

ON SOME CURVE SINGULARITY INVARIANTS AND REDUCTIONS

JULIO JOSÉ MOYANO-FERNÁNDEZ

ABSTRACT. This paper deals with the study of the behaviour of the value semigroup of a curve singularity defined over a global field reduced modulo a maximal ideal. We also define a global zeta function of the curve by means of motivic integration over a suitable ring of adèles, whose reduction modulo a maximal ideal will coincide with already known zeta functions of the singularity.

1. PRELIMINARIES

Resolution of singularities is a well-known topic since the pioneers Brill and Noether solved the case of complex curves more than a century ago. It became a central problem in algebraic geometry when mathematicians asked for generalisations in the two usual ways, namely resolution for varieties in higher dimensions and resolution in arbitrary ground fields. The introduction of basis change methods in scheme theory allowed to define the concept of reduction modulo a maximal ideal of an algebraic variety defined over a number field, which is nothing else than an attempt to preserve as many good properties as possible when one extends the variety to a scheme over the ring of integers of the field the curve is defined over.

Let C be a complex curve singularity. Many equivalent invariants, both topological and analytic, are since a long time well understood. We remark among others the associated value semigroup and the Poincaré series, profusely studied by v.g. Bayer [1], García [10], Kunz [12], Waldi [20], and Campillo, Delgado and Gusein-Zade [2]–[5], [7], as well as by

1991 *Mathematics Subject Classification*. Primary 13D40; Secondary 16W50, 20M99.

Key words and phrases. adèles, motivic integration, reduction of algebraic curves, zeta functions.

The author was partially supported by the Spanish Government Ministerio de Educación y Ciencia (MEC) grant MTM2007-64704 in cooperation with the European Union in the framework of the funds “FEDER”, and by the Deutsche Forschungsgemeinschaft (DFG).

Stöhr [17]–[19] and Zúñiga Galindo [21], [22] in the last years. The question this note deals with, is to describe the behaviour of these two singularity invariants under reduction modulo a maximal ideal \mathfrak{P} when the singularity is defined over a number field. The first issue to do is to understand resolution of singularities under such a reduction.

For the convenience of the reader we repeat the relevant material from [9], thus making our exposition self-contained.

1.1. Reduction modulo \mathfrak{P} . Let K be an arbitrary field of characteristic 0. Let R be a discrete valuation ring with quotient field K and maximal ideal \mathfrak{P} .

Let C be a plane curve defined over the field K given by an equation $f(x_1, x_2) \in K[x_1, x_2]$. Let us consider the schemes $X = \text{Spec } K[x_1, x_2]$, $\tilde{X} = \text{Spec } R[x_1, x_2]$, $\bar{X} = \text{Spec } \bar{K}[x_1, x_2]$ and

$$D = \text{Spec } (K[x_1, x_2]/(f(x_1, x_2))) \subset X.$$

Let Z be a closed subscheme of $\mathbb{P}_X^k := \text{Proj } K[x_1, x_2, \dots, x_k]$ for some nonnegative integer $k > 0$. We will denote by \tilde{Z} the scheme-theoretic closure of Z in $\mathbb{P}_{\tilde{X}}^k$. If Z is reduced (resp. integral), then \tilde{Z} is reduced (resp. integral). Moreover, \tilde{Z} is flat over $\text{Spec } R$.

Definition 1.1. The reduction modulo \mathfrak{P} of Z is the scheme

$$\bar{Z} = Z \bmod \mathfrak{P} := \tilde{Z} \times_R \text{Spec } \bar{K}.$$

Notice that $Z \bmod \mathfrak{P}$ can be considered a closed subscheme of \tilde{Z} .

1.2. Resolution of singularities. Let us recall the basics on resolution of curve singularities. By the sack of simplicity we will assume the singularities to be totally rational (i.e., all field extension degrees involved are assumed to be 1).

Definition 1.2. A resolution for the curve C over K is a pair (Y, π) , where Y is a closed integral scheme of \mathbb{P}_X^k for some nonnegative integer k and $\pi : Y \rightarrow X$ is a morphism such that

- (i) Y is smooth over $\text{Spec } K$;
- (ii) the restriction $\pi : Y \setminus \pi^{-1}(D) \rightarrow X \setminus D$ is an isomorphism;
- (iii) the reduced scheme $(\pi^{-1}(D))_{\text{red}}$ associated to $\pi^{-1}(D)$ has only normal crossings as subscheme of Y .

The scheme $(\pi^{-1}(D))_{\text{red}}$ will be called the exceptional divisor of π . Its irreducible components will be denoted by E_i , for $i \in T$.

Next result is well-known in singularity theory:

Theorem 1.3. *Any plane curve singularity has a resolution.*

Definition 1.4 ([9]). We say that (Y, π) is a resolution for C over K with good resolution modulo \mathfrak{P} if (Y, π) is a resolution for C over K satisfying

- (i) \bar{Y} is smooth over $\text{Spec } \bar{K}$;
- (ii) \bar{E}_i is smooth over $\text{Spec } \bar{K}$ for each $i \in T$, and $\bigcup_{i \in T} \bar{E}_i$ has only normal crossings as a subscheme of \bar{Y} ;
- (iii) \bar{E}_i and \bar{E}_j have no common irreducible components for $i \neq j$.

Remark 1.5. Note that “reduction modulo \mathfrak{P} ” does not change the set T , that is, it does not change the number of components of the exceptional divisor.

If C is a plane curve having a closed singular point, we can reach a resolution for C just blowing-up closed points successively to get a chain

$$Y := X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \dots \longrightarrow X_3 \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 =: X,$$

where π_i is the blowing-up of a closed point $p_{i-1} \in X_{i-1}$, for $1 \leq i \leq N$. Then Y is achieved by blowing-up successively closed points $p_i \in X_i$, $0 \leq i \leq N-1$. This process is called the *resolution process* of C . Every such a closed point p_i corresponds to a local ring $R_i := \mathcal{O}_{X_i, p_i}$. Since the curve C is assumed to be totally rational, the degree of this field extension is equal to 1. Finally, set $\pi := \pi_N \circ \pi_{N-1} \circ \dots \circ \pi_2 \circ \pi_1$. Further details on curve singularities over arbitrary fields can be founded in [11]; for another recent account see [13].

On the other hand, let us take $\pi : \tilde{X}_i \rightarrow \tilde{X}_{i-1}$ for any $i \in \{1, \dots, N\}$. Then we can consider the restriction to \tilde{X}_i of the projection $\mathbb{P}_{\tilde{X}}^k \rightarrow \tilde{X}_{i-1}$ for some $k > 0$, namely

$$\tilde{\pi}_i : \tilde{X}_i \rightarrow \tilde{X}_{i-1}.$$

Even more, using base extension we can get morphisms

$$\bar{\pi}_i : \bar{X}_i \rightarrow \bar{X}_{i-1}$$

with $1 \leq i \leq N$. Set $\bar{\pi} := \bar{\pi}_N \circ \bar{\pi}_{N-1} \circ \dots \circ \bar{\pi}_2 \circ \bar{\pi}_1$.

Theorem 1.6 ([9]). *If (Y, π) is a resolution for C over K with good reduction modulo \mathfrak{P} , then $(\overline{Y}, \overline{\pi})$ is a resolution for C modulo \mathfrak{P} over \overline{K} .*

Proof. First of all, \overline{Y} is a closed integral scheme of $\text{Proj } \overline{K}[x_1, \dots, x_k]$ for some $k > 0$ (see [9], Proposition 2.6 (b)). By Definition 1.4, condition (ii), \overline{Y} is smooth over $\text{Spec } \overline{K}$. Considering now the reduced structure of the exceptional divisor of $\overline{\pi}$, i.e., $\bigcup_{i \in T} \overline{E}_i$, it has only normal crossings as subscheme of \overline{Y} (again by Definition 1.4). Last, the restriction $\overline{Y} \setminus \bigcup_{i \in T} \overline{E}_i \rightarrow \overline{X} \setminus D$ is an isomorphism, since the morphism $\overline{Y} \rightarrow \overline{X}$ is birational (cf. [9], Proposition 2.6 (i)). Then $(\overline{Y}, \overline{\pi})$ is a resolution for $C \bmod \mathfrak{P}$. \square

Definition 1.7. Let C (resp. C') be a plane curve with resolution (Y, π) (resp. (Y', π')) over a field K (resp. K'). Let

$$Y := X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \dots \longrightarrow X_3 \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 =: X,$$

be the resolution process for C , and let

$$Y' := X'_{N'} \xrightarrow{\pi'_{N'}} X'_{N'-1} \xrightarrow{\pi'_{N'-1}} \dots \longrightarrow X'_3 \xrightarrow{\pi'_3} X'_2 \xrightarrow{\pi'_2} X'_1 \xrightarrow{\pi'_1} X'_0 =: X',$$

be the resolution process for C' . The curves C and C' are said to have the same resolution process if $N = N'$ and the strict transforms $\widetilde{C}^{(i)}$ (resp. $(\widetilde{C}')^{(i)}$) of C (resp. C') via π_i (resp. π'_i) have the same multiplicity.

Proposition 1.8. *Let C be a plane curve. Let (Y, π) be the resolution process of a totally rational closed point of C , and $(\overline{Y}, \overline{\pi})$ the corresponding process of the curve \overline{C} . These two resolution processes agree.*

Proof. By construction of $\overline{\pi}$ and Remark 1.5, both (Y, π) and $(\overline{Y}, \overline{\pi})$ consist of the same number of blowing-ups. Moreover, since the curve C is assumed to be totally rational, the multiplicities of the strict transforms of C and \overline{C} coincide. \square

The following theorem, due to Denef (cf. [9, Theorem 2.4]), ensures the existence of a resolution with good reduction modulo \mathfrak{P} for plane curves over number fields.

Theorem 1.9. *Let F be a number field, A its ring of algebraic integers. Let C be a plane curve over F and (Y, π) a resolution for C over F . Then (Y, π) is a resolution for C over F with good reduction modulo $\mathfrak{P}A_{\mathfrak{P}}$ for all except a finite number of maximal ideals \mathfrak{P} of A .*

2. THE VALUE SEMIGROUP AND POINCARÉ SERIES OF A PLANE CURVE SINGULARITY AND REDUCTION MODULO \mathfrak{P}

After introducing some elementary machinery, we will prove a first remarkable result: the value semigroup does not change under reduction modulo \mathfrak{P} .

Let F be a number field. Let A be the ring of integers in F . Assume that C is a complete, geometrically irreducible, algebraic curve defined over F . Let P be a rational and singular point of C . Assume that $\widehat{\mathcal{O}}_{P,C}$ is totally rational. Then $\widehat{\mathcal{O}}_{P,C}$ can be presented in the form

$$\widehat{\mathcal{O}}_{P,C} := \left\{ \left(\sum_{i=0}^{\infty} a_{i,1} t_1^i, \dots, \sum_{i=0}^{\infty} a_{i,d} t_d^i \right) \in \widetilde{\mathcal{O}}_{P,C} \mid \Delta = 0 \right\},$$

where $\widetilde{\mathcal{O}}_{P,C} \cong F[[t_1]] \times \dots \times F[[t_d]]$ and $\Delta = 0$ is a homogeneous system of linear equations with coefficients in F .

2.1. Value semigroup. Every factor in $\widetilde{\mathcal{O}}_{P,C}$ yields a discrete valuation v_i , for every $1 \leq i \leq d$, and thus a vector $\underline{v}(\underline{z}) = (v_1 z_1, \dots, v_d z_d)$ for any nonzero divisor $\underline{z} = (z_1, \dots, z_d) \in \widetilde{\mathcal{O}}_{P,C}$.

Definition 2.1. The value semigroup S of $\widehat{\mathcal{O}}_{P,C}$ consists of all the elements of the form $\underline{v}(\underline{z}) = (v_1(z_1), \dots, v_d(z_d)) \in \mathbb{N}^d$ for all the nonzero divisors $\underline{z} \in \widetilde{\mathcal{O}}_{P,C}$.

Observe that the value semigroup of $\widehat{\mathcal{O}}_{P,C}$ coincides with that of $\mathcal{O}_{P,C}$ (cf. [15, §2]).

The *degree of singularity* of the ring $\mathcal{O}_{P,C}$ is defined as

$$\delta(\mathcal{O}_{P,C}) = \delta_P = \dim_k \widetilde{\mathcal{O}}_{P,C} / \mathcal{O}_{P,C},$$

where $\widetilde{\mathcal{O}}_{P,C}$ is the normalisation of the ring $\mathcal{O}_{P,C}$. By [16, Theorem 1], $\delta_P < \infty$.

We set $\underline{1} := (1, \dots, 1) \in \mathbb{N}^d$ and, for $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, we consider the vector $\|\underline{n}\| := n_1 + \dots + n_d$. We introduce a partial order in \mathbb{N}^d , *the product order*, by taking $\underline{n} \geq \underline{m}$, if $n_i \geq m_i$ for every $i = 1, \dots, d$.

Definition 2.2. Let C be a plane curve. We define a complex model for C to be a plane curve C' over \mathbb{C} having the same resolution process.

The following theorem is due to Campillo (see [2, Proposition 4.3.12]):

Theorem 2.3. *Two given plane curves, defined over fields which are different in general, have the same value semigroup if and only if they have a complex model in common. In particular, the value semigroup of a curve and that of any of its models agree.*

Let \mathfrak{P} be a maximal ideal of A with residue field \mathbb{F}_q . We define the reduction mod \mathfrak{P} of $\widehat{\mathcal{O}}_{P,C}$ as

$$\widehat{\mathcal{O}}_{P,C \bmod \mathfrak{P}} := \left\{ \left(\sum_{i=0}^{\infty} b_{i,1} t_1^i, \dots, \sum_{i=0}^{\infty} b_{i,d} t_d^i \right) \in \widetilde{\mathcal{O}}_{P,C \bmod \mathfrak{P}} \mid \overline{\Delta} = 0 \right\},$$

where $\widetilde{\mathcal{O}}_{P,C \bmod \mathfrak{P}} \cong \mathbb{F}_q[[t_1]] \times \dots \times \mathbb{F}_q[[t_d]]$ and $\overline{\Delta} = 0$ denotes the reduction mod \mathfrak{P} of $\Delta = 0$.

We can now compare the value semigroup of a curve over a number field of characteristic 0 and its reduction modulo \mathfrak{P} :

Theorem 2.4. *For all except a finite number of maximal ideals \mathfrak{P} of A one has*

$$S(\mathcal{O}_{P,C}) = S(\mathcal{O}_{P,C \bmod \mathfrak{P}}).$$

Proof. It is just to apply previous results:

- (1) We have a field number of characteristic 0, then we have a resolution (Y, π) of C over F with good reduction modulo \mathfrak{P} for almost all \mathfrak{P} , by Theorem 1.9.
- (2) Then $(\overline{Y}, \overline{\pi})$ is a resolution of $\overline{C} = C \bmod \mathfrak{P}$ over \overline{K} for almost all \mathfrak{P} , by Theorem 1.6, and with the same resolution process as (Y, π) by Proposition 1.8.
- (3) Then C and \overline{C} have the same complex model for almost all \mathfrak{P} .
- (4) Last, by Theorem 2.3 the curves C and \overline{C} have the same value semigroup, for almost all \mathfrak{P} .

□

2.2. Generalised Poincaré series and Zeta functions. Let Var_k denote the category of k -algebraic varieties, and by $K_0(Var_k)$ the corresponding Grothendieck ring. It is the ring generated by symbols $[V]$, for V an algebraic variety, with the relations $[V] = [W]$ if V is isomorphic to W , $[V] = [V \setminus Z] + [Z]$ if Z is closed in V , and $[V \times W] = [V][W]$. We denote $\mathbf{1} := [\text{point}]$, $\mathbb{L} := [\mathbb{A}_k^1]$ and $\mathcal{M}_k := K_0(Var_k)[\mathbb{L}^{-1}]$ the ring obtained by localization with respect to the multiplicative set generated by \mathbb{L} .

For $\underline{n} \in S$ we set

$$\mathcal{I}_{\underline{n}} := \{I \subseteq \mathcal{O}_{P,C} \mid I = z\mathcal{O}_{P,C}, \text{ with } v(\underline{z}) = \underline{n}\},$$

and for $m \in \mathbb{N}$,

$$\mathcal{I}_m := \bigcup_{\substack{\underline{n} \in S \\ \|\underline{n}\|=m}} \mathcal{I}_{\underline{n}}.$$

Definition 2.5. We associate to $\mathcal{O}_{P,C}$ the two following zeta functions:

$$(1) \quad Z(t_1, \dots, t_d, \mathcal{O}_{P,C}) := \sum_{\underline{n} \in S} [\mathcal{I}_{\underline{n}}] \mathbb{L}^{-\|\underline{n}\|} t^{\underline{n}} \in \mathcal{M}_k[[t_1, \dots, t_d]],$$

where $t^{\underline{n}} := t_1^{n_1} \cdots t_d^{n_d}$, and

$$(2) \quad Z(t, \mathcal{O}_{P,C}) := Z(t, \dots, t, \mathcal{O}_{P,C}).$$

This zeta function was introduced in [15]; it coincides—up to a factor—with the generalised Poincaré series $P_g(t_1, \dots, t_d)$ defined by Campillo, Delgado and Gusein-Zade in [5] and was studied under a more general setting by the author in [14]:

Lemma 2.6.

$$Z(t_1, \dots, t_d) = \mathbb{L}^{\delta+1} P_g(t_1, \dots, t_d).$$

The generalised Poincaré series associated to a ring $\mathcal{O}_{P,C}$ only depends on the value semigroup of \mathcal{O} , hence by Theorem 2.4 we have

Corollary 2.7.

$$\begin{aligned} Z(t_1, \dots, t_d; \mathcal{O}_{P,C}) &= Z(t_1, \dots, t_d; \mathcal{O}_{P,C} \bmod \mathfrak{p}) \\ P_g(t_1, \dots, t_d; \mathcal{O}_{P,C}) &= P_g(t_1, \dots, t_d; \mathcal{O}_{P,C} \bmod \mathfrak{p}) \end{aligned}$$

3. MOTIVIC INTEGRATION OVER THE RING OF ADÈLES

The goal of this section is to define a global zeta function on the curve by using the theory of Adèles such that it can be expressed—roughly speaking—as a product of local zeta functions (Poincaré series), from which the ones corresponding to singular points are non-trivial. A good general reference here is the book of Cassels and Frölich [6].

Let X be a curve defined over k . Consider the family of local rings $\{\mathcal{O}_{P,X}\}_{P \in X}$. From now on we will write \mathcal{O}_P instead of $\mathcal{O}_{P,X}$. If $k(P)$ is the residue field at each point $P \in X$, we have

$$\begin{aligned} \widehat{\mathcal{O}}_P &\cong k(P)[[T]] \\ \widehat{K}_P &\cong k(P)((T)) \end{aligned}$$

for every $P \in X$. We want to establish a measure in \widehat{K}_P for $P \in X$. Take the class \mathcal{A}_P of subsets of X which are bounded from below; i.e.,

$Z \in \widehat{K}_P$ is said to be bounded from below if “ $\text{ord}_T(x) \geq \text{constant}$ ” for every $x \in Z$.

Let $n \in \mathbb{Z}$. Consider the map

$$\pi_n : \widehat{K}_P \rightarrow k(P)((T))/(T^{n+1}).$$

Notice that

$$k(P)((T))/(T^{n+1}) \cong \left\{ \sum_{k=-c}^n a_k T^k \mid a_k \in k(P) \right\} \cong k(P)^{n+c+1}$$

where the latter isomorphism holds because the subsets are bounded from below.

We say that a subset $Z \subseteq \widehat{K}_P$, $Z \in \mathcal{A}_P$ is measurable or cylindric if there exists $n \in \mathbb{Z}$ such that $Z = \pi_n^{-1}(Y)$, for $Y \subseteq k(P)((T))/(T^{n+1})$, Y constructible. We define the measure of such a Z as

$$\mu_P(Z) := [Y]\mathbb{L}^{-n}.$$

We define the ring of adèles \mathbb{A}_X of X as the restricted product of the \widehat{K}_P 's with respect to the $\widehat{\mathcal{O}}_P$'s, i.e., $x \in \mathbb{A}_X$ if and only if $x = (x_P)_{P \in X}$, $x_P \in \widehat{\mathcal{O}}_P$ for almost all P . Let S be a finite subset of points. Define

$$U_S := Z \times \prod_{P \notin S} \widehat{\mathcal{O}}_P.$$

We declare U_S as an open subset, where Z is a cylindric subset of $\prod_{P \in S} \widehat{K}_P$, and take as measure

$$\mu_S(Z) := \mu_S(\underline{Z}),$$

where $\mu_S(\underline{Z})$ is defined as follows: let

$$\underline{Z} \subseteq \prod_{P \in S} \widehat{K}_P$$

with \underline{Z} bounded from below (i.e., the order in T of each component is bounded from below); we say that \underline{Z} is measurable or cylindric if there exists $\underline{n} = (n_P)_{P \in S}$, $n_P \in \mathbb{Z}$ so that, for the map

$$\pi_{\underline{n}} : \prod_{P \in S} \widehat{K}_P \rightarrow \prod_{P \in S} k(P)((T))/(T^{n_P+1})$$

we have $\underline{Z} = \pi_{\underline{n}}^{-1}(\underline{Y})$, with \underline{Y} a constructible subset of

$$\prod_{P \in S} k(P)((T))/(T^{n_P+1}).$$

The measure of \underline{Z} is defined to be

$$\mu(\underline{Z}) := \mu_S(\underline{Z}) := [\underline{Y}] \mathbb{L}^{-\sum_{P \in S} n_P}.$$

We declare U_S as the open subsets and endow \mathbb{A}_X with a topology. Take the Borel σ -algebra generated by the subsets U_S which are bounded from below. Consider the maps

$$\phi : \mathbb{A}_X \rightarrow \mathbb{Z}.$$

The map ϕ is said to be cylindric if $\phi^{-1}(n) \subseteq \mathbb{A}_X$ is a cylindric subset and ϕ is bounded from below. Last we define the integral of a cylindric function $\phi : \mathbb{A}_X \rightarrow \mathbb{Z}$ to be

$$\int_{\mathbb{A}_X} T^\phi d\mu := \sum_{n \in \mathbb{Z}} \mu(\phi^{-1}(n)) T^n,$$

whenever the sum makes sense; in such a case the function ϕ is said to be integrable.

We give now the projective versions of the above definitions. Let $\mathbb{P}\widehat{K}_P$ be the projectivization of \widehat{K}_P with respect to the field K , that is, $\mathbb{P}\widehat{K}_P := (\widehat{K}_P \setminus \{0\}) / \sim$, where the equivalence relation \sim is defined as follows: for every $a, b \in \widehat{K}_P$, we say that $a \sim b$ if and only if there exists $\lambda \in K \setminus \{0\}$ so that $a = \lambda b$. From now on, all projectivizations we use will be referred to K .

Let $n \in \mathbb{Z}$. Let us consider the projectivization $\mathbb{P}(k(P)((T))/(T^{n+1}))$ and let us adjoin one point, that is,

$$\mathbb{P}^*(k(P)((T))/(T^{n+1})) := \mathbb{P}(k(P)((T))/(T^{n+1})) \cup \{*\},$$

with $*$ representing the added point making sense the definition, so that the map

$$\pi_n : \mathbb{P}\widehat{K}_P \rightarrow \mathbb{P}^*(k(P)((T))/(T^{n+1}))$$

is well-defined; i.e., for each $g \in \widehat{K}_P$, we denote $[g]$ the class of g in $\mathbb{P}\widehat{K}_P$. If $g \in \widehat{K}_P \setminus (T^{n+1})$, then $\pi_n([g]) \in \mathbb{P}(k(P)((T))/(T^{n+1}))$, and if $g \in (T^{n+1})$, then $\pi_n([g]) \in \{*\}$.

Notice that if $Z \in \mathcal{A}_X$, then $\mathbb{P}Z \in \mathcal{A}_X$, because

$$\text{ord}_T(x) = \text{ord}_T(\lambda x)$$

for all $x \in Z$ and all $\lambda \neq 0$. Thus, given a subset $Z \subseteq \mathbb{P}\widehat{K}_P$, $Z \in \mathcal{A}_X$, Z is said to be cylindric or measurable if there exists $n \in \mathbb{Z}$ so that

$$Z = \pi_n^{-1}(Y)$$

with $Y \subseteq \mathbb{P}k(P)((T))/(T^{n+1})$ constructible. Then we define the measure of such a Z as

$$\mu_P(Z) := [Y]\mathbb{L}^{-n}.$$

Let us take now a finite subset of points S of X and define

$$V_S = Z \times \mathbb{P} \prod_{P \notin S} \widehat{\mathcal{O}}_P,$$

where Z is a cylindric subset of $\mathbb{P} \prod_{P \in S} \widehat{K}_P$, and declare V_S as an open subset of $\mathbb{P}\mathbb{A}_X$, where $\mathbb{P}\mathbb{A}_X$ is the projectivization of the ring of adèles \mathbb{A}_X of X . As measure we take

$$\mu_S(Z) := \mu_S(\underline{Z}),$$

where $\mu_S(\underline{Z})$ is defined as follows. Let $\underline{Z} \in \mathbb{P} \prod_{P \in S} \widehat{K}_P$, $\underline{Z} = \mathbb{P}\underline{Z}'$ with $\underline{Z}' \in \prod_{P \in S} \widehat{K}_P$ such that the order function is bounded from below in every component. We say that \underline{Z} is measurable or cylindric if there exists $\underline{n} = (n_P)_{P \in S}$, $n_P \in \mathbb{Z}$ such that $\underline{Z} = \pi_{\underline{n}}^{-1}(\underline{Y})$ for

$$\pi_{\underline{n}} : \mathbb{P} \left(\prod_{P \in S} \widehat{K}_P \right) \rightarrow \mathbb{P} \left(\prod_{P \in S} k(P)((T))/(T^{n_P+1}) \right)$$

and \underline{Y} a constructible subset of $\mathbb{P} \left(\prod_{P \in S} k(P)((T))/(T^{n_P+1}) \right)$. Hence we define

$$\mu_S(\underline{Z}) := \mu(\underline{Z}) := [\underline{Y}]\mathbb{L}^{-\sum_{P \in S} n_P}.$$

The projectivization $\mathbb{P}\mathbb{A}_X$ is endowed with the topology inherited from \mathbb{A}_X .

Analogous to the non projectivised case, a map

$$\varphi : \mathbb{P}\mathbb{A}_X \rightarrow \mathbb{Z}$$

is said to be cylindric if $\varphi^{-1}(n) \subseteq \mathbb{P}\mathbb{A}_X$ is a cylindric subset and φ is bounded from below. Lastly, we define the integral of a cylindric function $\varphi : \mathbb{P}\mathbb{A}_X \rightarrow \mathbb{Z}$ to be

$$\int_{\mathbb{P}\mathbb{A}_X} T^\varphi d\mu := \sum_{n \in \mathbb{Z}} \mu(\varphi^{-1}(n))T^n,$$

whenever the sum makes sense; in such a case the function φ is said to be integrable.

Notice that the function $\phi : \mathbb{A}_X \rightarrow \mathbb{Z}$ is cylindric if and only if the function $\varphi : \mathbb{P}\mathbb{A}_X \rightarrow \mathbb{Z}$ is cylindric, and we have

$$(\mathbb{L} - 1) \int_{\mathbb{P}\mathbb{A}_X} T^\varphi d\mu = \int_{\mathbb{A}_X} T^\phi d\mu.$$

(cf. [15, Remark 4]).

Let us take the cylindric function $T^{|\underline{v}(\cdot)|} : \mathcal{O} \rightarrow \mathbb{Z}[[T]]$ defined by $\underline{z} \mapsto T^{|\underline{v}(\underline{z})|}$, with $T^{|\underline{v}(\underline{z})|} := 0$ if $\underline{v}(\underline{z}) = \infty$.

Definition 3.1. The integral

$$\zeta(X, T) := \int_{\mathbb{A}_X} T^{|\underline{v}(\underline{z})|} d\mu$$

will be called the adélic zeta function associated with X .

We want now to show that this global zeta function decomposes into the product of the adélic zeta function of the normalisation with the local zeta functions corresponding to the singular points of the curve, which generalises slightly a result of Zúñiga (cf. [22, Corollary 2.2]).

Let $P \in S = \{P_1, \dots, P_r\}$. Let us write $Z_P = \pi_{n_P}^{-1}(Y_P)$ for Y_P a constructible subset of $k(P)((T))/(T^{n_P+1})$. Then one has

$$\begin{aligned} \mu(\underline{Z}) &= [Y_{P_1} \times \dots \times Y_{P_r}] \mathbb{L}^{-n_{P_1} - \dots - n_{P_r}} = [Y_{P_1}] \mathbb{L}^{-n_{P_1}} \cdot \dots \cdot [Y_{P_r}] \mathbb{L}^{-n_{P_r}} \\ &= \mu(Z_{P_1}) \cdot \dots \cdot \mu(Z_{P_r}). \end{aligned}$$

Denoting ϕ_i the restriction $\phi|_{\widehat{\mathcal{O}}_{P_i}} : \widehat{\mathcal{O}}_{P_i} \rightarrow \mathbb{Z}$, then we have

$$\int_Z T^\phi d\mu = \int_{\widehat{\mathcal{O}}_{P_1}} T^{\phi_1} d\mu \cdot \dots \cdot \int_{\widehat{\mathcal{O}}_{P_r}} T^{\phi_r} d\mu.$$

Because of the multiplicativity of the measure $\mu(\cdot)$ one has

$$\mu(\mathbb{A}_X) = \mu(Z \times \prod_{P \notin S} \mathcal{O}_P) = \mu(Z) \mu(\prod_{P \notin S} \mathcal{O}_P),$$

hence

$$\begin{aligned} \zeta(X, T) &= \int_{Z \times \prod_{P \notin S} \mathcal{O}_P} T^\phi d\mu = \int_Z T^\phi d\mu \cdot \int_{\prod_{P \notin S} \mathcal{O}_P} T^\phi d\mu \\ &= \int_Z T^\phi d\mu \cdot \int_{\tilde{X}} T^\phi d\mu. \end{aligned}$$

Furthermore, by [15, Corollary 4] one has

$$Z(\mathcal{O}_P, T) = \frac{1}{(\mathbb{L} - 1) \mathbb{L}^{-\delta_P - 1}} \int_{\mathcal{O}_P} T^{|\underline{v}(\underline{z})|} d\mu.$$

We have thus proved:

Theorem 3.2. *Let $\delta := \delta_{P_1} + \dots + \delta_{P_r}$. We have*

$$\mathbb{L}^\delta \zeta(X, T) = (1 - \mathbb{L}^{-1})^r \cdot \zeta(\tilde{X}, T) \cdot \prod_{i=1}^r Z(T, \mathcal{O}_{P_i}).$$

By taking reduction modulo \mathfrak{P} we obtain:

Corollary 3.3. *Let $X' := X \bmod \mathfrak{P}$. We have:*

$$\mathbb{L}^\delta \zeta(X', T) = (1 - \mathbb{L}^{-1})^r \cdot \zeta(\tilde{X}', T) \cdot \prod_{i=1}^r Z(T, \mathcal{O}_{P_i, X'}).$$

Theorem 3.2 extends to the result of [22] if one specialises $[\cdot]$ to the additive invariant $\sharp : \text{Var}_k \rightarrow \mathbb{Z}$ given by counting points by taking a finite field \mathbb{F}_q as ground field.

Notice that our definition differs from the one introduced by Zúñiga in [22]; he considered there a Dirichlet series $Z(\text{Ca}(X), T)$ associated to the effective Cartier divisors on the algebraic curve defined over \mathbb{F}_q . Both series are related by means of the equality

$$\sharp(\zeta(X, T)) = \frac{(1 - q^{-1})^r}{q^\delta} Z(\text{Ca}(X), T).$$

We can express the formula given by Corollary 3.3 in another nice way. Let P be a point on X . Let us consider the group $U_{\tilde{\mathcal{O}}_P}$ of the units of the normalisation of the local ring \mathcal{O}_P . The subgroup $U_{\mathcal{O}_P}$ of the group $U_{\tilde{\mathcal{O}}_P}$ is of finite index, say $(U_{\tilde{\mathcal{O}}_P} : U_{\mathcal{O}_P})$. In fact, by taking the normalisation morphism $\pi : \tilde{X} \rightarrow X$, one has $\pi^{-1}(P) = \{Q_1, \dots, Q_m\}$, where Q_i are the branches of X centered at \mathcal{O}_P and correspond to maximal ideals \mathfrak{m}_i of the semilocal ring $\tilde{\mathcal{O}}_P$; furthermore the following well-known equality holds (cf. [18, p.182]):

$$(U_{\tilde{\mathcal{O}}_P} : U_{\mathcal{O}_P}) = \frac{q^{\delta_P}}{(1 - q^{-r_P})} \prod_{i=1}^m (1 - q^{-\deg(Q_i)}),$$

where r_P is the degree of the residue field of \mathcal{O}_P over \mathbb{F}_q , and the degree $\deg(Q_i) := \dim \tilde{\mathcal{O}}_P / \mathfrak{m}_i$, for every $i \in \{1, \dots, m\}$. Since we are assuming the local rings \mathcal{O}_P to be totally rational, both the integers r_P and $\deg(Q_i)$, for all $i \in \{1, \dots, m\}$, are equal to 1. By applying Corollary 3.3 we get:

Corollary 3.4.

$$\sharp(\zeta(X, T)) = \sharp(\zeta(\tilde{X}, T)) \cdot \prod_{P \in S} \prod_{Q \in \pi^{-1}(P)} \frac{1 - q^{-1}}{(U_{\tilde{\mathcal{O}}_P} : U_{\mathcal{O}_P})} \cdot Z(T, \mathcal{O}_{P_i}).$$

REFERENCES

- [1] Bayer, V.: *Semigroup of two irreducible algebroid plane curves*. Manuscripta Math. **39** (1985), 207–241.
- [2] Campillo, A.: *Algebroid curves in positive characteristic*. Lecture Notes in Math. **813**. Springer, Berlin-Heidelberg, 1980.
- [3] Campillo A., Delgado, F., and Gusein-Zade, S.M.: *The Alexander polynomial of a plane curve singularity via the ring of functions on it*. Duke Math. J. **117** (2003), no. 1, 125–156.
- [4] Campillo A., Delgado, F., and Gusein-Zade, S.M.: *The Alexander polynomial of a plane curve singularity and integrals with respect to the Euler characteristic*. Internat. J. Math. **14** (2003), no. 1, 47–54.
- [5] Campillo A., Delgado, F., and Gusein-Zade, S.M.: *Multi-index filtrations and Generalized Poincaré series*. Monatshefte für Math **150** (2007), 193–209.
- [6] Cassels J.W.L., Frölich, A.: *Algebraic Number Theory*. Thomson Book Company Inc., Washington D.C., 1967.
- [7] Delgado de la Mata, F.: *Gorenstein curves and symmetry of the semigroup of values*. Manuscripta Math. **61** (1988), no. 3, 285–296.
- [8] Delgado de la Mata, F., and Moyano-Fernández, J.J.: *On the relation between the generalized Poincaré series and the Stöhr zeta function*. Proc. Am. Math. Soc. **137** (2009), no. 1, 51–59.
- [9] Denef, J.: *On the degree of Igusa’s local zeta function*. American Journal of Math. **109**, (1987), no. 6, 991–1008.
- [10] García, A.: *Semigroups associated to singular points of plane curves*. J. reine angew. Math. **336** (1982), 165–184.
- [11] Kiyek, K., Vicente, J. L.: *Resolution of Curve and Surface Singularities in Characteristic Zero*, Kluwer, Dordrecht, 2004.
- [12] Kunz, E.: *The value-semigroup of a one-dimensional Gorenstein ring*. Proc. Am Math. Soc. **25** (1970), 748–751.
- [13] Moyano-Fernández, J.J.: *Curvettes and clusters of infinitely near points*. Rev. Mat. Complut. **24**, (2011), 439–463.
- [14] Moyano-Fernández, J.J.: *Generalised Poincaré series and embedded resolution of curves*. Monatshefte Math. Online first publication (2010), DOI 10.1007/s00605-010-0259-z, 24 pp.
- [15] Moyano-Fernández, J.J., Zuniga-Galindo, W.A.: *Zeta functions for curve singularities*. Nagoya Math. J. **198** (2010), 47–75.
- [16] Rosenlicht, M.: *Equivalence relations on algebraic curves*. Ann. of Math. (2) **56**, (1952), 169–191.
- [17] Stöhr, K.-O.: *On the poles of regular differentials of singular curves*. Bol. Soc. Brasil. Mat. (N.S.) **24** (1993), no. 1, 105–136.
- [18] Stöhr, K.-O.: *Local and Global Zeta Functions of Singular Algebraic Curves*. Journal of Number Theory **71**, (1998), 172–202.
- [19] Stöhr, K.-O.: *Multi-variable Poincaré series of algebraic curve singularities over finite fields*. Math. Zeitschrift **262**, (2008), no. 4, 849–866, DOI: 10.1007/s00209-008-0402-x
- [20] Waldi, R.: *Wertehalbgruppe und Singularität einer ebenen algebroiden Kurve*. Ph.D. Thesis, Universität Regensburg, Regensburg, Germany, 1972.
- [21] Zúñiga Galindo, W.A.: *Zeta functions and Cartier divisors on singular curves over finite fields*. Manuscripta Math. **94** (1997), no. 1, 75–88.

- [22] Zúñiga-Galindo, W.A.: *Zeta functions of singular curves over finite fields*. Rev. Colombiana Mat. **31** (1997), no. 2, 115–124.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT OSNABRÜCK. ALBRECHTSTRASSE
28A, D-49076 OSNABRÜCK, GERMANY

E-mail address: `jmojanof@uni-osnabrueck.de`