AN INEQUALITY FOR THE DISTANCE BETWEEN DENSITIES OF FREE CONVOLUTIONS

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Abstract

It is shown that under some conditions the distance between densities of free convolutions of two pairs of probability measures is smaller than the maximum of the Levy distance between the corresponding measures in these pairs. In particular, weak convergence $\mu_n \to \mu$, $\nu_n \to \nu$ implies that the density of $\mu_n \boxplus \nu_n$ is defined for all sufficiently large n and converges to the density of $\mu \boxplus \nu$. Some applications are provided including (i) a new proof of the local version of the free central limit theorem, (ii) a local limit theorem for sums of free projections, and (iii) a local limit theorem for sums of $\boxplus$-stable random variables. In addition, a local limit law for eigenvalues of a sum of N-by-N random matrices is proved.

1. Introduction

This paper contributes to the study of free convolution of probability measures by showing that under some conditions, if measures μ_i and ν_i, $i = 1, 2$, are close to each other in terms of the Levy metric and if free convolution $\mu_1 \boxplus \mu_2$ is absolutely continuous, then $\nu_1 \boxplus \nu_2$ is also absolutely continuous and the densities of measures $\nu_1 \boxplus \nu_2$ and $\mu_1 \boxplus \mu_2$ are close to each other.

The study of the free convolution is motivated by numerous applications of this operation. It first arose in the study of operator algebras ([20]). If two self-adjoint operators A and B with spectral measures μ_A and μ_B are free, then their sum $A+B$ has the spectral measure $\mu_A \boxplus \mu_B$, where $\boxplus$ denotes the free convolution operation. A concrete example of free operators is given by $P(g_1)$ and $Q(g_2)$, where g_1 and g_2 are operators of left multiplication by generators g_1 and g_2 in the group algebra $G(\mathbb{Z} * \mathbb{Z})$, and P, Q are two self-adjoint polynomials. Here $\mathbb{Z} * \mathbb{Z}$ denotes the free group with two generators g_1 and g_2.

Surprisingly, free convolution also appears prominently in the theory of large random matrices ([21], [17], [16]). Roughly speaking, if A_N and B_N are two independent sequences of Hermitian N-by-N matrices with empirical distribution of eigenvalues μ_{A_N} and μ_{B_N}, and if $\mu_{A_N} \to \mu_\alpha$ and $\mu_{B_N} \to \mu_\beta$ as $N \to \infty$, then the empirical distribution of eigenvalues of $A_N + B_N$ converges to $\mu_\alpha \boxplus \mu_\beta$. This link with random matrices brought some breakthrough

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results in the theory of operator algebras ([23], [10]), and attracted the attention of physicists ([25], [8]). A recent illustration of the connection between free probability techniques and random matrices is a proof of the so-called single ring theorem for random matrices ([9]).

Another surprising application of free convolution is to the representation theory of the symmetric group S_n ([7]). Recall that irreducible representations of S_n are indexed by Young diagrams (i.e., partitions of n). Every Young diagram corresponds in a one-to-one fashion to a certain probability measure, the transition measure of the diagram. Biane discovered that for large-dimensional representations the measure of the outer product of irreducible representations is close to the free convolution of the measures of original representations.

These applications suggest that free convolution of probability measures deserves a detailed study. Such a study was initiated by Voiculescu in [20]. It was noted that free convolution has strong smoothing properties, and in particular, the free convolution of discrete measures is often absolutely continuous. More precisely, it was proved in [4] that $\mu_1 \boxplus \mu_2$ has an atom at x if and only if there are y and z such that $x = y + z$, and $\mu_1(\{y\}) + \mu_2(\{z\}) > 1$. In [1], it was shown that $\mu_1 \boxplus \mu_2$ can have a singular component if and only if one of the measures is concentrated on one point and the other has a singular component (so that the resulting free convolution is simply a translation of the measure with the singular component). Moreover, in the same paper it was shown that the density of the absolutely continuous part is analytic wherever the density is positive and finite.

In earlier work, some quantitative versions of this smoothing property of free convolution were given. In particular, in [22] it was shown that if μ_1 is absolutely continuous with density $v_1 \in L^p(\mathbb{R})$, ($p \in (1, \infty]$), then the free convolution $\mu_1 \boxplus \mu_2$ is absolutely continuous with density $v_\boxplus \in L^p(\mathbb{R})$, and $\|v_\boxplus\|_p \leq \|v_1\|_p$. In particular, the supremum of the density $v_\boxplus$ is less than or equal to the supremum of the density of v_1.

Another important property of free convolution is that it is continuous with respect to weak convergence of measures. In particular, by a result in [2], if $\mu_N \to \mu$ and $\nu_N \to \nu$ as $N \to \infty$, then $\mu_N \boxplus \nu_N \to \mu \boxplus \nu$. In fact, Theorem 4.13 in [2] says that $d_L(\mu \boxplus \nu, \mu' \boxplus \nu') \leq d_L(\mu, \mu') + d_L(\nu, \nu')$, where d_L denotes the Levy metric on probability measures on $\mathbb{R}$, which metrizes weak convergence of measures.

The main result of this paper establishes a strengthened version of this property. If distances $d_L(\mu, \mu')$ and $d_L(\nu, \nu')$ are sufficiently small and $\mu \boxplus \nu$ is absolutely continuous, then $\mu' \boxplus \nu'$ is also absolutely continuous and the distance between the densities of $\mu \boxplus \nu$ and $\mu' \boxplus \nu'$ can be bounded in terms of the Levy distances between the original measures.

In particular, this result shows that free convolution transforms the weak convergence of measures $\mu_N \to \mu$ and $\nu_N \to \nu$ into the convergence of probability densities of $\mu_N \boxplus \nu_N$ to the density of $\mu \boxplus \nu$.

We prove this result under an additional assumption imposed on measures μ and ν, which we call the smoothness of the pair (μ, ν) at a point of its support E. This assumption holds if $\mu = \nu$ and the density of $\mu \boxplus \mu$ is absolutely continuous and positive at E. In the case when $\mu \neq \nu$, this assumption should be checked directly. We envision that in applications μ and

ν are fixed measures for which this assumption can be directly checked, and μ_N and ν_N are (perhaps random) measures for which it can be checked that they are close to μ and ν in Levy distance.

In order to formulate our main result precisely, we introduce several definitions. Let μ_A and μ_B be two probability measures on $\mathbb{R}$ with the Stieltjes transforms $m_A(z)$ and $m_B(z)$, where the Stieltjes transform of a probability measure μ is defined by the formula

$$m(z) := \int_{\mathbb{R}} \frac{\mu(dx)}{x-z}.$$

Then, the free convolution $\mu_A \boxplus \mu_B$ is defined as the probability measure with the Stieltjes transform, $m_{\boxplus}(z)$, which satisfies the following system of equations:

$$\begin{aligned} m_{\boxplus}(z) &= m_A(z - S_B(z)), \\ m_{\boxplus}(z) &= m_B(z - S_A(z)), \\ z + \frac{1}{m_{\boxplus}(z)} &= S_A(z) + S_B(z), \end{aligned} \tag{1}$$

where $S_A(z)$ and $S_B(z)$ are auxiliary functions.

This is not the standard definition of free convolution. Usually, one defines the R-transform of measure μ_A by the formula $R(t) = m_A^{(-1)}(-t) - 1/t$, where $m_A^{(-1)}$ is the functional inverse of m, and similarly for $R_A(t)$. Then $R = R_A + R_B$ is the R-transform of a probability measure, which is $\mu_A \boxplus \mu_B$. (Our references for free probability are [18] and [15]). It is easy to check that our definition is equivalent to the standard with the relation between functions $S_{A,B}$ and R-transforms given by

$$S_A(z) = R_A(-m_{\boxplus}(z)),$$

and similarly for $S_B(z)$ and $R_B(z)$.

It is an important and non-trivial fact that $S_{A,B}(z)$ is analytic everywhere in $\mathbb{C}^+$ and maps $\mathbb{C}^+$ to $\mathbb{C}^-$, where $\mathbb{C}^+ = \{z : \mathrm{Im} z \geq 0\}$ and $\mathbb{C}^- = \{z : \mathrm{Im} z \leq 0\}$ ([6]). These functions are called *subordination functions* for the pair (μ_A, μ_B).

We define $t_A(z) = z - S_A(z)$ and similarly for t_B. Then $t_{A,B}(z)$ are analytic everywhere in $\mathbb{C}^+$ and $\mathrm{Im} t_{A,B}(z) \geq \mathrm{Im} z$.

From the definition of $t_{A,B}(z)$, and the third equation in system (1), it follows that $m_{\boxplus}(z) = (z - t_A(z) - t_B(z))^{-1}$, and the first and second equations of (1) imply a system of equations for t_A and t_B:

$$\begin{aligned} \frac{1}{z - t_A(z) - t_B(z)} &= m_A(t_B(z)), \\ \frac{1}{z - t_A(z) - t_B(z)} &= m_B(t_A(z)). \end{aligned} \tag{2}$$

For large z, $t_A(z) \sim z + O(1)$, and similarly for $t_B(z)$.

The analytic solutions of system (2) that satisfy this asymptotic condition at infinity are unique in $\mathbb{C}^+ = \{z | \mathrm{Im} z > 0\}$. This follows from the facts that the solutions are unique in the area $\mathrm{Im}z \geq \eta_0$ for sufficiently large η_0 and that the analytic continuation in a simply-connected domain is unique.

Definition 1.1. *A pair of probability measures on the real line* (μ_A, μ_B) *is called* smooth *at* E *if the limits* $t_A(E) = \lim_{\eta\downarrow 0} \mathrm{Im}t_A(E+i\eta)$ *and* $t_B(E) = \lim_{\eta\downarrow 0} \mathrm{Im}t_B(E+i\eta)$ *exist and if both* $\mathrm{Im}t_A(E)$ *and* $\mathrm{Im}t_B(E)$ *are positive.*

Definition 1.2. *A point* $E \in \mathbb{R}$ *is called* generic *with respect to measures* (μ_α, μ_β) *if the following equation holds:*

$$k(E) := \frac{1}{m_\alpha'(t_\beta(E))} + \frac{1}{m_\beta'(t_\alpha(E))} - (E - t_\alpha(E) - t_\beta(E))^2 \neq 0. \tag{3}$$

Let μ_1 and μ_2 be two probability measures on the real line, and let $F_1(t)$ and $F_2(t)$ be their cumulative distribution functions. The *Levy distance* between measures μ_1 and μ_2 is defined by the formula:

$$d_L(\mu_1, \mu_2) = \sup_x \inf \{s \geq 0 : F_2(x-s) - s \leq F_1(x) \leq F_2(x+s) + s\}.$$

The Levy distance is in fact a metric on the space of probability measures and the convergence with respect to this metric is equivalent to the weak convergence of measures.

Here is the main result of this paper.

Theorem 1.3. *Assume that a pair of probability measures* (μ_A, μ_B) *is smooth at* E *and that* E *is generic for* (μ_A, μ_B)*. Then, for some positive* s_0 *and* c *and all pairs of probability measures* (ν_A, ν_B) *such that* $d_L(\mu_A, \nu_A) < s \leq s_0$ *and* $d_L(\mu_B, \nu_B) < s \leq s_0$*, it is true that* $\nu_A \boxplus \nu_B$ *is absolutely continuous in a neighborhood of* E *with the density* $\rho_{\boxplus,\nu}$*, and*

$$|\rho_{\boxplus,\nu}(E) - \rho_{\boxplus}(E)| < cs.$$

This theorem will be proved as a corollary to Proposition 2.4 below. The assumptions of the theorem are sufficient but possibly not necessary. Of course, it is necessary to require that $\mu_A \boxplus \mu_B$ be absolutely continuous in a neighborhood of E so that the density $\rho_\boxplus(E)$ is defined. However, it is not clear if this assumption alone implies the statement of the theorem.

The constant c in the theorem can be bounded in terms of $t_A(E)$, $t_B(E)$ and $|k(E)|$. In particular, if $t_A(E)$, $t_B(E)$ and $|k(E)|$ are uniformly bounded away from zero for all $E \in (a,b)$, then $\sup_{E\in(a,b)} |\rho_{\boxplus,\nu}(E) - \rho_\boxplus(E)| < cs$ for some $c > 0$.

The main ideas of the proof of Theorem 1.3 are as follows. Let $m_{\boxplus,\nu}(z)$ denote the Stieltjes transform of $\nu_A \boxplus \nu_B$. (We also define $S_{A,\nu}$, $S_{B,\nu}$, $t_{A,\nu}$, and $t_{B,\nu}$ for the pair (ν_A, ν_B) similarly to corresponding objects for the pair (μ_A, μ_B), which were denoted S_A, S_B, t_A, and t_B.) First, we prove that the smallness of $d_L(\mu_A, \nu_A)$ and $d_L(\mu_B, \nu_B)$ implies that the differences $|m_{A,\nu} - m_A|$ and $|m_{B,\nu} - m_B|$ are small, as well as the differences between derivatives of the Stieltjes transforms. Then we show that this fact, together with system (2), implies that the difference between corresponding t functions is small. At this stage we need the assumption of smoothness of the pair of measures as well as condition (3). Finally, we check that if both Stieltjes transforms and t-functions are close to each other, then the Stieltjes transforms of

$\mu_A \boxplus \mu_B$ and $\nu_A \boxplus \nu_B$ are close to each other, which implies that the densities of these free convolutions are close to each other.

In the case when $\mu_A = \mu_B = \mu$, the smoothness of the pair (μ, μ) is easy to check by using the following proposition.

Proposition 1.4. *If $\mu \boxplus \mu$ is (Lebesgue) absolutely continuous in a neighborhood of E and the density of $\mu \boxplus \mu$ is positive at E, then (μ, μ) is smooth at E.*

Note that even if the measures μ_A and μ_B are discrete, their free convolution can be Lebesgue absolutely continuous everywhere. For example, this is true for $\mu_A = \mu_B = (\delta_{-1} + \delta_0 + \delta_1)/3$. Moreover, by results in [1], if measure $\mu_A \boxplus \mu_B$ is absolutely continuous in a neighborhood of x_0 and its density is positive at x_0, then the density is analytic.

The idea of the proof of this proposition is that if $t_A(z) = t_B(z) := t(z)$, then we can express it as $t(z) = (1/2)\left(z - m_{\boxplus}(z)^{-1}\right)$, and the smoothness of $\mu \boxplus \mu$ implies that $m_{\boxplus}(z)$ is bounded and has a limit when z approaches the real axis. Therefore, $\mathrm{Im} t(z)$ is bounded and positive as $\mathrm{Im} z \downarrow 0$, and the pair (μ, μ) is smooth. The details will be given in Section 3.

Another important case is when one of the probability measures is semicircle. The semicircle distribution λ is an absolutely continuous distribution supported on the interval $[-2, 2]$ and its density is

$$\lambda(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

on this interval.

Since λ is absolutely continuous, $\lambda \boxplus \mu$ is always absolutely continuous.

Proposition 1.5. *Let $\mu_A = \lambda$ be the semicircle distribution. If the density of $\lambda \boxplus \mu_B$ is positive at E, and $|m_{\boxplus}(E)| \neq 1$, then (λ, μ_B) is smooth at E.*

Theorem 1.3 can be applied to derive some new results about sums of free random variables and about eigenvalues of large random matrices.

If $X_1, \ldots, X_n$ are free, identically distributed self-adjoint random variables with finite variance σ^2, then it is known ([19],[14]) that $S_n := (X_1 + \ldots + X_n) / (\sigma\sqrt{n})$ converges in distribution to a r.v. X with the standard semicircle law. Bercovici and Voiculescu in [3] showed that the convergence in this limit law holds in a stronger sense. Namely, assuming that X_i are bounded, they showed that S_n has a density for all sufficiently large n and that the sequence of these densities converges uniformly to the density of the semicircle law. Recently, this result was generalized in [24] to the case of possibly unbounded X_i with finite variance. This result can be considered as a local limit version of the free CLT.

In the first application (Theorem 4.1), we give a short proof of this result by using Theorem 1.3.

In the second application (Theorem 4.2), we prove an analogous local limit result for the sums $S_n = X_{1,n} + \ldots + X_{n,n}$, where $X_{i,n}$ are free projection operators with parameters $p_{i,n}$ such that $\sum_{i=1}^{n} p_{i,n} \to \lambda$ and $\max_i p_{i,n} \to 0$ as $n \to \infty$. The classical analogue of this situation is the sum of independent indicator random variables, and the classical result states that the sums converge in distribution to the Poisson r.v. with parameter λ. A local version of this result is absent in the classical case because the Poisson r.v. is discrete. In the free case, the limit is a r.v. with the Marchenko-Pastur law, which is continuous with bounded density for $\lambda > 1$. We show that in this case the sum S_n has a density for all sufficiently large n and the sequence of these densities converges uniformly to the density of the Marchenko-Pastur law.

In the third application (Theorem 4.3), we show that a similar local limit result holds for sums of free $\boxplus$-stable random variables.

The fourth application (Theorem 4.4) is of a different kind and is concerned with eigenvalues of large random matrices. Let $H_N = A_N + U_N B_N U_N^*$, where A_N and B_N are N-by-N Hermitian matrices, and U_N is a random unitary matrix with the Haar distribution on the unitary group $\mathcal{U}(N)$. Let $\lambda_1^{(A)} \geq \ldots \geq \lambda_N^{(A)}$ be the eigenvalues of A_N. Similarly, let $\lambda_k^{(B)}$ and $\lambda_k^{(H)}$ be ordered eigenvalues of matrices B_N and H_N, respectively. Define the *spectral point measures* of A_N by $\mu_{A_N} := N^{-1} \sum_{k=1}^N \delta_{\lambda_k^{(A)}(H)}$, and define the spectral point measures of B_N and H_N similarly.

Assume that $\mu_{A_N} \to \mu_\alpha$ and $\mu_{B_N} \to \mu_\beta$. and that the support of μ_{A_N} and μ_{B_N} is unifomly bounded. Let the pair (μ_α, μ_β) be smooth at E, and E be generic for (μ_α, μ_β).

Define $\mathcal{N}_I := N\mu_{H_N}(I)$, the number of eigenvalues of H_N in interval I, and let $\mathcal{N}_\eta(E) := \mathcal{N}_{(E-\eta, E+\eta]}$. Finally, assume that $\eta = \eta(N)$ and $\frac{1}{\sqrt{\log(N)}} \ll \eta(N) \ll 1$.

Then, by using the author's previous results from [13], and Theorem 1.3, it is shown that

$$\frac{\mathcal{N}_\eta(E)}{\eta N} \to \rho_\boxplus(E),$$

with probability 1, where $\rho_\boxplus$ denotes the density of $\mu_\alpha \boxplus \mu_\beta$. This result can be interpreted as a local limit law for eigenvalues of a sum of random Hermitian matrices.

The rest of the paper is organized as follows. Section 2 is concerned with the proof of the main theorem, Section 3 contains proofs of Propositions 1.4 and 1.5, Section 4 contains applications, and Section 5 concludes.

2. PROOF OF THEOREM 1.3

Let $F_1(x)$ and $F_2(x)$ denote the cumulative distribution functions of measures μ_1 and μ_2, respectively.

Lemma 2.1. *Let $d_L(\mu_1, \mu_2) = s$. Assume that $h(x)$ is a C^1 real-valued function, such that $\int_{-\infty}^{\infty} |h(u)| < \infty$ and $\int_{-\infty}^{\infty} |h'(u)| < \infty$. Assume that $h(u)$ has a finite number of zeroes. Then,*

$$\Delta := \left| \int_{\mathbb{R}} h(u) F_2(\eta u) \, du - \int_{\mathbb{R}} h(u) F_1(\eta u) \, du \right| \leq cs \max\left\{1, \eta^{-1}\right\},$$

where $c > 0$ depends only on h.

Proof: Let A be the set where $h > 0$ and B the set where $h < 0$. These sets are unions of open intevals because h is a differentiable function. The boundary of these sets consists of the points where $h = 0$. By the definition of the Levy distance,

$$\begin{aligned}\Delta &\leq \int_A h(u)\left[F_1(\eta u + s) + s - F_1(\eta u)\right]du \\ &\quad + \int_B h(u)\left[F_2(\eta u) - F_2(\eta u + s) - s\right]du.\end{aligned}$$

The estimation of these two integrals is similar. Consider, for example, the first one.

First of all, note that

$$\int_A h(u)\, s du < cs$$

because $\int_{-\infty}^{\infty} |h(u)| < \infty$.

Next, let $\widetilde{A} = A + s/\eta$. Then,

$$\int_A h(u)\, F_1(\eta u + s)\, du = \int_{\widetilde{A}} h(t - s/\eta)\, F_1(\eta t)\, dt,$$

and therefore,

$$\begin{aligned}\int_A h(u)\left[F_1(\eta u + s) - F_1(\eta u)\right] du &\leq \int_{A\cap\widetilde{A}} \left[h(t - s/\eta) - h(t)\right] F_1(\eta t)\, dt \\ &\quad + \int_{A\triangle\widetilde{A}} \max\left(|h(t - s/\eta)|, |h(t)|\right) F_1(\eta t)\, dt.\end{aligned}$$

For the first integral in this estimate, we can use the fact that

$$h(t - s/\eta) - h(t) = -\int_{t-s/\eta}^{t} h'(\xi)\, d\xi,$$

and therefore,

$$\begin{aligned}\left|\int_{A\cap\widetilde{A}} \left[h(t - s/\eta) - h(t)\right] F_1(\eta t)\, dt\right| &\leq \int_{\mathbb{R}} \int_{t-s/\eta}^{t} \left|h'(\xi)\right| F_1(\eta t)\, d\xi dt \\ &= \int_{\mathbb{R}} \left|h'(\xi)\right| \left(\int_{\xi}^{\xi+s/\eta} F_1(\eta t)\, dt\right) d\xi \\ &\leq \frac{s}{\eta} \int_{\mathbb{R}} \left|h'(\xi)\right| d\xi.\end{aligned}$$

For the second integral, we note that

$$\begin{aligned}\int_{A\triangle\widetilde{A}} \max\left(|h(t - s/\eta)|, |h(t)|\right) F_1(\eta t)\, dt &\leq \sup|h(t)| \left|A \triangle \widetilde{A}\right| \\ &\leq \sup|h(t)|\, Ks/\eta,\end{aligned}$$

where K is the number of points in the border of A, that is, the number of zeroes of $h(t)$.

By using all these estimates, we obtain:

$$\Delta \leq cs \max\left\{1, \eta^{-1}\right\},$$

where c depends only on function $h(t)$. $\square$.

Now, let $m_1(z)$ and $m_2(z)$ denote the Stieltjes transforms of measures μ_1 and μ_2, respectively.

Lemma 2.2. *Let $d_L(\mu_1,\mu_2)=s$ and $z=E+i\eta$, where $\eta>0$.Then,*
(a) $|m_1(z)-m_2(z)|<cs\eta^{-1}\max\{1,\eta^{-1}\}$ where $c>0$ is a numeric constant, and
(b) $\left|\frac{d^r}{dz^r}(m_1(z)-m_2(z))\right|<c_r s\eta^{-1-r}\max\{1,\eta^{-1}\}$ where $c>0$ are numeric constants.

Proof: (a) By integration by parts, we can write,

$$m_1(z)=\int_{-\infty}^{\infty}\frac{F_1(\lambda)}{(\lambda-z)^2}d\lambda.$$

Hence,

$$\begin{aligned}\mathrm{Im}m_1(z) &= \int_{-\infty}^{\infty}\frac{2\eta(\lambda-E)}{\left((\lambda-E)^2+\eta^2\right)^2}F_1(\lambda)\,d\lambda\\ &= \frac{2}{\eta^2}\int_{-\infty}^{\infty}\frac{\frac{\lambda-E}{\eta}}{\left(\left(\frac{\lambda-E}{\eta}\right)^2+1\right)^2}F_1(\lambda)\,d\lambda\\ &= \frac{2}{\eta}\int_{-\infty}^{\infty}F_1(E+\eta u)\frac{udu}{(1+u^2)^2}.\end{aligned}$$

Similarly,

$$\mathrm{Re}m_1(z)=\frac{1}{\eta}\int_{-\infty}^{\infty}F_1(E+\eta u)\frac{\left(u^2-1\right)du}{(1+u^2)^2}.$$

By translating measures μ_1 and μ_2 by E, we can assume that $E=0$ in the formulas for $\mathrm{Im}m_1(z)$ and $\mathrm{Re}m_1(z)$. Hence, the claim (i) follows from Lemma 2.1. Claim (ii) can be derived similarly by using the fact that

$$\frac{d^r}{d\lambda^r}\frac{1}{\lambda-i\eta}=\frac{1}{(\lambda-i\eta)^{r+1}}.$$

□

Lemma 2.3. *Assume that the pair (μ_A,μ_B) is smooth at E. Suppose that (ν_A,ν_B) is another pair of probability measures such that $d_L(\mu_A,\nu_A)<s$ and $d_L(\mu_B,\nu_B)<s$. Let $z=E+i\eta$. Then,*

$$\left|\frac{1}{z-t_A(z)-t_B(z)}-m_{A,\nu}(t_B(z))\right|\le cs,$$

and

$$\left|\frac{1}{z-t_A(z)-t_B(z)}-m_{B,\nu}(t_A(z))\right|\le cs,$$

where c is a positive constant that depends only on the pair of measures μ_A and μ_B.

That is, if we substitute t_A and t_B in the system for $t_{A,\nu}$ and $t_{B,\nu}$, then the error is small.

Proof: t_A and t_B satisfy the equations of system (2), which implies that it is enough to show that

$$|m_{A,\nu}(t_B(z))-m_A(t_B(z))|<cs$$

and

$$|m_{B,\nu}(t_A(z))-m_B(t_A(z))|<cs$$

for all $z = E + i\eta$. This follows directly from Lemma 2.2 and the assumption that the pair (μ_A, μ_B) is smooth at E. Indeed, let $\eta_0(E) > 0$ denote $\min\{\mathrm{Im} t_A(E), \mathrm{Im} t_B(E)\}$. Then, by Lemma 2.2,

$$|m_{A,\nu}(t_B(z)) - m_A(t_B(z))| < cs \min\left\{\frac{1}{\eta_0(E)}, \frac{1}{\eta_0^2(E)}\right\},$$

and a similar estimate holds for the difference $|m_{B,\nu}(t_A(z)) - m_B(t_A(z))|$. □

Proposition 2.4. *Assume that a pair of probability measures (μ_A, μ_B) is smooth at E and that E is generic for (μ_A, μ_B). Then for some positive s_0 and c and all pairs of probability measures (ν_A, ν_B) such that $d_L(\mu_A, \nu_A) < s \le s_0$ and $d_L(\mu_B, \nu_B) < s \le s_0$, the limits $t_{A,\nu}(E) := \lim_{\eta\downarrow 0} t_{A,\nu}(E + i\eta)$ and $t_{B,\nu}(E) := \lim_{\eta\downarrow 0} t_{B,\nu}(E + i\eta)$ exist, and it is true that*

$$|t_{A,\nu}(E) - t_A(E)| < cs,$$

and

$$|t_{B,\nu}(E) - t_B(E)| < cs.$$

Corollary 2.5. *Assume that assumptions of Proposition 2.4 hold. Then, $\nu_A \boxplus \nu_B$ is absolutely continuous in a neighborhood of E with the density $\rho_{\boxplus,\nu}$, and*

$$|\rho_{\boxplus,\nu}(E) - \rho_{\boxplus}(E)| < cs.$$

Proof of Corollary: Since $m_{\boxplus,\nu}(z) = (z - t_{A,\nu}(z) - t_{B,\nu}(z))^{-1}$, hence Proposition 2.4 implies that the limit $m_{\boxplus,\nu}(E) = \lim_{\eta\downarrow 0} m_{\boxplus,\nu}(E + i\eta)$ exists and

$$|m_{\boxplus,\nu}(E) - m_{\boxplus}(E)| < cs.$$

Moreover, this is true for every x in a neighborhood of E with a uniform constant c. This implies that for all sufficiently small s, the measure $\nu_A \boxplus \nu_B$ is absolutely continuous in a neighborhood of E with the density $\rho_{\boxplus,\nu}$, and

$$|\rho_{\boxplus,\nu}(E) - \rho_{\boxplus}(E)| < cs.$$

□

Proof of Proposition 2.4: Let $F(t) : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by the formula:

$$F : \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \to \begin{pmatrix} (z - t_1 - t_2)^{-1} - m_{A,\nu}(t_2) \\ (z - t_1 - t_2)^{-1} - m_{B,\nu}(t_1) \end{pmatrix}.$$

Let us use the norm $\|(x_1, x_2)\| = \left(|x_1|^2 + |x_2|^2\right)^{-1/2}$. By Lemma 2.3, $\|F(t_A(z), t_B(z))\| \le cs$ for all $z = E + i\eta$ and $\eta \ge 0$.

The derivative of F with respect to t is

$$F' = \begin{pmatrix} (z - t_1 - t_2)^{-2} & (z - t_1 - t_2)^{-2} - m'_{A,\nu}(t_2) \\ (z - t_1 - t_2)^{-2} - m'_{B,\nu}(t_1) & (z - t_1 - t_2)^{-2} \end{pmatrix}.$$

The determinant of this matrix is $\left[m'_{A,\nu}\left(t_2\right)+m'_{B,\nu}\left(t_1\right)\right]\left(z-t_1-t_2\right)^{-2}-m'_{A,\nu}\left(t_2\right)m'_{B,\nu}\left(t_1\right).$ By assumption of smoothness and by Lemma 2.2, this is close (i.e., $< cs$ for some $c > 0$) to $\left[m'_A\left(t_2\right)+m'_B\left(t_1\right)\right]\left(z-t_1-t_2\right)^{-2}-m'_A\left(t_2\right)m'_B\left(t_1\right)$ at $(t_1,t_2)=\left(t_A\left(z\right),t_B\left(z\right)\right)$ for all z in a neighborhood of E. The latter expression is non-zero by assumption (3). In addition, the assumption of smoothness shows that $\left(z-t_1-t_2\right)^{-2}$ is bounded in a neighborhood of E. Hence, the entries of the matrix $[F']^{-1}$ are bounded at $\left(t_A\left(z\right),t_B\left(z\right)\right)$ for all z in a neighborhood of E. This shows that operator norm of $[F']^{-1}$ is bounded.

Similarly, the assumption of smoothness of (μ_A,μ_B) and Lemma 2.2 imply that the operator norm of F'' is bounded for all (t_1,t_2) in a neighborhood of $\left(t_A\left(z\right),t_B\left(z\right)\right)$ and for all values of parameter z in a neighborhood of E.

It follows by the Newton-Kantorovich theorem ([12]) that the solution of the equation $F(t)=0$, which is $(t_{A,\nu}(z),t_{B,\nu}(z))$, exists for all $z=E+i\eta$ with $\eta\geq 0$ and satisfies the inequalities:

$$\left|t_{A,\nu}\left(z\right)-t_A\left(z\right)\right|<cs$$

and

$$\left|t_{B,\nu}\left(z\right)-t_B\left(z\right)\right|<cs$$

for all z with η sufficiently close to 0.

Since $t_{A,\nu}\left(z\right)$ and $t_A\left(z\right)$ are analytic in $\mathbb{C}^+=\{z|\mathrm{Im}z>0\}$ and $\lim_{\eta\downarrow 0}t_A\left(E+i\eta\right)$ exists, the fact that $\left|t_{A,\nu}\left(z\right)-t_A\left(z\right)\right|$ is bounded implies that $t_{A,\nu}\left(E\right):=\lim_{\eta\downarrow 0}t_{A,\nu}\left(E+i\eta\right)$ exists. A similar argument shows the existence of $t_{B,\nu}\left(E\right)$. Finally, by taking the limit we find that

$$\left|t_{A,\nu}\left(E\right)-t_A\left(E\right)\right|<cs$$

and

$$\left|t_{B,\nu}\left(E\right)-t_B\left(E\right)\right|<cs.$$

□

3. PROOF OF PROPOSITIONS 1.4 AND 1.5

Recall that a function $f\left(x\right)$ is called Hölder continuous at x_0 if there exist positive constants α, C, and ε such that $\left|f(x)-f\left(x_0\right)\right|<C\left|x-x_0\right|^{\alpha}$ for all x such that $\left|x-x_0\right|<\varepsilon$.

Lemma 3.1. *Suppose that a probability measure μ has a density which is positive and Hölder continuous at E. Let $m\left(z\right)$ be the Stieltjes transform of μ. Then $\left|m\left(E+i\eta\right)\right|\leq M<\infty$ for all $\eta>0$.*

Proof: The results of Sokhotskyi, Plemelj, and Privalov ensure that the limit of $m\left(E+i\eta\right)$ exists when $\eta\downarrow 0$ (see Theorems 14.1b and 14.1c in [11]). Since $m\left(E+i\eta\right)$ is continuous in the upper half-plane and $\left|m\left(E+i\eta\right)\right|\leq 1/\eta$, the claim of the lemma follows. □

Proof of Proposition 1.4: Note that for the case $\mu_A=\mu_B=\mu$,

$$t\left(z\right)=\left(z-m_{\boxplus}\left(z\right)^{-1}\right)/2. \tag{4}$$

Since $\mu \boxplus \mu$ is absolutely continuous in a neighborhood of E, and the density $\rho_{\boxplus}$ is positive at E, hence by results in [1] $\rho_{\boxplus}$ is analytic and therefore uniformly Hölder continuous in a neighborhood of E. This implies that the limit $m_{\boxplus}(E) = \lim_{\eta \downarrow 0} m_{\boxplus}(E + i\eta)$ exists and that $\mathrm{Im} m_{\boxplus}(E) = \pi \rho_{\boxplus}(E)$. Since $\rho_{\boxplus}(E) > 0$ by assumption, it is clear from (4) that the limit $t(E) = \lim_{\eta \downarrow 0} t(E + i\eta)$ exists. Moreover, since

$$\mathrm{Im} t(z) = \frac{1}{2}\left(\eta + \frac{\mathrm{Im} m_{\boxplus}(z)}{|m_{\boxplus}(z)|^2}\right),$$

and by Lemma 3.1, $\mathrm{Re} m_{\boxplus}(z)$ is bounded uniformly in η, hence $\mathrm{Im} t(E) > 0$. It follows that (μ, μ) is smooth at E. This completes the proof of the proposition. □

Lemma 3.2. *If μ_A has the semicircle distribution, then*
(i) $t_B = z - t_A + \frac{1}{z - t_A}$;
(ii) $m_{\boxplus} = t_A - z$, and
(iii) t_A satisfies the equation

$$t_A = z + \int \frac{\mu_B(dx)}{x - t_A}.$$

Proof: (i) If μ_A has the semicircle distribution, then $m_A^{(-1)} = -\left(z + z^{-1}\right)$; hence the first equation in system (2) implies

$$t_B = -\left(\frac{1}{z - t_A - t_B} + z - t_A - t_B\right),$$

which simplifies to

$$t_B = z - t_A + \frac{1}{z - t_A}.$$

(ii) By using (i),

$$m_{\boxplus} = \frac{1}{z - t_A - t_B} = -(z - t_A).$$

(iii) The second equation in system (2) becomes

$$-(z - t_A) = \int \frac{\mu_B(dx)}{x - t_A}.$$

□

Proof of Proposition 1.5: From (ii) in Lemma 3.2, $\mathrm{Im} t_A(E) = \mathrm{Im} m_{\boxplus}(E) = \pi \rho_{\boxplus}(E) > 0$. From (i),

$$\begin{aligned} \mathrm{Im} t_B(E) &= \mathrm{Im} t_A(E)\left(-1 + \frac{1}{|E - t_A|^2}\right) \\ &= \mathrm{Im} t_A(E)\left(-1 + \frac{1}{|m_{\boxplus}(E)|^2}\right). \end{aligned}$$

If $|m_{\boxplus}(E)|^2 < 1$ and $\mathrm{Im} t_A(E) > 0$, then $\mathrm{Im} t_B(E) > 0$ and (λ, μ_B) is smooth. It is not possible that $|m_{\boxplus}(E)|^2 > 1$ and $\mathrm{Im} t_A(E) > 0$ because this would imply that $\mathrm{Im} t_B(E) < 0$ which is impossible by a general result of Biane. Hence, $\rho_{\boxplus}(E) > 0$ and $|m_{\boxplus}(E)|^2 \neq 1$ imply that (λ, μ_B) is smooth. □

4. APPLICATIONS

In the first application we re-prove the free local limit theorem which was first demonstrated in [3] for bounded random variables and later generalized in [24] to the case of unbounded variables with finite variance.

Let X_i be a sequence of self-adjoint random variables in the sense of free probability theory. Define $S_n = (X_1 + \ldots + X_n)/\sqrt{n}$, and let μ_n denote the spectral probability measure of S_n.

Theorem 4.1. *Suppose X_i are freely independent, identically distributed with zero mean and unit variance. Let $I_\varepsilon = [-2+\varepsilon, 2-\varepsilon]$. Then for all sufficiently large n, μ_n is (Lebesgue) absolutely continuous everywhere on I, and the density $d\mu_n/dx$ uniformly converges on I_ε to the density of the standard semicircle law.*

Note that the results in [3] imply that for every closed interval J outside of $[2,-2]$, the measure $\mu_n(J) = 0$ for all sufficiently large n, provided that μ_1 has bounded support. In addition, the uniform convergence on I_ε can be strengthened to the uniform convergence on $\mathbb{R}$ as in the proof of Theorem 3.4(iii) in [24].

Proof: Let $\nu_{A,n}$ be the distribution of $\left(X_1 + \ldots + X_{[n/2]}\right)/\sqrt{n}$ and $\nu_{B,n}$ be the distribution of $\left(X_{[n/2]+1} + \ldots + X_n\right)/\sqrt{n}$. By using the free CLT from [14] (which generalizes the result in [19]), we infer that $\nu_{A,n}$ and $\nu_{B,n}$ converge weakly to $\mu_A = \mu_B = \mu$, where μ is the semicircle law with variance $1/2$. It is easy to calculate

$$t_A = t_B = \frac{3z + \sqrt{z^2 - 4}}{4},$$

and therefore the pair (μ, μ) is smooth on I_ε. This also follows from Proposition 1.4. A calculation shows that condition (3) is satisfied for each $E \in I_\varepsilon$, and therefore the density of $\nu_{A,n} \boxplus \nu_{B,n}$ exists for all sufficiently large n, and converges to the density of $\mu \boxplus \mu$ at each $E \in I_\varepsilon$. A remark after Theorem 1.3 shows that the convergence is in fact uniform. Since $\nu_{A,n} \boxplus \nu_{B,n} = \mu_n$, this implies that the density of μ_n converges uniformly on I_ε to the density of the standard semicircle law. □

In a similar fashion, it is possible to prove the local limit law for the convergence to the free Poisson distribution.

Let $\{X_{n,i}\}_{i=1}^n$ be freely independent self-adjoint random variables with the distribution $\mu_{n,i} = p_{n,i}\delta_1 + (1 - p_{n,i})\,\delta_0$. Let $S_n = X_{n,1} + \ldots + X_{n,n}$ and let μ_n denote the spectral probability measure of S_n. Recall that the *Marchenko-Pastur law* with parameter $\lambda \geq 1$ is the probability measure with the density

$$p(x) = \frac{\sqrt{4x - (1 - \lambda + x)^2}}{2\pi x}$$

on the interval $[x_{\min}, x_{\max}] := \left[\left(1 - \sqrt{\lambda}\right)^2, \left(1 + \sqrt{\lambda}\right)^2\right]$. In the free probability literature, this distribution is called the free Poisson distribution.

Theorem 4.2. *Assume that* $\sum_{i=1}^{n} p_{n,i} \to \lambda > 1$ *and* $\max_i p_{n,i} \to 0$ *as* $n \to \infty$*. Let* $I_\varepsilon = [x_{\min} + \varepsilon, x_{\max} - \varepsilon]$*. Then for all sufficiently large* n*,* μ_n *is (Lebesgue) absolutely continuous everywhere on* I_ε*, and the density* $d\mu_n/dx$ *uniformly converges on* I_ε *to the density of the Marchenko-Pastur law with parameter* λ*.*

The proof of this theorem is similar to the proof of the previous one. The first step is the proof of the weak convergence of μ_n. In the case when $p_{n,i} = \lambda/n$ for all i, it can be found on page 34 in [18]. The general case is a minor adaptation of this case and we omit it. Next, we choose k_n so that

$$\sum_{i=1}^{k_n} p_{n,i} \leq \lambda/2 < \sum_{i=1}^{k_n+1} p_{n,i}$$

and define $\nu_{A,n}$ and $\nu_{B,n}$ as the spectral probability measures of $X_{n,1} + \ldots + X_{n,k_n}$ and $X_{n,k_n+1} + \ldots + X_{n,n}$, respectively. It is easy to see that both $\nu_{A,n}$ and $\nu_{B,n}$ converge weakly to $\mu_A = \mu_B = \mu$, the Marchenko-Pastur distribution with parameter $\lambda/2$. By using Proposition 1.4, we conclude that the pair (μ, μ) is smooth for every point in I_ε. Moreover, a direct calculation shows that

$$t_A(z) = t_B(z) = \frac{1}{4}\left(z + \lambda - 1 + \sqrt{(z - (1+\lambda))^2 - 4\lambda}\right),$$

and

$$m'_A = m'_B = \frac{1 - \lambda/2}{2z^2} + \frac{-z(1+\lambda/2) + (1-\lambda/2)^2}{2z^2\sqrt{(z - (1+\lambda/2))^2 - 2\lambda}}.$$

After some calculations the violation of the genericity condition (3) can be simplified to the following equation:

$$f(E,\lambda) := E^3 - \left(5 + \frac{5}{2}\lambda\right)E^2 + \left(7 + \frac{13}{2}\lambda + 2\lambda^2\right)E - (3 - 5\lambda + \frac{5}{4}\lambda^2 + \frac{1}{2}\lambda^3) = 0. \quad (5)$$

Figure 4 shows the contour plot of $f(E,\lambda)$. It can be seen from this plot and can be checked formally that for $\lambda > 1$, there is only one $E = E(\lambda)$ that satisfies equation (5). Figure 4 shows the zero set of $f(E,\lambda)$ for $\lambda > 1$, compared with the bounds on the support of the Marchenko-Pastur distribution. It can be seen from this graph and can be checked formally that $E(\lambda) < t_{\min}(\lambda) = \left(1 - \sqrt{\lambda}\right)^2$. Consequently, E is always generic for (μ, μ) and Theorem 1.3 applies. Hence, the density of μ_n converges uniformly on I_ε to the density of the Marchenko-Pastur distribution with parameter λ. $\square$.

Similar results can be established for other limit theorems except that it is more difficult to check the genericity of a point in the support of the limit distribution. Here is one more theorem of this type. Let measures μ and ν be called equivalent ($\mu \sim \nu$) if there exist such real a and b, with $a > 0$, that for every Borel set $S \subset \mathbb{R}$, $\mu(S) = \nu(aS + b)$. Recall that measure μ is called $\boxplus$-stable, if $\mu \boxplus \mu \sim \mu$. Measure ν belongs to the domain of attraction of a $\boxplus$-stable law μ, if there exist measures ν_n equivalent to ν such that

$$\underbrace{\nu_n \boxplus \nu_n \boxplus \ldots \boxplus \nu_n}_{n\text{-times}} \to \mu.$$

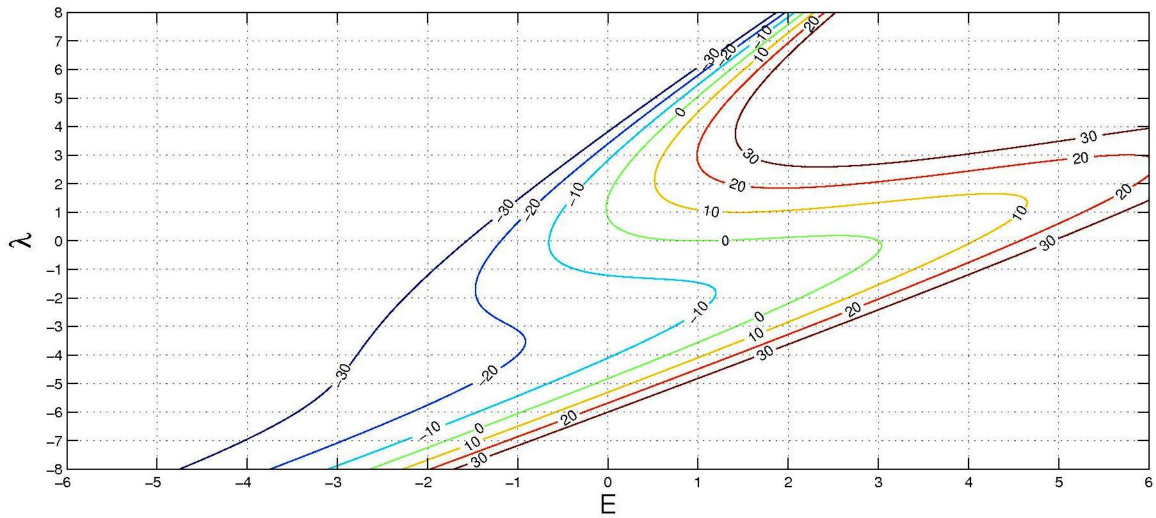

FIGURE 1. Contour plot of the right-hand side of equation 5

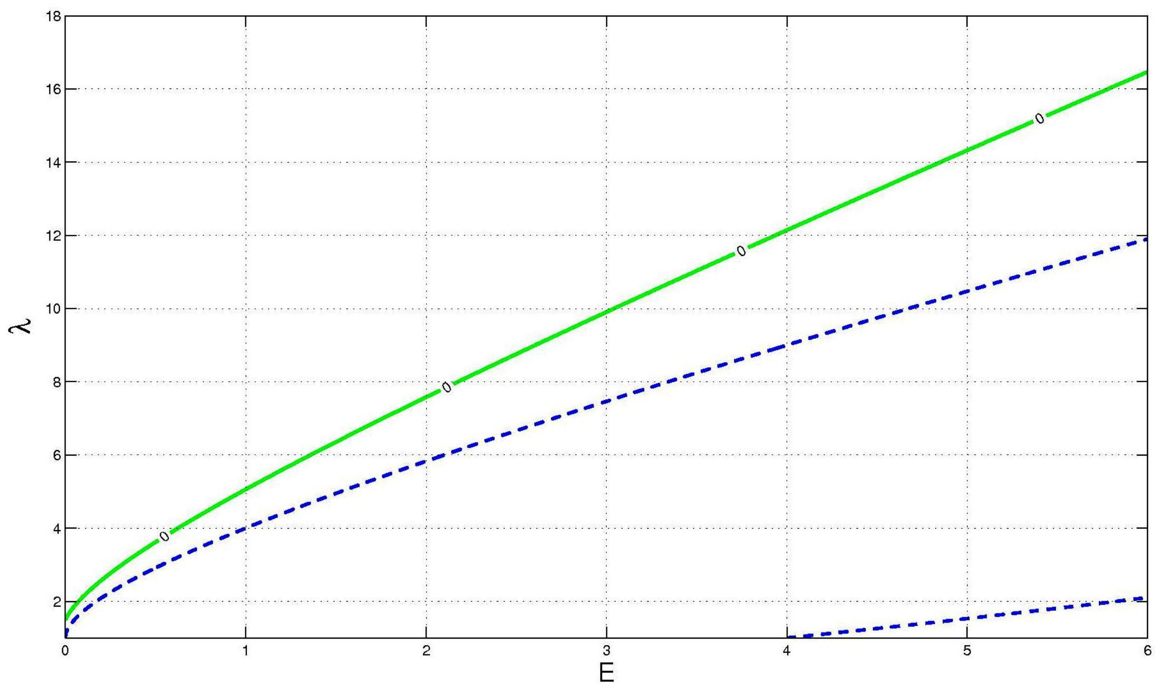

FIGURE 2. The zero set of the right-hand side of Equation 5 compared with the support bounds for E

Clearly, in this case there exists a sequence of real constants $B_n > 0$ and A_n such that if X_n are freely independent, identically distributed self-adjoint random variables with distribution ν, then

$$\frac{X_1 + \ldots + X_n}{B_n} - A_n \to X, \tag{6}$$

where X is a random variable with distribution μ. (More about the $\boxplus$-stability of probability measures and its relation to the classical stability of probability measures can be found in [5]).

Let μ_n denote the measure of

$$\frac{X_1 + \ldots + X_n}{B_n} - A_n.$$

Theorem 4.3. *Suppose a $\boxplus$-stable distribution μ is not equivalent to δ_0. Let J be a bounded closed interval such that the density of μ is strictly positive on J. Suppose that A_n, B_n, and μ_n are defined as in the limit law (6). Then μ_n is (Lebesgue) absolutely continuous on J for all sufficiently large n and there is a sequence of constants a_n and b_n such that the density $\frac{d\mu_n}{dx}(E)$ univormly converges to the density $\frac{d\mu_n}{dx}(E)$ at (Lebesgue) almost all $E \in J$.*

Proof: Let $J \subset I$, where the inclusion is strict and I is a larger bounded closed interval such that density of μ is strictly positive on I. It is possible to find such I because by the results of Biane in [5], μ is absolutely continuous with analytical density. As the first step, we will show that μ_n is (Lebesgue) absolutely continuous on I for all sufficiently large n and that there is a sequence of constants $\widetilde{a}_n$ and $\widetilde{b}_n$ such that the density $\frac{d\mu_n}{dx}\left(\widetilde{a}_n E + \widetilde{b}_n\right)$ converges to the density $\frac{d\mu_n}{dx}(E)$ at (Lebesgue) almost all $E \in I$. Note that by definition of μ_n, the convolution $\mu_n \boxplus \mu_n = \mu_{2n}(a_{2n} \cdot + b_{2n})$, where a_{2n} and b_{2n} are certain constants. Since $\mu \boxplus \mu = \mu(a \cdot + b)$ and μ has analytic density, therefore by Proposition 1.4 (μ, μ) is smooth at $E \in (I - b)/a$. Almost all points E are generic for (μ, μ), since otherwise $k(z)$ (in the genericity condition) would be exaclty 0 which is not possible. Hence, by Theorem 1.3 the weak convergence $\mu_n \to \mu$ implies that $\mu_{2n}(a_{2n} \cdot + b_{2n})$ has a density on $(I - b)/a$ for large n and that the density uniformly converges to the density of $\mu \boxplus \mu = \mu(a \cdot + b)$. It follows that the density of $\mu_{2n}\left(\widetilde{a}_{2n} \cdot + \widetilde{b}_{2n}\right)$ uniformly converges to the density of μ on I, where $\widetilde{a}_{2n}$ and $\widetilde{b}_{2n}$ are some constants. The case of μ_{2n+1} can be handled similarly by considering $\mu_n \boxplus \mu_{n+1}$

Since both $\mu_n\left(\widetilde{a}_n \cdot + \widetilde{b}_n\right)$ and μ_n converge to μ weakly ($\mu_n \to \mu$ by assumption and $\mu_n\left(\widetilde{a}_n \cdot + \widetilde{b}_n\right) \to \mu$ by the convergence of densities), we can conclude that the Levy distance between $\mu\left(\left(\cdot - \widetilde{b}_n\right)/\widetilde{a}_n\right)$ and μ becomes arbitrary small as $n \to \infty$. This implies that the sequences $\widetilde{a}_n$ and $\widetilde{b}_n$ converge to 1 and 0, respectively. Otherwise, we could select a subsequence of $\widetilde{a}_n$ and $\widetilde{b}_n$ that converge to a limit $(a, b) \neq (1, 0)$ and after taking the weak limit of measures $\mu\left(\left(\cdot - \widetilde{b}_n\right)/\widetilde{a}_n\right)$ we would find that $\mu((\cdot - b)/a) = \mu$, which is impossible.

Let f_n denote the density of μ_n and f the density of μ. By results of Biane in [5], density f is continuous at its support and approaches zero as $|x| \to \infty$, hence it is uniformly continuous. Moreover, for each $\varepsilon > 0$, we can find $\delta > 0$ such that if $|a_n - 1| < \delta$ and $b_n < \delta$, then

$|f(x) - f(a_n x + b_n)| < \varepsilon$ for all x. Hence, for every $\varepsilon > 0$, we can find $n_0 > 0$ such that for all $n > n_0$, $\left|f_n\left(\widetilde{a}_n x + \widetilde{b}_n\right) - f\left(\widetilde{a}_n x + \widetilde{b}_n\right)\right| < \varepsilon$ for all $x \in I \backslash S$ where S is a set of measure zero and $\widetilde{a}_n$ and $\widetilde{b}_n$ are the sequences defined above. After a change of variables we find that $|f_n(t) - f(t)| < \varepsilon$ for all $t \in \left(I \backslash S - \widetilde{b}_n\right)/\widetilde{a}_n$. By using the fact that the union of sets of measure zero has measure zero, we can conclude that $f_n(t)$ uniformly converges to $f(t)$ on $J \backslash \widetilde{S}$ where J is an arbitrary closed subinterval of I and $\widetilde{S}$ is a set of measure 0. □

Our next application is of a different kind and answers a question that arises in the theory of large random matrices.

Let $H_N = A_N + U_N B_N U_N^*$, where A_N and B_N are N-by-N Hermitian matrices, and U_N is a random unitary matrix with the Haar distribution on the unitary group $\mathcal{U}(N)$.

Let $\lambda_1^{(A)} \geq \ldots \geq \lambda_N^{(A)}$ be the eigenvalues of A_N. Similarly, let $\lambda_k^{(B)}$ and $\lambda_k^{(H)}$ be ordered eigenvalues of matrices B_N and H_N, respectively.

Define the *spectral point measures* of A_N by $\mu_{A_N} := N^{-1} \sum_{k=1}^{N} \delta_{\lambda_k^{(A)}(H)}$, and define the spectral point measures of A_N and H_N similarly. Let $\mathcal{N}_I := N\mu_{H_N}(I)$ denote the number of eigenvalues of H_N in interval I, and let $\mathcal{N}_\eta(E) := \mathcal{N}_{(E-\eta, E+\eta]}$.

Theorem 4.4. *Assume that*

(1) $\mu_{A_N} \to \mu_\alpha$ *and* $\mu_{B_N} \to \mu_\beta$.
(2) $\operatorname{supp}(\mu_{A_N}) \cup \operatorname{supp}(\mu_{B_N}) \subseteq [-K, K]$ *for all* N.
(3) *The pair* (μ_α, μ_β) *is smooth at* E, *and* E *is generic for* (μ_α, μ_β).
(4) $\frac{1}{\sqrt{\log(N)}} \ll \eta(N) \ll 1$.

Then,

$$\frac{\mathcal{N}_\eta(E)}{2\eta N} \to \rho_\boxplus(E),$$

with probability 1, where $\rho_\boxplus$ *denotes the density of* $\mu_\alpha \boxplus \mu_\beta$.

Previously, it was shown by Pastur and Vasilchuk in [16] that Assumption (1) together with a weaker version of Assumption (2) implies that $\mu_{H_N} \to \mu_\alpha \boxplus \mu_\beta$ with probability 1. Theorem 4.4 says that convergence of μ_{H_N} to $\mu_\alpha \boxplus \mu_\beta$ holds on the level of densities, so it can be seen as a local limit law for eigenvalues of the sum of random Hermitian matrices.

Proof: In Theorem 2 in [13], it was shown that the following claim holds. Suppose that $\eta = \eta(N)$ and $1/\sqrt{\log N} \ll \eta(N) \ll 1$. Assume that measure $\mu_{A_N} \boxplus \mu_{B_N}$ is absolutely continuous and its density is bounded by a constant T_N. Then, for all sufficiently large N,

$$P\left\{\sup_E \left|\frac{\mathcal{N}_\eta(E)}{2N\eta} - \varrho_{\boxplus,N}(E)\right| \geq \delta\right\} \leq \exp\left(-c\delta^2 \frac{(\eta N)^2}{(\log N)^2}\right), \tag{7}$$

where $c > 0$ depends only on $K_N := \max\{\|A_N\|, \|B_N\|\}$ and T_N. Here $\varrho_{\boxplus,N}$ denote the density of $\mu_{A_N} \boxplus \mu_{B_N}$. (The notation $f(N) \ll g(N)$ means that $\lim_{N\to\infty} g(N)/f(N) = +\infty$.)

This statement can be modified so that the supremum in the inequality holds for E in an interval, provided that the density of $\mu_{A_N} \boxplus \mu_{B_N}$ is bounded by a constant T_N in this interval:

$$P\left\{\sup_{E\in(a,b)}\left|\frac{\mathcal{N}_\eta(E)}{2N\eta}-\varrho_{\boxplus,N}(E)\right|\geq\delta\right\}\leq\exp\left(-c\delta^2\frac{(\eta N)^2}{(\log N)^2}\right). \tag{8}$$

Since Assumptions (1) and (3) hold, therefore we can use Theorem 1.3 and infer that $\varrho_{\boxplus,N}(E) \to \varrho_{\boxplus}(E)$. In particular, the sequence of densities $\varrho_{\boxplus,N}(E)$ is uniformly bounded by a constant T. This fact and Assumption (2) imply that the positive constant c in (7) can be chosen independently of N. By using the Borel - Cantelli lemma, we can conclude that

$$\frac{\mathcal{N}_\eta(E)}{2N\eta}\to\varrho_{\boxplus}(E)$$

with probability 1. □

5. Conclusion

We proved that if probability measures ν_A and ν_B are sufficiently close to probability measures μ_A and μ_B in the Levy distance, and if $\mu_A \boxplus \mu_B$ is (Lebesgue) absolutely continuous at E, then (under some mild additional conditions), $\nu_A \boxplus \nu_B$ is also absolutely continuous at E and its density is close to the density of $\mu_A \boxplus \mu_B$.

We applied this result to derive several local limit law results for sums of free random variables and for eigenvalues of a sum of random Hermitian matrices.

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