FROM A KINETIC EQUATION TO A DIFFUSION UNDER AN ANOMALOUS SCALING

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Abstract. A linear Boltzmann equation is interpreted as the forward equation for the probability density of a Markov process $(K(t), i(t), Y(t))$ on $(\mathbb{T}^2 \times \{1,2\} \times \mathbb{R}^2)$, where $\mathbb{T}^2$ is the two-dimensional torus. Here $(K(t), i(t))$ is an autonomous reversible jump process, with waiting times between two jumps with finite expectation value but infinite variance. $Y(t)$ is an additive functional of K, defined as $\int_0^t v(K(s))ds$, where $|v| \sim 1$ for small k . We prove that the rescaled process $(N \ln N)^{-1/2} Y(Nt)$ converges in distribution to a two-dimensional Brownian motion.

1. Introduction

The problem of energy transport in a solid is still quite far from being completly understood. One of the most interesting aspects is an anomalously large thermal conductivity observed in low dimensional materials (see [18], [6] for a general review; see also [16] for experimental data for graphene materials). So far very few results are obtained by a rigorous analysis of microscopic dynamics, and even crucial points, such as the law of divergence of thermal conductivity in dimension one, are still debated.

The theoretical approach proposed by Peierls in his seminal paper [22] in 1929 intended to compute heat conductivity in analogy with the kinetic theory of gases. The main idea is that at low temperatures the lattice vibrations responsible of energy transport can be described as a gas of interacting particles (phonons). The time-dependent distribution function of phonons solves a Boltzmann type equation, and the thermal conductivity is related to the average spreading of energy of these phonons. Over the last years, rigorous derivations of phonon-Boltzmann equations, starting form microscopic dynamics, have been achieved. In [2], [20], [23], the authors perform a kinetic limit of the Fermi-Pasta-Ulam (FPU) chain of anharmonic oscillators [11], and in

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[19] a linear Boltzmann equation was derived for the harmonic chain with random masses.

In [4] the authors consider a d-dimensional system of harmonic oscillators, perturbed by a conservative stochastic noise. In the kinetic limit, the following linear Boltzmann-type equation is deduced for the energy density distribution of the phonons, characterized by a vector valued wave-number $k \in \mathbb{T}^d$ (d-dimensional torus)

$$\begin{aligned} \partial_t W_\alpha(t,u,k) &+ v(k)\cdot\nabla W_\alpha(t,u,k) \\ &= \frac{1}{d-1}\sum_{\beta\neq\alpha}\int_{\mathbb{T}^d} dk' R(k,k')[W_\beta(t,u,k') - W_\alpha(t,u,k)], \end{aligned} \tag{1}$$

for every $\alpha = 1,..,d$. In dimension one the equation is similar, except for the mixing of the components. Since the kernel R is not negative, the equation describes the evolution of the probability density of a Markov process, $(K(t), i(t), Y(t))$, on $(\mathbb{T}^d \times \{1,..,d\}, \times\mathbb{R}^d)$. Here $(K(t), i(t))$ is a reversible jump process and $Y(t)$ is a vector valued additive functional of K, given by $Y(t) = \int_0^t ds\ v(K_s)$. K and i can be interpreted , respectively, as the wave number and the "polarization" of a phonon, while $Y(t)$ describes its trajectory.

It is well known that anomalous diffusion, related to the divergence of thermal conductivity, is connected with a macroscopic length of the mean free path of phonons with small wave number k (ballistic transport). This is exactly what happens here: the scattering kernel R behaves like $|k|^2$ for small k and like $|k'|^2$ for small k', and the velocity $|v| \to 1$ as $|k| \to 0$, thus phonons with small wave number k travel with finite velocity, but they have law probability to be scattered. Therefore, it is not surprising that the system exhibits anomalous conductance in dimension one and two [4], and we expect an anomalous spreading of the energy carried by the phonons. With this aim, in [3] we considered the process in dimension one, and we proved that the rescaled process, $N^{-2/3}Y(Nt)$, converges in distribution to a symmetric Lévy process, stable with index 3/2 (super-diffusion). Convergence of finite dimensional marginals is also proved in [15].

In the same spirit, in dimension two, we look at the Markov chain $\{X_i\}$ on $\mathbb{T}^2$ given by the sequence of states visited by $K(t)$, and at the waiting times $\{\tau(X_i)\}$, where $\tau(X_i)$ is the time that the process spends at the i-th visited state. The vector valued function $S_n = \sum_{i=1}^n \tau(X_i)v(X_i)$ gives the value of Y at the time of the n-th jump. According to the behavior of the rate R, the stationary distribution of the chain, $\pi(dk)$, is of the form $\pi(dk) \sim |k|^2 dk$, for k small, and thus the tail distribution of the random variables $\tau(X_i)v(X_i)$ behaves like

$$\pi\left[|\tau(X_i)v(X_i)| > \lambda\right] \sim \frac{1}{\lambda^2}. \tag{2}$$

In particular, the variables $\tau(X_i)v(X_i)$ have infinite variance with respect to the stationary measure, and if they were independent, they would be in the domain of attraction of a multivariate normal distribution. Looking at the behavior of the variance,

$$\pi\left[(\tau(X_i)v_\alpha(X_i))^2\mathbf{1}_{\{|\tau(X_i)v_\alpha(X_i)|\le\sqrt{\lambda}\}}\right]\sim\ln\lambda,\qquad \alpha\in\{1,2\},$$

it turns out that the proper scaling contains an extra factor $(\ln n)^{1/2}$. The rescaled process $(n\ln n)^{-1/2}S_{nt}$ has a central part, given by the sum of truncated variables $\tau(X_i)v_\alpha(X_i)\mathbf{1}_{\{|\tau(X_i)v_\alpha(X_i)|\le\sqrt{n}\}}$, with finite variance and an extremal part that goes to zero in probability, due to the extra term $(\ln n)^{-1/2}$. This is a standard argument used for a sum of i.i.d. random variables with tail distribution (2), introduced for the first time by Kolmogorov and Gnedenko in [14], that we can adapt also to this case of dependent variables.

Then we are reduced to the problem of convergence of a sum of dependent, bounded random variables (with zero mean) to a two dimensional Brownian motion. To do this, we will use an abstract theorem due to Durrett and Resnick [7], based on the invariance principle for martingale difference arrays with bounded variables proved by Freedman [12] [13], together with a random change of time (see, for example, Helland [17] and Billingsley [5]). The underlying central limit theorem for a martingale difference array can be found in Dvoretzky [8], [9]. See also [21], [17] and references therein. The multidimensional generalization is based a Cramér-Wold argument (see for example [5], [1], [24], and [17]).

With this strategy we prove convergence of $(n\ln n)^{-1/2}S_{n\cdot}$ to a two-dimensional Wiener process in the Skorokhod J_1-topology. Since $Y(t)$ is the piecewise interpolation of S_n at the random times $T_n=\sum_{i=1}^n\tau(X_i)$, which is a sum of positive variables with finite expectation, one can prove, using the same arguments as in [3], that the process $(n\ln n)^{-1/2}Y(n\cdot)$ converges to the same limit process in the uniform topology.

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2. The model

We consider the process $(K(t),i(t),Y(t))$, with values in $\mathbb{T}^2\times\{1,2\}\times\mathbb{R}^2$, described by equation (1). $v:\mathbb{T}^2\to\mathbb{R}^2$ and $R:\mathbb{T}^2\times\mathbb{T}^2\to R$ are the following:

$$v_\alpha(k) = \frac{\sin(\pi k_\alpha)\cos(\pi k_\alpha)}{\left(\sum_{\beta=1}^2\sin^2(\pi k_\beta)\right)^{1/2}},\qquad \forall\alpha\in\{1,2\} \tag{3}$$

$$R(k,k') = 16\sum_{\alpha=1}^2\sin^2(\pi k_\alpha)\sin^2(\pi k'_\alpha). \tag{4}$$

The jump process $(K(t), i(t))$ waits in the state (k, i) an exponential random time τ with parameter $\Phi(k, i)$, where

$$\begin{aligned}\Phi(k,i) = \Phi(k) &= \sum_{j=1}^{2}(1-\delta_{i,j}) \int_{\mathbb{T}^d} dk'\ R(k,k') \\ &= 8\sum_{\alpha=1}^{2} \sin^2(\pi k_\alpha).\end{aligned} \tag{5}$$

Then it jumps to another state (j, k') with probability

$$\nu\,[i,k;j,dk'] = (1-\delta_{i,j})\ P(k,dk'),$$

where

$$P(k,dk') := \Phi(k)^{-1}R(k,k')dk' = \frac{2\sum_\alpha \sin^2(\pi k_\alpha)\sin^2(\pi k'_\alpha)}{\sum_\beta \sin^2(\pi k_\beta)}dk'. \tag{6}$$

The process $Y(t)$, with value in $\mathbb{R}^2$, is an additive functional of $K(t)$, given by

$$Y(t) = Y(0) + \int_0^t ds\ v(K_s)ds.$$

We choose $Y(0) = 0$.

Disregarding the time, the stochastic sequence $\{X_n\}_{n\geq 0}$ of states visited by $K(t)$ is a Markov chain with value in $\mathbb{T}^2$, with probability kernel $P(k, dk')$. Since P is regular and strictly positive and defined on a compact set, it is ergodic, i.e. there exists a strictly positive probability distribution π such that $P^m(k, I) \to \pi(I) > 0$, $m \uparrow \infty$, for each I open interval of continuity of π ([10]). Here P^m, $m \geq 2$, denote the m-th convolution integral of P. By direct computation $\pi(dk) = \Phi(k)dk$.

We define two functions of the Markov chain $\{X_n\}_{n\geq 0}$, the clock, T_n, with values in $\mathbb{R}_+$ and the position, S_n, with values in $\mathbb{R}^2$, by

$$\begin{aligned} T_n &= \sum_{\ell=0}^{n-1} e_\ell\ \Phi(X_\ell)^{-1} \\ S_n &= \sum_{\ell=0}^{n-1} e_\ell\ v(X_\ell)\ \Phi(X_\ell)^{-1}. \end{aligned}$$

Here $\{e_\ell\}_{\ell\geq 0}$ are i.i.d. exponential random variables with parameter 1. The clock T_n is the time of the n-the jump of the process $K(t)$. It is a sum of positive random variables with finite expectation, as we will see below. The position S_n, is a two-components vector which gives the position of $Y(t)$ at time T_n, i.e. $S_n = Y(T_n)$.

Let T^{-1} denote the right-continuous inverse function of T_n, i.e.

$$T^{-1}(t) := \inf\{n : T_n \geq t\}. \tag{7}$$

We can represent the original processes, $(K(t), Y(t))$, as follows:

$$
\begin{aligned}
K(t) =& X_{[T^{-1}(t)-1]} \\
Y(t) =& S_{[T^{-1}(t)-1]} + v(X_{[T^{-1}(t)-1]})(t - T_{[T^{-1}(t)-1]}).
\end{aligned} \tag{8}
$$

In particular, $Y(t)$ is the (vector valued) function defined by linear interpolation between its values S_n at the random points T_n (we take $S_0 = 0$).

3. Main results.

We assume that the initial distribution, μ, of the process K_t satisfies the condition

$$
\int_{\mathbb{T}^2} d\mu(k)|k|^{-2} < \infty, \tag{9}
$$

which guarantees in particular that $\mathbb{E}_\mu \left[e_0 \phi(X_0)^{-1}\right] < \infty$.

We define the rescaled process

$$
\begin{aligned}
T_N(\theta) &= \frac{1}{N} T_{\lfloor N\theta \rfloor}, \quad T_N^{-1}(\theta) = \frac{1}{N} T^{-1}(N\theta), \\
Z_N(\theta) &= \tfrac{1}{\sqrt{N \ln N}} S_{\lfloor N\theta \rfloor} + (N\theta - \lfloor N\theta \rfloor) \tfrac{1}{\sqrt{N \ln N}} v\left(X_{\lfloor N\theta \rfloor - 1}\right)
\end{aligned} \tag{10}
$$

where $\lfloor \cdot \rfloor$ denotes the lower integer part. Since T_n is a sum of positive variables with finite expectation (in particular $\mathbb{E}_\pi[e_1 \Phi(X_1)^{-1}] = 1$), then we expect that both $T_N(\theta)$ and $T_N^{-1}(\theta)$ converge in probability (and thus in distribution) to the function θ, in the topology of uniform convergence on compact intervals.

Z_N is a two-dimensional continuous vector defined by linear interpolation between its values $\frac{1}{\sqrt{N \ln N}} S_n$ at the points n/N, where S_n is a sum of centered random vectors whose components show a tail behavior given in (2). Moreover, the covariance matrix of each of these random vectors is diagonal. Thus we expect that Z_N converges to a two-dimensional Wiener process. This is the content of the following theorem. Let us denote with

$$
c = \lim_{N \to \infty} \frac{1}{\ln N} \mathbb{E}_\pi \left[|e_1 v_1(X_1) \Phi(X_1)^{-1}|^2 \mathbf{1}_{\{|e_1 v_1(X_1)\Phi(X_1)^{-1}| \le \sqrt{N}\}} \right]. \tag{11}
$$

By symmetry, we can replace $v_1(X_1)$ with $v_2(X_1)$ in this definition. We use the notation $\bar{W}_c$ to denote the vector process $\bar{W}_c = (W_c^1, W_c^2)$, where W_c^1 and W_c^2 are independent Wiener process such that $W_c^\alpha(t) - W_c^\alpha(s) \sim \mathcal{N}(0, c(t-s))$, $\forall \alpha = 1, 2$.

Theorem 3.1. *Let Z_N be the process defined in (10). Then for any $0 < \mathcal{T} < \infty$, the process $\{Z_N(\theta)\}_{0 \le \theta \le \mathcal{T}}$ converges to the two-dimensional Wiener process $\{\bar{W}_c(\theta)\}_{0 \le \theta \le \mathcal{T}}$. Convergence is in distribution on the space of continuous functions $C\left([0, \mathcal{T}], \mathbb{R}^2\right)$ equipped with the uniform topology.*

We will prove that $\{T_N^{-1}(\theta)\}_{\theta\in[0,\mathcal{T}]}$ converges in distribution to the function θ. Combining these two results, we can show that $Z_N \circ T_N^{-1}$ converges in distribution to V. This implies our main theorem.

Theorem 3.2. *Let us denote with $Y_N(t)$ the process $Y_N(t) = (N\ln N)^{-1/2}\int_0^{Nt} ds\ v(K_s)$. Then for every $0 < \mathcal{T} < \infty$, $\{Y_N(t)\}_{0\le\mathcal{T}}$ converges to the two-dimensional Wiener process $\{\bar{W}_c(\theta)\}_{0\le t\le\theta\le\mathcal{T}}$. Convergence is in distribution on the space of continuous functions $C([0,\mathcal{T}],\mathbb{R}^2)$ equipped with the uniform topology.*

4. Sketch of the proof

4.1. **Theorem 3.1.** Define the two-dimensional random vector

$$\psi_n := \Phi(X_n)^{-1}v(X_n), \qquad n\in\mathbb{N}_0.$$

We will denote with ψ_n^α, $\alpha = 1,2$, the α-component of ψ_n.

We decompose Z_N, defined in (10) in two parts, i.e. $\forall\theta\ge 0$, $Z_N(\theta) = Z_N^>(\theta) + Z_N^<(\theta)$, where $\forall\alpha = 1,2$

$$\begin{aligned} Z_N^{\alpha>}(\theta) &= (N\ln N)^{-1/2}\sum_{n=0}^{\lfloor N\theta\rfloor-1} e_n\psi_n^\alpha\mathbf{1}_{\left\{e_n|\psi_n^\alpha|>\sqrt{N}\right\}} \\ &\quad +(N\ln N)^{-1/2}e_{\lfloor N\theta\rfloor}\psi_{\lfloor N\theta\rfloor}^\alpha\mathbf{1}_{\{e_{\lfloor N\theta\rfloor}|\psi_{\lfloor N\theta\rfloor}^\alpha|>\sqrt{N}\}}\left(N\theta-\lfloor N\theta\rfloor\right) \\ Z_N^{\alpha<}(\theta) &= (N\ln N)^{-1/2}\sum_{n=0}^{\lfloor N\theta\rfloor-1} e_n\psi_n^\alpha\mathbf{1}_{\{e_n|\psi_n^\alpha|\le\sqrt{N}\}} \\ &\quad +(N\ln N)^{-1/2}e_{\lfloor N\theta\rfloor}\psi_{\lfloor N\theta\rfloor}^\alpha\mathbf{1}_{\{e_{\lfloor N\theta\rfloor}|\psi_{\lfloor N\theta\rfloor}^\alpha|\le\sqrt{N}\}}\left(N\theta-\lfloor N\theta\rfloor\right). \end{aligned}$$

At first we will show that $Z_N^> \xrightarrow{P} 0$ when $N\to\infty$. It is enough to show that for every unitary vector $\lambda := (\lambda_1,\lambda_2)$

$$\lambda_1 Z_N^{1>} + \lambda_2 Z_N^{2>} \xrightarrow{P} 0, \quad N\to\infty.$$

This is stated in the next Lemma.

Lemma 4.1. *For every $\delta > 0$*

$$\lim_{N\to\infty}\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]}\left|\lambda_1 Z_N^{1>}(\theta)+\lambda_2 Z_N^{2>}(\theta)\right| > \delta\right] = 0, \tag{12}$$

$\forall\lambda\in\mathbb{R}^2$ such that $|\lambda| = 1$.

Proof. For every $\lambda \in \mathbb{R}^2$ with $|\lambda| = 1$

$$\begin{aligned}
&\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]} \left|\lambda_1 Z_N^{1>}(\theta) + \lambda_2 Z_N^{2>}(\theta)\right| > \delta\right] \\
&\leq \mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]} \left\{\left|Z_N^{1>}(\theta)\right| + \left|Z_N^{2>}(\theta)\right|\right\} > \delta\right] \\
&\leq \sum_{\alpha=1,2} \mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]} |Z_N^{\alpha>}(\theta)| > \frac{\delta}{2}\right]
\end{aligned}$$

For every $\theta \in [0, \mathcal{T}]$, $\forall \alpha = 1, 2$

$$|Z_N^{\alpha>}(\theta)| \leq \frac{1}{\sqrt{N \ln N}} \sum_{n=0}^{\lfloor N\mathcal{T}\rfloor - 1} e_n |\psi_n^\alpha| \, \mathbf{1}_{\left\{e_n|\psi_n^\alpha| > \sqrt{N}\right\}}.$$

Then, by Chebyshev's inequality

$$\begin{aligned}
\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]} |Z_N^{\alpha>}(\theta)| > \frac{\delta}{2}\right] &\leq \frac{2}{\delta}\frac{1}{\sqrt{N \ln N}} \sum_{n=0}^{\lfloor N\mathcal{T}\rfloor - 1} \mathbb{E}\left[e_n |\psi_n^\alpha| \, \mathbf{1}_{\left\{e_n|\psi_n^\alpha| > \sqrt{N}\right\}}\right] \\
&\leq \frac{2}{\delta}\frac{1}{\sqrt{\ln N}} C_0 \mathcal{T},
\end{aligned}$$

where in the last inequality we used the fact that $\forall n \geq 0$, $\forall \alpha = 1, 2$

$$\mathbb{E}\left[e_n |\psi_n^\alpha| \, \mathbf{1}_{\left\{e_n|\psi_n^\alpha| > \sqrt{N}\right\}}\right] \leq C_0 \frac{1}{\sqrt{N}},$$

as one can easily compute, using the upper bound for P^m (18) and the fact that $|k|^2 |\psi^\alpha(k)|$ is finite for every $k \in \mathbb{T}^2$, $\forall \alpha = 1, 2$.

□

Let us considre $Z_N^<$. As first step, we will prove that for every unitary vector $\lambda \in \mathbb{R}^2$, $\langle Z_N^<, \lambda\rangle := \lambda_1 Z_N^{1<} + \lambda_2 Z_N^{2<} \Rightarrow W_c$, where W_c is a one dimensional Wiener process such that $W_c(t) - W_c(s) \sim \mathcal{N}(0, c(t-s))$. This is stated in the following proposition.

Proposition 4.2. *Fix $\mathcal{T} > 0$. Then as $N \to \infty$, for every $\lambda \in \mathbb{R}^2$, with $|\lambda| = 1$, $\langle Z_N^<, \lambda\rangle$ converges weakly to the one dimensional Wiener process W_c. Convergence is in distribution on the space of continuous functions on $[0, \mathcal{T}]$ equipped with the uniform topology.*

The proof is postponed to the next section.

We follow the approach of [24] (see the proof of Lemma 4). The tightness of the sequence $\{Z_N^<\}_{N\geq 1}$ follows from the tightness of the sequence $\{\langle Z_N^<, \lambda\rangle\}_{N\geq 1}$, for every unitary vector λ. Thus we only have to prove the convergence of the finite dimensional distribution. In particular, we have to show the following:

(i) $Z_N^<(\theta) - Z_N^<(\zeta) \Rightarrow \bar{W}_c(\theta) - \bar{W}_c(\zeta)$, for every ζ, θ such that $0 \leq \zeta \leq \theta \leq \mathcal{T}$;

(ii) $Z_N^<(\zeta)$ and $(Z_N^<(\theta) - Z_N^<(\zeta))$ are independent, as $N \to \infty$, $\forall\, 0 \le \zeta \le \theta \le \mathcal{T}$.

In order to verify the first condition, we observe that the convergence of the process $\langle Z_N^<(\cdot), \lambda\rangle$ to $W_c(\cdot)$ implies that $(\langle Z_N^<(\zeta), \lambda\rangle, \langle Z_N^<(\theta), \lambda\rangle) \Rightarrow (W_c(\zeta), W_c(\theta))$, for every $\zeta, \theta \ge 0$. But $(W_c(\zeta), W_c(\theta))$ has the same law of $\big(\langle \bar{W}_c(\theta), \lambda\rangle, \langle \bar{W}_c(\zeta), \lambda\rangle\big)$, then

$$\langle Z_N^<(\theta), \lambda\rangle - \langle Z_N^<(\zeta), \lambda\rangle \Rightarrow \langle \bar{W}_c(\theta), \lambda\rangle - \langle \bar{W}_c(\zeta), \lambda\rangle$$

for all $\theta, \zeta \ge 0$, $\forall \lambda \in \mathbb{R}^2$ with $|\lambda| = 1$, and this implies (i).

In order to verify condition (ii) it is sufficient to prove that $Z_N^<(\zeta)$ and $Z_N^<(\theta) - Z_N^<(\zeta)$ are asymptotically jointly Gaussian and uncorrelated. This is stated in the next Lemma.

Lemma 4.3. *For all λ, $\mu \in \mathbb{R}^2$*

(13)
$$\langle Z_N^<(\zeta), \lambda\rangle + \langle (Z_N^<(\theta) - Z_N^<(\zeta)), \mu\rangle \Rightarrow \mathcal{N}\left(0, c^2\{|\lambda|^2\zeta + |\mu|^2(\theta - \zeta)\}\right),$$

$\forall\, 0 \le \zeta < \theta \le \mathcal{T}$.

We postpone the proof in section 5.2.

4.2. **Theorem 3.2.** Converge in probability of T_N^{-1} to the function χ, where $\chi(\theta) = \theta$, in a compact $[0, \mathcal{T}]$, can be proved as in [3] (Lemma 8.1 and Proposition 8.2). Then by Theorem 3.9 in [5], $(Z_N, T_N^{-1}) \Rightarrow (\bar{W}_c, \chi)$ and therefore $Z_N \circ T_N^{-1} \Rightarrow \bar{W}_c \circ \chi$ (Billingsley [5], Lemma pg. 151).

5. Details

We start with some preliminary results on P^m, the $m-$th convolution integral of P, the probability kernel defined in 6. By direct computation

$$P^m(k, dk') = \frac{2}{\sum_{\gamma=1}^2 \sin^2(\pi k_\gamma)} \sum_{\alpha=1}^2 \sum_{\beta=1}^2 \sin^2(\pi k_\alpha) A_{\alpha,\beta}^{(m)} \sin^2(\pi k'_\beta) dk' \tag{14}$$

where, $\forall \alpha, \beta \in \{1, 2\}$,

$$A_{\alpha,\beta}^{(1)} = \delta_{\alpha,\beta}, \qquad A_{\alpha,\beta}^{(m+1)} = [a^m]_{\alpha,\beta} \quad \forall m \ge 1. \tag{15}$$

Here a is a 2×2 real matrix with elements $\{a_{\alpha\beta}\}$ such that

$$a_{11} = a_{22} > 0, \quad a_{12} = a_{21} > 0, \quad a_{11} + a_{12} = 1. \tag{16}$$

Observe that the condition

$$\int_{\mathbb{T}^2} P^m(k, dk') = 1 \qquad \forall m \ge 1,$$

implies

$$\sum_{\beta=1}^2 A_{\alpha,\beta}^{(m)} = 1, \qquad \forall \alpha = 1, 2, \ \forall m \ge 1, \tag{17}$$

and thus

$$P^m(k,dk') \le 2 \sum_{\beta=1,2} \sin^2(\pi k'_\beta)dk', \qquad \forall k \in \mathbb{T}^2,\ \forall m \ge 1. \tag{18}$$

5.1. **Proof of Proposition 4.2.** Fix $\lambda := (\lambda_1, \lambda_2)$ with $\lambda_1^2 + \lambda_2^2 = 1$. We will follow the strategy of Durrett and Resnick [7] to prove that $\langle Z_N^{<}, \lambda\rangle := \lambda_1 Z_N^{1<} + \lambda_2 Z_N^{2<}$ converges weakly to a Wiener process W_c. They use a result of Freedman [12], pages 89-93, on martingale difference arrays with uniformly bounded variables. We start with the following

Definition 5.1. A collection of random variables $\{\xi_{N,i}\}$, $N \ge 1$, $i \ge 1$ and σ-fields $\mathcal{F}_{N,i}$, $i \ge 0$, $N \ge 1$ is a martingale difference array if

(i) for all $N \ge 1$, $\mathcal{F}_{N,i}$, $i \ge 0$ is a nondecreasing sequence of σ-fields;
(ii) for all $N \ge 1$, $i \ge 1$, $\xi_{N,i}$ is $\mathcal{F}_{N,i}$ measurable;
(iii) for all $N \ge 1$, $E\left[\xi_{N,i} | \mathcal{F}_{N,i-1}\right] = 0$ a.s.

We introduce the following notations:

$$\langle \lambda, \bar\Psi_{N,m}\rangle := \lambda_1 \frac{e_m \psi_m^1}{\sqrt{N \ln N}} \mathbf{1}_{\left\{e_m |\psi_m^1| \le \sqrt{N}\right\}} + \lambda_2 \frac{e_m \psi_m^2}{\sqrt{N \ln N}} \mathbf{1}_{\left\{e_m |\psi_m^2| \le \sqrt{N}\right\}},$$

$\forall N \ge 2, m \ge 0$, and, for $N = 1$,

$$\langle \lambda, \bar\Psi_{1,m}\rangle = \lambda_1 e_m \psi_m^1 \mathbf{1}_{\{e_m|\psi_m^1| \le 1\}} + \lambda_2 e_m \psi_m^2 \mathbf{1}_{\{e_m|\psi_m^2| \le 1\}},$$

for all $m \ge 0$.

For all $N \ge 1$, $m \ge 0$, we denote with $\mathcal{F}_{N,\,m} = \mathcal{F}_m$ the σ-field generated by $\{X_0, .., X_m\} \times \{e_0, .., e_m\}$, where $\{X_m\}_{m\ge0}$ is the Markov chain with value in $\mathbb{T}^2$. Then $\{\langle\lambda, \bar\Psi_{N,m}\rangle, \mathcal{F}_{N,\,m}\}_{N\ge1, m\ge1}$ is a martingale difference array. In particular, condition (iii) of 5.1 can be easily checked using the explicit form of probability kernel $P[k, dk']$.

By definition, the variables $\langle\lambda, \bar\Psi_{N,m}\rangle$ are uniformly bounded in m, i.e. for all $N \ge 1$ $\left|\langle\lambda, \bar\Psi_{N,m}\rangle\right| \le \varepsilon_N$, $\forall m \ge 0$, where $\varepsilon_N = \frac{2}{\sqrt{\ln N}}$ if $N \ge 2$, and $\varepsilon_1 = 2$. In particular $\varepsilon_N \downarrow 0$ when $N \to \infty$.

For every $N \ge 1$, $j \ge 1$, let us define

$$\langle\lambda, S_{N,j}\rangle = \sum_{m=1}^{j} \langle\lambda, \bar\Psi_{N,m}\rangle, \tag{19}$$

$$\langle\lambda, V_{N,j}\rangle = \sum_{m=1}^{j} E\left[\langle\lambda, \bar\Psi_{N,m}\rangle^2 | \mathcal{F}_{N,\,m-1}\right] \tag{20}$$

We will prove lemma 5.2 that $\mathbb{P}\left[\lim_{j\to\infty}\langle\lambda, V_{N,j}\rangle = \infty\right] = 1$, for all $N \ge 1$. Thus the martingale difference array $\{\langle\lambda, \bar\Psi_{N,m}\rangle, \mathcal{F}_{N,\,m}\}_{N\ge1, m\ge0}$, satisfies the hypotheses of Theorem 2.1 in [7]. Setting

$$j_{N,\lambda}(\theta) = \sup\{j | \langle\lambda, V_{N,j}\rangle \le \theta\},$$

we get that $\langle\lambda, S_{N, j_{N,\lambda}(\cdot)}\rangle$ converges weakly as a sequence of random elements of $D[0, \mathcal{T}]$ to a standard Wiener process W.

Let $\phi_{N,\lambda}(\theta) = \langle \lambda, V_{N,\lfloor N\theta \rfloor} \rangle$, $\forall \theta \in [0, \mathcal{T}]$. By definition

$$j_{N,\lambda} \circ \phi_{N,\lambda}(\theta) = \lfloor N\theta \rfloor.$$

In order to prove that $\phi_{N,\lambda}$ converges in probability to the function ϕ where $\phi(\theta) = c\theta$, it suffices to show that $\phi_{N,\lambda}(\theta) \stackrel{P}{\to} c\theta$, $\forall t \in [0, \mathcal{T}]$, since $c\theta$ is continuous and $\phi_{N,\lambda}$ is monotone. This is stated in Lemma 5.2. By Theorem 3.9 in [5], $(\langle \lambda, S_{N,j_{N,\lambda}} \rangle, \phi_{N,\lambda}) \Rightarrow (W, \phi)$, and therefore $\langle \lambda, S_{N,j_{N,\lambda}} \rangle \circ \phi_{N,\lambda} \Rightarrow W \circ \phi$ ([5], Lemma pg. 151). Thus $\langle \lambda, S_{N,\lfloor N\cdot \rfloor} \rangle = \langle \lambda, S_{N,j_N(\phi_N(\cdot))} \rangle \Rightarrow W_c$, where convergence is in distribution on the space $D[0, \mathcal{T}]$ equipped with the Skorohod J_1-topology.

Let us define $\langle \lambda, \tilde{S}_N(\theta) \rangle := \sum_{m=0}^{\lfloor N\theta \rfloor - 1} \langle \lambda, \bar{\Psi}_{N,m} \rangle$. Then also $\langle \lambda, \tilde{S}_N \rangle$ converges to W_c. For every $N \geq 2$, $\langle Z_N^<, \lambda \rangle = \lambda_1 Z_N^{1<} + \lambda_2 Z_N^{2<}$ is the continuous function defined by linear interpolation between its values $\langle \lambda, \tilde{S}_N(m/N) \rangle$ at points m/N. The two sequences $\{\langle \lambda, \tilde{S}_N(\theta) \rangle\}_{0 \leq \theta \leq \mathcal{T}}$ and $\{\langle Z_N(\theta), \lambda \rangle\}_{0 \leq \theta \leq \mathcal{T}}$ are asymptotically equivalent, i.e. if either converges in distribution as $N \to \infty$, then so does the other. In particular, for all $N \geq 2$, $\langle Z_N^<, \lambda \rangle$ is a continuous process, and since the limit process W_c is continuous too, convergence in distribution on the space $D[0, \mathcal{T}]$ equipped with the J_1-topology implies convergence in distribution on the space of continuous functions equipped with the uniform topology.

Lemma 5.2. *For every $N \geq 1$, for every unitary vector $\lambda \in \mathbb{R}^2$,*

$$\mathbb{P}\left[\lim_{j \to \infty} \langle \lambda, V_{N,j} \rangle = \infty\right] = 1. \tag{21}$$

Moreover, for every $\delta > 0$, for every unitary vector $\lambda \in \mathbb{R}^2$,

$$\lim_{N \to \infty} \mathbb{P}\Big[\big| \langle \lambda, V_{N,\lfloor N\theta \rfloor} \rangle - c\theta \big| > \delta \Big] = 0, \tag{22}$$

$\forall \theta \in [0, \mathcal{T}]$.

Proof. Fix $\lambda \in \mathbb{R}^2$, with $|\lambda|^2 = 1$. For every $N \geq 2$, let us denote

$$f_N(k) = \int_0^\infty dz\, e^{-z} \int_{\mathbb{T}^2} P(k, dk') \left(\sum_{\alpha=1,2} \lambda_\alpha \frac{z\, \psi^\alpha(k')}{\sqrt{N \ln N}} \mathbf{1}_{\{z\, |\psi^\alpha(k')| \leq \sqrt{N}\}} \right)^2, \tag{23}$$

$\forall k \in \mathbb{T}^2$. Using (14)-(16), by direct computation we get the lower bound $f_N(k) \geq C_0/N$, with $0 \leq C_0 < \infty$ independent of$k \in \mathbb{T}^2$. Since for every $m \geq 0$

$$f_N(X_m) = \mathbb{E}\left[\langle \bar{\Psi}_{N,m+1}, \lambda \rangle^2 | \mathcal{F}_m \right],$$

then, for all $N \geq 1$, $\langle \lambda, V_{N,j} \rangle \geq j\, C_0 N^{-1} \uparrow \infty$ for $j \to \infty$, a.s.

Now we focus on (22). By Chebychev inequality, for every $N \geq 1$

$$\begin{aligned}
&\mathbb{P}\Big[\big|\langle \lambda, V_{N,\lfloor N\theta \rfloor}\rangle - c\theta\big| > \delta\Big] \\
&\leq \mathbb{P}\left[\Big|\sum_{n=1}^{\lfloor N\theta\rfloor}\Big(\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\right] - \frac{c}{N}\Big)\Big| > \delta - \frac{1}{N}\right] \\
&\leq \frac{1}{\tilde\delta_N^2}\sum_{n=1}^{\lfloor N\theta\rfloor}\mathbb{E}\left[\Big(\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\right] - \frac{c}{N}\Big)^2\right] \\
&\quad + \frac{1}{\tilde\delta_N^2}\sum_{n=1}^{\lfloor N\theta\rfloor}\sum_{m\neq n}\mathbb{E}\left[\Big(\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\right] - \frac{c}{N}\Big)\Big(\mathbb{E}\left[\langle \bar\Psi_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\right] - \frac{c}{N}\Big)\right],
\end{aligned} \tag{24}$$

where $\tilde\delta_N = \delta - N^{-1}$. By (18), we see that $f_N(k) \leq \frac{C_0}{N}$, with C_0 finite, for all $k \in \mathbb{T}^2$. Thus

$$\sum_{n=1}^{\lfloor N\theta\rfloor}\mathbb{E}\left[\Big(\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\right] - \frac{c}{N}\Big)^2\right] \leq \frac{C_1\,T}{N}, \tag{25}$$

with C_1 finite.

Let us consider the second sum on the r.h.s. of (24)

$$\begin{aligned}
&\sum_{n=1}^{\lfloor N\theta\rfloor}\sum_{m\neq n}\mathbb{E}\left[\Big(\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\right] - \frac{c}{N}\Big)\Big(\mathbb{E}\left[\langle \bar\Psi_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\right] - \frac{c}{N}\Big)\right] \\
&= \sum_{n=1}^{\lfloor N\theta\rfloor}\sum_{m\neq n}\mathbb{E}\Big[\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\right]\mathbb{E}\left[\langle \bar\Psi_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\right]\Big] \\
&\quad - 2\frac{c}{N}(\lfloor N\theta\rfloor - 1)\sum_{n=1}^{\lfloor N\theta\rfloor}\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2\right] + \frac{c^2}{N^2}\lfloor N\theta\rfloor(\lfloor N\theta\rfloor - 1).
\end{aligned} \tag{26}$$

We observe that for $n > m$

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\right]\mathbb{E}\left[\langle \bar\Psi_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\right]\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\left[\langle \bar\Psi_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\right]\mathbb{E}\Big[\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\right]|\mathcal{F}_{m-1}\Big]\Big].
\end{aligned}$$

We set

$$g_N^{n-m}(X_{m-1}) := \mathbb{E}\Big[\mathbb{E}\left[\langle \bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\right]|\mathcal{F}_{m-1}\Big],$$

where, for every $l \geq 1$,

$$g_N^l(k) = \int_{\mathbb{T}^2} dk'\; P^l(k,dk')f_N(k'), \qquad \forall k \in \mathbb{T}^2,$$

with f_N defined in (23). Recalling the uniform bound on $f_N(k)$, we see that

$$g_N^l(k) \le \frac{C_0}{N}, \quad \forall k \in \mathbb{T}^2, \ \forall l \ge 1. \tag{27}$$

We fix M, $1 \le M < N$, and we split the first sum on the r.h.s. of (26), namely

$$\begin{aligned}
&\sum_{n=1}^{\lfloor N\theta \rfloor} \sum_{m \neq n} \mathbb{E}\Big[\mathbb{E}\left[\langle \bar{\Psi}_{N,n}, \lambda\rangle^2 | \mathcal{F}_{n-1}\right] \mathbb{E}\left[\langle \bar{\Psi}_{N,m}, \lambda\rangle^2 | \mathcal{F}_{m-1}\right]\Big] \\
&= 2\sum_{m=1}^{M} \sum_{n=m+1}^{\lfloor N\theta \rfloor} \mathbb{E}\Big[f_N(X_{m-1}) g_N^{n-m}(X_{m-1})\Big] \\
&\quad + 2\sum_{m=M+1}^{\lfloor N\theta \rfloor} \sum_{n=m+1}^{m+M} \mathbb{E}\Big[f_N(X_{m-1}) g_N^{n-m}(X_{m-1})\Big] \\
&\quad + 2\sum_{m=M+1}^{\lfloor N\theta \rfloor} \sum_{n=m+M+1}^{\lfloor N\theta \rfloor} \mathbb{E}\Big[f_N(X_{m-1}) g_N^{n-m}(X_{m-1})\Big].
\end{aligned}$$

By (27), the first and the second sum on the r.h.s. are bounded form above by $C\mathcal{T}M/N$, with C finite. We denote by μP^{m-1} the convolution integral of the initial measure μ and the probability P^{m-1}. For every $l \ge 1$,

$$\begin{aligned}
\mathbb{E}\Big[f_N(X_{m-1}) g_N^l(X_{m-1})\Big] =& \mathbb{E}_\pi\Big[f_N(X_{m-1}) g_N^l(X_{m-1})\Big] \\
&+ \int_{\mathbb{T}^2} \left[\mu P^{m-1}(dk) - \pi(dk)\right] f_N(k) g_N^l(k)
\end{aligned}$$

where the last term is bounded by $C' N^{-2} \int_{\mathbb{T}^2} |\mu P^{m-1}(dk) - \pi(dk)|$. Moreover, for every $l \ge 1$

$$\begin{aligned}
&\mathbb{E}_\pi\Big[f_N(X_{m-1}) g_N^l(X_{m-1})\Big] \\
&= \int_{\mathbb{T}^2} \pi(dk) f_N(k) \int_{\mathbb{T}^2} dk' P^l(k, dk') f_N(k') \\
&\le \left(\int_{\mathbb{T}^2} \pi(dk) f_N(k)\right)^2 + \frac{C'}{N^2} \int_{\mathbb{T}^2} \left|\mu P^{m-1}(dk) - \pi(dk)\right|.
\end{aligned}$$

We get

$$\begin{aligned}
&\sum_{n=1}^{\lfloor N\theta \rfloor} \sum_{m \neq n} \mathbb{E}\Big[\mathbb{E}\left[\langle \bar{\Psi}_{N,n}, \lambda\rangle^2 | \mathcal{F}_{n-1}\right] \mathbb{E}\left[\langle \bar{\Psi}_{N,m}, \lambda\rangle^2 | \mathcal{F}_{m-1}\right]\Big] \\
&\le \lfloor N\theta \rfloor (\lfloor N\theta \rfloor - 1) \left(\mathbb{E}_\pi\left[\langle \bar{\Psi}_{N,1}, \lambda\rangle^2\right]\right)^2 \\
&\quad + C\mathcal{T}\frac{M}{N} + C'\mathcal{T} \int_{\mathbb{T}^2} \left|\mu P^M(dk) - \pi(dk)\right|,
\end{aligned} \tag{28}$$

we C and C' finite. In the same way one can prove that

$$\sum_{n=1}^{\lfloor N\theta \rfloor} \mathbb{E}\left[\langle \bar{\Psi}_{N,n}, \lambda\rangle^2\right] \leq \lfloor N\theta\rfloor \mathbb{E}_\pi\left[\langle \bar{\Psi}_{N,n}, \lambda\rangle^2\right] + C\mathcal{T}\frac{M}{N} + C'\mathcal{T}\int_{\mathbb{T}^2} \left|\mu P^M(dk) - \pi(dk)\right|, \tag{29}$$

with some C, C' finite, and finally we get

$$\mathbb{P}\Big[\left|\langle \lambda, V_{N,\lfloor N\theta\rfloor}\rangle - c\theta\right| > \delta\Big] \leq \frac{1}{\tilde{\delta}_N^2}\, C\mathcal{T}\frac{M}{N} + \frac{1}{\tilde{\delta}_N^2}\, C'\mathcal{T}\int_{\mathbb{T}^2}\left|\mu P^M(dk) - \pi(dk)\right|,$$

where C, C' are finite. (22) is proved by sending $M, N \to \infty$ in such a way that $M/N \to 0$.

□

5.2. **Proof of Lemma (4.3).** We use the central limit theorem for martingale difference array ([8], Theorem 1; see also [9], [17]) which states the follows: fix $\theta > 0$, and let $\{\xi_{N,i}, \mathcal{F}_{N,i}\}_{N\geq 1,\, i\geq 0}$ be a martingale difference array such that

$$(i) \quad \sum_{i=1}^{\lfloor N\theta\rfloor} \mathbb{E}\left[\xi_{N,i}^2 | \mathcal{F}_{N,i-1}\right] \xrightarrow{P} c\theta, \qquad N\uparrow\infty;$$

$$(ii) \quad \sum_{i=1}^{\lfloor N\theta\rfloor} \mathbb{E}\left[\xi_{N,i}^2 \mathbf{1}_{\{|\xi_{N,i}|>\varepsilon\}} | \mathcal{F}_{N,i-1}\right] \xrightarrow{P} 0, \qquad N\uparrow\infty, \quad \forall\varepsilon > 0.$$

Then $\sum_{i=1}^{\lfloor N\theta\rfloor} \xi_{N,i} \Rightarrow \mathcal{N}(0, c\theta)$.

By definition of $Z_N^<$, $\forall\lambda \in \mathbb{R}^2$

$$\langle \lambda, Z_N^<(t)\rangle = \langle \lambda, S_{N,\lfloor Nt\rfloor}\rangle + (Nt - \lfloor Nt\rfloor)\langle \lambda, \bar{\Psi}_{\lfloor Nt\rfloor}\rangle, \tag{30}$$

$\forall t \in [0, \mathcal{T}]$, where $\langle \lambda, S_{N,\cdot}\rangle$ is defined in (19). The rightmost term in (30)goes to zero in probability by Chebyshev's inequality, thus we have to focus only on $S_{N,\lfloor N\cdot\rfloor}$.

We fix $\lambda, \mu \in \mathbb{R}^2$ and $0 \leq \zeta < \theta \leq \mathcal{T}$, and we define the following array of variables:

$$\tilde{\xi}_{N,i} = \begin{cases} \langle \bar{\Psi}_{N,\,i}, \lambda\rangle, & 0 \leq i \leq \lfloor N\zeta\rfloor - 1, \\ \langle \bar{\Psi}_{N,\,i}, \mu\rangle, & \lfloor N\zeta\rfloor \leq i \leq \lfloor N\theta\rfloor, \end{cases}$$

for every $N \geq 1$. In our case, $\mathcal{F}_{N,i}$ is the σ-algebra generated by $(X_0, ..., X_i)\times(e_0, .., e_i)$, for every $N \geq 1$, $\forall i \geq 0$. Then $\{\tilde{\xi}_{N,i}, \mathcal{F}_{N,i}\}_{N\geq 1,\, i\geq 0}$ is a martingale difference array. In particular, since $|\langle \bar{\Psi}_{N,\,i}, \nu\rangle| \leq 2(\ln N)^{-1/2}$ for every $i \geq 1$, for every unitary vector $\nu \in \mathbb{R}^2$, it follows that, for any $\varepsilon > 0$, there exists $\bar{N}$ such that $|\tilde{\xi}_{N,i}| < \varepsilon$, $\forall N \geq \bar{N}$, $\forall i \geq 1$. Therefore condition (ii) is trivially satisfied.

Moreover, with similar arguments of the proof of (22), one can prove that

$$\sum_{i=1}^{\lfloor N\theta \rfloor} \mathbb{E}\left[\tilde{\xi}_{N,i}^2 | \mathcal{F}_{N,i-1}\right] \xrightarrow{P} c|\lambda|^2\zeta + c|\mu|^2(\theta - \zeta),$$

with c defined in (11). Thus

$$\sum_{i=1}^{\lfloor N\zeta \rfloor -1} \langle \bar{\Psi}_{N,\,i}, \lambda \rangle + \sum_{i=\lfloor N\zeta \rfloor}^{\lfloor N\theta \rfloor -1} \langle \bar{\Psi}_{N,\,i}, \mu \rangle = \sum_{i=1}^{\lfloor N\theta \rfloor} \tilde{\xi}_{N,i} \Rightarrow \mathcal{N}(0, c|\lambda|^2\zeta + c|\mu|^2(\theta - \zeta)).$$

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