A STELLAR PROOF OF HIRZEBRUCH-RIEMANN-ROCH FOR TORIC VARIETIES

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Abstract. We give a simple proof of the Hirzebruch-Riemann-Roch theorem for smooth complete toric varieties, based on Ishida's result in [5] that the Todd genus of a smooth complete toric variety is one.

1. Introduction

The Hirzebruch-Riemann-Roch theorem relates the Euler characteristic of a coherent sheaf $\mathcal{F}$ on a smooth complete $n-$dimensional variety X to intersection theory, via the formula

$$\chi(\mathcal{F}) = \int ch(\mathcal{F})Td(\mathcal{T}_X). \tag{1}$$

In [2], Brion-Vergne prove an equivariant Hirzebruch-Riemann-Roch theorem for complete simplicial toric varieties. If the toric variety is actually smooth, it is possible to derive (1) from their result. In this note, we give a simple direct proof of (1) when X is a smooth complete toric variety. Such a variety is determined by a smooth complete rational polyhedral fan $\Sigma \subseteq N_{\mathbb{R}}$, where $N \simeq \mathbb{Z}^n$ is a lattice; we write X for the associated toric variety X_Σ. We will make use of the following standard facts about toric varieties. First,

$$Td(X_\Sigma) = \prod_{\rho \in \Sigma(1)} \frac{D_\rho}{1-e^{-D_\rho}}, \tag{2}$$

where $\Sigma(k)$ denotes the set of k-dimensional faces of Σ. For $\tau \in \Sigma(k)$ there is an associated torus invariant orbit $O(\tau)$, and we use $V(\tau)$ to denote the orbit closure $\overline{O(\tau)}$, which has dimension $n-k$. A key fact is that (see [4], Proposition 3.2.7)

$$V(\tau) = \overline{O(\tau)} \simeq X_{\mathrm{Star}(\tau)}. \tag{3}$$

Since Σ is smooth, all orbits are also smooth, and if ρ_i, ρ_j are distinct elements of $\Sigma(1)$, then (see [4], Lemma 12.5.7)

$$[D_{\rho_i}|_{V(\rho_j)}] = \begin{cases} V(\tau) & \tau = \rho_i + \rho_j \in \Sigma \\ 0 & \rho_i, \rho_j \text{ are not both in any cone in } \Sigma. \end{cases}$$

The final ingredient we need is a result of Ishida: building on work of Brion [1], in [5] Ishida shows that (1) holds for the structure sheaf of a smooth complete toric variety X:

$$1 = \int Td(\mathcal{T}_X) = \Big[\prod_{\rho \in \Sigma(1)} \frac{D_\rho}{1-e^{-D_\rho}} \Big]_n. \tag{4}$$

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2. The proof

For a smooth complete toric variety, any coherent sheaf has a resolution by line bundles [3], so it suffices to consider the case $\mathcal{F} = \mathcal{O}_X(D)$. Let $X = X_\Sigma$, and recall that $\mathrm{Pic}(X)$ is generated by the classes of the divisors D_ρ, $\rho \in \Sigma(1)$. We will show that if (1) holds for a divisor D, then it also holds for $D + D_\rho$ and $D - D_\rho$, for any $\rho \in \Sigma(1)$. We begin with the case $D - D_\rho$, and induct on the dimension of X.

A smooth complete toric variety of dimension one is simply $\mathbb{P}^1$, so the base case holds by Riemann-Roch for curves. Suppose the theorem holds for all smooth complete fans of dimension $< n$, and let Σ be a smooth complete fan of dimension n. When $D = 0$ the result holds by Ishida's theorem. Let $\rho \in \Sigma(1)$, and partition the rays of Σ as

$$\Sigma(1) = \rho \cup \Sigma'(1) \cup \Sigma''(1),$$

where the rays in $\Sigma'(1)$ are in one to one correspondence with the rays of the fan $\mathrm{Star}(\rho)$. Let $X' = X_{\mathrm{Star}(\rho)} \simeq V(\rho)$. Tensoring the standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D_\rho) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X'} \longrightarrow 0$$

with $\mathcal{O}_X(D)$ yields the sequence

$$0 \longrightarrow \mathcal{O}_X(D - D_\rho) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_{X'}(D) \longrightarrow 0.$$

From the additivity of the Euler characteristic, we have

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(D - D_\rho)) = \chi(\mathcal{O}_{X'}(D)).$$

Our hypotheses imply that

$$\begin{aligned} \int_{X'} e^D Td(\mathcal{T}_{X'}) &= \chi(\mathcal{O}_{X'}(D)) \\ \int_X e^D Td(\mathcal{T}_X) &= \chi(\mathcal{O}_X(D)), \end{aligned}$$

so it suffices to show that

$$\begin{aligned} \int_{X'} ch(D) Td(\mathcal{T}_{X'}) &= \int_X (e^D - e^{D - D_\rho}) Td(\mathcal{T}_X) \\ &= \int_X e^D \Big(\frac{1 - e^{-D_\rho}}{D_\rho}\Big) D_\rho Td(\mathcal{T}_X) \end{aligned} \tag{5}$$

Break the Todd class of X into two parts:

$$Td(\mathcal{T}_X) = \prod_{\gamma \in \Sigma'(1) \cup \rho} \frac{D_\gamma}{1 - e^{-D_\gamma}} \cdot \prod_{\gamma \in \Sigma''(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}}$$

In (5), the term $\frac{1 - e^{-D_\rho}}{D_\rho}$ cancels with the corresponding term in $Td(\mathcal{T}_X)$, so that

$$\begin{aligned} \int_X e^D \Big(\frac{1 - e^{-D_\rho}}{D_\rho}\Big) D_\rho Td(\mathcal{T}_X) &= \int_X e^D D_\rho \prod_{\gamma \in \Sigma'(1) \cup \Sigma''(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}} \\ &= \int_X e^D D_\rho \prod_{\gamma \in \Sigma'(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}}. \end{aligned} \tag{6}$$

The second equality follows since $D_\rho \cdot D_\gamma = 0$ if $\gamma \in \Sigma''(1)$. By smoothness, all intersections are either zero or one, and thus

$$\begin{aligned}\int_X e^D D_\rho \prod_{\gamma\in\Sigma'(1)} \frac{D_\gamma}{1-e^{-D_\gamma}} &= \Big[e^D D_\rho \prod_{\gamma\in\Sigma'(1)} \frac{D_\gamma}{1-e^{-D_\gamma}}\Big]_n \\ &= \Big[e^D\big|_{V(\rho)} \prod_{\gamma\in\Sigma'(1)} \frac{D_\gamma}{1-e^{-D_\gamma}}\Big]_{n-1} \\ &= \int_{X'} e^D \cdot Td(\mathcal{T}_{X'}).\end{aligned}$$

This proves the result for $D - D_\rho$. For $D + D_\rho$, the result follows using the substitution $e^{D_\rho} - 1 = e^{D_\rho}(1 - e^{-D_\rho})$.

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