega(y)|}{|x-y|^\delta} \tag{1.12}$$

denotes the δ-Holder seminorm, for $L>0$ fixed, and $\delta>0$. More precisely, we prove the following theorem:

Theorem 1.2. *Let u be a solution to* (1.1) - (1.3) *in the class* (1.4)*, for $s=\frac{5}{2}+\delta$. Assume that $\ell_\delta(t)$ is defined as above, and that*

$$\int_0^T \left(\ell_\delta(\tau)\right)^{-\frac{5}{2}} \, d\tau \;<\; \infty \,. \tag{1.13}$$

Then, there exists a finite positive constant $C_\delta = O(\delta^{-1})$ independent of u and t such that

$$\|u(\cdot,t)\|_{H^s} \;\leq\; \|u_0\|_{H^s} \, \exp\Big[\, C_\delta \, \|u_0\|_{L^2} \int_0^t \left(\ell_\delta(\tau)\right)^{-\frac{5}{2}} \, d\tau \,\Big]$$

holds for $0\leq t\leq T$.

The quantity $\ell_\delta(t)$ has the dimension of length, and was introduced by Constantin in [3] (see also the work of Constantin, Fefferman and Majda [5] where a criterion for loss of regularity in terms of the direction of vorticity was obtained), where it was observed that

$$\int_0^T \left(\ell_\delta(t)\right)^{-\frac{5}{2}} \, dt \;=\; \infty \tag{1.14}$$

is a necessary and sufficient condition for blow-up of Euler equations. In particular, the necessity of the condition follows from the inequality obtained in [3]

$$\|\omega(\cdot,t)\|_{L^\infty} \leq \|u(\cdot,t)\|_{L^2} \, \left(\ell_\delta(t)\right)^{-\frac{5}{2}} \,, \tag{1.15}$$

and Theorem 1.1 of Beale-Kato-Majda. This is so because Theorem 1.1 implies that if the solution cannot be continued to some time T, then $\int_0^T \|\omega(\cdot,t)\|_{L^\infty} \, dt = \infty$. As a consequence of (1.15), and conservation of energy

$$\|u(\cdot,t)\|_{L^2} \;=\; \|u_0\|_{L^2} \,, \tag{1.16}$$

this in turn implies (1.14). However, by invoking the result of Beale-Kato-Majda in this argument, one again obtains a double exponential bound on $\|u(\cdot,t)\|_{H^s}$ in terms of $\int_0^T \left(\ell_\delta(t)\right)^{-\frac{5}{2}} dt$. We refer to [4, 6] for recent developments in this and related areas.

In this paper, we observe that one can actually obtain a single exponential bound on the H^s-norm of the solution $u(t)$ in terms of $\int_0^T \left(\ell_\delta(t)\right)^{-\frac{5}{2}} dt$, as stated in Theorem 1.2. This is achieved by avoiding the use of the logarithmic inequality (1.8) from [1]. More precisely, we combine the energy bound (1.6) with a Calderon-Zygmund type bound on the symmetric and antisymmetric parts of Du.

Also, we obtain a lower bound on the blowup rate of solutions in $H^{\frac{5}{2}+\delta}$. Specifically, we prove:

Theorem 1.3. *Let u be a solution to* (1.1) – (1.3) *in the class*

$$C([0,T];H^{\frac{5}{2}+\delta})\cap C^1([0,T];H^{\frac{3}{2}+\delta}). \tag{1.17}$$

Suppose that there exists a time T^* such that the solution cannot be continued in the class (1.17) *to $T=T^*$. If T^* is the first such time then there exists a finite, positive constant $C(\delta,\|u_0\|_{L^2})$ such that*

$$\|u(\cdot,t)\|_{H^{\frac{5}{2}+\delta}} \,\geq\, C(\delta,\|u_0\|_{L^2})\left(\frac{1}{T^*-t}\right)^{1+\frac{2}{5}\delta}, \tag{1.18}$$

for all t sufficiently close to T^* (see the conditions (3.22) *and* (3.23) *below, with $t_0=t$).*

The proof of Theorem 1.3 can be outlined as follows. We assume that u is a solution in the class (1.17) that cannot be continued to $T=T^*$, and that T^* is the first such time. Invoking the local in time existence result, we derive a lower bound $T_{loc,t_1}>0$ on the time of existence of solutions to Euler equations in (1.17) for initial data $u(t_1)\in H^{\frac{5}{2}+\delta}$ at an arbitrary time $t_1<T^*$. By definition of T^*, we thus have

$$t_1 \,+\, T_{loc,t_1} \,<\, T^*\,. \tag{1.19}$$

Based on an energy bound on the $H^{\frac{5}{2}+\delta}$-norm of the solution, we obtain in Section 3 an expression for T_{loc,t_1} of the form $\frac{1}{C\|u(\cdot,t_1)\|_{H^{\frac{5}{2}+\delta}}}$, which together with (1.19) implies that

$$\|u(\cdot,t_1)\|_{H^{\frac{5}{2}+\delta}} > \frac{1}{C(T^*-t_1)}\,, \tag{1.20}$$

for all $t_1<T^*$. This is an "a priori" lower bound on the blowup rate. Subsequently, we improve (1.20) by a recursion argument in Theorem 1.3 for times t close to T^*, to yield the stronger bound (1.18).

After completing this work, V. Vicol called to our attention that in a recent work, D. Chae proved in [2] (see Theorem 1.1 part (i) of [2]) that for integer values of $s\in\mathbb{N}$ with $s>1+\frac{d}{2}$, and in dimensions $d\geq 2$,

$$\liminf_{t\to T^*}(T^*-t)\|D^s u(t)\|_{L^2(\mathbb{R}^d)}^{\frac{d+2}{s}} \,\geq\, \frac{K}{\|u_0\|_{L^2}^{\frac{d+2}{s}}} \tag{1.21}$$

is a necessary and sufficient condition for blowup at time T^*, where $K=K(d,s)$ is an absolute constant. In our estimate (1.18), we allow for real values of $s=\frac{5}{2}+\delta$, $\delta>0$, and provide a pointwise lower bound instead of an infimum limit.

2. Proof of theorem 1.2

First we recall that the full gradient of velocity Du can be decomposed into symmetric and antisymmetric parts,

$$Du \,=\, Du^+ \,+\, Du^- \tag{2.1}$$

where

$$Du^\pm \,=\, \frac{1}{2}\big(\,Du\pm Du^T\,\big)\,. \tag{2.2}$$

Du^+ is called the deformation tensor.

In the following lemma we recall important properties of Du^+ and Du^-. For the convenience of the reader, we give proofs of those properties, although some of them are available in the literature, see e.g. [3].

Lemma 2.1. *For both the symmetric and antisymmetric parts Du^+, Du^- of Du, the L^2 bound*

$$\|Du^{\pm}\|_{L^2} \;\leq\; C\|\omega\|_{L^2}\,. \tag{2.3}$$

holds.

The antisymmetric part Du^- satisfies

$$Du^- v \;=\; \frac{1}{2}\,\omega\wedge v \tag{2.4}$$

for any vector $v \in \mathbb{R}^3$. The vorticity ω satisfies the identity

$$\omega(\xi) \;=\; \frac{1}{4\pi} P.V. \int \sigma(\widehat{y})\,\omega(x+y)\,\frac{dy}{|y|^3}\,, \tag{2.5}$$

("P.V." denotes principal value) where $\sigma(\widehat{y}) = 3\,\widehat{y}\otimes\widehat{y} - \mathbf{1}$, with $\widehat{y} = \frac{y}{|y|}$. Notably,

$$\int_{S^2} \sigma(\widehat{y})\,d\mu_{S^2}(y) \;=\; 0\,, \tag{2.6}$$

where $d\mu_{S^2}$ denotes the standard measure on the sphere S^2.

The matrix components of the symmetric part have the form

$$Du^+_{ij} \;=\; \sum_\ell T^\ell_{ij}(\omega_\ell) \;=\; \sum_\ell \mathcal{K}^\ell_{ij} * \omega_\ell\,, \tag{2.7}$$

where ω_ℓ are the vector components of ω, and where the integral kernels $\mathcal{K}^\ell_{ij}$ have the properties

$$\mathcal{K}^\ell_{ij}(y) \;=\; \sigma^\ell_{ij}(\widehat{y})\,|y|^{-3} \tag{2.8}$$

$$\|\sigma^\ell_{ij}\|_{C^1(S^2)} \;\leq\; C \tag{2.9}$$

$$\int_{S^2} \sigma^\ell_{ij}(\widehat{y})\,d\mu_{S^2}(y) \;=\; 0\,. \tag{2.10}$$

Thus in particular, T^ℓ_{ij} is a Calderon-Zygmund operator, for every $i,j,\ell \in \{1,2,3\}$.

Proof. An explicit calculation shows that the Fourier transform of Du as a function of $\widehat{\omega}$ is given by

$$\widehat{Du}(\xi) \;=\; -[(\partial_i(\Delta^{-1}\nabla\wedge\omega)_j)\widehat{\ }(\xi)]_{i,j} \;=\; \widehat{G}(\xi) \,+\, \widehat{H}(\xi) \tag{2.11}$$

where

$$\widehat{G}(\xi) \;:=\; \frac{1}{2|\xi|^2}\begin{bmatrix} \xi_1\xi_2\widehat{\omega}_3 - \xi_1\xi_3\widehat{\omega}_2 & -\xi_2\xi_3\widehat{\omega}_2 & \xi_2\xi_3\widehat{\omega}_3 \\ \xi_1\xi_3\widehat{\omega}_1 & \xi_2\xi_3\widehat{\omega}_1 - \xi_1\xi_2\widehat{\omega}_3 & -\xi_1\xi_3\widehat{\omega}_3 \\ -\xi_1\xi_2\widehat{\omega}_1 & \xi_1\xi_2\widehat{\omega}_2 & \xi_1\xi_3\widehat{\omega}_2 - \xi_2\xi_3\widehat{\omega}_1 \end{bmatrix} \tag{2.12}$$

and

$$\widehat{H}(\xi) \;:=\; \frac{1}{2|\xi|^2}\begin{bmatrix} 0 & \xi_2^2\widehat{\omega}_3 & -\xi_3^2\widehat{\omega}_2 \\ -\xi_1^2\widehat{\omega}_3 & 0 & \xi_3^2\widehat{\omega}_1 \\ \xi_1^2\widehat{\omega}_2 & -\xi_2^2\widehat{\omega}_1 & 0 \end{bmatrix}\,, \tag{2.13}$$

using the notation $\widehat{\omega}_j \equiv \widehat{\omega}_j(\xi)$ for brevity.

Clearly, every component of G is given by a sum of Fourier multiplication operators with symbols of the form $\frac{\xi_i\xi_j}{|\xi|^2}$, $i \neq j$, applied to a component of ω. For instance,

$$G_{21}(x) \;=\; const.\; P.V. \int \widehat{y}_1\widehat{y}_3\,\omega_1(x+y)\,\frac{dy}{|y|^3} \tag{2.14}$$

corresponds to the component G_{21}. It is easy to see that every component G_{ij} is a sum of Calderon-Zygmund operators applied to components of ω, with kernel satisfying the asserted properties (2.8) $\sim$ (2.10). The same is true for the symmetric part, $G^+ = \frac{1}{2}(G + G^T)$.

The symmetric part of $\widehat{H}(\xi)$ is given by

$$\widehat{H}^+(\xi) \;=\; \frac{1}{2|\xi|^2}\left[\begin{array}{ccc} 0 & (\xi_2^2-\xi_1^2)\widehat{\omega}_3 & (\xi_1^2-\xi_3^2)\widehat{\omega}_2 \\ (\xi_2^2-\xi_1^2)\widehat{\omega}_3 & 0 & (\xi_3^2-\xi_2^2)\widehat{\omega}_1 \\ (\xi_1^2-\xi_3^2)\widehat{\omega}_2 & (\xi_3^2-\xi_2^2)\widehat{\omega}_1 & 0 \end{array}\right] \tag{2.15}$$

so that each component defines a Fourier multiplication operator with symbol of the form $\frac{\xi_i^2-\xi_j^2}{|\xi|^2}$, $i \neq j$, acting on a component of ω (with associated kernel of the form $\frac{x_i^2-x_j^2}{|x|^{n+2}}$). That is, for instance,

$$H_{12}^+(x) \;=\; const\; P.V. \int (\widehat{y}_2^2 - \widehat{y}_1^2)\,\omega_3(x+y)\,\frac{dy}{|y|^3}\,. \tag{2.16}$$

The properties (2.8) $\sim$ (2.10) follow immediately.

The Fourier transforms of the integral kernels $\mathcal{K}_{ij}^{\ell}$ can be read off from the components $\widehat{G}_{ij}^+ + \widehat{H}_{ij}^+$. In position space, one finds that $\sigma_{ij}^{\ell}(\widehat{y})$ is obtained from a sum of terms proportional to terms of the form $\widehat{y}_{i_1}\widehat{y}_{j_1}$ and $(\widehat{y}_{i_2}^2 - \widehat{y}_{j_2}^2)$.

For the antisymmetric part Du^-, one generally has $Du^-v = \frac{1}{2}(\nabla\wedge u)\wedge v$ for any $v \in \mathbb{R}^3$, and from $u = -\Delta^{-1}\nabla\wedge\omega$, we get $Du^-v = \frac{1}{2}\omega\wedge v$, using that $\nabla\cdot u = 0$.

As a side remark, we note that while H^- does not by itself exhibit the properties (2.8) $\sim$ (2.10), it combines with G^- in a suitable manner to yield the stated properties of Du^-, thanks to the condition $\nabla\cdot\omega = 0$. $\square$

Next, Lemma 2.2 below provides an upper bound in terms of the quantity $\ell_\delta(t)$ on singular integral operators applied to ω of the type appearing in (2.7). We note that similar bounds were used in [3] and [5] for the antisymmetric part Du^-. Here, we observe that they also hold for the symmetric part Du^+. As shown in [5] for Du^-, the proof of such a bound follows standard steps based on decomposing the singular integral into an inner and outer contribution. The inner contribution can be bounded based on a certain mean zero property, while the outer part is controlled via integration by parts.

Lemma 2.2. *For $L > 0$ fixed, and $\delta > 0$, let $\ell_\delta(t)$ be defined as above. Moreover, let ω_ℓ, $\ell = 1,2,3$, denote the components of the vorticity vector $\omega(t)$. Then, any* *singular integral operator*

$$T\omega_\ell(x) \;=\; \frac{1}{4\pi}P.V.\int \sigma_T(\widehat{y})\,\omega_\ell(x+y)\,\frac{dy}{|y|^3} \tag{2.17}$$

with

$$\int_{S^2}\sigma_T(\widehat{y})d\mu_{S^2}(y) \;=\; 0 \quad , \quad \|\sigma_T\|_{C^1(S^2)} \;<\; C\;, \tag{2.18}$$

satisfies

$$\|T\omega_\ell\|_{L^\infty} \;\leq\; C(\delta)\,\|u_0\|_{L^2}\,\ell_\delta(t)^{-\frac{5}{2}} \tag{2.19}$$

for $\ell\in\{1,2,3\}$*, for a constant* $C(\delta)=O(\delta^{-1})$ *independent of* u *and* t*.*

Proof. Let $\chi_1(x)$ be a smooth cutoff function which is identical to 1 on $[0,1]$, and identically 0 for $x>2$. Moreover, let $\chi_R(x)=\chi_1(x/R)$, and $\chi_R^c=1-\chi_R$.

We consider

$$\int_{|y|>\epsilon}\sigma_T(\widehat{y})\,\omega_\ell(x+y)\,\frac{dy}{|y|^3} \;=\; (I)\;+(II) \tag{2.20}$$

for $\epsilon>0$ arbitrary, where

$$(I) \;:=\; \int_{|y|>\epsilon}\sigma_T(\widehat{y})\,\omega_\ell(x+y)\,\chi_{\ell_\delta(t)}(|y|)\;\frac{dy}{|y|^3} \tag{2.21}$$

and

$$(II) \;:=\; \int \sigma_T(\widehat{y})\,\omega_\ell(x+y)\,\chi^c_{\ell_\delta(t)}(|y|)\;\frac{dy}{|y|^3}\,. \tag{2.22}$$

From the zero average property (2.18), we find

$$\begin{aligned}
\|(I)\|_{L^\infty} \;&=\; \Big|\int_{|y|>\epsilon}\sigma_T(\widehat{y})\,(\omega_\ell(x+y)-\omega_\ell(x))\,\chi_{\ell_\delta(t)}(|y|)\;\frac{dy}{|y|^3}\Big| \\
&\leq\; \|\omega_\ell\|_{C^\delta}\int_{|y|<2\ell_\delta(t)}\frac{dy}{|y|^{3-\delta}} \\
&\leq\; \frac{C}{\delta}(\ell_\delta(t))^\delta\,\|\omega_\ell\|_{C^\delta} \\
&\leq\; C\,\delta^{-1}\,\|u_0\|_{L^2}\,(\ell_\delta(t))^{-\frac{5}{2}}
\end{aligned} \tag{2.23}$$

since from the definition of $\ell_\delta(t)$,

$$\|\omega_\ell\|_{C^\delta} \;\leq\; \|u_0\|_{L^2}\,(\ell_\delta(t))^{-\delta-\frac{5}{2}} \tag{2.24}$$

follows straightforwardly. We can send $\epsilon\searrow 0$, since the estimates are uniform in ϵ.

On the other hand,

$$(II) \;\;=\;\; \int \sigma_T(\widehat{y})\,(\partial_{y_i}u_j-\partial_{y_j}u_i)(x+y)\,\chi^c_{\ell_\delta(t)}(|y|)\;\frac{dy}{|y|^3}\,. \tag{2.25}$$

It suffices to consider one of the terms in the difference,

$$\begin{aligned}
&\Big| \int \sigma_T(\widehat{y})\, \partial_{y_i} u_j(x+y)\, \chi^c_{\ell_\delta(t)}(|y|)\ \frac{dy}{|y|^3} \Big| \\
&= \ \Big| \int dy\, u_j(x+y)\, \partial_{y_i}\Big(\sigma_T(\widehat{y})\, \chi^c_{\ell_\delta(t)}(|y|)\ \frac{1}{|y|^3} \Big) \Big| \\
&\leq \ C\, \|u_j\|_{L^2} \Big\| \partial_{y_i}\Big(\sigma_T(\widehat{y})\, \chi^c_{\ell_\delta(t)}(|y|)\ \frac{1}{|y|^3} \Big) \Big\|_{L^2} \\
&\leq \ C\, \|u_0\|_{L^2}\, (\ell_\delta(t))^{-\frac{5}{2}} \qquad (2.26)
\end{aligned}$$

where to obtain the last line we used the conservation of energy (1.16) and the following three bounds:

(i)

$$\begin{aligned}
\Big\| \Big(\partial_{y_i} \chi^c_R(|y|) \Big)\ \frac{\sigma_T(\widehat{y})}{|y|^3} \Big\|^2_{L^2} \ &\leq \ C\, \frac{1}{R^2} \int_{R<|y|<2R} \frac{dy}{|y|^6} \\
&\leq \ C\, R^{-5}\,, \qquad (2.27)
\end{aligned}$$

for $R = \ell_\delta(t)$.

(ii)

$$\begin{aligned}
\Big\| \sigma_T(\widehat{y})\, \chi^c_R(|y|)\ \partial_{y_i} \frac{1}{|y|^3} \Big\|^2_{L^2} \ &\leq \ C \int_{|y|>R} \frac{dy}{|y|^8} \\
&\leq \ C\, R^{-5}\,. \qquad (2.28)
\end{aligned}$$

(iii)

$$\begin{aligned}
\Big\| \chi^c_R(|y|)\, \frac{1}{|y|^3}\ \partial_{y_i}\, \sigma_T(\widehat{y}) \Big\|^2_{L^2} \ &\leq \ C \int_{|y|>R} \frac{1}{|y|^6}\, \frac{1}{|y|^2}\, dy \\
&\leq \ C\, R^{-5}\,, \qquad (2.29)
\end{aligned}$$

where we used that

$$\begin{aligned}
\Big| \nabla_y \sigma_T(\widehat{y}) \Big| \ &= \ \Big| \frac{1}{|y|}\, (\nabla_z \sigma_T(z_1, z_2, z_3)) \Big|_{z=\widehat{y}} \Big| \\
&\leq \ \frac{1}{|y|}\, \|\sigma_T\|_{C^1(S^2)} \qquad (2.30)
\end{aligned}$$

holds.

Summarizing, we arrive at

$$\|T\omega_\ell\|_{L^\infty} \ \leq \ C(\delta)\, \|u_0\|_{L^2}\, \ell_\delta(t)^{-\frac{5}{2}} \qquad (2.31)$$

for $C(\delta) = O(\delta^{-1})$, which is the asserted bound. □

The form of the singular integral operator that appears in the statement of Lemma 2.2 is suitable for application to Du^+ and Du^-, as we shall see in the following corollary.

Corollary 2.3. *There exists a finite, positive constant $C_\delta = O(\frac{1}{\delta})$ independent of u and t such that the estimate*

$$\|Du^+\|_{L^\infty} + \|Du^-\|_{L^\infty} \ \leq \ C_\delta\, \|u_0\|_{L^2}\, \ell_\delta(t)^{-\frac{5}{2}} \qquad (2.32)$$

holds.

Proof. According to Lemma 2.1, the matrix components of both Du^+ and Du^- have the form (2.17).

Accordingly, Lemma 2.2 immediately implies the assertion. □

Now we are ready to give a proof of Theorem 1.2, which is based on combining an energy estimate for Euler equations with Corollary 2.3.

For $s \geq 3$ integer-valued, the energy bound (1.6)

$$\frac{1}{2}\partial_t \|u(t)\|_{H^s}^2 \leq \|Du(t)\|_{L^\infty} \|u(t)\|_{H^s}^2 \tag{2.33}$$

was proven in [1]. For fractional $s > \frac{5}{2}$, we recall the definitions of the homogenous and inhomogenous Besov norms for $1 \leq p, q \leq \infty$,

$$\|u\|_{\dot{B}^s_{p,q}} = \Big(\sum_{j \in \mathbb{Z}} 2^{jqs} \|u_j\|_{L^p}^q \Big)^{\frac{1}{q}}, \tag{2.34}$$

respectively,

$$\|u\|_{B^s_{p,q}} = \Big(\|u\|_{L^p}^q + \|u\|_{\dot{B}^s_{p,q}}^q \Big)^{\frac{1}{q}}, \tag{2.35}$$

where $u_j = P_j u$ is the Paley-Littlewood projection of u of scale j. In analogy to (1.6), we obtain the bound on the $B^s_{2,2}$ Besov norm of $u(t)$ given by

$$\frac{1}{2}\partial_t \|u(t)\|_{B^s_{2,2}}^2 \leq \|Du(t)\|_{L^\infty} \|u(t)\|_{B^s_{2,2}}^2 , \tag{2.36}$$

from a straightforward application of estimates obtained in [9]; details are given in the Appendix. Accordingly, since the left hand side yields

$$\partial_t \|u(t)\|_{B^s_{2,2}}^2 = 2\|u(t)\|_{B^s_{2,2}} \partial_t \|u(t)\|_{B^s_{2,2}} , \tag{2.37}$$

we get

$$\partial_t \|u(t)\|_{B^s_{2,2}} \leq \|Du(t)\|_{L^\infty} \|u(t)\|_{B^s_{2,2}} . \tag{2.38}$$

However, Corollary 2.3 implies that

$$\begin{aligned} \|Du(t)\|_{L^\infty} &\leq \|Du^+(t)\|_{L^\infty} + \|Du^-(t)\|_{L^\infty} \\ &\leq C_\delta \|u_0\|_{L^2} (\ell_\delta(t))^{-\frac{5}{2}} . \end{aligned} \tag{2.39}$$

Therefore, by combining (2.38) and (2.39) we obtain

$$\partial_t \|u(t)\|_{B^s_{2,2}} \leq C_\delta \|u_0\|_{L^2} (\ell_\delta(t))^{-\frac{5}{2}} \|u(t)\|_{B^s_{2,2}} ,$$

which implies that

$$\begin{aligned} \|u(t)\|_{H^s} &\sim \|u(t)\|_{B^s_{2,2}} \\ &\leq \|u_0\|_{B^s_{2,2}} \exp\Big[C_\delta \|u_0\|_{L^2} \int_0^t \ell_\delta(s)^{-\frac{5}{2}} ds \Big] \\ &\sim \|u_0\|_{H^s} \exp\Big[C_\delta \|u_0\|_{L^2} \int_0^t \ell_\delta(s)^{-\frac{5}{2}} ds \Big] , \end{aligned}$$

for $s \geq 0$, where we recall from (2.23) that $C_\delta = O(\delta^{-1})$ (see also [10] for a related bound, but without (2.39)).

This completes the proof of Theorem 1.2. □

3. Lower bounds on the blowup rate

In this section, we prove Theorem 1.3.

Recalling the energy bound (2.38),

$$\partial_t \|u(t)\|_{B^s_{2,2}} \leq \|Du(t)\|_{L^\infty} \|u(t)\|_{B^s_{2,2}} , \tag{3.1}$$

we invoke the Sobolev embedding

$$\begin{aligned} \|Du\|_{L^\infty} &\leq \|\widehat{Du}\|_{L^1} \\ &\leq \Big(\int d\xi \, \langle \xi \rangle^{-3-2\delta} \Big)^{\frac{1}{2}} \|Du\|_{H^{\frac{3}{2}+\delta}} \\ &\leq C_\delta \, \|u\|_{H^{\frac{5}{2}+\delta}} \\ &\sim C_\delta \, \|u\|_{B^{\frac{5}{2}+\delta}_{2,2}} , \end{aligned} \tag{3.2}$$

with $C_\delta = O(\delta^{-\frac{1}{2}})$, to get, for $s = \frac{5}{2} + \delta$,

$$\partial_t \|u(t)\|_{B^s_{2,2}} \leq C_\delta \, (\|u(t)\|_{B^s_{2,2}})^2 . \tag{3.3}$$

Straightforward integration implies

$$- \Big(\frac{1}{\|u(t)\|_{B^s_{2,2}}} - \frac{1}{\|u(t_0)\|_{B^s_{2,2}}} \Big) \leq C_\delta (t - t_0) . \tag{3.4}$$

Hence,

$$\begin{aligned} \|u(t)\|_{H^s} &\sim \|u(t)\|_{B^s_{2,2}} \\ &\leq \frac{\|u(t_0)\|_{B^s_{2,2}}}{1 - (t-t_0) C_\delta \|u(t_0)\|_{B^s_{2,2}}} \\ &\sim \frac{\|u(t_0)\|_{H^s}}{1 - (t-t_0) C_\delta \|u(t_0)\|_{H^s}} , \end{aligned} \tag{3.5}$$

where a possible trivial modification of C_δ is implicit in passing to the last line. This implies that the solution $u(t)$ is locally well-posed in H^s, with $s = \frac{5}{2} + \delta$, for

$$t_0 \leq t < t_0 + \frac{1}{C_\delta \|u(t_0)\|_{H^s}} . \tag{3.6}$$

In particular, this infers that if T^* is the first time beyond which the solution u cannot be continued, one necessarily has that

$$T^* > t_0 + \frac{1}{C_\delta \|u(t_0)\|_{H^s}} . \tag{3.7}$$

This in turn implies an a priori lower bound on the blowup rate given by

$$\|u(t)\|_{H^s} > \frac{1}{C_\delta \, (T^* - t)} \tag{3.8}$$

for all $0 \leq t < T^*$.

Next, we derive the lower bound on the blowup rate stated in Theorem 1.3 which is stronger than (3.8). To begin with, we note that

$$\begin{aligned}\|\omega(t)\|_{C^\delta} &\leq C_\delta \|\omega(t)\|_{H^{\frac{3}{2}+\delta}} \\ &\leq C_\delta \|u(t)\|_{H^{\frac{5}{2}+\delta}} \\ &\leq \frac{C_\delta \|u(t_0)\|_{H^{\frac{5}{2}+\delta}}}{1-(t-t_0)C_\delta \|u(t_0)\|_{H^{\frac{5}{2}+\delta}}} . \end{aligned} \tag{3.9}$$

That is, local well-posedness of u in $H^{\frac{5}{2}+\delta}$ implies δ-Holder continuity of the vorticity.

The parameter L in the definition (1.11) of $\ell_\delta(t)$ is arbitrary. Thus, in view of (3.9), we may now let $L \to \infty$ for convenience. Then,

$$\begin{aligned}\ell_\delta(t)^{-\frac{5}{2}} &= \Big(\frac{\|\omega(t)\|_{C^\delta}}{\|u_0\|_{L^2}}\Big)^{\frac{2}{2\delta+5}\cdot\frac{5}{2}} \\ &\leq \Big(\frac{C_\delta \, \|u(t)\|_{H^{\frac{5}{2}+\delta}}}{\|u_0\|_{L^2}}\Big)^{1-\tilde{\delta}} \\ &\leq \Big(\frac{C_\delta}{\|u_0\|_{L^2}}\Big)^{1-\tilde{\delta}}\Big(\frac{\|u(t_0)\|_{H^s}}{1-(t-t_0)C_\delta\|u(t_0)\|_{H^s}}\Big)^{1-\tilde{\delta}} , \end{aligned} \tag{3.10}$$

where

$$\tilde{\delta} := \frac{2\delta}{5+2\delta} \text{ and } s = \frac{5}{2}+\delta . \tag{3.11}$$

We note that while the right hand side of (3.10) diverges as t approaches

$$t_1 := t_0 + \frac{1}{C_\delta \|u(t_0)\|_{H^s}} , \tag{3.12}$$

the integral

$$\begin{aligned}\int_{t_0}^{t_1} \ell_\delta(t)^{-\frac{5}{2}} dt &\leq \Big(\frac{C_\delta}{\|u_0\|_{L^2}}\Big)^{1-\tilde{\delta}} \int_{t_0}^{t_1} \Big(\frac{\|u(t_0)\|_{H^s}}{1-(t-t_0)C_\delta\|u(t_0)\|_{H^s}}\Big)^{1-\tilde{\delta}} dt \\ &=: B_0(\delta) \end{aligned} \tag{3.13}$$

converges for $\delta > 0$ ($\Leftrightarrow \widetilde{\delta} > 0$). This implies that the solution $u(t)$ for $t \in [t_0, t_1)$ can be extended to $t > t_1$.

In particular, we obtain that

$$\begin{aligned}\|u(t_1)\|_{H^{\frac{5}{2}+\delta}} &\leq \|u(t_0)\|_{H^{\frac{5}{2}+\delta}} \exp\Big(C_\delta \, \|u_0\|_{L^2} \int_{t_0}^{t_1} (\ell_\delta(t))^{-\frac{5}{2}} dt \Big) \\ &\leq \|u(t_0)\|_{H^{\frac{5}{2}+\delta}} \exp\Big(C_\delta \, \|u_0\|_{L^2} B_0(\delta) \Big) \end{aligned} \tag{3.14}$$

from Theorem 1.2.

We may now repeat the above estimates with initial data $u(t_1)$ in $H^{\frac{5}{2}+\delta}$, thus obtaining a local well-posedness interval $[t_1, t_2]$. Accordingly, we may set t_2 to be given by

$$t_2 := t_1 + \frac{1}{C_\delta \|u(t_1)\|_{H^s}} . \tag{3.15}$$

More generally, we define the discrete times t_j by

$$t_{j+1} := t_j + \frac{1}{C_\delta \|u(t_j)\|_{H^s}} . \tag{3.16}$$

We then have

$$\|u(t_{j+1})\|_{H^s} \leq \exp\Big(C_\delta \|u_0\|_{L^2} B_j(\delta) \Big) \|u(t_j)\|_{H^s} , \tag{3.17}$$

where $B_j(\delta)$ is defined by

$$\begin{aligned} & C_\delta \|u_0\|_{L^2} B_j(\delta) \\ := \quad & C_\delta \|u_0\|_{L^2} \Big(\frac{C_\delta}{\|u_0\|_{L^2}} \Big)^{1-\tilde{\delta}} \int_{t_j}^{t_{j+1}} \Big(\frac{\|u(t_j)\|_{H^s}}{1-(t-t_j)C_\delta \|u(t_j)\|_{H^s}} \Big)^{1-\tilde{\delta}} dt \\ = \quad & \frac{1}{\tilde{\delta}} C_\delta^{1-\tilde{\delta}} \Big(\frac{\|u_0\|_{L^2}}{\|u(t_j)\|_{H^s}} \Big)^{\tilde{\delta}} \\ =: \quad & b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_j)\|_{H^s}} \Big)^{\tilde{\delta}} . \end{aligned} \tag{3.18}$$

Letting

$$\rho_j := \exp\Big(b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_j)\|_{H^s}} \Big)^{\tilde{\delta}} \Big) , \tag{3.19}$$

we have

$$\|u(t_j)\|_{H^s} \leq \rho_{j-1} \|u(t_{j-1})\|_{H^s} , \tag{3.20}$$

and we remark that $(\rho_j)_j$ satisfy the recursive estimates

$$\begin{aligned} \rho_j \quad & \geq \quad \exp\Big(b_\delta \Big(\frac{\|u_0\|_{L^2}}{\rho_{j-1} \|u(t_{j-1})\|_{H^s}} \Big)^{\tilde{\delta}} \Big) \\ & = \quad (\rho_{j-1})^{\rho_{j-1}^{-\tilde{\delta}}} \\ & = \quad \exp\Big(\rho_{j-1}^{-\tilde{\delta}} \ln \rho_{j-1} \Big) . \end{aligned} \tag{3.21}$$

We note that from its definition, $\rho_j > 1$ for all j.

We shall now assume that $T^* > 0$ is the first time beyond which the solution $u(t)$ cannot be continued. Thus, by choosing t_0 close enough to T^*, (3.8) implies that $\|u(t_0)\|_{H^s}$ can be made sufficiently large that the following hold:

(1) The quantity

$$b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \Big)^{\tilde{\delta}} \ll 1 \tag{3.22}$$

is small.

(2) There is a positive, finite constant $\widetilde{C}$ independent of j such that

$$\|u(t_j)\|_{H^s} \geq \widetilde{C} \|u(t_0)\|_{H^s} \tag{3.23}$$

holds for all $j \in \mathbb{N}$. Without any loss of generality (by a redefinition of the constant b_δ if necessary), we can assume that $\widetilde{C} = 1$.

Accordingly, (3.23) with $\widetilde{C} = 1$ implies that $\rho_j \leq \rho_0$ for all j. Then, for any $N \in \mathbb{N}$,

$$\begin{aligned}
T^* - t_0 &\geq \sum_{j=0}^{N} (t_{j+1} - t_j) \\
&= \frac{1}{C_\delta} \Big(\frac{1}{\|u(t_0)\|_{H^s}} + \cdots + \frac{1}{\|u(t_N)\|_{H^s}} \Big) \\
&= \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \Big(1 + \frac{\|u(t_0)\|_{H^s}}{\|u(t_1)\|_{H^s}} + \cdots + \frac{\|u(t_0)\|_{H^s}}{\|u(t_N)\|_{H^s}} \Big) \\
&\geq \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \Big(1 + \frac{1}{\rho_0} + \cdots + \frac{1}{\rho_0 \cdots \rho_N} \Big) \\
&\geq \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \Big(1 + \frac{1}{\rho_0} + \cdots + \frac{1}{\rho_0^N} \Big) \qquad (3.24)
\end{aligned}$$

from $\frac{1}{\rho_j} \geq \frac{1}{\rho_0}$ for all j, and the fact that $\rho_0 > 1$ since the argument in the exponent (3.19) is positive.

Then, letting $N \to \infty$, we obtain

$$\begin{aligned}
\frac{1}{T^* - t_0} &\leq C_\delta \|u(t_0)\|_{H^s} \Big(1 - \frac{1}{\rho_0} \Big) \\
&= C_\delta \|u(t_0)\|_{H^s} \Big(1 - \exp\Big(- b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \Big)^{\tilde{\delta}} \Big) \Big). \qquad (3.25)
\end{aligned}$$

Next, we deduce a lower bound on the blowup rate.

Invoking (3.22), we obtain

$$\begin{aligned}
\frac{1}{T^* - t_0} &\leq C_\delta \|u(t_0)\|_{H^s} \Big(1 - \exp\Big(- b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \Big)^{\tilde{\delta}} \Big) \Big) \\
&\approx C_\delta \|u(t_0)\|_{H^s} b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \Big)^{\tilde{\delta}} \\
&= C_\delta \, b_\delta \|u_0\|_{L^2}^{\tilde{\delta}} \|u(t_0)\|_{H^s}^{1-\tilde{\delta}}. \qquad (3.26)
\end{aligned}$$

This implies a lower bound on the blowup rate of the form

$$\begin{aligned}
\|u(t_0)\|_{H^{\frac{5}{2}+\delta}} &\geq C(\delta, \|u_0\|_{L^2}) \Big(\frac{1}{T^* - t_0} \Big)^{\frac{1}{1-\tilde{\delta}}} \\
&= C(\delta, \|u_0\|_{L^2}) \Big(\frac{1}{T^* - t_0} \Big)^{\frac{2\delta+5}{5}}, \qquad (3.27)
\end{aligned}$$

under the condition that (3.22) and (3.23) hold.

This concludes our proof of Theorem 1.3. □

Appendix A. Proof of inequality (2.38) for $s > \frac{5}{2}$

In this Appendix, we prove (2.38) which follows from (2.36),

$$\frac{1}{2} \partial_t \|u(t)\|_{B^s_{2,2}}^2 \lesssim \|Du(t)\|_{L^\infty} \|u(t)\|_{B^s_{2,2}}^2, \qquad (A.1)$$

for $s > \frac{5}{2}$. We invoke Eq. (26) in the work [9] of F. Planchon, which is valid for $s > 1 + \frac{n}{2}$ in n dimensions (thus, $s > \frac{5}{2}$ in our case of $n = 3$), for parameter values $p = q = 2$ in the notation of that paper. It yields

$$\begin{aligned} \frac{1}{2}\partial_t 2^{2js}\|u_j\|_{L^2}^2 \quad &\lesssim \quad 2^{2js}\sum_{k\sim j}\|S_{j+1}Du\|_{L^\infty}\,\|u_k\|_{L^2}\,\|u_j\|_{L^2} \\ &\qquad + 2^{2js}\sum_{j\lesssim k\sim k'}\|u_k\|_{L^2}\,\|u_{k'}\|_{L^2}\,\|Du_j\|_{L^\infty} \qquad \text{(A.2)} \end{aligned}$$

where $u_k = P_k u$ is the Paley-Littlewood projection of u at scale k, and $S_j = \sum_{j'\leq j} P_{j'}$ is the Paley-Littlewood projection to scales $\leq j$.

Summing over j,

$$\begin{aligned} \frac{1}{2}\partial_t\sum_j 2^{2js}\|u_j\|_{L^2}^2 \quad &\lesssim \quad \sup_j\|S_{j+1}Du\|_{L^\infty}\Big(\sum_j 2^{2js}\sum_{k\sim j}\|u_k\|_{L^2}\,\|u_j\|_{L^2} \\ &\qquad + \sum_j\sum_{k\sim k'\gtrsim j} 2^{2s(j-k)}2^{ks}\,\|u_k\|_{L^2}\,2^{k's}\,\|u_{k'}\|_{L^2}\Big) \\ &\lesssim \quad \|Du\|_{L^\infty}\Big(\sum_j 2^{2js}\|u_j\|_{L^2}^2 \\ &\qquad + \sum_k\Big(\sum_{j\lesssim k}2^{2s(j-k)}\Big)2^{ks}\,\|u_k\|_{L^2}^2\Big) \\ &\lesssim \quad \|Du\|_{L^\infty}\sum_j 2^{2js}\|u_j\|_{L^2}^2\,. \qquad \text{(A.3)} \end{aligned}$$

To pass to the second inequality, we used that

$$\|S_{j+1}Du\|_{L^\infty} \;=\; \|m_{j+1}*Du\|_{L^\infty} \;\lesssim\; \|Du\|_{L^\infty}\,\|m_{j+1}\|_{L^1}\,, \qquad \text{(A.4)}$$

where $\widehat{m_j}$ is the symbol of the Fourier multiplication operator S_j, and the fact that $\|m_j\|_{L^1}\sim 1$ uniformly in j. Accordingly, we get

$$\frac{1}{2}\partial_t\|u(t)\|_{\dot B^s_{2,2}}^2 \lesssim \|Du(t)\|_{L^\infty}\,\|u(t)\|_{\dot B^s_{2,2}}^2\,. \qquad \text{(A.5)}$$

From

$$\|u(t)\|_{B^s_{2,2}}^2 \;=\; \|u(t)\|_{L^2}^2 \,+\, \|u(t)\|_{\dot B^s_{2,2}}^2\,, \qquad \text{(A.6)}$$

and energy conservation, $\partial_t\|u(t)\|_{L^2}^2 = 0$, we obtain

$$\begin{aligned} \frac{1}{2}\partial_t\|u(t)\|_{B^s_{2,2}}^2 \quad &= \quad \frac{1}{2}\partial_t\|u(t)\|_{\dot B^s_{2,2}}^2 \\ &\lesssim \quad \|Du(t)\|_{L^\infty}\,\|u(t)\|_{\dot B^s_{2,2}}^2 \\ &\lesssim \quad \|Du(t)\|_{L^\infty}\,\|u(t)\|_{B^s_{2,2}}^2\,. \qquad \text{(A.7)} \end{aligned}$$

This proves (A.1). □

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