Joint Extremal Behavior of Hidden and Observable Time Series

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Abstract

We analyze the joint extremal behavior of two real-valued processes $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ which can be interpreted as an observable and an unobservable time series. Our analysis is motivated by the well-known GARCH model which correspondingly represents both the observable log returns of an asset as well as the hidden volatility sequence. In particular, we study the behavior of $(Y_t)_{t\in\mathbb{Z}}$ under an extreme event of the observable process $(X_t)_{t\in\mathbb{Z}}$ where our results complement the findings of Segers [J. Segers, Multivariate regular variation of heavy-tailed Markov chains, arXiv:math/0701411 (2007). Available online: http://arxiv.org/abs/math/0701411] and Smith [R. L. Smith, The extremal index for a Markov chain. J. Appl. Prob. (1992)] for a single time series. We show that under suitable assumptions their concept of a tail chain as a limiting process is also applicable to our setting. Furthermore, we discuss existence and uniqueness of a limiting process under some weaker assumptions. Finally, we explore connections of our approach with the notion of multivariate regular variation.

Keywords: tail chain, time series, joint extremal behavior, ARCH processes, GARCH processes, multivariate regular variation
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1. Introduction

An extensive class of financial time series models is composed of two interrelated processes. In particular, many models entail an unobservable part that reflects a certain regime or volatility of the process. A well-known example is given by the GARCH family. It is typically chosen in order to model financial log-returns where the observable price of an asset is driven by an unobservable volatility process. In the following, let $(X_t)_{t\in\mathbb{Z}}$ denote such a visible process and $(Y_t)_{t\in\mathbb{Z}}$ its unobservable counterpart. Let both $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ be univariate. A common approach for the analysis of the extremal behavior of such interrelated processes focusses on the joint sequence $(Z_t)_{t\in\mathbb{Z}} := (X_t, Y_t)_{t\in\mathbb{Z}}$. More precisely, the process is studied under the condition $\{\|Z_0\| > x\}$ for $x \to \infty$ and an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^2$. The connection of this approach to the concept of multivariate regular variation has been discussed extensively in [2]. We shall follow a more natural point of view if the process $(Y_t)_{t\in\mathbb{Z}}$ is unobservable. That is, we analyze its limiting behavior under the (observable) event $\{|X_0| > x\}$ as $x \to \infty$. Hence, for $-\infty < m \le n < \infty$ we are interested in the limit distribution of

$$\mathcal{L}\left(\frac{Y_m}{x}, \dots, \frac{Y_n}{x} \,\middle|\, |X_0| > x\right) \tag{1.1}$$

as $x \to \infty$. To be in line with the assumptions in [12] and [13] we assume $(Y_t)_{t\in\mathbb{Z}}$ to be of a simple Markovian structure, i.e.

$$Y_t = \Phi(Y_{t-1}, \epsilon_t), \quad t \in \mathbb{Z}, \tag{1.2}$$

for some measurable mapping $\Phi : \mathbb{R}^2 \to \mathbb{R}$ and some sequence $(\epsilon_t)_{t\in\mathbb{Z}}$ of i.i.d. innovations. Additionally, we will require the sequence of innovations $(\epsilon_t)_{t>s}$ to be independent of $(Y_t)_{t\le s}$ for all $s \in \mathbb{Z}$. Based on $(Y_t)_{t\in\mathbb{Z}}$ and the innovations let the observable process be given by

$$X_t = \Psi(Y_t, \epsilon_{t-s_-}, \dots, \epsilon_{t+s_+}), \quad t \in \mathbb{Z}, \tag{1.3}$$

for some measurable mapping $\Psi : \mathbb{R}^{s_-+s_++2} \to \mathbb{R}$ with $s_+ \ge 0, s_- \ge -1$ and $s_+ \ge -s_-$. We will always assume that a stationary solution to (1.2) and (1.3) exists. Now, by Ψ as well as by s_- and s_+ we have a simple, but flexible way to model the dependence between $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$. A scheme of the connection between the two processes is given in Figure 1. Note that from the recursive definition in (1.2) we may find a function $\tilde{\Psi} : \mathbb{R}^{s_-+s_++2} \to \mathbb{R}$

such that $X_t = \tilde{\Psi}(Y_{t-s_--1}, \epsilon_{t-s_-}, \dots, \epsilon_{t+s_+})$, $t \in \mathbb{Z}$, and $\tilde{s}_+ = s_- + s_+ + 1$ with $\tilde{s}_- = -1$. Consequently, our original definition entails a slight redundancy. Still, it is often easier to work with and will therefore be kept.

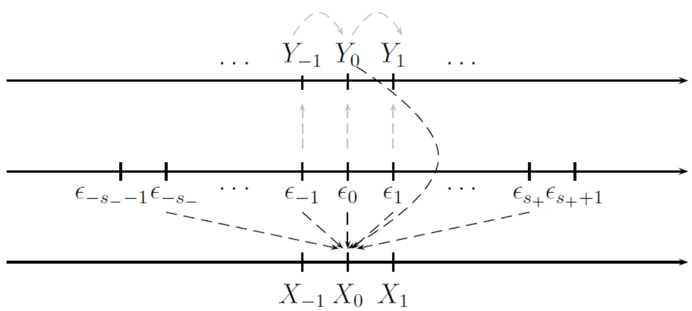

Figure 1: Connection between $(\epsilon_t)_{t\in\mathbb{Z}}, (Y_t)_{t\in\mathbb{Z}}$ and $(X_t)_{t\in\mathbb{Z}}$

As indicated above, a prominent example for a model defined by (1.2) and (1.3) is the GARCH$(1,1)$ process [3, 14], i.e.

$$\zeta_t = \sigma_t z_{t+1}, \quad t \in \mathbb{Z}, \tag{1.4}$$

and

$$\sigma_t = \sqrt{\alpha_0 + \alpha_1 \sigma_{t-1}^2 z_t^2 + \beta_1 \sigma_{t-1}^2}, \quad t \in \mathbb{Z}, \tag{1.5}$$

for suitable constants $\alpha_0 > 0$ and $\alpha_1, \beta_1 \geq 0$. Here, the sequence $(\zeta_t)_{t\in\mathbb{Z}}$ is the observable part, e.g. a model for financial log-returns, and the series $(\sigma_t)_{t\in\mathbb{Z}}$ describes the conditional standard deviation of the process at time $t \in \mathbb{Z}$. In the basic setup the innovation sequence $(z_t)_{t\in\mathbb{Z}}$ is assumed to be i.i.d. standard normal. Note that the above GARCH(1,1) satisfies (1.2) and (1.3) with

$$\Phi(x,e) = \sqrt{\alpha_0 + \alpha_1 x^2 e^2 + \beta_1 x^2}, \quad \Psi(x,e) = xe, \quad s_- = -1, \ s_+ = 1.$$

We remark that for $\beta_1 = 0$ in (1.5) the GARCH$(1,1)$ setup includes the ARCH(1) as a special case, cf. [8]. It is well-known [1] that under quite general assumptions about the distribution of the z_t, $t \in \mathbb{Z}$, and about the parameters α_0, α_1 and β_1 the stationary solutions to (1.4) and (1.5) share a common regularly varying (heavy tailed) behavior. Accordingly, we will assume regular variation for the stationary solutions to both (1.2) and (1.3), cf. Condition 1 below. The rest of the paper will be organized as follows. As it is not clear whether the limit in (1.1) exists at all nor whether it is unique we will discuss those questions in more detail in Sections 2 and 3. Under some further assumptions in Section 4 we will show that the limiting distribution in (1.1) has a particularly simple form which can be seen as an extension to similar findings in [12]. More precisely, our results will allow for a representation of the limit process in (1.1) as a multiplicative random walk, at least outside of the period $\{-s_- - 1, \ldots, 0, \ldots, s_+\}$. In Section 5 we will analyze connections of our results with multivariate regular variation of the time series $(X_t, Y_t)_{t\in\mathbb{Z}}$.

2. Existence of a Limiting Distribution

In the following, we will assume that the stationary distribution of $Y_t = \Phi(Y_{t-1}, \epsilon_t)$, $t \in \mathbb{Z}$, cf. (1.2), is regularly varying with index $\alpha > 0$ and tail-balanced, i.e. for all $u > 0$ we have

$$\lim_{x\to\infty} \frac{P(|Y_0| > ux)}{P(|Y_0| > x)} = u^{-\alpha}, \quad \lim_{x\to\infty} \frac{P(Y_0 > x)}{P(|Y_0| > x)} = p \in [0,1]. \tag{2.1}$$

We will study the joint extremal behavior of (1.2) and (1.3) under the assumption that X_0 shares the tail behavior of Y_0, i.e. there exists a constant $C > 0$ such that

$$\lim_{x\to\infty} \frac{P(|X_0| > x)}{P(|Y_0| > x)} = C. \tag{2.2}$$

Throughout, we shall use some conventions for abbreviation. We will say that *Condition 1.a* holds if the time series $(Y_t)_{t\in\mathbb{Z}}$ satisfies (2.1). Accordingly, *Condition 1.b* holds if the time series $(X_t)_{t\in\mathbb{Z}}$ satisfies this equation with Y_0 replaced by X_0. Furthermore, if both conditions are satisfied and if (2.2) holds in addition, then we will say that *Condition 1* holds.

Proposition 2.1. *Let $(Y_t)_{t\in\mathbb{Z}}$ and $(X_t)_{t\in\mathbb{Z}}$ be stationary time series given by* (1.2) *and* (1.3) *and let Condition 1 be satisfied. Then the family*

$$\mathcal{L}\left(\frac{Y_m}{x}, \ldots, \frac{Y_n}{x} \,\middle|\, |X_0| > x\right), \quad x > 1,$$

of conditional distributions is tight for all $-\infty < m \le n < \infty$.

Proof. Let $u > 0$. Then $P\left(\bigcup_{i=m}^{n}\left\{\frac{|Y_i|}{x} > u\right\} \,\middle|\, |X_0| > x\right)$
$\leq \sum_{i=m}^{n} P\left(\frac{|Y_i|}{x} > u \,\middle|\, |X_0| > x\right) \leq \sum_{i=m}^{n} \frac{P(|Y_i|>ux)}{P(|X_0|>x)} = \sum_{i=m}^{n} \frac{P(|Y_0|>ux)}{P(|Y_0|>x)} \frac{P(|Y_0|>x)}{P(|X_0|>x)}$.
By Condition 1 the r.h.s. is bounded by $(n-m+1)\cdot 2\cdot u^{-\alpha}\cdot\frac{1}{C}$ for x large. □

Therefore, a weak limit point of the family of distributions exists. The following lemma shows, however, that it is not necessarily unique.

Lemma 2.2. *There exist time series* $(Y_t)_{t\in\mathbb{Z}}$ *and* $(X_t)_{t\in\mathbb{Z}}$ *of the form* (1.2)-(1.3) *such that Condition 1 is satisfied but the weak limit in* (1.1) *is not unique.*

Proof. Let $\epsilon_t \overset{i.i.d.}{\sim} \mathrm{Par}(1)$, i.e. $P(\epsilon_t > x) = x^{-1}, x \geq 1$. With $\Phi(Y_{t-1}, \epsilon_t) = \epsilon_t$ we have $Y_t = \epsilon_t, t \in \mathbb{Z}$. Let $s_- = -1, s_+ = 1$ and $\Psi(Y_t, \epsilon_{t+1}) = f(\epsilon_{t+1})$ for a continuous function $f : \mathbb{R} \to \mathbb{R}$ to be described below. Thus $X_t = f(\epsilon_{t+1}), t \in \mathbb{Z}$. By independence, any weak limit of $\mathcal{L}\left(\frac{Y_0}{x}, \frac{Y_1}{x} \,\middle|\, |X_0| > x\right)$ equals $\delta_0 \times \mu$ for some weak limit μ of $\mathcal{L}\left(\frac{Y_1}{x} \,\middle|\, |X_0| > x\right)$. With $Y := \epsilon_1 \sim \mathrm{Par}(1)$ we will construct f such that $\mathcal{L}\left(\frac{Y}{x} \,\middle|\, |f(Y)| > x\right)$ has a continuum of weak limits.
Let $f(t) = t, t \leq 1$. For the sequence $z_i = 5^i, i \in \mathbb{N}_0$, each interval $[z_i, 5z_i]$ is mapped onto itself by f. On $[4z_i, 5z_i]$ it interpolates linearly between the values z_i and $5z_i$, and on $[3z_i, 4z_i]$ between $3z_i$ and z_i. The function f can be extended on each interval $[z_i, 3z_i]$ such that $f([z_i, 3z_i]) \subset [z_i, 5z_i]$ and $f(Y) \sim \mathrm{Par}(1)$. We omit the rather tedious details of the extension of f which are not used below. To give a rough picture, we mention the following properties which also uniquely determine f on $[z_i, 3z_i]$: On $[z_i, \frac{21}{8}]$ the function f is strictly increasing (from value z_i to $5z_i$), strictly decreasing on $[\frac{21}{8}z_i, 3z_i]$ (from value $5z_i$ to $3z_i$) and symmetric w.r.t. $\frac{21}{8}z_i$.
We first exhibit two different weak limits. Since $f(Y)$ is non-negative we drop the absolut values. Along the sequence $x_i = 5^i, i \in \mathbb{N}_0$, we have $\{f(Y) > x_i\} = \{Y > x_i\}$. Thus for $b \geq 1$ it holds $P\left(\frac{Y}{x_i} > b \,\middle|\, f(Y) > x_i\right) = b^{-1}$. Thus $\mathcal{L}\left(\frac{Y_1}{x_i} \,\middle|\, |X_0| > x_i\right) = \mathrm{Par}(1)$ for all i.
Now suppose that $x_i = 3\cdot 5^i, i \in \mathbb{N}_0$. By construction $x_i < Y < \frac{3}{2}x_i$ implies $f(Y) < x_i$, thus $P\left(\frac{Y}{x_i} \in (1, \frac{3}{2}) \,\middle|\, f(Y) > x_i\right) = 0$. This leads (at least along a subsequence) to a necessarily different weak limit. For $b \geq \frac{3}{2}$ one still has $P\left(\frac{Y}{x_i} > b \,\middle|\, f(Y) > x_i\right) = 1/b$, since $Y > \frac{3}{2}x_i$ implies $f(Y) > x_i$.
Adapting the above argument shows that each sequence $x_i = c\cdot 5^i, 3 \leq$

$c < 5$ leads to a different weak limit μ_c (at least along a subsequence) with $\mu_c((1, b_c)) = 0$ and $\mu_c([b, \infty)) = 1/b$ for all $b \geq b_c = \frac{15+c}{4c}$. $\square$

In order to study the properties of the limit in (1.1) in more detail we will add further assumptions about the functional form of Φ and Ψ which are based on those given in [12]. There, the single time series $(Y_t)_{t\in\mathbb{Z}}$ is analyzed and both the existence and the form of the weak limit

$$\lim_{x\to\infty} \mathcal{L}\left(\frac{Y_m}{x}, \ldots, \frac{Y_n}{x} \,\middle|\, |Y_0| > x\right)$$

for all $-\infty < m \leq n < \infty$ are discussed. Under Condition 1.a and an additional assumption (cf. Condition 2.a below) this so-called tail chain bears resemblance to a multiplicative random walk. Note that our Condition 2.a is a slightly stronger version of [12, Condition 2.2] that will allow to simplify some of our proofs in Section 4.

Condition 2.a. *There exists a function $\phi : \mathbb{R}^d \times \{-1, 1\} \to \mathbb{R}$ such that*

$$\lim_{y\to\infty} \frac{\Phi(yw(y), v(y))}{y} = w\phi(v, \operatorname{sign}(w))$$

for all $w(y) \to w \in \mathbb{R}, v(y) \to v \in \mathbb{R}^d$. Here, $\operatorname{sign}(w) = 2 \cdot \mathbb{1}_{[0,\infty)}(w) - 1$, *where $\mathbb{1}_{\{\cdot\}}(\cdot)$ denotes the indicator function.*

The following propositions are taken from [12] and will be fundamental to our subsequent analysis.

Proposition 2.3 (cf. [12], Theorem 2.3)**.** *Let $(Y_t)_{t\in\mathbb{Z}}$ (not necessarily stationary) be given by* (1.2) *and let Conditions 1.a and 2.a hold. Then for $n \in \mathbb{N}$, as $y \to \infty$,*

$$\mathcal{L}\left(\frac{|Y_0|}{y}, \frac{Y_0}{|Y_0|}, \epsilon_1, \frac{Y_1}{|Y_0|} \ldots, \epsilon_n, \frac{Y_n}{|Y_0|} \,\middle|\, |Y_0| > y\right) \xrightarrow{w} \mathcal{L}(Y, M_0, \epsilon_1^Y, M_1, \ldots, \epsilon_n^Y, M_n),$$

with

$$M_j = h(M_{j-1}, A_j, B_j), \quad j = 1, 2, \ldots,$$

where $h : \mathbb{R}^3 \to \mathbb{R},\ h(y, a, b) := y\left(a\mathbb{1}_{(0,\infty)}(y) + b\mathbb{1}_{(-\infty,0)}(y)\right)$ and $Y,\ M_0, \epsilon_1^Y$, $\epsilon_2^Y, \ldots$ are independent with

(i) $Y \sim Par(\alpha)$,

(ii) $P(M_0 = 1) = p = 1 - P(M_0 = -1)$,

(iii) $\epsilon_i^Y, i \in \mathbb{N}$, *are i.i.d. with* $\mathcal{L}(\epsilon_1^Y) = \mathcal{L}(\epsilon_1)$ *and*

$$(A_i, B_i) = (\phi(\epsilon_i^Y, 1), \phi(\epsilon_i^Y, -1)), \quad i \in \mathbb{N}.$$

Note that by embedding the ϵ_i, $i \in \mathbb{N}$, for later reference, the formulation of Proposition 2.3 differs slighty from its analog in [12]. The proof is analogous to the proof of [12, Theorem 2.3]. The joint limit distribution in Proposition 2.3 will be an important building block for the derivation of (1.1). If $(Y_t)_{t\in\mathbb{Z}}$ is addtionally assumed to be stationary there exists a so-called "backward tail chain" which also has a surprisingly simple form.

Proposition 2.4 (cf. [12], Theorem 5.2)**.** *Let* $(Y_t)_{t\in\mathbb{Z}}$ *be a stationary process given by* (1.2) *and let Conditions 1.a and 2.a hold. Then, for all* $m, n \in \mathbb{N}$, *as* $y \to \infty$,

$$\mathcal{L}\left(\frac{|Y_0|}{y}, \frac{Y_{-m}}{|Y_0|}, \ldots, \frac{Y_n}{|Y_0|} \;\middle|\; |Y_0| > y\right) \xrightarrow{w} \mathcal{L}(Y, M_{-m}, \ldots, M_n), \tag{2.3}$$

with

(i) $Y \sim Par(\alpha)$, *independent of* $(M_t)_{t\in\mathbb{Z}}$,

(ii) $(M_t)_{t\in\mathbb{Z}}$ *is a BFTC*(α, μ) *where* $\mu = \mathcal{L}(M_0, M_1)$ *with* (M_0, M_1) *as in Proposition 2.3.*

The abbreviation BFTC stands for back-and-forth tail chain and is defined as follows.

Definition 2.5 (cf. [12], Definition 4.1). A discrete-time process $(M_t)_{t\in\mathbb{Z}}$ is said to be a back-and-forth tail chain with index $0 < \alpha < \infty$ and forward transition law μ, denoted by BFTC(α, μ), if

(i) $\mathcal{L}(M_0, M_1) = \mu$,

(ii) $\mu^* := \mathcal{L}(M_0, M_{-1})$ is *adjoint* to μ, i.e.

$$E[(xM_0)_+^\alpha \wedge (yM_1)_+^\alpha] = E[(xM_{-1})_+^\alpha \wedge (yM_0)_+^\alpha], \quad \forall\, x, y \in \mathbb{R}, \tag{2.4}$$

(iii.a) for all integer $t \geq 1$ and all real $x_{t-1}, x_{t-2}, \dots$,

$$\mathcal{L}(M_t | M_{t-1} = x_{t-1}, M_{t-2} = x_{t-2}, \dots) = \mathcal{L}(h(x_{t-1}, A_1, B_1)),$$

(cf. Proposition 2.3 for the definition of h) where A_1 and B_1 are independent with

$$\mathcal{L}(A_1) = \mathcal{L}\left(\frac{M_1}{M_0} \,\middle|\, M_0 = 1\right), \quad \mathcal{L}(B_1) = \mathcal{L}\left(\frac{M_1}{M_0} \,\middle|\, M_0 = -1\right),$$

(iii.b) for all integer $t \geq 1$ and all real $x_{-t+1}, x_{-t+2}, \dots$,

$$\mathcal{L}(M_{-t} | M_{-t+1} = x_{-t+1}, M_{-t+2} = x_{-t+2}, \dots) = \mathcal{L}(h(x_{-t+1}, A_{-1}, B_{-1})),$$

where A_{-1} and B_{-1} are independent with

$$\mathcal{L}(A_{-1}) = \mathcal{L}\left(\frac{M_{-1}}{M_0} \,\middle|\, M_0 = 1\right), \quad \mathcal{L}(B_{-1}) = \mathcal{L}\left(\frac{M_{-1}}{M_0} \,\middle|\, M_0 = -1\right).$$

Propositions 2.3 and 2.4 show that the assumption about the asymptotic behavior of Φ leads to a very simple form of the tail process for $(Y_t)_{t\in\mathbb{Z}}$. In order to discuss similar results for the above case of two connected time series we introduce an analogous condition for Ψ.

Condition 2.b. *There exists a function* $\psi : \mathbb{R}^{s_-+s_++1} \times \{-1, 1\} \to \mathbb{R}$ *such that*

$$\lim_{y\to+\infty} \frac{\Psi(yw(y), v(y))}{y} = w\psi(v, \operatorname{sign}(w))$$

for all $w(y) \to w \in \mathbb{R}, v(y) \to v \in \mathbb{R}^{s_-+s_++1}$.

If both Conditions 2.a and 2.b hold we will say that *Condition 2* is satisfied.

3. Uniqueness of the Limit Distribution

In the following, we will investigate the uniqueness of the limit in (1.1) under Conditions 1 and 2. It will turn out that the behavior of the univariate distribution $\mathcal{L}(Y_{-s_--1}/x \,|\, |X_0| > x)$ as $x \to \infty$ leads to a sufficient condition.

Proposition 3.1. *Let* $(Y_t)_{t\in\mathbb{Z}}$ *and* $(X_t)_{t\in\mathbb{Z}}$ *be stationary time series given by* (1.2) *and* (1.3) *and let Conditions 1 and 2 hold. Equivalent are*

(i) *the weak limit of* (1.1) *is unique, and*

$$\mathcal{L}(Y^X_{-s_--1}) := \lim_{x\to\infty} \mathcal{L}\left(\frac{Y_{-s_--1}}{x} \,\middle|\, |X_0| > x \right)$$

has no mass in zero,

(ii) *there exists a weak limit point* $\mathcal{L}(\hat{Y}^X_{-s_--1})$ *of* $\mathcal{L}(Y_{-s_--1}/x \,|\, |X_0| > x)$ *with* $\hat{Y}^X_{-s_--1} \neq 0$ *a.s.*

Consequently, the uniqueness of the limit in (1.1) may well be derived from *any* weak limit. The following lemma will be used in the proof of Proposition 3.1. In addition, it is also of interest in its own right as it provides a criterion for property (ii) of Proposition 3.1.

Lemma 3.2. *Let the assumptions of Proposition 3.1 hold and let* $\mathcal{L}(\hat{Y}^X_{-s_--1})$ *be any weak limit point of* $\mathcal{L}(Y_{-s_--1}/x \,|\, |X_0| > x)$*. Then*

$$P(\hat{Y}^X_{-s_--1} = 0) = 1 - \frac{1}{C} E(|\chi|^\alpha),$$

with

$$\chi = \chi(M_0, \epsilon^Y_1, \ldots, \epsilon^Y_{s_-+s_++1}) = M_{s_-+1} \cdot \psi(\epsilon^Y_1, \ldots, \epsilon^Y_{s_-+s_++1}, \operatorname{sign}(M_{s_-+1})),$$

where $M_0, M_{s_-+1}, \epsilon^Y_1, \ldots, \epsilon^Y_{s_-+s_++1}$ *are defined as in Proposition 2.3.*

Proof. For $a > 0$ we have (apply stationarity for the second equality)

$$\begin{aligned} & P\left(\frac{|Y_{-s_--1}|}{x} > a \,\middle|\, |X_0| > x \right) \\ = \; & P\left(|X_0| > x \,\middle|\, |Y_{-s_--1}| > ax \right) \cdot \frac{P(|Y_{-s_--1}| > ax)}{P(|X_0| > x)} \\ = \; & P\left(\frac{|X_{s_-+1}|}{ax} > \frac{1}{a} \,\middle|\, |Y_0| > ax \right) \cdot \frac{P(|Y_{-s_--1}| > ax)}{P(|X_0| > x)}. \end{aligned}$$

With $x \to \infty$ the second term converges to $\frac{a^{-\alpha}}{C}$ by Condition 1. For the first term we analyze the limit of

$$\mathcal{L}\left(|X_{s_-+1}|(ax)^{-1} \,\middle|\, |Y_0| > ax \right)$$

$$= \mathcal{L}\left(\left|\Psi\left(ax\frac{Y_{s_-+1}}{ax}, \epsilon_1, \ldots, \epsilon_{s_-+s_++1}\right)\right|(ax)^{-1}\,\middle|\,|Y_0| > ax\right) \tag{3.1}$$

as $x \to \infty$. By an application of the continuous mapping theorem (cf. [10], Theorem 4.27) combined with Condition 2 and Proposition 2.3, this converges to $\mathcal{L}(|Y \cdot \chi|)$. Since Y is Pareto distributed and independent of χ we may rule out point mass of $|Y \cdot \chi|$ in $1/a$, and conclude that

$$\lim_{x\to\infty} P\left(\frac{|X_{s_-+1}|}{ax} > \frac{1}{a}\,\middle|\,|Y_0| > ax\right) = P\left(|Y \cdot \chi| > \frac{1}{a}\right).$$

Therefore, for a sequence $a_n \searrow 0$ which avoids possible point masses of $\hat{Y}^X_{-s_--1}$, it follows that

$$P(|\hat{Y}^X_{-s_--1}| > a_n) = P\left(|Y \cdot \chi| > \frac{1}{a_n}\right) \cdot \frac{a_n^{-\alpha}}{C} = \frac{P\left(Y \cdot |\chi| > \frac{1}{a_n}\right)}{P\left(Y > \frac{1}{a_n}\right)}\frac{1}{C}.$$

The result now follows with $a_n \searrow 0$ if we show that

$$\lim_{x\to\infty} \frac{P(Y \cdot |\chi| > x)}{P(Y > x)} = E(|\chi|^\alpha), \tag{3.2}$$

which can be seen as a kind of extension of Breiman's Theorem (cf. [5]) for the special case of $Y \sim \mathrm{Par}(\alpha)$. It follows because $P(Y \cdot |\chi| > x) = P(Y \cdot |\chi| > x,\, 0 \le |\chi| \le x) + P(Y \cdot |\chi| > x, |\chi| > x)$, where the first term equals $\int_0^x P(Y > \frac{x}{z})dP^{|\chi|}(z) = \int_0^x \left(\frac{z}{x}\right)^\alpha dP^{|\chi|}(z)$. The second term equals $P(|\chi| > x)$, since $Y \ge 1$ a.s. Thus, $P(Y \cdot |\chi| > x)/P(Y > x) = x^\alpha P(Y \cdot |\chi| > x) = \int_0^\infty z^\alpha \mathbb{1}_{[0,x]}(z) + x^\alpha \mathbb{1}_{(x,\infty)}(z)dP^{|\chi|}(z)$ and (3.2) follows from monotone convergence. This gives the result. □

Proof of Proposition 3.1. We show that (ii) implies (i). Let ν_1 and ν_2 denote two weak limit points. In the following, let $a \ge 0, a_1, \ldots, a_{s_-+s_++1} \in \mathbb{R}$, and $A = A(a_1, \ldots, a_{s_-+s_++1}) = (a_1, \infty) \times \ldots \times (a_{s_-+s_++1}, \infty)$. We will show that $\nu_k((a, \infty) \times A)$ and $\nu_k([-a, \infty) \times A)$ do not depend on k. Here, we shall use that (ii) implies $C = E(|\chi|^\alpha)$ by Lemma 3.2, which in turn implies that $\nu(\{0\} \times \mathbb{R}^{s_-+s_++1}) = 0$ for any weak limit point ν. Since the above sets form a generating π-system, any two weak limit points coincide. By tightness (cf. Proposition 2.1) this implies weak convergence.

Consider first $a > 0$ and $a_1, \dots, a_{s_-+s_++1} \neq 0$ that avoid the at most countably many point masses of the coordinate projections of ν_1 and ν_2. Then, $\nu_k((a,\infty) \times A)$ is the limit of

$$P\left(\frac{Y_{-s_--1}}{x} > a, \frac{Y_{-s_-}}{x} > a_1, \dots, \frac{Y_{s_+}}{x} > a_{s_-+s_++1} \;\middle|\; |X_0| > x\right)$$

along a subsequence depending on k. For general x, insert the term $\frac{|Y_{-s_--1}|}{x} > a$. By a similar computation as in the proof of Lemma 3.2 this equals

$$P\left(\frac{Y_0}{ax} > 1, \frac{Y_1}{ax} > \frac{a_1}{a}, \dots, \frac{Y_{s_-+s_++1}}{ax} > \frac{a_{s_-+s_++1}}{a}, \frac{|X_{s_-+1}|}{ax} > \frac{1}{a} \;\middle|\; |Y_0| > ax\right)$$
$$\cdot \frac{P(|Y_{-s_--1}| > ax)}{P(|X_0| > x)} .$$

By Conditions 1 and 2, and since the variables have point masses at most at zero, this converges to

$$P\left(YM_0 > 1, YM_1 > \frac{a_1}{a}, \dots, YM_{s_-+s_++1} > \frac{a_{s_-+s_++1}}{a}, |X^Y_{s_-+1}| > \frac{1}{a}\right) \cdot \frac{1}{C} a^{-\alpha}.$$

We have shown that $\nu_k((a,\infty) \times A)$ does not depend on k. Approximation from inside extends this to all $a \geq 0$ and $a_1, \dots, a_{s_-+s_++1} \in \mathbb{R}$. Replacing $\frac{Y_{-s_--1}}{x} > a$ by $\frac{Y_{-s_--1}}{x} < -a$ the same computation followed by an approximation argument shows the same for $\nu_k((-\infty,-a) \times A)$. Combining these two results for $a = 0$ with $\nu_k(\{0\} \times \mathbb{R}^{s_-+s_++1}) = 0$ shows that $\nu_k(\mathbb{R} \times A)$ does not depend on k. Thus, the same holds for the sets $[-a,\infty) \times A = (\mathbb{R} \times A) \setminus ((-\infty,-a) \times A)$. $\square$

Remark 3.3. *Lemma 3.2 shows that Prop 3.1 (ii), and thus (i), holds if and only if $C = E(|\chi|^\alpha)$. We give some examples. Suppose that $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are nonnegative processes and $X_t = Y_t \cdot \psi(\epsilon_{t+1})$, thus $\chi = \psi$. Then, $E(\psi(\epsilon_0)^{\alpha+\delta}) < \infty$ for some $\delta > 0$ implies $C = E(|\chi|^\alpha) < \infty$ by Breiman's Theorem (cf. [5]) and Condition 2 holds if ψ is continuous. If $P(Y_0 > x) \sim c \cdot x^{-\alpha}$ for some $c > 0$, then already $E(\psi(\epsilon_0)^\alpha) < \infty$ suffices (cf. e.g. [9], Lemma 2.1). For the special case $Y_0 \sim Par(\alpha)$ cf. the end of the proof of Lemma 3.2. For further generalizations of Breiman's Theorem see [7].*

Remark 3.4. *By similar computations it can be shown that under the assumptions of Proposition 3.1 uniqueness of the weak limit in* (1.1) *is also ensured by* $P(M_{-1} = 0) = 0$*, with* M_{-1} *as in Proposition 2.4. A key step in the argument shows that this condition implies weak convergence of*

$$\mathcal{L}\left(\frac{Y_{-m}}{y}, \epsilon_{-m+1} \ldots, \frac{Y_{-1}}{y}, \epsilon_0, \frac{Y_0}{y}, \epsilon_1, \frac{Y_1}{y}, \ldots, \epsilon_n, \frac{Y_n}{y} \;\middle|\; |Y_0| > y\right)$$

as $y \to \infty$ *for all* $m \geq 1$ *and* $n \geq 0$*. We give an example with* $P(M_{-1} = 0) = 0$ *but* $P(Y^X_{-s_-+1} = 0) > 0$*, i.e.* $P(M_{-1} = 0) = 0$ *may ensure uniqueness even if property (ii) in Proposition 3.1 fails. Let therefore* Y_0 *and* $\epsilon_t, t \in \mathbb{Z}$*, be nonnegative i.i.d. random variables with* $P(Y_0 > x) = \frac{1}{x \cdot \ln(x)^2}$ *for* $x \geq c$*, where* $c \approx 2.02$ *solves* $x \cdot \ln(x)^2 = 1$*. With* $\Phi(y, v) = y$*, let* $Y_t = Y_0$ *for all* $t \in \mathbb{Z}$*. Then,* $Y_{-1} = Y_0$ *implies* $Y \cdot M_{-1} \sim Y \sim Par(1)$*, thus* $P(M_{-1} = 0) = 0$*. For* $s_- = -1, s_+ = 1$ *let* $X_t = Y_t \cdot \epsilon_{t+1}, t \in \mathbb{Z}$*. Careful calculations, inspired by [7], show that* $C = \lim_{x\to\infty} \frac{P(X_0>x)}{P(Y_0>x)} = 2(c + \sqrt{c})$*. But with* $\alpha = 1$ *it holds that* $E(|\chi|^\alpha) = E(\epsilon_1) = \int_0^\infty P(\epsilon_1 > x)dx = c + \frac{1}{\ln(c)} = c + \sqrt{c}$*, thus* $P(Y^X_{-s_-+1} = 0) = 1/2$ *by Lemma 3.2.*

4. Structure of the Limit Process

While the existence of a limit in (1.1) has been analyzed in the last chapter we will now deal with its particular form. For easy reference we shall introduce the following condition.

Condition 3. *There exists a random vector* $(Y^X_{-s_--1}, \ldots, Y^X_0, \ldots, Y^X_{s_+})$ *such that*

$$\lim_{x\to\infty} \mathcal{L}\left(\frac{Y_{-s_--1}}{x}, \ldots, \frac{Y_0}{x}, \ldots, \frac{Y_{s_+}}{x} \;\middle|\; |X_0| > x\right) = \mathcal{L}(Y^X_{-s_--1}, \ldots, Y^X_0, \ldots, Y^X_{s_+}).$$

We assume that the limit distribution is unique in order to simplify the statement of the proposition below. Note, however, Remark 4.4 at the end of this chapter for a generalization to the case of non-uniqueness. We will use Conditions 1 to 3 to derive a result for the form of the limit in (1.1) which is similar to Proposition 2.4.

While Conditions 1 and 2 bear a natural resemblance to the assumptions made in [12], Condition 3 is necessary to ensure that a "starting point" for a tail chain exists that covers the time span from $-s_- - 1$ to s_+ where the $\epsilon_{-s_-}, \dots, \epsilon_{s_+}$ and therefore $Y_{-s_--1}, \dots, Y_{s_+}$ are directly influenced by the event $\{|X_0| > x\}$. We will see that outside of this range the behavior of the process $(Y_t)_{t\in\mathbb{Z}}$ corresponds to Proposition 2.4.

Proposition 4.1. *Let $(Y_t)_{t\in\mathbb{Z}}$ and $(X_t)_{t\in\mathbb{Z}}$ be stationary time series given by* (1.2) *and* (1.3) *and let Conditions 1, 2 and 3 hold. Then, for all integers $m > 0$ and $n \geq 0$ we have*

$$\lim_{x\to\infty} \mathcal{L}\left(\frac{Y_{-s_--m}}{x}, \dots, \frac{Y_{s_++n}}{x} \,\middle|\, |X_0| > x\right) \quad = \quad \mathcal{L}(Y^X_{-s_--m}, \dots, Y^X_{s_++n})$$

with $(Y^X_{-s_--1}, \dots, Y^X_{s_+})$ as in Condition 3, and

$$\begin{aligned} Y^X_t &= h(Y^X_{t-1}, A_t, B_t), && t > s_+, \\ Y^X_{-t} &= h(Y^X_{-t+1}, A_{-t}, B_{-t}), && t > s_- + 1. \end{aligned}$$

Here, $(A_t, B_t), t \in \mathbb{Z}$, are independent, and independent of $(Y^X_{-s_--1}, \dots, Y^X_{s_+})$ with

$$\mathcal{L}(A_t, B_t) = \mathcal{L}(A_1, B_1),\ t \geq 1, \quad \mathcal{L}(A_t, B_t) = \mathcal{L}(A_{-1}, B_{-1}),\ t \leq -1.$$

Further, $\mathcal{L}(A_1, B_1)$ and $\mathcal{L}(A_{-1}, B_{-1})$ are as in Definition 2.5.

The proof is predecessed by a lemma and a corollary where we only assume that Conditions 1 and 2 hold.

Lemma 4.2. *Let $m > 1$. For any $\eta > 0$ there is $\delta_0(\eta) > 0$ such that for x large enough*

$$P\left(\frac{|Y_{-s_--m}|}{x} > \eta, \frac{|Y_{-s_--1}|}{x} \leq \delta \,\middle|\, |X_0| > x\right) < \eta$$

for all $\delta < \delta_0(\eta)$.

Proof. The l.h.s. equals $P\left(\frac{|Y_{-s_--1}|}{x} \leq \delta, |X_0| > x \,\middle|\, \frac{|Y_{-s_--m}|}{x} > \eta\right) \cdot \frac{P(|Y_{-s_--m}| > \eta x)}{P(|X_0| > x)}$. The second factor converges to $\frac{1}{C} \cdot \eta^{-\alpha}$ by Condition 1. It suffices to show that the first factor becomes small for $\delta \to 0$. To this end, note that by stationarity the first factor equals

$$P\left(\frac{|Y_{m-1}|}{\eta x} \leq \frac{\delta}{\eta}, \frac{|X_{s_-+m}|}{\eta x} > \frac{1}{\eta} \,\middle|\, |Y_0| > \eta x\right)$$

which, by the definition of X_{s_-+m}, equals

$$P\left(\frac{|Y_{m-1}|}{\eta x}\leq\frac{\delta}{\eta},\frac{\left|\Psi\left(\eta x\frac{Y_{s_-+m}}{\eta x},\epsilon_m,\ldots,\epsilon_{m+s_-+s_+}\right)\right|}{\eta x}>\frac{1}{\eta}\,\middle|\,|Y_0|>\eta x\right).$$

We proceed as in the proof of Lemma 3.2. By an application of the continuous mapping theorem with Condition 2 and Proposition 2.3 this converges to $P\left(|YM_{m-1}|\leq\delta/\eta, Y|\chi_{s_-+m}|>1/\eta\right)$ with

$$\chi_{s_-+m}:=M_{s_-+m}\psi(\bar{\epsilon}^Y_{s_-+m},\operatorname{sign}(M_{s_-+m}))\quad\text{and}\quad\bar{\epsilon}^Y_{s_-+m}=(\epsilon^Y_m,\ldots,\epsilon^Y_{m+s_-+s_+}).$$

Again, we use that the two limit random variables have $Y\sim\operatorname{Par}(\alpha)$ as an independent factor which excludes point masses on the positive axis. Now, with $M_{s_-+m}=M_{m-1}\cdot\prod_{i=m}^{m+s_-}\phi(\epsilon^Y_i,\operatorname{sign}(M_{i-1}))$ we note that the set $\left\{|YM_{m-1}|\leq\delta/\eta, Y|\chi_{s_-+m}|>1/\eta\right\}$ is contained in

$$\bigcup_{s_{m-1},\ldots,s_{m+s_-}\in\{-1,+1\}}\left\{\prod_{i=m}^{m+s_-}\phi(\epsilon^Y_i,s_{i-1})\cdot\psi(\bar{\epsilon}^Y_{s_-+m},s_{m+s_-})>\frac{1}{\delta}\right\}.$$

For $0<\delta<\delta_0(\eta)$, $\delta_0(\eta)$ small, all these events have small probability. □

Corollary 4.3. *Let $m>1, n\geq 0$ and f be a bounded uniformly continuous function on $\mathbb{R}^{n+1}$ with $f(0,\ldots)=0$. For any $\epsilon>0$ there is $\delta_0(\epsilon)>0$ such that for x large enough*

$$E\left(f\left(\frac{Y_{-s_--m}}{x},\ldots,\frac{Y_{-s_--m+n}}{x}\right)\cdot\mathbb{1}_{\{|Y_{-s_--1}|\leq\delta x\}}\,\middle|\,|X_0|>x\right)<\epsilon$$

for all $0<\delta<\delta_0(\epsilon)$.

Proof. Since f is bounded and uniformly continuous there is some $\eta>0$ such that

$$||f||_\infty\cdot\eta+\sup\{|f(y_{-s_--m},\ldots,y_{-s_--m+n})|\,|\,|y_{-s_--m}|\leq\eta\}<\epsilon.$$

Choose δ_0 as in Lemma 4.2. For $\delta<\delta_0$ split $\mathbb{1}_{\{|Y_{-s_--1}|\leq\delta x\}}$ into

$$\mathbb{1}_{\{|Y_{-s_--m}|>\eta x,|Y_{-s_--1}|\leq\delta x\}}+\mathbb{1}_{\{|Y_{-s_--m}|\leq\eta x,|Y_{-s_--1}|\leq\delta x\}}.$$

The first integral is bounded by $||f||_\infty\cdot\eta$ and the second by

$$\sup\{|f(y_{-s_--m},\ldots,y_{-s_--m+n})|\,|\,|y_{-s_--m}|\leq\eta\}.$$

□

Proof of Proposition 4.1. Note that the case $m = 1$ and $n \geq 0$ is analogous to the proof of Proposition 2.3 in [12]. Since $(\epsilon_{s_++1}, \epsilon_{s_++2}, \dots)$ is independent of $(X_0, Y_{-s_--1}, \dots, Y_{s_+})$ the continuous mapping theorem can be applied to derive (4.1) and leads to the multiplicative structure with independent increments.

Let now $m > 1$ and $n \geq 0$, and let us assume that Proposition 4.1 holds for $(Y^X_{-s_--m+1}, \dots, Y^X_{s_++n})$. Let $f : \mathbb{R}^{s_-+s_++m+n+1} \to \mathbb{R}$ be bounded and continuous. We will show that

$$\begin{aligned} & \lim_{x\to\infty} E\left(f\left(\frac{Y_{-s_--m}}{x}, \dots, \frac{Y_{s_++n}}{x}\right) \,\middle|\, |X_0| > x\right) \\ = \; & E\left(f(Y^X_{-s_--m}, \dots, Y^X_{s_++n})\right) \end{aligned} \tag{4.1}$$

with $(Y^X_{-s_--m}, \dots, Y^X_{s_++n})$ as defined in the statement of the proposition. Let us further assume that $f(x) = 0$ as soon as the first component of x equals 0. Note that an arbitrary function $f : \mathbb{R}^{s_-+s_++m+n+1} \to \mathbb{R}$ can be split up additively into two functions f_1 and f_2 with

$$\begin{aligned} f_1(x_{-s_--m}, \dots, x_{s_++n}) &= f(x_{-s_--m}, \dots, x_{s_++n}) - f(0, x_{-s_--m+1}, \dots, x_{s_++n}), \\ f_2 &= f(0, x_{-s_--m+1}, \dots, x_{s_++n}), \end{aligned}$$

such that the first function satisfies the aforementioned assumption and the second function depends merely on $(x_{-s_--m+1}, \dots, x_{s_++n})$. Since the induction hypothesis implies that (4.1) is satisfied by a function of $(x_{-s_--m+1}, \dots, x_{s_++n})$ the assumption about the structure of f is no loss of generality.

The idea of the proof is to substitute the condition $\{|X_0| > x\}$ by a corresponding event in $(Y_t)_{t\in\mathbb{Z}}$. Let $\epsilon > 0$. Then, for x large enough

$$\begin{aligned} &\Bigg| E\left(f\left(\frac{Y_{-s_--m}}{x}, \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_++n}}{x}\right) \,\middle|\, |X_0| > x\right) \\ &- E\left(f\left(\frac{Y_{-s_--m}}{x}, \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_++n}}{x}\right) \mathbb{1}_{\{|Y_{-s_--1}| > \delta x\}} \,\middle|\, |X_0| > x\right) \Bigg| < \epsilon, \end{aligned}$$

for all $0 < \delta < \delta_0(\epsilon)$, where $\delta_0(\epsilon)$ is chosen according to Corollary 4.3. We have

$$\begin{aligned} & E\left(f\left(\frac{Y_{-s_--m}}{x}, \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_++n}}{x}\right) \mathbb{1}_{\{|Y_{-s_--1}| > \delta x\}} \,\middle|\, |X_0| > x\right) \\ = \; & \frac{E\left(f\left(\frac{Y_{-s_--m}}{x}, \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_++n}}{x}\right) \mathbb{1}_{\{|X_0|>x\}} \mathbb{1}_{\{|Y_{-s_--1}| > \delta x\}}\right)}{P(|X_0| > x)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{P(|Y_{-s_--1}|>\delta x)}{P(|X_0|>x)} E\left(f\left(\frac{Y_{-s_--m}}{x},\dots,\frac{Y_0}{x},\dots,\frac{Y_{s_++n}}{x}\right)\right. \\
&\qquad\qquad \left. \mathbb{1}_{\{|X_0|>x\}} \,\middle|\, |Y_{-s_--1}|>\delta x\right) \\
&= \frac{P(|Y_{-s_--1}|>\delta x)}{P(|X_0|>x)} E\left(f\left(\frac{Y_{-m+1}}{x},\dots,\frac{Y_{s_-+1}}{x},\dots,\frac{Y_{s_++n+s_-+1}}{x}\right)\right. \\
&\qquad\qquad \left. \mathbb{1}_{\{|X_{s_-+1}|>x\}} \,\middle|\, |Y_0|>\delta x\right) \\
&= \frac{P(|Y_{-s_--1}|>y)}{P(|X_0|>y/\delta)} E\left(f\left(\delta\frac{Y_{-m+1}}{y},\dots,\delta\frac{Y_{s_-+1}}{y},\dots,\delta\frac{Y_{s_++n+s_-+1}}{y}\right)\right. \\
&\qquad\qquad \left. \mathbb{1}_{\{\delta\cdot|\Psi(Y_{s_-+1},\epsilon_1,\dots,\epsilon_{s_-+s_++1})|>y\}} \,\middle|\, |Y_0|>y\right),
\end{aligned}
$$

where stationarity has been used for the penultimate equality. Here, the first term converges by Condition 1. Furthermore, an application of the continuous mapping theorem in connection with Propositions 2.3 and 2.4 yields that the whole expression converges to

$$
\frac{\delta^{-\alpha}}{C} E\Big(f\left(\delta Y M_{-m+1},\dots,\delta Y M_{s_-+1},\dots,\delta Y M_{s_++n+s_-+1}\right) \\
\mathbb{1}_{\{\delta\cdot|YM_{s_-+1}\psi(\epsilon_1^Y,\dots,\epsilon^Y_{s_-+s_++1})|>1\}}\Big)
$$

with Y, $\epsilon_i^Y, i\in\mathbb{N}$, and $M_n, n\in\mathbb{Z}$, as in Propositions 2.3 and 2.4. Defining new variables $(\tilde{A}_{-m+1},\tilde{B}_{-m+1})$ with the same distribution as (A_{-m+1},B_{-m+1}) in the statement of the proposition and independent of $Y,\epsilon_1^Y,\dots,\epsilon^Y_{s_-+s_++1}$, $M_{-m+2},\dots,M_0$, we may write

$$
\frac{\delta^{-\alpha}}{C} E\Big(f\left(h(\delta Y M_{-m+2},\tilde{A}_{-m+1},\tilde{B}_{-m+1}),\dots,\delta Y M_{s_-+1},\dots,\delta Y M_{s_++n+s_-+1}\right) \\
\mathbb{1}_{\{\delta\cdot|YM_{s_-+1}\psi(\epsilon_1^Y,\dots,\epsilon^Y_{s_-+s_++1})|>1\}}\Big).
$$

Next, note that by the continuous mapping theorem the above expression equals

$$
\lim_{y\to\infty}\frac{\delta^{-\alpha}}{C} E\Big(f\left(h\left(\delta\frac{Y_{-m+2}}{y},\tilde{A}_{-m+1},\tilde{B}_{-m+1}\right),\dots,\delta\frac{Y_{s_-+1}}{y},\dots,\delta\frac{Y_{s_++n+s_-+1}}{y}\right)
$$

$$\mathbb{1}_{\{\delta \cdot |X_{s_-+1}|>y\}} \;\Bigg|\; |Y_0| > y \Bigg).$$

Replacing y by δx and using stationarity this equals

$$\lim_{x\to\infty} \frac{\delta^{-\alpha}}{C} E\Bigg(f\left(h\left(\frac{Y_{-s_--m+1}}{x}, \tilde{A}_{-s_--m}, \tilde{B}_{-s_--m} \right), \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_++n}}{x} \right)$$

$$\mathbb{1}_{\{|X_0|>x\}} \;\Bigg|\; |Y_{-s_--1}| > \delta x \Bigg).$$

Again with Condition 1 we get

$$\lim_{x\to\infty} E\Bigg(f\left(h\left(\frac{Y_{-s_--m+1}}{x}, \tilde{A}_{-s_--m}, \tilde{B}_{-s_--m} \right), \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_++n}}{x} \right)$$

$$\mathbb{1}_{\{|Y_{-s_--1}|>\delta x\}} \;\Bigg|\; |X_0| > x \Bigg).$$

Since both h and f are uniformly continuous with $f(0,\dots) = 0$ and $h(0,\dots) = 0$, the complementary expression

$$\lim_{x\to\infty} E\Bigg(f\left(h\left(\frac{Y_{-s_--m+1}}{x}, \tilde{A}_{-s_--m}, \tilde{B}_{-s_--m} \right), \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_++n}}{x} \right)$$

$$\mathbb{1}_{\{|Y_{-s_--1}|\le\delta x\}} \;\Bigg|\; |X_0| > x \Bigg)$$

is bounded by

$$\sup f(h(y_1, a, b), y_1, \dots, y_{s_-+s_++m+n}), \quad y_1 < \delta, a, b, y_2, \dots, y_{s_-+s_++m+n} \in \mathbb{R},$$

which tends to 0 as δ does. We may now conclude that

$$\begin{aligned}
&\lim_{x\to\infty} E\left(f\left(\frac{Y_{-s_--m}}{x}, \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_++n}}{x} \right) \;\Big|\; |X_0| > x \right) \\
= &\lim_{x\to\infty} E\Bigg(f\left(h\left(\frac{Y_{-s_--m+1}}{x}, \tilde{A}_{-s_--m}, \tilde{B}_{-s_--m} \right), \dots, \right.
\end{aligned}$$

$$\left.\left.\frac{Y_0}{x}, \ldots, \frac{Y_{s_++n}}{x}\right) \,\right|\, |X_0| > x\right).$$

An application of the continuous mapping theorem in connection with the induction hypothesis yields that the latter expression equals

$$E\left(f\left(h\left(Y^X_{-s_--m+1}, A_{-s_--m}, B_{-s_--m}\right), \ldots, Y^X_0, \ldots, Y^X_{s_++n}\right)\right),$$

with (A_{-s_--m}, B_{-s_--m}) as in the statement of the proposition. Since $Y_{-s_--m} = h\left(Y^X_{-s_--m+1}, A_{-s_--m}, B_{-s_--m}\right)$ this finishes the proof. □

Remark 4.4. *If $(\hat{Y}^X_{-s_--1}, \ldots, \hat{Y}^X_0, \ldots, \hat{Y}^X_{s_+})$ is a random vector such that for a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \infty$ the relation*

$$\lim_{n\to\infty} \mathcal{L}\left(\frac{Y_{-s_--1}}{x_n}, \ldots, \frac{Y_0}{x_n}, \ldots, \frac{Y_{s_+}}{x_n} \,\middle|\, |X_0| > x_n\right) = \mathcal{L}(\hat{Y}^X_{-s_--1}, \ldots, \hat{Y}^X_0, \ldots, \hat{Y}^X_{s_+})$$

holds instead of Condition 3, a statement analogous to Proposition 4.1 holds true along the sequence $(x_n)_{n\in\mathbb{N}}$. The existence of such sequences is guaranteed by Condition 1, cf. Proposition 2.1.

5. Multivariate Regular Variation

In this chapter we will show that Condition 3 is closely related to the theory of multivariate regular variation. The latter is well explored for large classes of common time series models such as the aforementioned GARCH family.

From the equivalent definitions of multivariate regular variation given in the literature we choose the one used in [12]. Recall that a measurable function $U : \mathbb{R} \to \mathbb{R}^+$ is said to be univariate regularly varying with index $\alpha \in \mathbb{R}$ if $\lim_{x\to\infty} U(\tau x)/U(x) = \tau^\alpha$ for all $\tau > 0$. We call a random vector $\mathbf{Z} \in \mathbb{R}^d$ multivariate regularly varying if there exists a univariate regularly varying function $U : \mathbb{R}_+ \to \mathbb{R}_+$ with index $-\alpha$ and a non-degenerate, non-zero Radon measure ν on $\mathbb{E} = [-\infty, \infty]^d \setminus \{\mathbf{0}\}$ such that

$$P\left(\mathbf{Z} \in x\cdot\right)/U(x) \xrightarrow{v} \nu(\cdot), \qquad x \to \infty, \tag{5.1}$$

where $\xrightarrow{v}$ stands for vague convergence (cf. [11]) in $M_+(\mathbb{E})$, the space of all non-negative Radon measures on $\mathbb{E}$. One can show that the limit measure

ν is necessarily homogeneous, i.e. $\nu(xA) = x^{-\alpha}\nu(A)$ holds for all $x > 0$ and all Borel sets $A \subset \mathbb{E}$ (cf. [11]). The measure ν and, consequently, the extremal behavior of $\mathbf{Z}$ are thus completely described by the index α of regular variation, a constant $c > 0$ and a probability measure S on $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d | \|x\| = 1\}$. The latter is the so-called spectral measure. All three components together satisfy that

$$\nu\left(\left\{x \in \mathbb{E} : \|x\| > a, \frac{x}{\|x\|} \in \cdot\right\}\right) = c \cdot a^{-\alpha} \cdot S(\cdot)$$

holds for all $a > 0$ (cf. [11]).

It has been shown by [2] and [4] (cf. also [6]) that under mild assumptions about the distribution of $\epsilon_t, t \in \mathbb{Z}$, a stationary GARCH$(p,q)$ process is multivariate regularly varying, i.e. for $m, n \geq 0$ the vector $\mathbf{Z} = (\sigma^2_{-m}, \zeta^2_{-m}, \ldots, \sigma^2_n, \zeta^2_n)$ with σ_t and ζ_t as defined in (1.4) and (1.5) satisfies (5.1). Furthermore, one can easily show that the same holds for the vector $(\sigma_{-m}, |\zeta_{-m}|, \ldots, \sigma_n, |\zeta_n|)$. Now, knowing that a certain vector derived from the processes $(Y_t)_{t\in\mathbb{Z}}$ and $(X_t)_{t\in\mathbb{Z}}$ is multivariate regularly varying will be useful in the verification of Condition 3 as is shown in the following.

Let us again assume that $(Y_t, X_t)_{t\in\mathbb{Z}}$ is stationary and given by (1.2) and (1.3). Note that Condition 3 is equivalent to

$$\begin{aligned} & \lim_{x\to\infty} P\left(\left(\frac{Y_{-s_--1}}{x}, \ldots, \frac{Y_0}{x}, \ldots, \frac{Y_{s_+}}{x}\right) \in A \,\middle|\, |X_0| > x\right) \\ = \quad & \lim_{x\to\infty} \frac{P\left(\left(\frac{Y_{-s_--1}}{x}, \ldots, \frac{Y_0}{x}, \ldots, \frac{Y_{s_+}}{x}\right) \in A, |X_0| > x\right)}{P(|X_0| > x)} \\ = \quad & P\left(\left(Y^X_{-s_--1}, \ldots, Y^X_{s_+}\right) \in A\right) \end{aligned} \tag{5.2}$$

for a random vector $(Y^X_{-s_--1}, \ldots, Y^X_{s_+})$ and all $A \in \mathbb{R}^{s_-+s_++2}$ such that $P((Y^X_{-s_--1}, \ldots, Y^X_{s_+}) \in \partial A) = 0$.

In the following, we will assume multivariate regular variation of $(|X_0|, Y_{-s_--1}, \ldots, Y_{s_+})$ on $\mathbb{C} = \left(\bar{\mathbb{R}}_{+,0} \times \bar{\mathbb{R}}^{s_-+s_++2}\right) \setminus \{\mathbf{0}\}$, and show how this concept relates strongly to Condition 3. By continuity from below it suffices to look at such A which are bounded away from $\mathbf{0}$ in order to derive Condition 3 from (5.2). The assumption of multivariate regular variation of $(|X_0|, Y_{-s_--1}, \ldots, Y_{s_+})$ guarantees the existence of a function $U : \mathbb{R}_+ \to \mathbb{R}_+$

such that

$$
\begin{aligned}
& \lim_{x\to\infty} \frac{P\left(|X_0|>x, \left(\frac{Y_{-s_--1}}{x}, \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_+}}{x}\right) \in A\right)}{P(|X_0|>x)} \\
= & \lim_{x\to\infty} \frac{P\left(\left(\frac{Y_{-s_--1}}{x}, \dots, \frac{Y_0}{x}, \dots, \frac{Y_{s_+}}{x}\right) \in A, |X_0|>x\right)}{U(x)} \frac{U(x)}{P(|X_0|>x)} \\
= & \frac{\nu((1,\infty)\times A)}{\nu((1,\infty)\times \mathbb{R}^{s_-+s_++2})} \qquad (5.3)
\end{aligned}
$$

if the denominator is positive (it is necessarily finite since $(1,\infty)\times\mathbb{R}^{s_-+s_++2}$ is bounded away from the origin). One easily checks that (5.3) defines a probability measure for $A \in \mathbb{B}^{s_-+s_++2}$ and may be set as the law of the random vector $(Y^X_{-s_--1}, \dots, Y^X_{s_+})$ if $\nu((1,\infty)\times\mathbb{R}^{s_-+s_++2}) > 0$. Because of the aforementioned homogeneity of ν we note the equivalence

$$
\nu((1,\infty)\times\mathbb{R}^{s_-+s_++2}) = 0 \Leftrightarrow \nu((\delta,\infty)\times\mathbb{R}^{s_-+s_++2}) = 0 \ \ \forall\,\delta>0. \qquad (5.4)
$$

Thus, $\nu((1,\infty)\times\mathbb{R}^{s_-+s_++2}) = 0$ implies that the mass of ν is concentrated on the hyperplane $\{0\}\times\mathbb{R}^{s_-+s_++2}$. Note that this is not excluded by the definition of regular variation. Nevertheless, since ν is non-degenerate and since the process $(Y_t)_{t\in\mathbb{Z}}$ is stationary we find that

$$
\nu((1,\infty)\times\mathbb{R}^{s_-+s_++2}) = 0 \Rightarrow \lim_{x\to\infty} \frac{P(|Y_0|>x)}{U(x)} > 0.
$$

Now, $\nu((1,\infty)\times\mathbb{R}^{s_-+s_++2}) = 0$ implies that

$$
\lim_{x\to\infty} \frac{P(|X_0|>x)}{U(x)} = 0.
$$

Hence, $\nu((1,\infty)\times\mathbb{R}^{s_-+s_++2}) = 0$ entails that $|X_0|$ and $|Y_0|$ are not tail equivalent. This contradicts Condition 1 and leads to the following proposition.

Proposition 5.1. *Let $(|X_0|, Y_{-s_--1}, \dots, Y_{s_+}) \in \mathbb{R}_+\times\mathbb{R}^{s_-+s_++2}$ be a multivariate regularly varying vector with index α and let Condition 1 hold. Then Condition 3 is satisfied.*

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