A Multi-level Correction Scheme for Eigenvalue Problems *

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Abstract

In this paper, a new type of multi-level correction scheme is proposed for solving eigenvalue problems by finite element method. With this new scheme, the accuracy of eigenpair approximations can be improved after each correction step which only needs to solve a source problem on finer finite element space and an eigenvalue problem on the coarsest finite element space. This correction scheme can improve the efficiency of solving eigenvalue problems by finite element method.

Keywords. Eigenvalue problem, multi-level correction scheme, finite element method, multi-space, multi-grid.

AMS subject classifications. 65N30, 65B99, 65N25, 65L15

1 Introduction

The purpose of this paper is to propose a new type of multi-level correction scheme based on finite element discretization to solve eigenvalue problems. The two-grid method for solving eigenvalue problems has been proposed and analyzed by Xu and Zhou in [21]. The idea of the two-grid comes from [19, 20] for nonsymmetric or indefinite problems and nonlinear elliptic equations. Since then, there have existed many numerical methods for solving eigenvalue problems based on the idea of two-grid method ([1, 6, 17]).

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In this paper, we present a new type of multi-level correction scheme for solving eigenvalue problems. With the new proposed method solving eigenvalue problem will not be much more difficult than the solution of the corresponding source problem. Our method is some type of operator iterative method ([11, 21, 23]). The correction method for eigenvalue problem in this paper is based on a series of finite element spaces with different approximation properties which are related to the multilevel method ([18]).

The standard Galerkin finite element method for eigenvalue problem has been extensively investigated, e.g. Babuška and Osborn [2, 3], Chatelin [5] and references cited therein. Here we adopt some basic results in these papers in our analysis. The finite element method for eigenvalue problem has been developed well and many high efficient methods have also been proposed and analyzed for different types of eigenvalue problems ([6, 7, 10, 13, 14, 16, 17, 21]).

The corresponding error estimates of the type of multi-level correction scheme which is introduced here will be analyzed. Based on the analysis, the new method can improve the convergence rate of the eigenpair approximations after each correction step. The multi-level correction procedure can be described as follows: (1) solve the eigenvalue problem in the coarsest finite element space; (2) solve an additional source problem in an augmented space using the previous obtained eigenvalue multiplying the corresponding eigenfunction as the load vector; (3) solve eigenvalue problem again on the finite element space which is constructed by combining the coarsest finite element space with the obtained eigenfunction approximation in step (2). Then go to step (2) for next loop.

Similarly to [21], in order to describe our method clearly, we give the following simple Laplace eigenvalue problem to illustrate the main idea in this paper with multi-grid implementation way (see section 5).

Find(λ, u) such that

$$\left\{\begin{array}{rcll} -\Delta u &=& \lambda u, & \text{in } \Omega,\\ u &=& 0, & \text{on } \partial\Omega,\\ \int_\Omega u^2 d\Omega &=& 1, & \end{array}\right. \tag{1.1}$$

where $\Omega \subset \mathcal{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and Δ denote the Laplace operator.

Let V_H denote the coarsest linear finite element defined on the coarsest mesh $\mathcal{T}_H$. Additionally, we also need to construct a series of finite element spaces V_{h_2}, V_{h_3}, $\cdots$, V_{h_n} which are defined on the corresponding series of meshes $\mathcal{T}_{h_k}$ $(k = 2, 3, \cdots n)$ such that $V_H \subset V_{h_2} \subset \cdots \subset V_{h_n}$ ([4, 8]). Our multi-level correction algorithm to obtain the approximation of the eigenpair can be defined as follows (see section 3 and section 4):

1. Solve an eigenvalue problem in the coarsest space V_H:

Find $(\lambda_H, u_H) \in \mathcal{R} \times V_H$ such that $\|u_H\|_0 = 1$ and

$$\int_\Omega \nabla u_H \nabla v_H d\Omega \;=\; \lambda_H \int_\Omega u_H v_H d\Omega, \quad \forall v_H \in V_H.$$

2. Set $h_1 = H$ and Do $k = 1, \cdots, n-2$

   - Solve the following auxiliary source problem:
     Find $\widetilde{u}_{h_{k+1}} \in V_{h_{k+1}}$ such that

     $$\int_\Omega \nabla \widetilde{u}_{h_{k+1}} \nabla v_{h_{k+1}} d\Omega \;=\; \lambda_{h_k} \int_\Omega u_{h_k} v_{h_{k+1}} d\Omega, \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}.$$

   - Define a new finite element space $V_{H,h_{k+1}} = V_H + \mathrm{span}\{\widetilde{u}_{h_{k+1}}\}$ and solve the following eigenvalue problem:
     Find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{H,h_{k+1}}$ such that $\|u_{h_{k+1}}\|_0 = 1$ and

     $$\int_\Omega \nabla u_{h_{k+1}} \nabla v_{H,h_{k+1}} d\Omega = \lambda_{h_{k+1}} \int_\Omega u_{h_{k+1}} v_{H,h_{k+1}} d\Omega, \quad \forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}.$$

   end Do

3. Solve the following auxiliary source problem:
   Find $\widetilde{u}_{h_n} \in V_{h_n}$ such that

   $$\int_\Omega \nabla u_{h_n} \nabla v_{h_n} d\Omega \;=\; \lambda_{h_{n-1}} \int_\Omega u_{h_{n-1}} v_{h_n} d\Omega, \quad \forall v_{h_n} \in V_{h_n}.$$

   Then compute the Rayleigh quotient

   $$\lambda_{h_n} = \frac{\|\nabla u_{h_n}\|_0^2}{\|u_{h_n}\|_0^2}.$$

If, for example, λ_H is the first eigenvalue of the problem at the first step and Ω is a convex domain, then we can establish the following results (see section 3 and section 4 for details)

$$\|\nabla(u - u_{h_n})\|_0 = \mathcal{O}\Big(\sum_{k=1}^{n} h_k H^{n-k}\Big), \quad \text{and} \quad |\lambda_{h_n} - \lambda| = \mathcal{O}\Big(\sum_{k=1}^{n} h_k^2 H^{2(n-k)}\Big).$$

These two estimates means that we can obtain asymptotic optimal errors by taking $H = \sqrt[n]{h_n}$ and $h_k = H^k$ $(k = 2, \cdots, n-1)$.

In this method, we replace solving eigenvalue problem on the finest finite element space by solving a series of boundary value problems in the corresponding series of finite element spaces and a series of eigenvalue problems in the coarsest finite element space. As we know, there exists multigrid method for solving boundary value problems efficiently. So this correction method can improve the efficiency of solving eigenvalue problems.

An outline of the paper goes as follows. In Section 2, we introduce finite element method for eigenvalue problem and the corresponding error estimates. A type of one correction step is given in section 3. In Section 4, we propose a type of multi-level correction algorithm for solving eigenvalue problem by finite element methods. In section 5, two numerical examples are presented to validate our theoretical analysis and some concluding remarks are given in the last section.

2 Discretization by finite element method

In this section, we introduce some notation and error estimates of the finite element approximation for eigenvalue problems. In this paper, the letter C (with or without subscripts) denotes a generic positive constant which may be different at different occurrences. For convenience, the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. That $x_1 \lesssim y_1, x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh sizes.

Let $(V, \|\cdot\|)$ be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, respectively. Let $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ be two symmetric bilinear forms on $X \times X$ satisfying

$$a(w,v) \lesssim \|w\|\|v\|, \quad \forall w \in V \text{ and } \forall v \in V, \tag{2.1}$$

$$\|w\|^2 \lesssim a(w,w), \quad \forall w \in V \text{ and } 0 < b(w,w), \quad \forall w \in V \text{ and } w \neq 0. \tag{2.2}$$

From (2.1) and (2.2), we know that $\|\cdot\|_a := a(\cdot,\cdot)^{1/2}$ and $\|\cdot\|$ are two equivalent norms on V. We assume that the norm $\|\cdot\|$ is relatively compact with respect to the norm $\|\cdot\|_b := b(\cdot,\cdot)^{1/2}$. We shall use $a(\cdot,\cdot)$ and $\|\cdot\|_a$ as the inner product and norm on V in the rest of this paper.

Set

$$W := \text{the completion of } V \text{ with respect to } \|\cdot\|_b.$$

Thus W is a Hilbert space with the inner product $b(\cdot,\cdot)$ and compactly imbedded in V. Construct a "negative space" by V' = the dual of V with a norm $\|\cdot\|_{-a}$ given by

$$\|w\|_{-a} = \sup_{v \in V, \|v\|_a = 1} b(w,v). \tag{2.3}$$

Then $W \subset V'$ compactly, and for $v \in V$, $b(w,v)$ has a continuous extension to $w \in V'$ such that $b(w,v)$ is continuous on V' by Hahn-Banach theorem ([9]). We assume that $V_h \subset V$ is a family of finite-dimensional spaces that satisfy the following assumption:

For any $w \in V$

$$\lim_{h\to 0}\inf_{v\in V_h}\|w-v\|_a = 0. \tag{2.4}$$

Let P_h be the finite element projection operator of V onto V_h defined by

$$a(w-P_hw, v) = 0, \quad \forall w \in V \text{ and } \forall v \in V_h. \tag{2.5}$$

Obviously

$$\|P_hw\|_a \leq \|w\|_a, \quad \forall w \in V. \tag{2.6}$$

For any $w \in V$, by (2.4) we have

$$\|w-P_hw\|_a \;=\; o(1), \quad \text{as } h \to 0. \tag{2.7}$$

Define $\eta_a(h)$ as

$$\eta_a(h) = \sup_{f\in V, \|f\|_a=1}\inf_{v\in V_h}\|Tf-v\|_a, \tag{2.8}$$

where the operator $T : V' \mapsto V$ is defined as

$$a(Tf, v) \;=\; b(f, v), \quad \forall f \in V' \text{ and } \forall v \in V. \tag{2.9}$$

In order to derive the error estimate of eigenpair approximation in negative norm $\|\cdot\|_{-a}$, we need the following negative norm error estimate of the finite element projection operator P_h.

Lemma 2.1. *([3, Lemma 3.3 and Lemma 3.4])*

$$\eta_a(h) \;=\; o(1), \quad \text{as } h \to 0, \tag{2.10}$$

and

$$\|w-P_hw\|_{-a} \;\lesssim\; \eta_a(h)\|w-P_hw\|_a, \quad \forall w \in V. \tag{2.11}$$

In our methodology description, we are concerned with the following general eigenvalue problem:

Find $(\lambda, u) \in \mathcal{R} \times V$ such that $b(u, u) = 1$ and

$$a(u, v) \;=\; \lambda b(u, v), \quad \forall v \in V. \tag{2.12}$$

For the eigenvalue λ, there exists the following Rayleigh quotient expression ([2, 3, 21])

$$\lambda = \frac{a(u, u)}{b(u, u)}. \tag{2.13}$$

From [3, 5], we know the eigenvalue problem (2.12) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \cdots, u_k, \cdots,$$

where $b(u_i, u_j) = \delta_{ij}$. In the sequence $\{\lambda_j\}$, the λ_j are repeated according to their geometric multiplicity.

Now, let us define the finite element approximations of the problem (2.12). First we generate a shape-regular decomposition of the computing domain $\Omega \subset \mathcal{R}^d$ ($d = 2, 3$) into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K. The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh $\mathcal{T}_h$, we can construct a finite element space denoted by $V_h \subset V$. In order to do multi-level correction method, we start the process on the original mesh $\mathcal{T}_H$ with the mesh size H and the original coarsest finite element space V_H defined on the mesh $\mathcal{T}_H$.

Then we can define the approximation of eigenpair (λ, u) of (2.12) by the finite element method as:

Find $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ such that $b(u_h, u_h) = 1$ and

$$a(u_h, v_h) = \lambda_h b(u_h, v_h), \qquad \forall v_h \in V_h. \tag{2.14}$$

From (2.14), we can know the following Rayleigh quotient expression for λ_h holds ([2, 3, 21])

$$\lambda_h = \frac{a(u_h, u_h)}{b(u_h, u_h)}. \tag{2.15}$$

Similarly, we know from [3, 5] the eigenvalue problem (2.12) has eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{k,h} \leq \cdots \leq \lambda_{N_h,h},$$

and the corresponding eigenfunctions

$$u_{1,h}, u_{2,h}, \cdots, u_{k,h}, \cdots, u_{N_h,h},$$

where $b(u_{i,h}, u_{j,h}) = \delta_{ij}, 1 \leq i, j \leq N_h$ (N_h is the dimension of the finite element space V_h).

From the minimum-maximum principle ([2, 3]), the following upper bound result holds

$$\lambda_i \leq \lambda_{i,h}, \quad i = 1, 2, \cdots, N_h.$$

Let $M(\lambda_i)$ denote the eigenspace corresponding to the eigenvalue λ_i which is defined by

$$M(\lambda_i) = \big\{w \in V : w \text{ is an eigenvalue of (2.12) corresponding to } \lambda_i \text{ and } \|w\|_b = 1\big\}. \quad (2.16)$$

Then we define

$$\delta_h(\lambda_i) = \sup_{w\in M(\lambda_i)} \inf_{v\in V_h} \|w - v\|_a. \quad (2.17)$$

For the eigenpair approximations by finite element method, there exist the following error estimates.

Proposition 2.1. *([2, Lemma 3.7, (3.29b)],[3, P. 699] and [5])*
(i) For any eigenfunction approximation $u_{i,h}$ of (2.14) ($i = 1, 2, \cdots, N_h$), there is an eigenfunction u_i of (2.12) corresponding to λ_i such that $\|u_i\|_b = 1$ and

$$\|u_i - u_{i,h}\|_a \leq C_i\delta_h(\lambda_i). \quad (2.18)$$

Furthermore,

$$\|u_i - u_{i,h}\|_{-a} \leq C_i\eta_a(h)\|u_i - u_{i,h}\|_a. \quad (2.19)$$

(ii) For each eigenvalue, we have

$$\lambda_i \leq \lambda_{i,h} \leq \lambda_i + C_i\delta_h^2(\lambda_i) \quad (2.20)$$

Here and hereafter C_i is some constant depending on i but independent of the mesh size h.

3 One correction step

In this section, we present a type of correction step to improve the accuracy of the current eigenvalue and eigenfunction approximations. This correction method contains solving an auxiliary source problem in the finer finite element space and an eigenvalue problem on the coarsest finite element space. For simplicity of notation, we set $(\lambda, u) = (\lambda_i, u_i)$ ($i = 1, 2, \cdots, k, \cdots$) and $(\lambda_h, u_h) = (\lambda_{i,h}, u_{i,h})$ ($i = 1, 2, \cdots, N_h$) to denote an eigenpair of problem (2.12) and (2.14), respectively.

To analyze our method, we introduce the error expansion of eigenvalue by the Rayleigh quotient formula which comes from [2, 3, 16, 21].

Theorem 3.1. *Assume (λ, u) is the true solution of the eigenvalue problem (2.12), $0 \neq \psi \in V$. Let us define*

$$\widehat{\lambda} = \frac{a(\psi, \psi)}{b(\psi, \psi)}. \quad (3.1)$$

Then we have

$$\widehat{\lambda}-\lambda \;=\; \frac{a(u-\psi,u-\psi)}{b(\psi,\psi)}-\lambda\frac{b(u-\psi,u-\psi)}{b(\psi,\psi)}. \tag{3.2}$$

Proof. First from (2.13), (3.1) and direct computation, we have

$$\begin{aligned}
\widehat{\lambda}-\lambda &= \frac{a(\psi,\psi)-\lambda b(\psi,\psi)}{b(\psi,\psi)}\\
&= \frac{a(\psi-u,\psi-u)+2a(\psi,u)-a(u,u)-\lambda b(\psi,\psi)}{b(\psi,\psi)}\\
&= \frac{a(\psi-u,\psi-u)+2\lambda b(\psi,u)-\lambda b(u,u)-\lambda b(\psi,\psi)}{b(\psi,\psi)}\\
&= \frac{a(\psi-u,\psi-u)-\lambda b(\psi-u,\psi-u)}{b(\psi,\psi)}. \qquad (3.3)
\end{aligned}$$

Then we obtain the desired result (3.2). ☐

Assume we have obtained an eigenpair approximation $(\lambda_{h_1},u_{h_1})\in\mathcal{R}\times V_{h_1}$. Now we introduce a type of correction step to improve the accuracy of the current eigenpair approximation (λ_{h_1},u_{h_1}). Let $V_{h_2}\subset V$ be a finer finite element space such that $V_{h_1}\subset V_{h_2}$. Based on this finer finite element space, we define the following correction step.

Algorithm 3.1. *One Correction Step*

1. *Define the following auxiliary source problem:*

   Find $\tilde{u}_{h_2}\in V_{h_2}$ such that

$$a(\widetilde{u}_{h_2},v_{h_2}) \;=\; \lambda_{h_1}b(u_{h_1},v_{h_2}), \quad \forall v_{h_2}\in V_{h_2}. \tag{3.4}$$

   Solve this equation to obtain a new eigenfunction approximation $\tilde{u}_{h_2}\in V_{h_2}$.

2. *Define a new finite element space $V_{H,h_2}=V_H+\mathrm{span}\{\widetilde{u}_{h_2}\}$ and solve the following eigenvalue problem:*

   Find $(\lambda_{h_2},u_{h_2})\in\mathcal{R}\times V_{H,h_2}$ such that $b(u_{h_2},u_{h_2})=1$ and

$$a(u_{h_2},v_{H,h_2}) \;=\; \lambda_{h_2}b(u_{h_2},v_{H,h_2}), \quad \forall v_{H,h_2}\in V_{H,h_2}. \tag{3.5}$$

Define $(\lambda_{h_2},u_{h_2})=Correction(V_H,\lambda_{h_1},u_{h_1},V_{h_2})$.

Theorem 3.2. *Assume the current eigenpair approximation $(\lambda_{h_1},u_{h_1})\in\mathcal{R}\times V_{h_1}$ has the following error estimates*

$$\|u-u_{h_1}\|_a \;\lesssim\; \varepsilon_{h_1}(\lambda), \tag{3.6}$$

$$\begin{aligned}
\|u-u_{h_1}\|_{-a} &\lesssim \eta_a(H)\|u-u_{h_1}\|_a, & (3.7)\\
|\lambda-\lambda_{h_1}| &\lesssim \varepsilon_{h_1}^2(\lambda). & (3.8)
\end{aligned}$$

Then after one correction step, the resultant approximation $(\lambda_{h_2}, u_{h_2}) \in \mathcal{R}\times V_{h_2}$ *has the following error estimates*

$$\begin{aligned}
\|u-u_{h_2}\|_a &\lesssim \varepsilon_{h_2}(\lambda), & (3.9)\\
\|u-u_{h_2}\|_{-a} &\lesssim \eta_a(H)\|u-u_{h_2}\|_a, & (3.10)\\
|\lambda-\lambda_{h_2}| &\lesssim \varepsilon_{h_2}^2(\lambda), & (3.11)
\end{aligned}$$

where $\varepsilon_{h_2}(\lambda) := \eta_a(H)\varepsilon_{h_1}(\lambda)+\varepsilon_{h_1}^2(\lambda)+\delta_{h_2}(\lambda)$.

Proof. From problems (2.5), (2.12) and (3.4), and (3.6), (3.7) and (3.8), the following estimate holds

$$\begin{aligned}
\|\widetilde{u}_{h_2}-P_{h_2}u\|_a^2 &\lesssim a(\widetilde{u}_{h_2}-P_{h_2}u,\widetilde{u}_{h_2}-P_{h_2}u) = b(\lambda_{h_1}u_{h_1}-\lambda u,\widetilde{u}_{h_2}-P_{h_2}u)\\
&\lesssim \|\lambda_{h_1}u_{h_1}-\lambda u\|_{-a}\|\widetilde{u}_{h_2}-P_{h_2}u\|_a\\
&\lesssim (|\lambda_{h_1}-\lambda|\|u_{h_1}\|_{-a}+\lambda\|u_{h_1}-u\|_{-a})\|\widetilde{u}_{h_2}-P_{h_2}u\|_a\\
&\lesssim \big(\varepsilon_{h_1}^2(\lambda)+\eta_a(H)\varepsilon_{h_1}(\lambda)\big)\|\widetilde{u}_{h_2}-P_{h_2}u\|_a.
\end{aligned}$$

Then we have

$$\|\widetilde{u}_{h_2}-P_{h_2}u\|_a \lesssim \varepsilon_{h_1}^2(\lambda)+\eta_a(H)\varepsilon_{h_1}(\lambda). \tag{3.12}$$

Combining (3.12) and the error estimate of finite element projection

$$\|u-P_{h_2}u\|_a \lesssim \delta_{h_2}(\lambda),$$

we have

$$\|\widetilde{u}_{h_2}-u\|_a \lesssim \varepsilon_{h_1}^2(\lambda)+\eta_a(H)\varepsilon_{h_1}(\lambda)+\delta_{h_2}(\lambda). \tag{3.13}$$

Now we come to estimate the eigenpair solution (λ_{h_2}, u_{h_2}) of problem (3.5). Based on the error estimate theory of eigenvalue problem by finite element method ([2, 3]), the following estimates hold

$$\|u-u_{h_2}\|_a \lesssim \sup_{w\in M(\lambda)}\inf_{v\in V_{H,h_2}}\|w-v\|_a \lesssim \|u-\widetilde{u}_{h_2}\|_a, \tag{3.14}$$

and

$$\|u-u_{h_2}\|_{-a} \lesssim \widetilde{\eta}_a(H)\|u-u_{h_2}\|_a, \tag{3.15}$$

where

$$\widetilde{\eta}_a(H) = \sup_{f\in V,\|f\|_a=1}\inf_{v\in V_{H,h_2}}\|Tf-v\|_a \le \eta_a(H). \tag{3.16}$$

From (3.13), (3.14), (3.15) and (3.16), we can obtain (3.9) and (3.10). The estimate (3.11) can be derived by Theorem 3.1 and (3.9). □

4 Multi-level correction scheme

In this section, we introduce a type of multi-level correction scheme based on the *One Correction Step* defined in Algorithm 3.1. This type of correction method can improve the convergence order after each correction step which is different from the two-grid method in [21].

Algorithm 4.1. *Multi-level Correction Scheme*

1. *Construct a coarse finite element space V_H and solve the following eigenvalue problem:*

   Find $(\lambda_H, u_H) \in \mathcal{R} \times V_H$ such that $b(u_H, u_H) = 1$ and

$$a(u_H, v_H) \;=\; \lambda_H b(u_H, v_H), \qquad \forall v_H \in V_H. \tag{4.1}$$

2. *Set $h_1 = H$ and construct a series of finer finite element spaces $V_{h_2}, \cdots, V_{h_n}$ such that $\eta_a(H) \gtrsim \delta_{h_1}(\lambda) \geq \delta_{h_2}(\lambda) \geq \cdots \geq \delta_{h_n}(\lambda)$.*

3. *Do $k = 0, 1, \cdots, n-2$*

   Obtain a new eigenpair approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}$ by a correction step

$$(\lambda_{h_{k+1}}, u_{h_{k+1}}) = Correction(V_H, \lambda_{h_k}, u_{h_k}, V_{h_{k+1}}). \tag{4.2}$$

   end Do

4. *Solve the following source problem:*

   Find $u_{h_n} \in V_{h_n}$ such that

$$a(u_{h_n}, v_{h_n}) \;=\; \lambda_{h_{n-1}} b(v_{h_{n-1}}, v_{h_n}), \quad \forall v_{h_n} \in V_{h_n}. \tag{4.3}$$

   Then compute the Rayleigh quotient of u_{h_n}

$$\lambda_{h_n} \;=\; \frac{a(u_{h_n}, u_{h_n})}{b(u_{h_n}, u_{h_n})}. \tag{4.4}$$

Finally, we obtain an eigenpair approximation $(\lambda_{h_n}, u_{h_n}) \in \mathcal{R} \times V_{h_n}$.

Theorem 4.1. *After implementing Algorithm 4.1, the resultant eigenpair approximation (λ_{h_n}, u_{h_n}) has the following error estimate*

$$\|u_{h_n} - u\|_a \;\lesssim\; \varepsilon_{h_n}(\lambda), \tag{4.5}$$

$$|\lambda_{h_n} - \lambda| \;\lesssim\; \varepsilon_{h_n}^2(\lambda), \tag{4.6}$$

where $\varepsilon_{h_n}(\lambda) = \sum\limits_{k=1}^{n} \eta_a(H)^{n-k}\delta_{h_k}(\lambda)$.

Proof. From $\eta_a(H) \gtrsim \delta_{h_1}(\lambda) \geq \delta_{h_2}(\lambda) \geq \cdots \geq \delta_{h_n}(\lambda)$ and Theorem 3.2, we have

$$\varepsilon_{h_{k+1}}(\lambda) \lesssim \eta_a(H)\varepsilon_{h_k}(\lambda) + \delta_{h_{k+1}}(\lambda), \quad \text{for } 1 \leq k \leq n-2. \tag{4.7}$$

Then by recursive relation, we can obtain

$$\begin{aligned}
\varepsilon_{h_{n-1}}(\lambda) &\lesssim \eta_a(H)\varepsilon_{h_{n-2}}(\lambda) + \delta_{h_{n-1}}(\lambda) \\
&\lesssim \eta_a(H)^2\varepsilon_{h_{n-3}}(\lambda) + \eta_a(H)\delta_{h_{n-2}}(\lambda) + \delta_{h_{n-1}}(\lambda) \\
&\lesssim \sum_{k=1}^{n-1} \eta_a(H)^{n-k-1}\delta_{h_k}(\lambda).
\end{aligned} \tag{4.8}$$

Based on the proof in Theorem 3.2 and (4.8), the final eigenfunction approximation u_{h_n} has the error estimate

$$\begin{aligned}
\|u_{h_n} - u\|_a &\lesssim \varepsilon_{h_{n-1}}^2(\lambda) + \eta_a(H)\varepsilon_{h_{n-1}}(\lambda) + \delta_{h_n}(\lambda) \\
&\lesssim \sum_{k=1}^{n} \eta_a(H)^{n-k}\delta_{h_k}(\lambda).
\end{aligned} \tag{4.9}$$

This is the estimate (4.5). From Theorem 3.1 and (4.9), we can obtain the estimate (4.6). ☐

5 The application to second order elliptic eigenvalue problem

In this section, for example, the multi-level correction method presented in this paper is applied to the second order elliptic eigenvalue problem. We also discuss two possible ways to implement the multi-level correction Algorithm 4.1. The first way is the "two-grid method" of Xu and Zhou introduced and studied in [21]. The second one proposed and studied by Andreev and Racheva in [1, 17] uses the same mesh but higher order finite elements.

In (2.12), the second order elliptic eigenvalue problem can be defined by

$$a(u,v) = \int_\Omega \nabla u \cdot \mathcal{A}\nabla v d\Omega, \qquad b(u,v) = \int_\Omega \rho u v d\Omega,$$

where $\Omega \subset \mathcal{R}^d$ $(d = 2,3)$ is a bounded domain, $\mathcal{A} \in \big(W^{1,\infty}(\Omega)\big)^{d\times d}$ a uniformly positive definite matrix on Ω and $\rho \in W^{0,\infty}(\Omega)$ is a uniformly positive function on Ω. We pose Dirichlet boundary condition to the problem and it means here $V = H_0^1(\Omega)$ and $W = L^2(\Omega)$. In order to use the finite element discretization method, we employ the meshes defined in section 3.

Here, we introduce two ways to implement the multi-level correction Algorithm 4.1. The first way uses finer meshes to construct the series of finite element spaces.

The advantage of this approach is that it uses the same finite element and does not require higher regularity of the exact eigenfunctions ([17]). The second way is based on the same finite element mesh but using higher order finite elements. In order to improve the convergence order, the higher regularity of the exact eigenfunctions is required.

Let us discuss the methods to construct the series of finite element spaces $V_{h_k}(k = 2, 3, \cdots, n)$ for implementing the multi-level correction method.

Way 1. ("Multi-grid method"): In this case, V_{h_k} $(k = 2, 3, \cdots, n)$ is the same type of finite element as V_H on the finer mesh $\mathcal{T}_{h_k}$ with smaller mesh size h_k. Here $\mathcal{T}_{h_k}$ is a finer mesh of Ω which can be generated by the refinement just as in the multigrid method ([21]) from $\mathcal{T}_{h_{k-1}}$ such that $h_k = \eta_a(H)h_{k-1}$. Assume the computing domain Ω is a convex domain. Then $\eta_a(H) = \mathcal{O}(H)$ and $\delta_{h_k} = \mathcal{O}(h_k) = \mathcal{O}(H^k)$ $(k = 1, 2, \cdots, n)$, and we can obtain the following error estimate for (λ_{h_n}, u_{h_n})

$$|\lambda - \lambda_{h_n}| \lesssim \eta_a(H)^{2n-2}\delta_H^2(\lambda) = \mathcal{O}(H^{2n}), \tag{5.1}$$

$$\|u - u_{h_n}\|_a \lesssim \eta_a(H)^{n-1}\delta_H(\lambda) = \mathcal{O}(H^n). \tag{5.2}$$

From the error estimates above, we can find that the multi-level correction scheme can obtain the accuracy as same as solving the eigenvalue problem on the finest mesh $\mathcal{T}_{h_n}$. This improvement costs solving the source problems on the finer finite element spaces V_{h_k} $(k = 2, 3, \cdots, n)$ and the eigenvalue problems in coarse spaces V_{H,h_k} $(k = 2, 3, \cdots, n-1)$. This is better than solving the eigenvalue problem on the finest finite element space directly, because solving source problem needs much less computation than solving the corresponding eigenvalue problem.

Way 2. ("Multi-space method"): In this case, V_{h_k} is defined on the same mesh $\mathcal{T}_H$ but using higher order finite element than $V_{h_{k-1}}$. In order to describe the scheme simply, we suppose the exact eigenfunction has sufficient regularity. We use the linear finite element space to solve the original eigenvalue problem (2.12) on V_H, and solve the source problem (3.4) in higher order finite element space with the way that the order of V_{h_k} is one order higher than $V_{h_{k-1}}$. Then we have the following error estimate for the final eigenpair approximation (λ_{h_n}, u_{h_n})

$$|\lambda - \lambda_{h_n}| \lesssim \eta_a(H)^{2n-2}\delta_H^2(\lambda) = \mathcal{O}(H^{2n}), \tag{5.3}$$

$$\|u - u_{h_n}\|_a \lesssim \eta_a(H)^{n-1}\delta_H(\lambda) = \mathcal{O}(H^n). \tag{5.4}$$

The improved error estimates above just cost solving the source problems on the same mesh but in higher order finite element spaces and eigenvalue problems in the lowest order finite element space.

6 Numerical results

In this section, we give two numerical examples to illustrate the efficiency of the multi-level correction algorithm proposed in this paper. We solve the model eigenvalue problem (1.1) on the unit square $\Omega = (0, 1) \times (0, 1)$.

6.1 Multi-space way

Here we give the numerical results of the multi-level correction scheme in which the finer finite element spaces are constructed by improving the finite element orders on the same mesh. We first solve the eigenvalue problem (1.1) in linear finite element space on the mesh $\mathcal{T}_H$. Then do the first correction step with quadratic element and cubic element for the second one.

Here, we adopt the meshes which are produced by regular refinement from the initial mesh generated by Delaunay method to investigate the convergence behaviors. Figure 1 shows the initial mesh. Figure 2 gives the corresponding numerical results for the first eigenvalue $\lambda_1 = 2\pi^2$ and the corresponding eigenfunction. From Figure

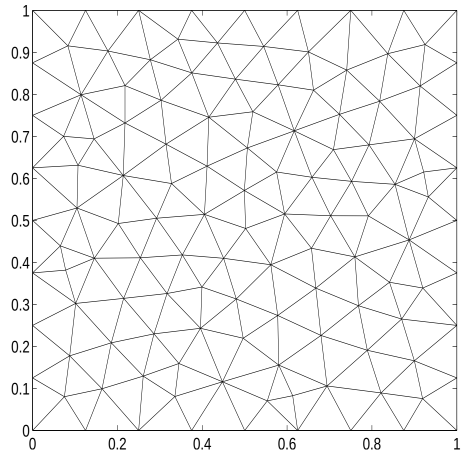

Figure 1: Initial mesh for multi-space way

2, we can find each correction step can improve the convergence order by two for the eigenvalue approximations and one for the eigenfunction approximations with multi-space way when the exact eigenfunction is smooth.

6.2 Multi-grid way

Here we give the numerical results of the multi-level correction scheme where the finer finite element spaces are constructed by refining the existed mesh. We first solve the eigenvalue problem (1.1) by linear finite element space on the mesh $\mathcal{T}_H$. Then refine the mesh by the regular way such that the size of the resultant mesh $h_k = O(H^k)$ to obtain the mesh $\mathcal{T}_{h_k}$ $(k = 2, \cdots, n)$ and solve the auxiliary source problem (3.4) in the finer linear finite element space V_{h_k} defined on $\mathcal{T}_{h_k}$ and the corresponding eigenvalue problem in V_{H,h_k}.

Figure 3 gives the corresponding numerical results for the first eigenvalue $\lambda = 2\pi^2$ and the corresponding eigenfunction on the uniform meshes. From Figure 3, we can also find each correction step can improve the convergence order by two for the

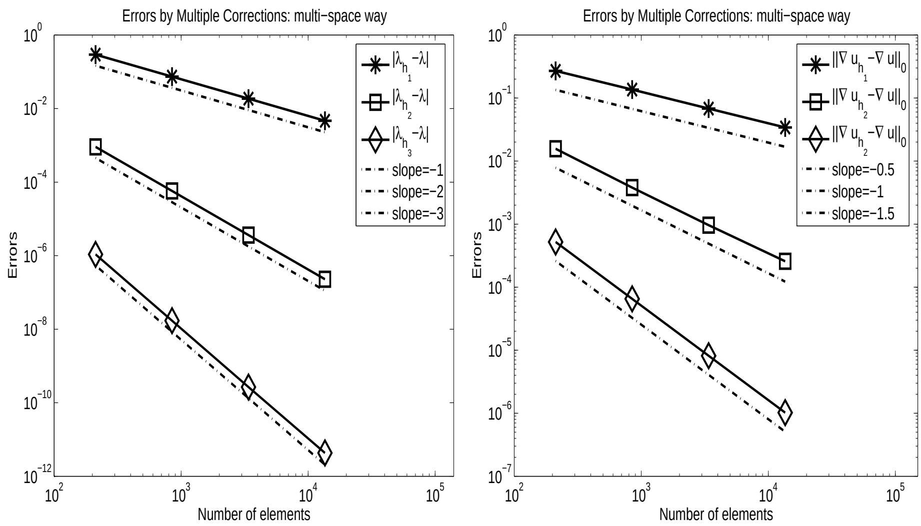

Figure 2: The errors for the eigenvalue approximations by multi-level correction algorithm for the first eigenvalue $2\pi^2$ and the corresponding eigenfunction with multi-space way

eigenvalue approximations and one for the eigenfunction approximations with the multi-grid way.

7 Concluding remarks

In this paper, we give a new type of multi-level correction scheme to improve the accuracy of the eigenpair approximations. We can use the better eigenvalue and eigenfunction approximation (λ_{h_n}, u_{h_n}) to construct an a posteriori error estimator of the eigenpair approximation for the eigenvalue problem ([7, 15]).

Furthermore, our multi-level correction scheme can be coupled with the multigrid method to construct a type of multigrid and parallel method for eigenvalue problems ([22]). It can also be combined with the adaptive refinement technique for the singular eigenfunction cases. These will be our future work.

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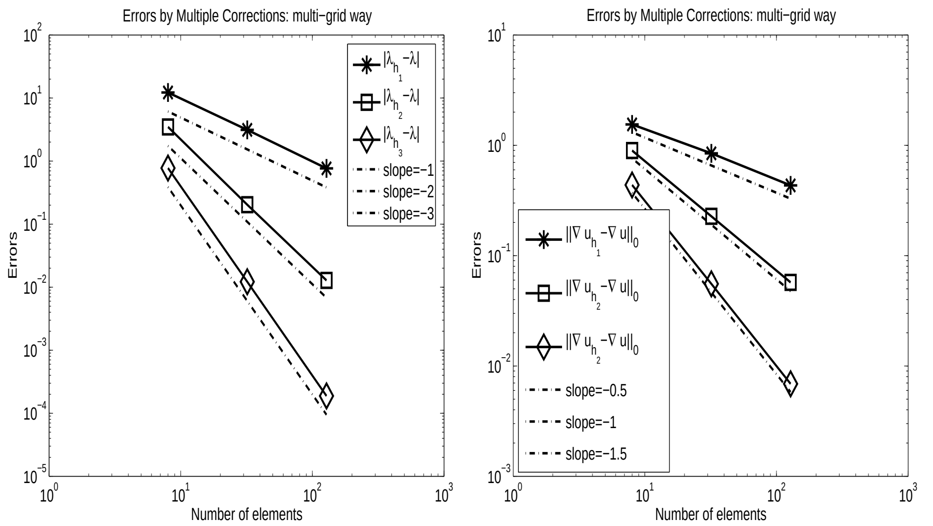

Figure 3: The errors for the eigenvalue approximations by multi-level correction algorithm for the first eigenvalue $2\pi^2$ with multi-grid way

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