Stability of waves using the Fredholm determinant

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Abstract

We present a new method for reducing the Fredholm determinant associated with an underlying Birman-Schwinger operator to a finite dimensional determinant. Moreover, we compute explicitly the connection between the Fredholm determinant and the Evans function for travelling wave problems of all orders, in one dimension.

1 Introduction

The purpose of this paper, is to determine the spectrum of a variable coefficient linear differential operator $\mathcal{L} = \mathcal{L}_0 + v(x)$ obtained from linearising a nonlinear partial differential equation (PDE) about its stationary solution $\phi(x)$ satisfying $\lim_{|x|\to\infty} \phi(x) = \phi^{\pm}$ with $\phi^+ \neq \phi^-$, where $\mathcal{L}_0$ is a constant coefficient differential operator of order $n \geq 2$, and v is the Jacobian evaluated at ϕ of the nonlinear part associated to the original problem. While there are various methods to solve this type of problem, we limit ourselves to the Evans function and the Fredholm determinant. Both of these are analytic functions whose zeros coincide in location and multiplicity with eigenvalues of $\mathcal{L}$. The Evans function, as introduced in [7] and extended for instance in [1], is defined as the Wronskian of the set of solutions $u^{\pm}$ of the associated problem $(\mathcal{L}-\lambda)u = 0$ which decay at $x = \pm\infty$, respectively. On the other hand, the Fredholm determinant is defined as the determinant of trace class operators that differ by the identity operator, has been extensively used, in mathematical physics not only to locate a pure point spectrum but also for counting resonances, for details see [29, 8, 16].

Our motivation for this article is based on the fact that numerically we have observed that the eigenvalue $\lambda = 0$, associated to the translational invariance and embedded in the essential spectrum, might be in some situations a pole for the Fredholm determinant or might not be. Therefore, this implies that the Fredholm determinant is the ratio of two analytic functions, which when evaluated at $\lambda = 0$ are zero. So the behaviour of the Fredholm determinant near $\lambda = 0$ depends entirely on the rate of decay of these functions. It turns out that the analytic numerator and denominator of the Fredholm determinant are simply the Evans function and the Wronskian of the set of solutions corresponding to the unperturbed problem $(\mathcal{L}_0 - \lambda)u = 0$. Hence the Fredholm determinant which is an infinite determinant is equal, up to a non-zero analytic function of the spectral parameter, to a finite dimensional determinant.

Reducing the Fredholm determinant to a finite dimensional determinant is not a new see for instance [29, 11, 12, 25, 10, 16, 19, 24]. However, in some of this literature the reduction which has been presented is that of the scalar Schrödinger operator on the whole [29] or half real line [19], for which the finite dimensional determinant is the Evans or Jost function. While in other literature [10, 25, 24], the reduction of the Fredholm determinant associated with an underlying Birman-Schwinger operator, is that of a first order system of differential equations. In this case, the finite dimensional determinant obtained is of dimension $n_1 < n$. In [10, 25, 24], the key point in the reduction is based on the observation that the kernel of the Birman-Schwinger operator is semi-separable, hence a decomposition into a finite rank and a Volterra operator is possible. Our approach of achieving this result is *new* and it does not need the decomposition of the Birman-Schwinger operator, instead we exploit the interpretation of the Evans function which measures the linear dependence of the subspaces $Y^{\pm}$ defined by the set of solutions of the operator $\mathcal{L}$ decaying at

$\pm\infty$. By this, we mean that if the subspace Y^- decaying at $-\infty$ (respectively $+\infty$) is orthogonal to a given subspace decaying at $+\infty$ (respectively $-\infty$) defined by the set of solutions of the adjoint problem, then necessary we must have that Y^- and Y^+ are linearly dependent . Thus the finite dimensional determinant that one obtains by means of the Hodge star operator is of dimension equal to one of these subspaces Y^+ or Y^-. From this, it follows that the Evans function is equal to the Fredholm determinant up to a non-zero analytic function of the spectral parameter. Consequently this result implies that the Fredholm determinant associated to the Birman-Schwinger operator written as a first order system or in the scalar form for the travelling wave problem are identical. Therefore we have, if the Birman-Schwinger operator corresponding to the scalar problem is of trace-class then the same holds for the first order system. It then follows that, despite the jump discontinuity along the diagonal of the Birman-Schwinger operator associated to the first order system, one can explicitly compute its trace. Our result generalising the connection between the Fredholm determinant and the Evans function on the whole real line for $n > 2$ is *new* since the only explicit computation proving that result we are aware of is the one found in [29] for $n = 2$. The organisation of the paper is as follows, in Section 2 and 3 we construct the appropriate Green's function and Green's matrix for the travelling wave problem. In Section 4 we show that for $m \in \{0, \ldots, n-1\}$ the Birman-Schwinger operator corresponding to our problem is of trace-class, therefore one can compute its Fredholm determinant. This property enables us to prove that the Birman-Schwinger operator associated to the first order system is also of trace-class. Having that, we can establish the connection between the Fredhlom determinant and the Evans function. In Section 5, we construct the appropriate Fredholm determinant when $\lim_{x\to\pm\infty}\phi(x) = \phi^\pm$ with $\phi^+ \neq \phi^-$. Finally in Section 6, we conclude and look to future computation work of the Fredholm determinant for travelling wave problem using the Nyström method as found in [2].

2 Green's Functions

A differential operator $\mathcal{A}$ defined on $L^2(\mathbb{R},\mathbb{C})$ with domain $\mathrm{dom}(\mathcal{A}) = L^2(\mathbb{R},\mathbb{C})$ is *invertible* if it is one-to-one and if its range, i.e $\mathrm{Im}(\mathcal{A})$, is the entire space $L^2(\mathbb{R},\mathbb{C})$. This because we wish $\mathcal{A}$ to have both left and right inverses with these being the same. The operator is one-to-one if only if its null space has dimension zero that is $\dim(\mathrm{Ker}(\mathcal{A})) = 0$. The range of $\mathcal{A}$ is all of $L^2(\mathbb{R},\mathbb{C})$ if and only if the null space of the adjoint $\mathcal{A}^*$ has dimension zero. Thus assuming $\mathcal{A}$ is invertible, both the dimension of the null spaces of $\mathcal{A}$ and $\mathcal{A}^*$ must be zero, where $\mathcal{A}^*$ satisfies

$$\langle \mathcal{A}u, v\rangle = \langle u, \mathcal{A}^* v\rangle, \quad \forall u \in \mathrm{dom}(\mathcal{A}), v \in \mathrm{dom}(\mathcal{A}^*).$$

In this case, we write

$$\mathcal{A}^{-1}u(x) = \int_{\mathbb{R}} g(x,y)v(y)dy, \quad \forall v \in L^2(\mathbb{R},\mathbb{C}),$$

where $g(x,y)$ is the *Green's function*. Throughout this paper, by invertibility we mean the existence of the left and the right inverse, i.e

$$\mathcal{A}^{-1}\mathcal{A} = \mathcal{A}\mathcal{A}^{-1} = I_{L^2}.$$

Let $\mathcal{L}$ be an nth order differential given by

$$\mathcal{L} = a_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n} + \ldots + a_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + a_0(x).$$

Theorem 2.1. *[20] The Green's function $g(x,y)$ satisfies $\mathcal{L}g(x,y) = \delta(x-y)$ in the sense of distributions if and only if :*

1. *$\mathcal{L}g(x,y) = 0$ for $x \neq y$;*
2. *$g(x,y)$ is $n-2$ times continuously differentiable in x at $x = y$;*
3. *$\left.\frac{d^{n-1}g(x,y)}{dx^{n-1}}\right|_{x=y^-}^{x=y^+} = \frac{1}{a_n(x)}$ (jump condition).*

If the Green's function exists, then the solution of $\mathcal{L}u(x) = f(x)$ is given by

$$u(x) = Kf(x) = \int_a^b g(x,y)f(y)\mathrm{d}y,$$

where K denotes the inverse of the operator $\mathcal{L}$.

The maximal operator $\mathcal{L}_{\max}$ corresponding to $\mathcal{L}$ is such that

$$\begin{aligned}\mathrm{dom}(\mathcal{L}_{\max}) &= \{u \in AC(I) \cap L^2(I) \colon\ \mathcal{L}u \in L^2(I)\},\\ \mathcal{L}_{\max}u &= \mathcal{L}u,\end{aligned}$$

where $I \subseteq \mathbb{R}$ and $AC(I)$ denotes the set of absolutely continuous functions on I.

Proposition 2.2. *[25, Chap XIV. 3] The maximal operator corresponding to $\mathcal{L}$ and any interval is closed.*

The Green's function of a closed ordinary differential operator on a domain I is *semi-separable*, this means that $g(x,y)$ has the following form [25, Chap XIV. 3]

$$g(x,y) = \begin{cases} F_1(x)G_1(y), & y \leq x, \quad x, y \in I; \\ F_2(x)G_2(y), & x < y, \end{cases} \tag{1}$$

where $F_j(x) \in L^2(I, \mathbb{C}^{n \times n_j})$ and $G_j(x) \in L^2(I, \mathbb{C}^{n_j \times n})$ for $j = 1, 2$ and some $n_1, n_2 \in \mathbb{N}$.
Consider the differential operator L obtained from linearising a nonlinear partial differential equation (PDE) on $\mathbb{R}$ about its travelling wave solution $\phi(x)$. The operator L is defined on $L^2(\mathbb{R}, \mathbb{C})$ and it is given by

$$L = L_0 + v(x), \quad x \in \mathbb{R}, \tag{2}$$

where

$$L_0 = \frac{\mathrm{d}^n}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \ldots + a_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + a_0(x), \quad x \in \mathbb{R}, \tag{3}$$

and $v(x) = V(\phi(x))$ where V is the Jacobian corresponding to the nonlinear part associated to the PDE. In what follows, we assume that $\lim_{|x|\to\infty} \phi(x) = \phi^\infty$, where $\phi^\infty \in \mathbb{R}$. For simplicity, we consider the case $\phi^\infty = 0$, however the analysis that we will carry out is the same when $\phi^\infty \neq 0$ or more generally when $\phi^{-\infty} \neq \phi^{+\infty}$. Instead of $(L_0 - \lambda I_{L^2})$ and $v(x)$, we will consider $(L_0 + v^\infty(x) - \lambda I_{L^2})$ and $(v(x) - v^\infty(x))$, where $v^\infty(x) = V(\phi^\infty)$, respectively. Note that if $v(x)$ is a variable-coefficient differential operator then $v^\infty(x)$ becomes a constant differential operator. The eigenvalue problem reads

$$\begin{cases} Lu = \lambda u \\ u(\pm\infty) = 0. \end{cases} \tag{4}$$

Our objective is to determine the values of λ for which $T(\lambda) = (L - \lambda I_{L^2})$ is Fredholm of index zero. We recall that an operator $\mathcal{A}$ is Fredholm if its range is closed and the dimension of its kernel and cokernel are finite.

Hypothesis 2.1. *Assume that the a_i are constant for $i = 0, \ldots, n-1$, and*

$$\mathcal{P}_n(\mu) = \mu^n + a_{n-1}\mu^{n-1} + \ldots + a_1\mu + a_0 - \lambda = 0, \quad \forall \lambda \in \Omega, \tag{5}$$

has k roots $\kappa_i^+ = \mu_i^+(\lambda)$, $i = 1, \ldots, k$ with $\mathrm{Re}(\kappa_i^+) > 0$ and $n-k$ roots $\kappa_i^- = \mu_i^-(\lambda)$, $i = k+1, \ldots, n$ with $\mathrm{Re}(\kappa_i^-) < 0$ for a suitable $\Omega \subseteq \mathbb{C}$ which may contain the right-half complex plane.

Since $\lambda = 0$ corresponds to the eigenfunction $\phi'(x)$, then $\dim(\mathrm{Ker}(L)) \neq 0$, i.e L is not invertible. However, instead of constructing the pseudo-inverse of L achieved by solving for the Green's function in the sense of least squares, we consider Hypothesis 2.1 and take $\mathrm{dom}(L_0) = H^n(\mathbb{R}, \mathbb{C})$ the Sobolev space so that L_0 is closed and hence $L_0 - \lambda I_{L^2}$ too. Therefore, since we know how to construct the Green's function of a closed differential operator, we can take advantage of this to study the stability spectrum of the operator L. The underlying idea is that for $\lambda \in \Omega$, we have

$$\det\left(L_0 + v(x) - \lambda I_{L^2}\right) = \det\left(L_0 - \lambda I_{L^2}\right)\det\left(I_{L^2} + (L_0 - \lambda I_{L^2})^{-1} v(x)\right). \tag{6}$$

Hence λ is eigenvalue of the operator L if only if

$$\det\left(I_{L^2} + (L_0 - \lambda I_{L^2})^{-1} v(x)\right) = 0, \tag{7}$$

since under Hypothesis 2.1

$$\det(L_0 - \lambda I_{L^2}) \neq 0, \quad \forall \lambda \in \Omega. \tag{8}$$

Thus, by K we denote the inverse of $(L_0 - \lambda I_{L^2})$ given by

$$K(\lambda)u(x,\lambda) = \int_{\mathbb{R}} g(x,\xi,\lambda)u(\xi,\lambda)\mathrm{d}\xi, \qquad \forall u \in L^2(\mathbb{R},\mathbb{C}).$$

Furthermore $L_0 - \lambda I_{L^2}$ is closed, thus from Proposition 2.2, Theorem 2.1 and (1), we give an explicit expression of its Green's function .

Definition 2.3. For $\lambda \in \Omega$, we define the Green's function $g(x,\xi,\lambda) \in L^2(\mathbb{R}^2,\mathbb{C})$ of the operator $L_0 - \lambda I_{L^2}$ by

$$g(x,\xi,\lambda) = \begin{cases} \displaystyle\sum_{i=1}^{k} \alpha_i(\lambda) e^{\kappa_i^+(x-\xi)}, & x \leq \xi, \qquad x,\xi \in \mathbb{R}; \\ \displaystyle\sum_{i=k+1}^{n} \alpha_i(\lambda) e^{\kappa_i^-(x-\xi)}, & \xi < x, \end{cases} \tag{9}$$

where $\alpha_i(\lambda),\ i = 1,\dots,n$ are solutions of the following matrix equation

$$\begin{pmatrix} 1 & \dots & 1 & -1 & \dots & -1 \\ \kappa_1^+ & \dots & \kappa_k^+ & -\kappa_{k+1}^- & \dots & -\kappa_n^- \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (\kappa_1^+)^{n-1} & \dots & (\kappa_k^+)^{n-1} & -(\kappa_{k+1}^-)^{n-1} & \dots & -(\kappa_n^-)^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1(\lambda) \\ \alpha_2(\lambda) \\ \vdots \\ \alpha_n(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}. \tag{10}$$

3 Green's Matrix

In order to construct the *Green's matrix* associated with the corresponding first order system of ordinary differential equation, we set $y_j(x) = u^{(j-1)}(x), \quad j = 1,\dots,n$, so (4) becomes

$$Y' = (A(\lambda) + R(x))Y \tag{11}$$

where

$$A(\lambda) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & 0 \\ 0 & 0 & & & 1 \\ \lambda - a_0 & -a_1 & & \cdots & -a_{n-1} \end{pmatrix} \tag{12}$$

and

$$R(x) = \begin{pmatrix} 0_{(n-1)\times(n-1)} & 0 \\ V(\phi(x))_{(1\times(m+1))} & 0 \end{pmatrix}, \tag{13}$$

for some fixed $m \in \{0,\dots,n-1\}$. Consider the differential operator $\tilde{L}_0$ defined by

$$\begin{aligned} \tilde{L}_0 : H^1(\mathbb{R},\mathbb{C}^n) &\to L^2(\mathbb{R},\mathbb{C}^n) \\ Y &\mapsto Y' - A(\lambda)Y \end{aligned}$$

Assume Hypothesis 2.1, then $\sigma(A(\lambda)) \cap i\mathbb{R} = \emptyset$ where $\sigma(A(\lambda))$ denotes the spectrum of $A(\lambda)$. In other words, $\tilde{L}_0$ has an exponential dichotomy on $\mathbb{R}$, hence it is Fredholm with non-zero index [34]. Therefore, $\tilde{L}_0$ is invertible with inverse K_0 given by

$$K_0(\lambda)Y(x,\lambda) = \int_{\mathbb{R}} G(x,\xi,\lambda)Y(\xi,\lambda)\mathrm{d}\xi, \qquad \forall\, Y \in L^2(\mathbb{R},\mathbb{C}^n), \tag{14}$$

where $G(x,\xi,\lambda) \in L^2(\mathbb{R}^2,\mathbb{C}^{n\times n})$ is the *Green's matrix* satisfying

$$\mathcal{L}G(x,\xi,\lambda) = 0, \quad \text{for} \quad x \neq \xi, \tag{15a}$$
$$G(\xi^+,\xi,\lambda) - G(\xi^-,\xi,\lambda) = I_n, \tag{15b}$$
$$G(x,x_-,\lambda) - G(x,x_+,\lambda) = I_n, \tag{15c}$$
$$G(\pm\infty,\xi,\lambda) = 0. \tag{15d}$$

Consider the adjoint operator $\tilde{L}_0^*$ of $\tilde{L}_0$ defined by

$$\begin{aligned} \tilde{L}_0^* : H^1(\mathbb{R},\mathbb{C}^n) &\to L^2(\mathbb{R},\mathbb{C}^n) \\ Z &\mapsto -Z' - ZA(\lambda) \end{aligned}$$

Let $U(x,\lambda)$ be the $n\times n$ fundamental matrix that is, the column vectors of $U(x,\lambda)$ are linearly independent solutions of $\tilde{L}_0 Y(x,\lambda) = 0$. We write $U(x,\lambda) = \begin{pmatrix} Y_0^-(x,\lambda) & Y_0^+(x,\lambda)\end{pmatrix}$ where $Y_0^-(x,\lambda)$ and $Y_0^+(x,\lambda)$ are $n\times k$ and $n\times(n-k)$ matrix-valued functions with columns satisfying

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Y_0^\pm(x,\lambda)\right)_j = A(\lambda)\left(Y_0^\pm(x,\lambda)\right)_j, \quad \left(Y_0^\pm(\pm\infty,\lambda)\right)_j = 0,$$

where for Y_0^-, $j = 1,\ldots,k$ and for Y_0^+, $j = 1\ldots,n-k$.
Since $\tilde{L}_0$ has non-zero Fredholm index then $U(x,\lambda)$ is invertible. Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}x}U^{-1}(x,\lambda) = -U^{-1}(x,\lambda)A(\lambda). \tag{16}$$

If $V(x,\lambda)$ is the fundamental matrix of $\tilde{L}_0^* Z = 0$, then its row vectors are linearly independent and it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}V(x,\lambda) = -V(x,\lambda)A(\lambda). \tag{17}$$

Equation (16) and (17) imply that

$$U^{-1}(x,\lambda) = V(x,\lambda),$$

where $V(x,\lambda) = \begin{pmatrix} Z_0^+(x,\lambda) & Z_0^-(x,\lambda)\end{pmatrix}^T$ with $Z_0^+(x,\lambda)$ and $Z_0^-(x,\lambda)$ are $k\times n$ and $(n-k)\times n$ matrix-valued functions satisfying

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Z_0^\pm(x,\lambda)\right)_j = -\left(Z_0^\pm(x,\lambda)\right)_j A(\lambda), \quad \left(Z_0^\pm(\pm\infty,\lambda)\right)_j = 0.$$

Since $V(x,\lambda)U(x,\lambda) = U(x,\lambda)V(x,\lambda) = I_n$ we write

$$\begin{pmatrix} Z_0^+(x,\lambda) \\ Z_0^-(x,\lambda)\end{pmatrix}\begin{pmatrix} Y_0^-(x,\lambda) & Y_0^+(x,\lambda)\end{pmatrix} = Y_0^-(x,\lambda)Z_0^+(x,\lambda) + Y_0^+(x,\lambda)Z_0^-(x,\lambda) = I_n. \tag{18}$$

Consequently for all $x\in\mathbb{R}$ and $\lambda\in\Omega$, we have

$$Z_0^+(x,\lambda)Y_0^-(x,\lambda) = I_k, \tag{19a}$$
$$Z_0^-(x,\lambda)Y_0^+(x,\lambda) = I_{n-k}, \tag{19b}$$
$$Z_0^+(x,\lambda)Y_0^+(x,\lambda) = 0_{(n-k)\times k}, \tag{19c}$$
$$Z_0^-(x,\lambda)Y_0^-(x,\lambda) = 0_{k\times(n-k)}. \tag{19d}$$

Definition 3.1. For $\lambda \in \Omega$, we define the Green's matrix $g(x,\xi,\lambda) \in L^2(\mathbb{R}^2, \mathbb{C}^{n\times n})$ of the operator $\tilde{L}_0$ by

$$G(x,\xi,\lambda) = \begin{cases} -Y_0^-(x,\lambda)Z_0^+(\xi,\lambda), & x \leq \xi, \quad x,\xi \in \mathbb{R}; \\ Y_0^+(x,\lambda)Z_0^-(\xi,\lambda), & \xi < x. \end{cases} \tag{20}$$

The Green's matrix $G(x,\xi,\lambda)$ constructed in (20) satisfies (15).
Since a constant differential operator has no eigenvalue [6, Chap IX] then the spectrum of L_0 is reduced to that of the essential spectrum $\sigma_e(L_0)$, which is obtained by applying the Fourier transform to $L_0 u = \lambda u$. Thus we have

$$\sigma_e(L_0) = \overline{\{\lambda \in \mathbb{C}\colon\ \lambda = \sum_{l=0}^{n} c_l \zeta^l \colon \zeta \in \mathbb{R},\ c_l = i^l a_l \text{ and } c_n = i^n\}}, \tag{21}$$

or equivalently

$$\sigma_e(\tilde{L}_0) = \overline{\{\lambda \in \mathbb{C}\colon\ \det\left(i\zeta I_n - A(\lambda)\right) = 0,\ \zeta \in \mathbb{R}\}}.$$

From the invariance property of the essential spectrum under perturbations [6, Chap IX, sec. 2], we have $\sigma_e(L) = \sigma_e(L_0)$. Hence

$$\Omega = \mathbb{C} \setminus \sigma_e(L). \tag{22}$$

Note that Ω is in fact the resolvent set of L_0, and $0 \in \sigma_e(L_0)$ or $\sigma_e(\tilde{L}_0)$, though $\lambda = 0$ is an eigenvalue of L.

4 Fredholm determinant and Evans function

4.1 The scalar Schrödinger problem

We present a new method for reducing the Fredholm determinant associated to the Birman-Schwinger operator into a finite dimensional determinant, by means of the interpretation of the Evans function. Our method exploits the fact that if Y^- and Y^+ defined by the set of solutions of (11) that decay at $\pm\infty$, respectively, are linearly dependent as x goes to $+\infty$, then Y^- and Y_0^+ must also be linearly dependent. This can be seen as follows, since Y_0^+ and Z_0^+ are orthogonal then if Y^- and Z_0^+ are also orthogonal as x goes to $+\infty$, then there must exist a bounded solution in L^2. By following this reasoning, it turns out that the Gramian matrix generated by the elements of Y^- and Z_0^+ is independent of the variable x, and so a reduction of the Fredholm determinant to a finite dimensional determinant is possible. So far, the only reduction we are aware of is that of the scalar Schrödinger problem on the whole or half real line [29, 19] for which the connection between the Fredholm determinant and the Evans function is explicitly computed. Therefore our result is new in the sense that we reduce the Fredholm determinant associated to a nth order scalar problem to a finite dimensional determinant which in turn is the ratio of the Evans function and a nonzero analytic function of the spectral parameter λ. However, to achieve our result we convert the problem to a first order system where the corresponding Birman-Schwinger operator is of trace class. This induces a matrix-valued semi-separable kernel, for which its Fredholm determinant can be reduced to a finite dimensional determinant, this can be found in [11, 12] and [25, 24, chap. IX]. Thus our result of reducing the Fredholm determinant to a finite dimensional determinant in itself is not new but explicitly connecting it to the Evans function is new. Furthermore, our method of obtaining the reduction appear to be new since unlike the method presented in [11, 12] and [25, 24, chap. IX], we do not decompose the integral operator into a finite rank operator and a Volterra equation in order to have the reduction.
To introduce our idea, we begin with the scalar Schrödinger problem written as first order system.

$$\begin{aligned} Y'(x,\lambda) &= \left(A(\lambda) + R(x)\right) Y(x,\lambda), \qquad x \in \mathbb{R}; \\ Y(\pm\infty,\lambda) &= 0, \end{aligned} \tag{23}$$

where

$$A(\lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}, \qquad R(x) = \begin{pmatrix} 0 & 0 \\ V(x) & 0 \end{pmatrix}$$

and $V(x) \in L^1(\mathbb{R}, e^{\beta|x|}\mathrm{d}x)$ for some $\beta > 0$. Following [29], we set $\lambda = -\tau^2$. Suppose that $u_-(x,\tau)$ and $u_+(x,\tau)$ are solutions for the scalar Schrödinger problem such that $u_-(x,\tau) \sim e^{\tau x}$ and $u_+(x,\tau) \sim e^{-\tau x}$ as x tends to $\mp\infty$, respectively. Then we have [20, chap 7 sec 7.5]

$$u_-(x,\tau) = c(\tau)u_+(x,\tau) + d(\tau)u_+(x,-\tau). \tag{24}$$

Multiplying by $e^{-\tau x}$ and taking the limit when x goes to $+\infty$ in (24), we get

$$\lim_{x\to+\infty} e^{-\tau x}u_-(x,\tau) = d(\tau). \tag{25}$$

Now, we introduce $Y^-(x,\tau) = \begin{pmatrix} u_-(x,\tau) & \frac{\mathrm{d}}{\mathrm{d}x}u_-(x,\tau)\end{pmatrix}^T$ and $Y^+(x,\tau) = \begin{pmatrix} u_+(x,\tau) & \frac{\mathrm{d}}{\mathrm{d}x}u_+(x,\tau)\end{pmatrix}^T$ the Jost solutions [19] of the problem (23)

$$Y^\pm(x,\tau) = Y_0^\pm(x,\tau) + \int_{\pm\infty}^x U(x)U^{-1}(y)R(y)Y^\pm(y,\tau)\mathrm{d}y,$$

where $U(x) = \begin{pmatrix} Y_0^-(x,\tau) & Y_0^+(x,\tau)\end{pmatrix}$ and $Y_0^\pm(x,\tau) = e^{\mp\tau x}\begin{pmatrix} 1 & \mp\tau\end{pmatrix}^T$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}Y_0^\pm(x,\tau) = A(\tau)Y_0^\pm(x,\tau).$$

Alternatively, we may write $d(\tau)$ as follows

$$d(\tau) = \lim_{x\to\infty}\langle \left(Z_0^+(x,\tau)\right)^*, Y^-(x,\tau)\rangle_{\mathbb{R}^2} = \lim_{x\to\infty} Z_0^+(x,\tau)Y^-(x,\tau), \tag{26}$$

where $Z_0^+(x,\tau) = \frac{e^{-\tau x}}{2\tau}\begin{pmatrix}\tau & 1\end{pmatrix}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}Z_0^+(x,\tau) = -Z_0^+(x,\tau)A(\tau).$$

Indeed the right-hand side of (26) gives,

$$\begin{aligned}\lim_{x\to\infty} Z_0^+(x,\tau)Y^-(x,\tau) &= \lim_{x\to\infty}\frac{1}{2\tau}\left(\tau e^{-\tau x}u_-(x,\tau) + e^{-\tau x}\frac{\mathrm{d}}{\mathrm{d}x}u_-(x,\tau)\right)\\ &= \lim_{x\to\infty}\frac{1}{2\tau}\left(\tau e^{-\tau x}u_-(x,\tau) + \frac{\mathrm{d}}{\mathrm{d}x}(e^{-\tau x}u_-(x,\tau)) + \tau e^{-\tau x}u_-(x,\tau)\right)\\ &= \lim_{x\to+\infty} e^{-\tau x}u_-(x,\lambda) = d(\tau),\end{aligned}$$

where

$$u_-(x,\tau) = e^{\tau x} + \frac{1}{\tau}\int_{-\infty}^x \sinh\left(\tau(y-x)\right)V(y)u_-(y,\tau)\mathrm{d}y. \tag{27}$$

Substitute (27) into (25), we obtain

$$d(\tau) = 1 + \frac{1}{2\tau}\int_{\mathbb{R}} e^{-\tau y}V(y)u_-(y,\tau)\mathrm{d}y, \quad \mathrm{Re}(\tau) > 0. \tag{28}$$

Replacing the Neumann series corresponding to (27) into (28) one obtains

$$d(\tau) = 1 + \frac{1}{2\tau}\int_{\mathbb{R}} V(x)\mathrm{d}x + \frac{1}{2\tau^2}\int_{\mathbb{R}}\int_{-\infty}^x e^{-\tau x}V(x)\sinh\left(\tau(x_1 - x)\right)V(x_1)e^{\tau x_1}\mathrm{d}x\mathrm{d}x_1 + \cdots,$$

which is the Fredholm determinant [29] of a trace-class operator $K(\tau)$ given by

$$K(\tau)u(x,\tau) = \frac{1}{2\tau}\int_{\mathbb{R}} V(x)^{1/2}e^{-\tau|x-y|}|V(y)|^{1/2}u(y,\tau)\mathrm{d}y.$$

The Fredholm determinant and the Evans function $E(\tau)$ for problem (23) are related by [29, 16],

$$d(\lambda) = \det(1 - K(\tau)) = -\frac{E(\tau)}{2\tau}, \tag{29}$$

where

$$E(\tau) = \det_{\mathbb{R}^{2\times2}}\begin{pmatrix} Y^-(x,\tau) & Y^+(x,\tau)\end{pmatrix}. \tag{30}$$

Remark. Geometrically, equation (26) means that since $\langle Y_0^+(x,\lambda), \left(Z_0^+(x,\lambda)\right)^*\rangle = 0$ for all $x \in \mathbb{R}$, and in particular when $x \to +\infty$. Then, if $\langle Y^-(x,\lambda), \left(Z_0^+(x,\lambda)\right)^*\rangle = 0$ for some $\lambda_0 \in \mathbb{C}$ as x goes to $+\infty$, we must have $Y^-(x,\lambda_0) = \alpha Y_0^+(x,\lambda_0)$. In other words, if Y^- and Y^+ are linearly dependent for all $x \in \mathbb{R}$ and for some fixed $\lambda_0 \in \mathbb{C}$, so are Y^- and Y_0^+ when x tends to $+\infty$.

4.2 The Birman-Schwinger operator for scalar problem

With the above remark in mind, we extend the idea of connecting the Fredholm determinant and the Evans function to the travelling wave problem given in (4). To this end, we list some hypotheses and show that the Birman-Schwinger operator associated to our problem is of trace-class. Let us denote by $\mathcal{C}(\mathcal{H})$ the set of compact operators in a separable Hilbert space $\mathcal{H}$, and by $\sigma_n(A)$ the singular values of a compact operator A. Then the Schatten-von Neumann classes of compact operators are defined as

$$\mathcal{J}_p(\mathcal{H}) = \left\{ A \in \mathcal{C}(\mathcal{H})\colon\ \left[\sum_{n=1}^{\infty} \sigma_n^p(A)\right]^{1/p} < \infty \right\}, \quad (1 \le p < \infty)$$

with the corresponding norm

$$\|A\|_p = \left[\sum_{n=1}^{\infty} \sigma_n^p(A)\right]^{1/p},$$

and the usual convention for $p = \infty$.
The travelling wave problem which concern us is given, for some fixed $m \in \{0, \dots, n-1\}$ by

$$L_0 u + \frac{\mathrm{d}^m}{\mathrm{d}x^m}(\phi(x)u) = \lambda u, \tag{31}$$

where L_0 is given in (3) with a_l constants for all $l = 0, \dots, n-1$.

Hypothesis 4.1. *Assume Hypothesis 2.1 and that the travelling wave $\phi(x)$ satisfies $\lim_{|x|\to\infty} \phi^{(l)}(x) = 0$ for all $l \ge 0$, and $|\phi(x)| \le Ke^{-\beta|x|}$ for some $\beta > 0$ and $K \in \mathbb{R}$.*

For $\lambda \in \Omega$ defined in (22), equation (8) holds, and then we can write equation (7) as follows

$$u(x,\lambda) = -K(\lambda)v_0(x) = -\int_{\mathbb{R}} h(x,\xi,\lambda)v_0(\xi)\mathrm{d}\xi, \tag{32}$$

where h is given in (9) and $v_0(x) = \dfrac{\mathrm{d}^m}{\mathrm{d}x^m}(\phi(x)u)$.

4.2.1 The case when $m = 0$

In the case $m = 0$, equation (32) becomes

$$u(x,\lambda) = -K(\lambda)\left(\phi(x)u(x,\lambda)\right) = -\int_{\mathbb{R}} h(x,\xi,\lambda)\phi(\xi)u(\xi,\lambda)\mathrm{d}\xi. \tag{33}$$

Let $\tilde{\phi}(x) = |\phi(x)|^{1/2}e^{i\arg(\phi(x))}$, then equation (33) is equivalent to

$$\psi(x,\lambda) = -\mathcal{B}(\lambda)\psi(x,\lambda),$$

where $\psi(x,\lambda) = |\phi(x)|^{1/2}e^{i\arg(\phi(x))}u(x,\lambda)$, and $\mathcal{B}(\lambda)$ is the Birman-Schwinger operator

$$\mathcal{B}(\lambda) = \tilde{\phi}(x)K(\lambda)|\phi|^{1/2}, \tag{34}$$

with integral kernel

$$\tilde{g}(x,\xi,\lambda)=\begin{cases}\tilde{\phi}(x)\displaystyle\sum_{i=1}^{k}\alpha_i(\lambda)e^{\kappa_i^+(x-\xi)}|\phi(\xi)|^{1/2}, & x\le\xi,\quad x,\xi\in\mathbb{R};\\ \tilde{\phi}(x)\displaystyle\sum_{i=k+1}^{n}\alpha_i(\lambda)e^{\kappa_i^-(x-\xi)}|\phi(\xi)|^{1/2}, & \xi<x.\end{cases} \tag{35}$$

The Birman-Schwinger principle states that L has eigenvalue λ if and only if $\mathcal{B}(\lambda)$ has minus one as an eigenvalue [29].

Theorem 4.1. *[30, Theorem 4.1 Chap 4] If $f\in L^p$ and $g\in L^p$ with $2\le p<\infty$, then $f(x)g(-i\nabla)$ is in $\mathcal{J}_p$ and*

$$\|f(x)g(-i\nabla)\|_{\mathcal{J}_p}\le(2\pi)^{-1/p}\|f\|_{L^p}\|g\|_{L^p}. \tag{36}$$

For $\lambda\in\Omega$, we write $\mathcal{B}(\lambda)=f(x).g(q)=\tilde{\phi}(x).\hat{h}(q)|\phi|^{1/2}$ where $q=\frac{\mathrm{d}}{\mathrm{d}x}$, and

$$\hat{h}(q)=\frac{1}{\displaystyle\sum_{j=0}^{n}a_jq^j-\lambda}, \tag{37}$$

with $\hat{h}$ being the inverse of the Fourier transform associated to the constant coefficient differential operator $L_0-\lambda I_{L^2}$ defined in (3). Since $\phi\in L^1(\mathbb{R},e^{\beta|x|}\mathrm{d}x)$ then the function $\tilde{g}(x,\xi,\lambda)$ given in (35) is in L^2 in $\mathbb{R}\times\mathbb{R}$. Hence $\mathcal{B}(\lambda)$ is a Hilbert-Schmidt operator with norm satisfying (36).

Theorem 4.2. *For $\lambda\in\Omega$, the Birman-Schwinger operator $\mathcal{B}(\lambda)$ is a trace class operator.*

Proof. Observe that $\hat{h}$ given in (37) is the product of n Hilbert-Schmidt operators, that is

$$\hat{h}(q)=\prod_{j=1}^{n}\hat{h}_j(q)=\prod_{j=1}^{n}(q-\kappa_j)^{-1},$$

where

$$\kappa_i=\begin{cases}\kappa_i^+, & \text{for } i=1,\dots,k,\\ \kappa_i^-, & \text{for } i=k+1,\dots,n.\end{cases} \tag{38}$$

Hence, $\hat{h}$ is trace-class operator [24, Theorem 11.2 Chap IV] with norm

$$\|\hat{h}\|_{\mathcal{J}_1}\le\prod_{j=1}^{n}\|\hat{h}_j\|_{\mathcal{J}_2}.$$

It follows that $\mathcal{B}(\lambda)$ is of trace-class, where $\mathcal{B}(\lambda)$ satisfies

$$\|\mathcal{B}(\lambda)\|_{\mathcal{J}_1}\le\|\hat{h}\|_{\mathcal{J}_1}\|\phi\|_{L^1}.$$

□

For a trace-class operator $\mathcal{B}(\lambda)$, we can compute the Fredholm determinant

$$d(\lambda)=\det(I_{L^2}+\mathcal{B}(\lambda)), \tag{39}$$

where $d(\lambda)$ is given by

$$d(\lambda)=1+\sum_{n=1}^{\infty}d_n(\lambda),$$

with

$$d_n(\lambda) = \frac{1}{n!}\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \det([\tilde{g}(x_i, x_j, \lambda)]_{i,j=1,\dots,n}) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

For example, note that when $n = 1$, we have

$$d_1(\lambda) = \mathrm{Tr}\ \mathcal{B}(\lambda) = \sum_{i=1}^{k} \alpha_i(\lambda) \int_{\mathbb{R}} \phi(x)\mathrm{d}x = \sum_{i=k+1}^{n} \alpha_i(\lambda) \int_{\mathbb{R}} \phi(x)\mathrm{d}x. \tag{40}$$

4.2.2 The case when $m \neq 0$

Suppose that the integer $m \in \{1, \dots, n-1\}$. Then integration by parts of equation (32) gives

$$u(x, \lambda) = (-1)^{m+1} \int_{\mathbb{R}} \tilde{g}_m(x, \xi, \lambda)\phi(\xi)u(\xi, \lambda)d\xi, \tag{41}$$

where

$$\tilde{g}_m(x, \xi, \lambda) = \begin{cases} \displaystyle\sum_{i=1}^{k} (-1)^m (\kappa_i^+)^m \alpha_i(\lambda) e^{\kappa_i^+(x-\xi)}, & x \leq \xi, \quad x, \xi \in \mathbb{R}; \\ \displaystyle\sum_{i=k+1}^{n} (-1)^m (\kappa_i^-)^m \alpha_i(\lambda) e^{\kappa_i^-(x-\xi)}, & \xi < x. \end{cases}$$

As in (34), we rewrite (41) in the following form

$$\psi(x, \lambda) = (-1)^{m+1} B_m(\lambda)\psi(x, \xi),$$

where the Birman-Schwinger operator $B_m(\lambda)$ is given by

$$B_m(\lambda) = \tilde{\phi}(x) K_m(\lambda)|\phi|^{1/2}$$

with $K_m(\lambda)$ the resolvent operator corresponding to the constant differential operator $c_m(\lambda)\left(L_0 - \lambda I_{L^2}\right)$, where $c_m(\lambda)$ is a non-zero constant.

Remark. Recall that the Green's function of $L_0 - \lambda I_{L^2}$ has a jump discontinuity along the diagonal in the $(n-1)$th derivative. Therefore we split the problem into the case when $m \in \{1, \dots, n-2\}$ and $m = n-1$.

Corollary 1. *For all $m \in \{1, \dots, n-2\}$, the operators $B_m(\lambda)$ with $B_0(\lambda) := \mathcal{B}(\lambda)$ given in (34) are of trace class.*

Proof. Write $B_m(\lambda) = \tilde{\phi}(x)\hat{h}_m(q)|\phi|^{1/2}$ with $\hat{h}_0 := \hat{h}$. From the continuity property of the Green's function corresponding to $L_0 - \lambda I_{L^2}$, that is for all $m \in \{1, \dots, n-2\}$,

$$\sum_{i=k+1}^{n} \alpha_i(\lambda)(\kappa_i^-)^m = \sum_{i=1}^{k} \alpha_i(\lambda)(\kappa_i^+)^m.$$

We have

$$\hat{h}_m(q) = \hat{h}(q)\left(\prod_{i=1}^{n} \kappa_i^m\right),$$

where κ_i given in (38). Then it follows from Theorem 4.2 that $B_m(\lambda)$ is trace class operator for $m \in \{1, \dots, n-2\}$. $\square$

Theorem 4.3. *When $m = n-1$, the operator $B_{n-1}(\lambda)$ with kernel given by*

$$\tilde{g}_{n-1}(x, \xi, \lambda) = \begin{cases} \tilde{\phi}(x) \displaystyle\sum_{i=1}^{k} (-1)^{n-1} (\kappa_i^+)^{n-1} \alpha_i(\lambda) e^{\kappa_i^+(x-\xi)} |\phi(\xi)|^{1/2}, & x \leq \xi, \quad x, \xi \in \mathbb{R}; \\ \tilde{\phi}(x) \displaystyle\sum_{i=k+1}^{n} (-1)^{n-1} (\kappa_i^-)^{n-1} \alpha_i(\lambda) e^{\kappa_i^-(x-\xi)} |\phi(\xi)|^{1/2}, & \xi < x \end{cases} \tag{42}$$

is of trace-class.

Proof. For $m = n-1$, we have $B_{n-1}(\lambda) = \tilde{\phi}(x)\hat{h}_{n-1}(q)|\phi|^{1/2}$. The integral kernel $\tilde{g}_{n-1}(x,\xi,\lambda)$ is L^2 in $\mathbb{R}\times\mathbb{R}$, so it is Hilbert-Schmidt. Using the the jump condition of the Green's function associated to $L_0 - \lambda I_{L^2}$

$$\sum_{i=k+1}^{n} \alpha_i(\lambda)(\kappa_i^-)^{n-1} - \sum_{i=1}^{k} \alpha_i(\lambda)(\kappa_i^+)^{n-1} = 1,$$

and for an i_0 chosen, for instance in $\{k+1,\dots,n\}$ such that

$$\alpha_{i_0}(\lambda)(\kappa_{i_0}^-)^{n-1} = 1 + \sum_{i=1}^{k} \alpha_i(\lambda)(\kappa_i^+)^{n-1} - \sum_{\substack{i=k+1\\ i\neq i_0}}^{n} \alpha_i(\lambda)(\kappa_i^-)^{n-1}.$$

We have

$$\begin{aligned}
\hat{h}_{n-1}(\zeta) &= \sum_{i=1}^{k} \frac{\alpha_i(\lambda)(\kappa_i^+)^{n-1}}{\kappa_i^+ - i\zeta} - \sum_{i=k+1}^{n} \frac{\alpha_i(\lambda)(\kappa_i^-)^{n-1}}{\kappa_i^- - i\zeta} \\
&= -\frac{1}{\kappa_{i_0}^- - i\zeta} + \sum_{i=1}^{k} \alpha_i(\lambda)(\kappa_i^+)^{n-1} \frac{\kappa_{i_0}^- - \kappa_i^+}{(\kappa_{i_0}^- - i\zeta)(\kappa_i^+ - i\zeta)} + \sum_{\substack{i=k+1\\ i\neq i_0}}^{n} \alpha_i(\lambda)(\kappa_i^-)^{n-1} \frac{\kappa_i^- - \kappa_{i_0}^-}{(\kappa_i^- - i\zeta)(\kappa_{i_0}^- - i\zeta)} \\
&= \hat{h}^-(\zeta) + \hat{h}_{n-1}^-(\zeta) + \hat{h}_{n-1}^{\pm}(\zeta).
\end{aligned}$$

Hence,

$$B_{n-1}(\lambda) = B_1^-(\lambda) + B_{n-1}^-(\lambda) + B_{n-1}^{\pm}(\lambda),$$

where

$$\begin{aligned}
B_1^-(\lambda) &= \tilde{\phi}(x)\hat{h}^-(q)|\phi|^{1/2}, \\
B_{n-1}^-(\lambda) &= \tilde{\phi}(x)\hat{h}_{n-1}^-(q)|\phi|^{1/2}, \\
B_{n-1}^{\pm}(\lambda) &= \tilde{\phi}(x)\hat{h}_{n-1}^{\pm}(q)|\phi|^{1/2}.
\end{aligned}$$

The operators $B_1^-(\lambda)$ and $B_{n-1}^-(\lambda)$ induced a smooth kernel in $L^2(\mathbb{R}_+^2)$, hence they are trace-class operators. On the other hand, each integral operator associated to the operator $B_{n-1}^{\pm}(\lambda)$ is a product of two Hilbert-Schmidt operators, so $B_{n-1}^{\pm}(\lambda)$ is of trace-class. It follows that $B_{n-1}(\lambda)$ is of trace-class.

□

Remark. The operator $B_{n-1}(\lambda)$ having a jump discontinuity along the diagonal is, according to our analysis, a trace-class operator hence Tr $B_{n-1}(\lambda) < \infty$. However with the discontinuity, one does not have an explicit expression of its trace in terms of its kernel unlike the case of a continuous kernel induced by a trace-class operator. Nevertheless from the decomposition of $B_{n-1}(\lambda)$ into a sum of trace-class operators for which the kernel of each operator is continuous on its domain respectively, we thus have a formula for computing its trace. We call this formula the *regularised integral trace* in accordance to Remark 4.2 found in [10]. Furthermore the operator $B_{n-1}(\lambda)$ of trace-class satisfies $\int|\tilde{g}_{n-1}(x,x,\lambda)|\mathrm{d}x < \infty$, where $\tilde{g}_{n-1}$ is given in (42). Therefore $B_{n-1}(\lambda)$ can be considered as an example to the quotation found in [30, p. 25].

From Corollary 1 and Theorem 4.3, we have the following result.

Proposition 4.4. *If $\mathcal{B}(\lambda)$ is of trace class then $B_m(\lambda)$ is also of trace class, when $m \in \{1,\dots,n-1\}$.*

In the travelling wave problem, it may happen that a transition to instability occurs see. [4, 14] due to an eigenvalue emerging from the essential spectrum. So, using the Fredholm determinant one can locate these values that become eigenvalues. This is accomplished by extending the subdomain Ω to some region in the essential spectrum wherein the determinant of the matrix given in (10) is non-zero.
Let Γ denote the extended subregion given by

$$\Gamma = \{\lambda \in \mathbb{C}\colon \prod_{1\le i<j\le n} (\kappa_i - \kappa_j) \neq 0 \text{ with } \mathrm{Re}(\kappa_j) \in \mathbb{R}\}. \tag{44}$$

Note that since $L_0 - \lambda I_{L^2}$ is a constant differential operator, then the product given in (44) is equal up to a non-zero constant to the determinant of $L_0 - \lambda I_{L^2}$, that is

$$\det(L_0 - \lambda I_{L^2}) = e^{-a_{n-1}x} \prod_{1\leq i<j\leq n} (\kappa_i - \kappa_j). \tag{45}$$

Remark. Observe that $\Omega \subset \Gamma$, and for $\lambda \in \Gamma$ we either have $\lambda \in \Gamma \setminus \Omega$ or $\lambda \in \Omega$. Since, the latter case is known from previous analysis, we focus on the first case. In this situation, as $\det(L_0 - \lambda I_{L^2}) \neq 0$ it implies that there exists n distinct complex roots κ of $\mathcal{P}_n$ such that the resolvent operator $(L_0 - \lambda I_{L^2})^{-1}$ is not an operator in $L^2(\mathbb{R})$. However, due to the exponential decreasing nature of the function v_0, the function $\tilde{\phi}(x)(L_0 - \lambda I_{L^2})^{-1}(x, y)|\phi(y)|^{1/2}$ is in $L^2(\mathbb{R}^2)$, so it is a Hilbert-Schmidt operator. Depending on the distribution of the roots in the complex plane, the function $\tilde{g}(x, \xi, \lambda)$ given in (35) might change completely, or change by the distribution function *sgn* if there exists at most one root $\kappa_{i_0} \equiv 0$ for some $i_0 \in \{1, \dots, n\}$, otherwise it remains unchanged.

We thus have the following result.

Proposition 4.5. *The Birman-Schwinger operator $\mathcal{B}(\lambda)$ is Hilbert-Schmidt and analytic for $\lambda \in \Gamma$.*

Note that $0 \notin \Gamma$, and for $\lambda \in \Gamma \setminus \Omega$ one can prove that $\mathcal{B}(\lambda)$ is of trace class and then compute the Fredholm determinant whose zeros are precisely these values that become eigenvalues.

4.3 The Birman-Schwinger operator for the first order system

In this section, we focus on the matrix formulation of the travelling wave problem given in (11), and we write $Y(x)$ instead of $Y(x, \lambda)$. Assume that all the conditions regarding the scalar problem (4) are satisfied, with Ω given in (22) then we write

$$Y(x) = -|R(x)|^{1/2} \left(K_0(\lambda)\widetilde{R}Y\right)(x) \tag{46}$$

where $\widetilde{R}(x) = U|R(x)|^{1/2}$ with U a unitary transformation so that $R(x) = \widetilde{R}(x)|R(x)|^{1/2}$. Let $\mathcal{K}(\lambda)$ denote the Birman-Schwinger matrix operator defined by

$$\mathcal{K}(\lambda) = |R(x)|^{1/2}K_0(\lambda)\widetilde{R}$$

with integral kernel given by

$$k(x, \xi, \lambda) = \begin{cases} -|R(x)|^{1/2}Y_0^-(x)Z_0^+(\xi)\widetilde{R}(\xi), & x \leq \xi, \quad x, \xi \in \mathbb{R}; \\ |R(x)|^{1/2}Y_0^+(x)Z_0^-(\xi)\widetilde{R}(\xi), & \xi < x. \end{cases} \tag{47}$$

For $\lambda \in \Omega$, $\mathcal{K}(\lambda)$ is a Hilbert-Schmidt operator.
Let us suppose that we can write equation (46) as follows

$$\begin{pmatrix} u(x,\lambda) \\ u'(x,\lambda) \\ \vdots \\ u^{(n-1)}(x,\lambda) \end{pmatrix} = - \begin{pmatrix} \mathcal{B}(\lambda) & 0 & \cdots & 0 \\ 0 & \mathcal{D}_1(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{D}_{n-1}(\lambda) \end{pmatrix} \begin{pmatrix} u(x,\lambda) \\ u'(x,\lambda) \\ \vdots \\ u^{(n-1)}(x,\lambda) \end{pmatrix} \tag{48}$$

with $\mathcal{D}_0(\lambda) := \mathcal{B}(\lambda)$, and for $i = 1, \dots, n-1$, $\mathcal{D}_i(\lambda)$ are integral operators to determine.
Then one can define for $p \geq 1$ a set M_p of $n \times n$ compact operator as follows

$$M_p = \{A \in \mathcal{C}(L^2(\mathbb{C}^{n\times n})),\ A_{ii} = D_i \text{ if } i = j \text{ and } A_{ij} = 0 \text{ if } i \neq j,\ \text{for } i, j = 0, \dots, n-1 \text{ and } D_i \in \mathcal{J}_p(\mathcal{H})\}.$$

It follows that when $p = 1$, one might define a norm and a trace of an operator $K \in M_1$ by

$$\|K\|_1 = \sum_{i=0}^{n-1} \|D_i\|_1$$

and

$$\text{Tr } K = \sum_{i=0}^{n-1} \text{Tr } D_i,$$

respectively, where $D_i \in \mathcal{J}_1(\mathcal{H})$. Note that $K \in \mathcal{J}_p(\mathcal{H})$ if and only if $D_i \in \mathcal{J}_p$, for $i = 0, \dots, n-1$ and $p \geq 1$. In general, the form given in (48) is not available but for the travelling wave problem, using the correct transformation, one can always put the Birman-Schwinger operator associated to the transformed problem in the form of (48).

Theorem 4.6. *For $\lambda \in \Omega$, the Birman-Schwinger matrix operator $\mathcal{K}(\lambda)$ is of trace-class.*

Proof. Note that when $m \in \{0, \dots, n-1\}$, the decomposition in (48) is such that $\mathcal{B}(\lambda) = B_m(\lambda)$, which is a trace class operator. Therefore, we only need to determine $\mathcal{D}_j(\lambda)$ and show that they are of trace class for all $j = 1, \dots, n-1$. However, for a fixed $m \in \{0, \dots, n-1\}$, $\mathcal{D}_j(\lambda) \equiv 0$ for all $j = 1, \dots, n-1$. This is due essentially to the structure of the matrix valued-function $R(x)$ given in (13). When $m = 0$ it is straightforward to observe that the operator $\mathcal{K}(\lambda)$ can be written in the form of (48) where $\mathcal{D}_j(\lambda) = 0$ for all $j = 1, \dots, n-1$. On the other hand, when $m \in \{1, \dots, n-1\}$ one can use a transformation essentially integration by parts in order to reduce the expression of $R(x)$ to the corresponding one when $m = 0$. Hence $\mathcal{K}(\lambda)$ is a trace class operator with norm and trace given by

$$\|\mathcal{K}(\lambda)\|_{\mathcal{J}_1} = \|\mathcal{B}(\lambda)\|_{\mathcal{J}_1}$$

and

$$\text{Tr } \mathcal{K}(\lambda) = \text{Tr } \mathcal{B}(\lambda), \tag{49}$$

respectively. $\square$

Since $\mathcal{K}(\lambda)$ is of trace-class, we have

$$\tilde{d}(\lambda) = \det(I_{L^2(\mathbb{C}^{n\times n})} + \mathcal{K}(\lambda)), \tag{50}$$

and since $\mathcal{D}_j(\lambda) = 0$ for $j = 1, \dots, n-1$, it follows that

$$\tilde{d}(\lambda) = \det(I_{L^2} + \mathcal{B}(\lambda)).$$

Proposition 4.7. *For $\lambda \in \Omega$, we have*

$$\text{Tr } \mathcal{K}(\lambda) = \text{Tr } \mathcal{K}^-(\lambda) = \text{Tr } \mathcal{K}^+(\lambda) = \int_{\mathbb{R}} \text{Tr}\,(k(x,x,\lambda))\, \mathrm{d}x,$$

where

$$\mathcal{K}^-(\lambda)Y(x) = -\int_x^{+\infty} |R(x)|^{1/2} Y_0^-(x) Z_0^+(\xi) \widetilde{R}(\xi) Y(\xi) \mathrm{d}\xi,$$

and

$$\mathcal{K}^+(\lambda)Y(x) = \int_{-\infty}^{x} |R(x)|^{1/2} Y_0^+(x) Z_0^-(\xi) \widetilde{R}(\xi) Y(\xi) \mathrm{d}\xi.$$

Proof. From equation (49), and using (18), we have

$$\begin{aligned}
\mathrm{Tr}\ \mathcal{B}(\lambda) =& \int_{\mathbb{R}} \mathrm{Tr}\left(k(x,x,\lambda)\right) \mathrm{d}x \\
=& \int_0^{+\infty} \mathrm{Tr}\left(|R(x)|^{1/2} Y_0^+(x) Z_0^-(x) \widetilde{R}(x)\right) \mathrm{d}x - \int_{-\infty}^0 \mathrm{Tr}\left(|R(x)|^{1/2} Y_0^-(x) Z_0^+(x) \widetilde{R}(x)\right) \mathrm{d}x \\
=& \int_0^{+\infty} \mathrm{Tr}\left(|R(x)|^{1/2} (I_n - Y_0^-(x) Z_0^+(x)) \widetilde{R}(x)\right) \mathrm{d}x - \int_{-\infty}^0 \mathrm{Tr}\left(|R(x)|^{1/2} Y_0^-(x) Z_0^+(x) \widetilde{R}(x)\right) \mathrm{d}x \\
=& \int_{\mathbb{R}} \mathrm{Tr}\left(\pm Y_0^{\pm}(x) Z_0^{\mp}(x) R(x)\right) \mathrm{d}x \\
=& \mathrm{Tr}\ \mathcal{K}^{\pm}(\lambda).
\end{aligned}$$

□

For the travelling wave problem given by (4), the Birman-Schwinger matrix operators is of trace-class if and only if its corresponding scalar problem is also of trace-class. Therefore, computing the Fredholm determinant whether for the scalar problem or the matrix problem yield the same.

4.4 The reduction

In this section, we prove that (50) can be reduced to a finite dimensional determinant under the assumption of all the previous hypotheses.

Hypothesis 4.2. *Assume Hypothesis 4.1, then we have*

$$\|R\|_{\mathbb{C}^{n\times n}} \in L^1(\mathbb{R}, e^{\beta|x|}\mathrm{d}x),$$

for some constant $\beta > 0$.

Let $\wedge^k(\mathbb{C}^n)$ denote the kth exterior power of the vector space $\mathbb{C}^n$ where $\wedge$ is the usual wedge product. Giving $U = U_1 \wedge \ldots \wedge U_k \in \wedge^k(\mathbb{C}^n)$ and $V = V_1 \wedge \ldots \wedge V_k \in \wedge^k(\mathbb{C}^n)$, then the inner product of U and V on $\wedge^k(\mathbb{C}^n)$ is defined by

$$[\![U, V]\!]_k = \det_{\mathbb{C}^{k\times k}}\left(\langle U_i, V_j\rangle_{\mathbb{C}^n}\right)\ i, j = 1, \ldots, k.$$

We introduce the Hodge star operator which is an isomorphism between $\wedge^{n-k}(\mathbb{C}^n)$ and $\wedge^k(\mathbb{C}^n)$.
By $\star$ we denote the Hodge star operator, so for any $U \in \wedge^k(\mathbb{C}^n)$ and $V \in \wedge^{n-k}(\mathbb{C}^n)$ we have

$$[\![U, \star V]\!]_k \mathcal{V} = U \wedge V,$$

where $\mathcal{V} = e_1 \wedge \ldots \wedge e_n \in \wedge^n(\mathbb{C}^n)$ is a volume form, and $(e_i)_{i=1\ldots n}$ a unitary basis for $\mathbb{C}^n$.

Definition 4.8. Let $U^-(x,\lambda) = Y_1^-(x,\lambda) \wedge \cdots \wedge Y_k^-(x,\lambda) \in \wedge^k(\mathbb{C}^n)$ and $S_0^+(x,\lambda) = \left(Z_0^+(x,\lambda)\right)_1^* \wedge \cdots \wedge \left(Z_0^+(x,\lambda)\right)_k^* \in \wedge^k(\mathbb{C}^n)$. We define the *matrix transmission coefficient* $D(\lambda)$ by

$$D(\lambda) = \lim_{x\to+\infty} \langle Y_i^-(x,\lambda), \left(Z_0^+(x,\lambda)\right)_j^*\rangle_{\mathbb{C}^n},\ \text{for } i, j = 1, \ldots, k,$$

where from Hypothesis 4.2 we have

$$Y_l^-(x,\lambda) = \left(Y_0^-(x,\lambda)\right)_l + \int_{-\infty}^x H(x,\xi,\lambda) R(\xi) Y_l^-(\xi,\lambda)\mathrm{d}\xi,\ \text{where } l = 1, \ldots, k$$

with $H(x,\xi,\lambda) = Y_0^-(x,\lambda) Z_0^+(\xi,\lambda) + Y_0^+(x,\lambda) Z_0^-(\xi,\lambda)$.

Theorem 4.9. *For $\lambda \in \Omega$, we have*

$$\det_{\mathbb{C}^{k\times k}} D(\lambda) = \lim_{x\to+\infty} [\![U^-(x,\lambda), S_0^+(x,\lambda)]\!]_k = d(\lambda).$$

Proof. Using equation (19) of Section 3, we have

$$\langle Y_i^-(x,\lambda), \left(Z_0^+(x,\lambda)\right)_j^* \rangle_{\mathbb{C}^n} = \left(\delta_{ij} + \int_{-\infty}^{x} \left(Z_0^+(\xi,\lambda)\right)_j R(\xi) Y_i^-(\xi,\lambda)\mathrm{d}\xi\right)_{i,j=1,\ldots,k}. \tag{51}$$

Observe that (51) is independent of the variable x, so we have

$$\begin{aligned}\det_{\mathbb{C}^{k\times k}} D(\lambda) &= \lim_{x\to+\infty}\left(\det_{\mathbb{C}^{k\times k}} \langle Y_i^-(x,\lambda), \left(Z_0^+(x,\lambda)\right)_j^* \rangle_{\mathbb{C}^n}\right)\\ &= \lim_{x\to+\infty} [\![U^-(x,\lambda), S_0^+(x,\lambda)]\!]_k.\end{aligned}$$

Taking the limit when $x \to +\infty$ in (51) we have

$$\lim_{x\to+\infty} [\![U^-(x,\lambda), S_0^+(x,\lambda)]\!]_k = \det_{\mathbb{C}^{k\times k}}\left(\delta_{ij} + \int_{\mathbb{R}} \left(Z_0^+(\xi,\lambda)\right)_j R(\xi) Y_i^-(\xi,\lambda)\mathrm{d}\xi\right) \text{ where } i,j = 1,\cdots,k.$$

The discrete Fredholm determinant expansion is given by [24]

$$\det_{\mathbb{C}^{N\times N}}(\delta_{jk} + b_{jk})_{j,k=1}^N = 1 + \sum_{m=1}^{N} \frac{1}{m!} \sum_{i_1,\ldots,i_m=1}^{N} \det\begin{pmatrix} b_{i_1 i_1} & \cdots & b_{i_1 i_m}\\ \vdots & & \vdots\\ b_{i_m i_1} & \cdots & b_{i_m i_m}\end{pmatrix}$$

where $b_{jk} \in \mathbb{C}$. Thus, by setting

$$b_{ij} = \int_{\mathbb{R}} \left(Z_0^+(x,\lambda)\right)_j R(x) Y_i^-(x,\lambda)\mathrm{d}x,$$

provided that $\left(Z_0^+(x,\lambda)\right)_j R(x) Y_i^-(x,\lambda)$ is continuous, we have

$$\det_{\mathbb{C}^{k\times k}} D(\lambda) = 1 + \sum_{m=1}^{k} \int_{\mathbb{R}} \left(Z_0^+(x,\lambda)\right)_m R(x) Y_m^-(x,\lambda)\mathrm{d}x + \sum_{m=2}^{k} \frac{1}{m!} \sum_{i_1,\ldots,i_m=1}^{k} \det(b_{i_p i_q})_{p,q=1}^k. \tag{52}$$

Substituting the Neumann series for $Y_l^-(x,\lambda)$ in (52) and using the invariance of trace for a cyclic rotation of three matrices, in particular $\mathrm{Tr}(Z_0^+ R Y^-) = \mathrm{Tr}(Y^- Z_0^+ R)$ [27, p.110], yields

$$\begin{aligned}\det_{\mathbb{C}^{k\times k}} D(\lambda) &= 1 + \int_{\mathbb{R}} \mathrm{Tr}\left(Y_0^-(x,\lambda) Z_0^+(x,\lambda) R(x)\right)\mathrm{d}x + \sum_{m=2}^{\infty} \frac{1}{m!} C_m(K(\lambda))\\ &= 1 + \mathrm{Tr}\,\mathcal{B}(\lambda) + \sum_{m=2}^{\infty} \frac{1}{m!} C_m(K(\lambda)),\end{aligned}$$

where

$$C_m(K(\lambda)) = \sum_{i_1,i_2,\ldots,i_m=1}^{k} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \det\left(k_{i_p i_q}(x_p,x_q,\lambda)\right)_{p,q=1}^m \mathrm{d}x_1\cdots\mathrm{d}x_m$$

with $k(x_p,x_q,\lambda)$ given in (47).
Hence we have $\det_{\mathbb{C}^{k\times k}} D(\lambda) = \det(I + K(\lambda))$. □

From Theorem 4.9, we can establish the connection between the Fredholm determinant and the Evans function. Let the Evans function $E(\lambda)$ be defined by

$$E(\lambda) = \det_{\mathbb{C}^{n\times n}}\left(Y^-(x,\lambda) \quad Y^+(x,\lambda)\right), \tag{53}$$

where Y^- and Y^+ are the set of solutions for the first order system (11) that decay at $-\infty$ and $+\infty$, respectively.

Theorem 4.10. *For $\lambda \in \Omega$, we have*

$$d(\lambda) = \frac{E(\lambda)}{c(\lambda)}, \tag{54}$$

where $c(\lambda) = \det_{\mathbb{C}^{n\times n}} \left(\begin{pmatrix} Y_0^-(x,\lambda) & Y_0^+(x,\lambda) \end{pmatrix} \right)$.

Proof. Consider the Jost solution of the adjoint problem

$$Z_l^+(x,\lambda) = \left(Z_0^+(x,\lambda)\right)_l + \int_x^{+\infty} Z_l^+(\xi,\lambda)R(\xi)H(\xi,x,\lambda)\mathrm{d}\xi, \quad l = 1,\ldots,k.$$

Observe that the Evans function

$$\begin{aligned} E(\lambda) &= \det_{\mathbb{C}^{n\times n}} \left(Y_1^-(x,\lambda)\cdots Y_k^-(x,\lambda) \quad Y_{k+1}^+(x,\lambda)\cdots Y_n^+(x,\lambda) \right) \\ &= [\![U^-(x,\lambda), \star S^+(x,\lambda)]\!]_k \\ &= [\![U^-(x,\lambda), W^+(x,\lambda)]\!]_k \\ &= \det_{\mathbb{C}^{k\times k}} \left(\langle Y_i^-(x,\lambda), W_j^+(x,\lambda)\rangle_{\mathbb{C}^n} \right), \text{ for } i,j = 1,\ldots,k \end{aligned} \tag{55}$$

where $S^+ = Y_{k+1}^+(x,\lambda) \wedge \cdots \wedge Y_n^+(x,\lambda)$, and $W^+(x,\lambda) \in \wedge^k(\mathbb{C}^n)$ satisfying [3, Proposition 2]

$$\frac{\mathrm{d}}{\mathrm{d}x} W^+(x,\lambda) = [\overline{\mathrm{Tr} A(x,\lambda)} I_k - (A^{(k)}(x,\lambda))^*] W^+(x,\lambda),$$

with $A^{(k)}(x,\lambda)$ is derivation associated with $A(x,\lambda)$ defined by the sum of (12) and (13).
It is shown in [3] that the two sets

$$\{W_1^+(x,\lambda),\ldots,W_k^+(x,\lambda)\}, \quad \{Z_1^+(x,\lambda),\ldots,Z_k^+(x,\lambda)\}$$

span the same space. Therefore

$$W^+(x,\lambda) = (Z^+(x,\lambda))^* C(\lambda),$$

where $C(\lambda)$ is a $k \times k$ invertible matrix depending analytically on $\lambda \in \Omega$.
We now compute the scalar product given in (55) :

$$\begin{aligned} \langle Y_i^-(x,\lambda), \left((Z^+(x,\lambda))^* C(\lambda)\right)_j \rangle_{\mathbb{C}^n} &= \langle (Y_0^-(x,\lambda))_i, \left((Z_0^+(x,\lambda))^* C(\lambda)\right)_j \rangle_{\mathbb{C}^n} \\ &+ \langle (Y_0^-(x,\lambda))_i, \int_x^{+\infty} H^*(x,\xi,\lambda)R^*(\xi)\left((Z^+(\xi,\lambda))^* C(\lambda)\right)_j \mathrm{d}\xi \rangle_{\mathbb{C}^n} \\ &+ \langle \int_{-\infty}^x H(x,\xi,\lambda)R(\xi)Y_i^-(\xi,\lambda)\mathrm{d}\xi, \left((Z_0^+(x,\lambda))^* C(\lambda)\right)_j \rangle_{\mathbb{C}^n} \\ + \langle \int_{-\infty}^x H(x,\xi,\lambda)R(\xi)Y_i^-(\xi,\lambda)\mathrm{d}\xi, &\int_x^{+\infty} H^*(x,\xi,\lambda)R^*(\xi)\left((Z^+(\xi,\lambda))^* C(\lambda)\right)_j \mathrm{d}\xi \rangle_{\mathbb{C}^n}. \end{aligned}$$

Observe that each term in the scalar product is independent of the variable x. Using the definition of the function $H(x,\xi,\lambda)$ given in Definition 4.8 and taking the limit $x \to +\infty$ in the scalar product, we get

$$\begin{aligned} \det_{\mathbb{C}^{k\times k}} \left(\langle Y_i^-(x,\lambda), W_j^+(x,\lambda)\rangle_{\mathbb{C}^n} \right) &= \det_{\mathbb{C}^{k\times k}} \left(\langle (Y_0^-(x,\lambda))_i, \left((Z_0^+(x,\lambda))^* C(\lambda)\right)_j \rangle_{\mathbb{C}^n} \right) \\ &\times \det_{\mathbb{C}^{k\times k}} \left(\delta_{ij} + \int_{-\infty}^{\infty} Z_0^+(\xi,\lambda)_j R(\xi) Y_i^-(\xi,\lambda)\mathrm{d}x \right), \end{aligned}$$

where the second and fourth terms on the right of the scalar product vanish as $x \to +\infty$. It follows from the isomorphism of the Hodge star operator that

$$E(\lambda) = \det_{\mathbb{C}^{n\times n}} \left(\begin{pmatrix} Y_0^-(x,\lambda) & Y_0^+(x,\lambda) \end{pmatrix} \right) d(\lambda). \tag{56}$$

□

Since $L_0 - \lambda I_{L^2}$ is a constant differential operator then equation (45) is equivalent to

$$\det(L_0 - \lambda I_{L^2}) = \det_{\mathbb{C}^{n\times n}} \left(\begin{pmatrix} Y_0^-(x,\lambda) & Y_0^+(x,\lambda) \end{pmatrix} \right).$$

By comparison of (6) and (56), we deduce that

$$E(\lambda) = \det(L_0 + v(x) - \lambda I_{L^2}).$$

5 Fredholm determinant for fronts

5.1 The scalar case

This section deals with the construction of an appropriate Fredholm determinant satisfying (56). Since under the hypotheses given previously, the result connecting the Evans function and the Fredholm determinant remains valid for fronts that is, $\lim_{x\to\pm\infty} \phi(x) = \phi^\pm$, where $\phi^\pm \in \mathbb{R}$ with $\phi^+ \neq \phi^-$.

Hypothesis 5.1. *For some fixed $m \in \{0,\dots,n-1\}$, consider equation (31) with a_i constants for all $i = 0,\dots,n-1$. Assume that the characteristic polynomial associated to the constant coefficient differential operator $L_0 + \phi^\pm \dfrac{\mathrm{d}^m}{\mathrm{d}x^m} - \lambda I_{L^2}$,*

$$\mathcal{P}_n^\pm(\mu) = \mu^n + a_{n-1}\mu^{n-1} + \ldots + a_1\mu + \phi^\pm \mu^m + a_0 - \lambda = 0, \quad \forall \lambda \in \Omega,$$

has k roots $\kappa_i^\pm = \mu_i^\pm(\lambda)$, $i = 1,\dots,k$ with $\mathrm{Re}(\kappa_i^\pm) > 0$ and $n-k$ roots $\tau_i^\pm = \mu_i^\pm(\lambda)$, $i = k+1,\dots,n$ with $\mathrm{Re}(\tau_i^\pm) < 0$ for $\Omega \subseteq \mathbb{C}$, where Ω is the region of interest.

The strategy for constructing the Fredholm determinant associated to the eigenvalue problem (31), where the travelling wave $\phi(x)$ admit different limits at $\pm\infty$, is to project the variable coefficient differential operator $(L_0 + v(x) - \lambda I_{L^2})$ onto the subspaces decaying at $\pm\infty$, associated to the constant coefficient differential operator $L_0 + \phi^\pm \dfrac{\mathrm{d}^m}{\mathrm{d}x^m} - \lambda I_{L^2}$, and construct a new constant coefficient differential operator $(\mathcal{L}_0 - \lambda I_{L^2})$. This yields the following variable coefficient differential operator

$$\mathcal{L}_0 + q(x) - \lambda I_{L^2},$$

where $\mathcal{L}_0$ is an nth order constant coefficient differential operator, and $q(x)$ satisfying

$$\lim_{|x|\to\infty} q(x) = 0, \tag{57}$$

is the projection of $v^\pm(x)$ onto the subspaces decaying at $\pm\infty$, as above. The functions $v^\pm(x)$ are given by

$$v^\pm(x) = v(x) - V(\phi^\pm),$$

where V defined in Section 2, is the Jacobian associated to the nonlinear part of the PDE.
The Green's function of the constant coefficient differential operator $(\mathcal{L}_0 - \lambda I_{L^2})$ is of the form of equation (9) of the Definition 2.3, and it is given by

$$g(x,\xi,\lambda) = \begin{cases} \displaystyle\sum_{i=1}^{k} \alpha_i(\lambda) e^{\kappa_i^-(x-\xi)}, & x \le \xi, \quad x,\xi \in \mathbb{R}; \\ \displaystyle\sum_{i=k+1}^{n} \alpha_i(\lambda) e^{\tau_i^+(x-\xi)}, & \xi < x, \end{cases}$$

where for $i = 1,\dots,n$, $\alpha_i(\lambda)$ satisfy

$$\begin{pmatrix} 1 & \dots & 1 & -1 & \dots & -1 \\ \kappa_1^- & \dots & \kappa_k^- & -\tau_{k+1}^+ & \dots & -\tau_n^+ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (\kappa_1^-)^{n-1} & \dots & (\kappa_k^-)^{n-1} & -(\tau_{k+1}^+)^{n-1} & \dots & -(\tau_n^+)^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1(\lambda) \\ \alpha_2(\lambda) \\ \vdots \\ \alpha_n(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}.$$

Observe that the zeros of the characteristic polynomial associated to the constant-coefficient differential operator $(\mathcal{L}_0 - \lambda I_{L^2})$ are κ_i^- for $i = 1, \ldots, k$ and τ_j^+ for $j = k+1, \ldots, n$. Also, by projecting the eigenvalue problem (31) onto the appropriate subspaces, we ensure that when integrating the eigenvalue problem from the far field, the solutions stay on the subspaces decaying at $-\infty$ and $+\infty$, respectively. Note that in the construction of the Green's function corresponding to the operator $(\mathcal{L}_0 - \lambda I_{L^2})$, one does not need to have the explicit expression of the differential operator $\mathcal{L}_0$.
For example if $m = 0$, the Birman-Schwinger operator is given by

$$\mathcal{B}(\lambda) = \tilde{\theta}\,(\mathcal{L}_0 - \lambda I_{L^2})^{-1}\,|\theta|^{1/2},$$

where $\theta(x)$ is the projection of $(\phi(x) - \phi^{\pm})$ as above, and $\tilde{\theta}(x) = |\theta(x)|^{1/2} \mathrm{e}^{i \arg(\theta(x))}$

Remark. Note that the function $q(x)$ satisfying (57) has a jump discontinuity at $x = 0$. This is essentially due to the function $\theta(x)$.

5.2 First order system case

Under Hypothesis 5.1, we construct the Birman-Schwinger operator corresponding to the first order system in the same manner described in the previous section. We denote by $U(x, \lambda) = \begin{pmatrix} Y_0^-(x, \lambda) & Y_0^+(x, \lambda) \end{pmatrix}$ a $n \times n$ fundamental matrix satisfying

$$\frac{\mathrm{d}}{\mathrm{d}x} U(x, \lambda) = B(\lambda) U(x, \lambda), \tag{58}$$

where the $n \times n$ constant matrix $B(\lambda)$ must satisfy

$$P^{-1}(\lambda) B(\lambda) P(\lambda) = \mathrm{diag}\left(\kappa_1^-, \ldots, \kappa_k^-, \tau_{k+1}^+, \ldots, \tau_n^+\right) \tag{59}$$

with

$$P(\lambda) = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \kappa_1^- & \cdots & \kappa_k^- & \tau_{k+1}^+ & \cdots & \tau_n^+ \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ (\kappa_1^-)^{n-1} & \cdots & (\kappa_k^-)^{n-1} & (\tau_{k+1}^+)^{n-1} & \cdots & (\tau_n^+)^{n-1} \end{pmatrix}$$

where κ_i^- for $i = 1, \ldots, k$ and τ_j^+ for $j = k+1, \ldots, n$ are eigenvalues of $A^-(\lambda)$ and $A^+(\lambda)$ with positive real and negative real parts, respectively. Hence we have

$$B(\lambda) = P(\lambda) \Lambda P^{-1}(\lambda),$$

where Λ is the $n \times n$ diagonal matrix given in (59). Consequently, the Green's matrix of the operator $\mathrm{d}/\mathrm{d}x - B(\lambda)$ is given by

$$G(x, \xi, \lambda) = \begin{cases} -Y_0^-(x, \lambda) Z_0^+(\xi, \lambda), & x \le \xi, \quad x, \xi \in \mathbb{R}; \\ Y_0^+(x, \lambda) Z_0^-(\xi, \lambda), & \xi < x, \end{cases} \tag{60}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}x} Z_0^{\pm}(x, \lambda) = -Z_0^{\pm}(x, \lambda) B(\lambda), \quad Z_0^{\pm}(\pm\infty, \lambda) = 0.$$

The Green's matrix constructed in (60) satisfies the conditions given in (15). In the same manner, the Birman-Schwinger operator is given by

$$\mathcal{K}(\lambda) = |Q(x)|^{1/2} \left(\frac{\mathrm{d}}{\mathrm{d}x} - B(\lambda) \right)^{-1} \widetilde{Q},$$

where $Q(x)$ satisfying

$$\lim_{|x| \to \infty} Q(x) = 0_{n \times n},$$

is the projection of $R^{\pm}(x) = R(x) - R^{\pm}$ onto the subspaces that decay at $\pm\infty$ associated to the operator $\mathrm{d}/\mathrm{d}x - A^{\pm}(\lambda)$, and $\widetilde{Q}(x) = U |Q(x)|^{1/2}$ with U a unitary transformation so that $Q(x) = \widetilde{Q}(x) |Q(x)|^{1/2}$.

Remark. Observe that $Y_0^{\pm}$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}Y_0^{\pm}(x,\lambda) = A^{\pm}(\lambda)Y_0^{\pm}(x,\lambda), \quad Y_0^{\pm}(\pm\infty,\lambda) = 0,$$

where

$$A^{\pm}(\lambda) = A(\lambda) - R^{\pm}$$

with $\lim_{x\to\pm\infty} R(x) = R^{\pm}$, $A(\lambda)$ and $R(x)$ given in (12) and (13), respectively.

The subregion Ω in the case $\lim_{x\to\pm\infty} \phi(x) = \phi^{\pm}$ with $\phi^- \neq \phi^+$, is given by [13, Lemma 2, p.138]

$$\Omega = \mathbb{C} \setminus \left(\sigma_e(L_0^-) \cup \sigma_e(L_0^+)\right), \tag{61}$$

where $L_0^{\pm} = L_0 + V(\phi^{\pm}) - \lambda I_{L^2}$ with $V(\phi^{\pm}) = \phi^{\pm}\frac{\mathrm{d}^m}{\mathrm{d}x^m}$, in our case.

Remark. Since $\mathcal{L}_0 + q(x) - \lambda I_{L^2}$ is the projection of $L_0 + v(x) - \lambda I_{L^2}$ onto the subspaces decaying at $\pm\infty$, we have

$$\det\left(L_0 + v(x) - \lambda I_{L^2}\right) = \tilde{c}(\lambda)\det\left(\mathcal{L}_0 + q(x) - \lambda I_{L^2}\right),$$

where $\tilde{c}(\lambda)$ is a non-zero function. On the other hand, for $\lambda \in \Omega$ defined in (61), we have

$$\det\left(\mathcal{L}_0 + q(x) - \lambda I_{L^2}\right) = \det\left(\mathcal{L}_0 - \lambda I_{L^2}\right) d(\lambda).$$

If follows from the previous analysis that

$$\det\left(\mathcal{L}_0 + q(x) - \lambda I_{L^2}\right) = E(\lambda).$$

Therefore, we have

$$\det\left(L_0 + v(x) - \lambda I_{L^2}\right) = \tilde{c}(\lambda)E(\lambda)$$

In conclusion, for the one dimensional case if the Birman-Schwinger operator corresponding to the travelling wave problem equation (4) is of trace-class, then the Fredholm determinant and the Evans function satisfy equation (54). This means that the extension of the Evans function to some region of the essential spectrum can be achieved using the Fredholm determinant instead of modifying the Evans function for such purpose. This of course applies under the assumption that $\phi \in L^1(\mathbb{R}, e^{\beta|x|}\mathrm{d}x)$

6 Conclusion

We have investigated the connection between the Evans function and the Fredholm determinant for the travelling wave problem in one dimension. This connection turned out to be natural in the sense that the Birman-Schwinger operator associated with the travelling wave problem is of trace class.
Following the approach described in [2] for computing the Fredholm determinant using the Nyström method, which lead to an exponential convergence for smooth kernels, computing the Fredholm determinant may be more efficient than the Evans function, in some cases. Therefore our next objective is to compute numerically the Fredholm determinant and also investigate its behaviour near the eigenvalue $\lambda = 0$ embedded in the essential spectrum.

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