SIMPLIFYING AND UNIFYING BRUHAT ORDER FOR $B\backslash G/B$, $P\backslash G/B$, $K\backslash G/B$, AND $K\backslash G/P$

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Abstract. This paper provides a unifying and simplifying approach to Bruhat order in which the usual Bruhat order, parabolic Bruhat order, and Bruhat order for symmetric pairs are shown to have combinatorially analogous and relatively simple descriptions. Such analogies are valuable as they permit the study of $P\backslash G/B$ and $K\backslash G/B$ by reducing to $B\backslash G/B$ rather than by introducing additional machinery. A concise definition for reduced expressions and a simple proof of the exchange condition for $P\backslash G/B$ are provided as applications of this philosophy. A parametrization of $K\backslash G/P$ is a simple consequence of understanding the Bruhat order of $K\backslash G/B$ restricted to a P-orbit.

1. Introduction

Bruhat order is an important tool in many branches of representation theory, in part because of the importance of studying orbits on the flag variety. Category $\mathcal{O}$ and the category of Harish-Chandra modules are two categories for which representations are related to orbits on the flag variety. Let G be a complex reductive linear algebraic group with Lie algebra $\mathfrak{g}$, θ a Cartan involution of G (specifying a real form), $K = G^\theta$, and B a θ-stable Borel subgroup. Using Beilinson-Bernstein's geometric construction, irreducible representations in Category $\mathcal{O}$ of trivial infinitesimal character are known to be in correspondence with B-orbits on the flag variety while irreducible Harish-Chandra modules of trivial infinitesimal character are in bijection with K-equivariant local systems on K-orbits on the flag variety. The module constructions may be modified suitably to produce modules for other infinitesimal characters.

Multiplicities of irreducible composition factors in standard modules for each of these categories can be computed using Kazhdan-Lusztig-Vogan polynomials. Finding efficient means of computing such polynomials is a heavily studied problem. Since the recursion formulas for computing Kazhdan-Lusztig-Vogan polynomials are expressed in terms of the Bruhat order on orbits and on local systems, we hope that the simplifications to Bruhat order for $P\backslash G/B$ and $K\backslash G/B$ contained in this paper may lead to a deeper understanding of the Kazdhan-Lusztig-Vogan polynomials in these categories, and the relationships among them. (Recall that local systems are parametrized by certain orbits in pairs of flag varieties.)

Within this paper, we discuss how Bruhat order can be described by the simple relations. Simple relations for Bruhat order are examined from the following perspectives for each of $B\backslash G/B$, $P\backslash G/B$, and $K\backslash G/B$:

- topological (closure order)
- cross actions and Cayley transforms
- roots and the Weyl group

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- roots and pullbacks.

Strong analogies are drawn between the different cases $B\backslash G/B$, $P\backslash G/B$, and $K\backslash G/B$. This permits definitions of standard objects and proofs of properties to be simplified for $P\backslash G/B$ and for $K\backslash G/B$: it is more efficient to exploit similarities with $B\backslash G/B$ than it is to introduce new machinery to accommodate the differences. The simplest combinatorial descriptions of the simple Bruhat relations are Theorems 3.5, 4.6, and 7.24.

Beyond parametrizing representations of various categories, orbits on the flag variety and Bruhat order appear in geometry (symmetric spaces, spherical homogeneous spaces) and in number theory problems in which one studies the fixed points of an involution. Bruhat order is ubiquitous in mathematics and is of fundamental importance.

This paper is structured as follows. Section 2 contains notation and the setup which will remain fixed for the duration of the paper. Sections 3 and 4 discuss Bruhat order for $B\backslash G/B$ and for $P\backslash G/B$, respectively.

We discuss reduced expressions and the exchange property in sections 5 and 6 for each of $B\backslash G/B$ and $P\backslash G/B$.

In section 7, Bruhat order for $K\backslash G/B$ is simplified and shown to be analogous to Bruhat orders for $B\backslash G/B$ and for $P\backslash G/B$. Motivated by Adams-duCloux, the definition of type I and type II roots is shown to have a natural conjugation invariant definition independent of the real form. Using this definition, whether or not a cross action is trivial is easy to understand.

In section 8, we give a simple combinatorial parametrization of $K\backslash G/P$ and discuss how the monoidal action descends (or fails to descend) from $K\backslash G/B$ to $K\backslash G/P$.

The final section contains a discussion of future work.

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2. Setup and Notation

The following notation will be fixed for the duration of this paper.

- G: complex reductive linear algebraic group
- $\theta \in \mathrm{Aut}(G)$: a Cartan involution (i.e. holomorphic involution)
- B: a θ-stable Borel (exists by Steinberg, see [**?**] 2.3)
- $B = TU$: the θ-stable Levi decomposition. Since B is θ-stable, therefore T is maximally compact (see Lemma 2.6 of [**?**]).
- $\mathfrak{g}$: the Lie algebra of G. Analogous notation will be used for Lie algebras of other groups.
- $\Delta(\mathfrak{g}, \mathfrak{t})$: the roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$
- $W = W_G$: the Weyl group $N_G(T)/T$ of G
- Π: the set of simple roots corresponding to B
- S: the set of simple reflections corresponding to Π
- P_J: the standard parabolic subgroup corresponding to $J \subset \Pi$
- $P = LN$: the T-stable Levi decomposition of P
- $I \subset \Pi$: the subset corresponding to P
- W_L: the Weyl Group $N_L(T)/T$ of L

- $x_\alpha : \mathbb{R} \to G$: an appropriately selected one-parameter subgroup of G associated to α
- $\phi_\alpha : SL_2 \to G$ the group homomorphism satisfying

$$x_\alpha(m) = \phi_\alpha \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \qquad x_{-\alpha}(m) = \phi_\alpha \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \qquad \alpha^\vee(t) = \phi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

- $\dot{s}_\alpha = x_\alpha(-1)x_{-\alpha}(1)x_\alpha(-1) = \phi_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in N(T)$ ($= n_\alpha$ in the notation of [**?**])
- $\dot{w}$: Given $w = s_{i_1}s_{i_2}\cdots s_{i_k} \in W$ a reduced expression, define $\dot{w} = \dot{s}_{i_1}\dot{s}_{i_2}\cdots\dot{s}_{i_k} \in N(T)$. It is known that $\dot{w}$ is independent of the reduced expression chosen.
- $H_g := gHg^{-1}$ for every $g \in G$, $H \leq G$
- $\mathscr{B}$:= the variety of Borel subgroups of G. This is in bijection with G/B where $B_g \leftrightarrow gB$.

3. Bruhat Order for $B\backslash G/B$

3.1. **Reducing to Simple Relations.** It is well-known that:

Proposition 3.1. *Bruhat Decomposition:*

$$G = \amalg_{w\in W} B\dot{w}B.$$

Therefore $B\backslash G/B \leftrightarrow W$.

Studying the closures of double cosets of $B\backslash G/B$ leads to a geometric definition for the Bruhat order on the Weyl group:

$$\overline{B\dot{w}B} = \cup_{y\leq w} B\dot{y}B.$$

Utilizing the bijection between the double cosets and the Weyl group, which as a reflection group has a simple description in terms of generators and relations, Bruhat order may be defined more tangibly as follows:

Definition 3.2. For $u, v \in W$, t a (not necessarily simple) reflection, we write $u \xrightarrow{t} v$ if $v = ut$ and $\ell(u) < \ell(v)$. Bruhat order on W is defined by $u \leq v$ if there exists a sequence of elements $w_0, w_1, \ldots, w_k \in W$ such that $u = w_0 \to w_1 \to \cdots \to w_k = v$.

In [**?**], Deodhar discusses four equivalent definitions of Bruhat order. We are most interested in description II, which we now repeat.

Definition 3.3. Let $u, v \in W$ and s be a simple reflection. *Property $Z(s, u, v)$* is satisfied if whenever $\ell(us) \leq \ell(u)$ and $\ell(vs) \leq \ell(v)$, then

$$u \leq v \iff us \leq v \iff us \leq vs.$$

Proposition 3.4. *([**?**]) Bruhat order is the unique partial ordering $\leq$ on W such that*

(1) *for all $w \in W$, $w \leq 1 \iff w = 1$;*
(2) *$\leq$ has Property $Z(s, u, v)$.*

We focus on this particular definition of Bruhat order since it allows us to consider only $u \xrightarrow{t} v$ where t is a simple reflection (although we aim to be more general where possible). We consider different formulations of the simple Bruhat relations.

3.2. **Weyl Group and Roots.** Another means of describing (not necessarily simple) Bruhat relations is:

Theorem 3.5. *Let α be a positive root and $w \in W$. Then Bruhat order is generated by the relations:*

$$\begin{aligned} w &\xrightarrow{s_\alpha} ws_\alpha \text{ if } w\alpha \in \Delta^+ = \Delta(\mathfrak{u}, \mathfrak{t}) \\ w &\xleftarrow{s_\alpha} ws_\alpha \text{ if } w\alpha \in \Delta^- = \Delta(\mathfrak{u}^-, \mathfrak{t}). \end{aligned}$$

Proof. This is known if α is a simple root ([?], Lemma 1.6). In general, suppose $\ell(w) < \ell(ws_\alpha)$. Let $ws_\alpha = s_1 s_2 \cdots s_r$ be a reduced expression for ws_α. By the strong exchange condition, $w = s_1 s_2 \cdots \hat{s}_i \cdots s_r$. Then $s_\alpha = w^{-1} s_1 s_2 \cdots s_r = s_r s_{r-1} \cdots s_{i+1} s_i s_{i+1} \cdots s_{r-1} s_r$. We conclude that $\alpha = s_r s_{r-1} \cdots s_{i+1} \alpha_i$. (Note that since $s_1 s_2 \cdots s_r$ is a reduced expression, $\alpha = s_r s_{r-1} \cdots s_{i+1} \alpha_i$ is a positive root, [?] p. 14.) Then $w\alpha = s_1 \cdots \hat{s}_i \cdots s_r s_r \cdots s_{i+1} \alpha_i = s_1 \cdots s_{i-1} \alpha_i > 0$ since $s_1 s_2 \cdots s_r$ is a reduced expression.

Similarly, if $\ell(w) > \ell(ws_\alpha)$, then $w\alpha < 0$. □

3.3. **Roots and Pullbacks.** Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ and $\Delta^+ = \Delta(\mathfrak{b}, \mathfrak{t})$. Define T_g and U_g by $B = TU \overset{\mathrm{int}(g)}{\longmapsto} gBg^{-1} = (gTg^{-1})(gUg^{-1}) =: T_g U_g$. Considering Lie algebras, $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u} \overset{\mathrm{Ad}(g)}{\longmapsto} \mathfrak{b}_g = \mathfrak{t}_g \oplus \mathfrak{u}_g$. We have the map between Cartan subalgebras $\mathrm{Ad}(g^{-1}) : \mathfrak{t}_g \to \mathfrak{t}$, while pullback allows us to map between duals of Cartan subalgebras: $\mathrm{Ad}(g^{-1})^* : \mathfrak{t}^* \to \mathfrak{t}_g^*$:

$$(\mathrm{Ad}(g^{-1})^* \alpha)(t) = \alpha(\mathrm{Ad}(g^{-1})t) \quad \text{for all } t \in \mathfrak{t}_g.$$

(Or view α as a group homomorphism and use $\mathrm{int}(g^{-1})$.) For $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$, for all $x \in \mathfrak{g}_\alpha$, and for all $t \in \mathfrak{t}_g$:

$$\begin{aligned} [t, \mathrm{Ad}(g)x] &= \mathrm{Ad}(g)[\mathrm{Ad}(g^{-1})t, x] \\ &= \alpha(\mathrm{Ad}(g^{-1})t)(\mathrm{Ad}(g)x) \qquad \text{since } \mathrm{Ad}(g) \text{ is linear and } \mathrm{Ad}(g^{-1})t \in \mathfrak{t} \\ &= (\mathrm{Ad}(g^{-1})^* \alpha)(t)\, \mathrm{Ad}(g)x \end{aligned}$$

Thus letting $\alpha_g = \mathrm{Ad}(g^{-1})^* \alpha$,

$$\begin{aligned} \mathrm{Ad}(g) : \mathfrak{u} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \quad &\to \quad \mathfrak{u}_g = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\alpha_g} \\ \text{with} \qquad \mathrm{Ad}(g)\mathfrak{g}_\alpha \quad &= \quad \mathfrak{g}_{\alpha_g}. \end{aligned}$$

It is straightforward to prove (use int rather than Ad):

Lemma 3.6. *For $w \in W$ and $n \in N(T)$ any representative of w, $w\alpha = \alpha_n$. In particular, $w\alpha = \alpha_{\dot{w}}$.*

Using pullbacks, Bruhat order may be reformulated as follows:

Proposition 3.7. *Let α be a positive root and $w \in W$. Then:*

$$\begin{aligned} w &\xrightarrow{s_\alpha} ws_\alpha \text{ if } \alpha_{\dot{w}} \in \Delta(\mathfrak{u}, \mathfrak{t}) \\ w &\xleftarrow{s_\alpha} ws_\alpha \text{ if } \alpha_{\dot{w}} \in \Delta(\mathfrak{u}^-, \mathfrak{t}). \end{aligned}$$

This may also be written

$$\begin{aligned} B\dot{w}B &\xrightarrow{\alpha} B\dot{w}\dot{s}_\alpha B \text{ if } \alpha_{\dot{w}} \in \Delta(\mathfrak{u}, \mathfrak{t}) \\ B\dot{w}B &\xleftarrow{\alpha} B\dot{w}\dot{s}_\alpha B \text{ if } \alpha_{\dot{w}} \in \Delta(\mathfrak{u}^-, \mathfrak{t}). \end{aligned}$$

Remark 3.8. Pullbacks turn out to be particularly useful in studying Bruhat order in more general situations: for example, if one of the subgroups with respect to which you take double cosets is twisted by conjugation or if it is a more general spherical subgroup. See [?] for details.

3.4. **Cross Actions.**

Definition 3.9. The cross action of W on $B\backslash G/B$ is the action generated by

$$s_\alpha \times B\dot{w}B := B\dot{w}\dot{s}_\alpha^{-1}B$$

where α is a positive root.

Under the correspondence between $B\backslash G/B$ and W, cross action corresponds to the natural left action of W on itself by right multiplication by the inverse.

It follows immediately:

Theorem 3.10. *Let α be a positive root and $w \in W$. Then Bruhat order is generated by the relations:*

$$\begin{array}{lcl} B\dot{w}B & \stackrel{\alpha}{\rightarrow} & s_\alpha \times B\dot{w}B = B\dot{w}\dot{s}_\alpha^{-1}B = B\dot{w}\dot{s}_\alpha B \text{ if } w\alpha \in \Delta(\mathfrak{u}, \mathfrak{t}) \\ B\dot{w}B & \stackrel{\alpha}{\leftarrow} & s_\alpha \times B\dot{w}B = B\dot{w}\dot{s}_\alpha^{-1}B = B\dot{w}\dot{s}_\alpha B \text{ if } w\alpha \in \Delta(\mathfrak{u}^-, \mathfrak{t}). \end{array}$$

3.5. **Closure Order.**

Definition 3.11. The closure order on $B\backslash G/B$ is defined by $\mathcal{O}_1 \preceq^B \mathcal{O}_2$ if $\mathcal{O}_1 \subset \overline{\mathcal{O}}_2$.

As discussed above, it is well-known that closure order and Bruhat order on $B\backslash G/B$ are the same. We discuss the proof in order to highlight how orbit dimensions change by at most one under simple relations.

Definition 3.12. Given α a simple root, we define $P_\alpha = P_{\{\alpha\}}$ (the standard parabolic of type α containing B). Then we have the canonical projection:

$$\pi_\alpha : G/B \to G/P_\alpha.$$

which may be viewed as a projection from $\mathscr{B}$ to $\mathscr{P}_\alpha$, the variety of parabolics of type α.

The set of Borel subgroups contained in P_α is in bijection with $\mathbb{P}^1$. Therefore we see that:

Lemma 3.13. *([?]) π_α is a $\mathbb{P}^1$-fibration: $\pi_\alpha^{-1}\pi_\alpha(x) \simeq \mathbb{P}^1$ for all $x \in \mathscr{B}$.*

$$\begin{array}{ccc} \mathbb{P}^1 & \rightarrow & G/B \\ & & \downarrow \pi_\alpha \\ & & G/P_\alpha \end{array}.$$

Lemma 3.14. *If $\ell(ws_\alpha) = \ell(w) + 1$ then:*

1. $\pi_\alpha^{-1}\pi_\alpha(B\dot{w}B) = B\dot{w}B \cup B\dot{w}\dot{s}_\alpha B$;
2. $\dim B\dot{w}\dot{s}_\alpha B = \dim B\dot{w}B + 1$;
3. $\overline{\pi_\alpha^{-1}\pi_\alpha(B\dot{w}B)} = \pi_\alpha^{-1}\pi_\alpha(\overline{B\dot{w}B})$;
4. $B\dot{w}B \subset \overline{B\dot{w}\dot{s}_\alpha B}$.

First, $\pi_\alpha^{-1}\pi_\alpha(B\dot{w}B) = B\dot{w}P_\alpha/B = B\dot{w}B \cup B\dot{w}\dot{s}_\alpha B$ by the parabolic Bruhat decomposition (see the following section) and by Lemma 8.3.7 of [**?**]. Relating the dimension of the B-orbit of $\dot{w}B \in G/B$ to the dimension of its stabilizer $B_{\dot{w}} \cap B$, (2) follows from the relationship between lengths of Weyl group elements and the number of positive roots sent to negative roots by the elements. The third statement follows from purely pointset topological considerations relying upon the definition of the quotient topology: as we will see in the proof of Proposition 8.2, $\overline{\pi_\alpha(B\dot{w}B)} = \pi_\alpha(\overline{B\dot{w}B})$, whence $\overline{\pi_\alpha^{-1}\pi_\alpha(B\dot{w}B)} = \pi_\alpha^{-1}\pi_\alpha(\overline{B\dot{w}B})$. The final statement follows immediately from the third.

Notation 3.15. Let $\alpha \in \Pi$ and $w \in W$ with $\ell(ws_\alpha) = \ell(w) + 1$. Then write

$$B\dot{w}B \preceq_\alpha^B B\dot{w}\dot{s}_\alpha B.$$

Theorem 3.16. *For $w, w' \in W$, α simple,*

$$B\dot{w}B \xrightarrow{\alpha} B\dot{w'}B \iff B\dot{w}B \preceq_\alpha^B B\dot{w'}B.$$

Therefore, $\mathcal{O}_u \preceq^B \mathcal{O}_v \iff u \le v$.

Proof. Since Bruhat order depended only upon simple relations and the previous lemma states that simple Bruhat relations and simple closure relations are the same, we conclude that

$$u \le v \Rightarrow BuB \subset \overline{BvB}.$$

To prove that Bruhat order and closure order are the same, it suffices to prove that $\cup_{u \le v} BuB$ is closed. See Proposition 8.5.5 of [**?**]. $\square$

In subsequent sections, the $\mathbb{P}^1$-bundle accounts for the dimension change of 1 in the simple Bruhat relations.

4. Bruhat Order on $P\backslash G/B$

4.1. **Reducing to Simple Relations.** It is well-known that:

Proposition 4.1. *Bruhat Decomposition:*

$$P = \amalg_{w \in W_L} B\dot{w}B$$

and thus $P\backslash G/B \leftrightarrow W_L\backslash W_G$.

Definition 4.2. Bruhat order on both $P\backslash G/B$ and $W_L\backslash W_G$ is the partial order induced from Bruhat order on both $B\backslash G/B$ and W_G. That is, $W_L u \le W_L v$ if there are coset representatives u_0 and v_0, respectively, such that $u_0 \le v_0$.

That is, if $u, v \in W_G$ are such that $u \le v$, then $W_L u \le W_L v$. The converse holds for minimal length coset representatives:

Proposition 4.3. *Let $u, v \in W_G$ be minimal length coset representatives for $W_L u$ and for $W_L v$. Then*

$$W_L u \le W_L v \iff u \le v.$$

Proof. This is a special case of [**?**], Lemma 3.5. $\square$

Because Bruhat order for $P\backslash G/B$ may be described using Bruhat order for $B\backslash G/B$ restricted to minimal length coset representatives, therefore we make the analogous definition for property Z:

Definition 4.4. Let $u, v \in W_G$ be minimal length coset representatives for $W_L u$ and for $W_L v$. Let s be a simple reflection. Then *property* $Z(s, W_L u, W_L v)$ is satisfied if whenever $\ell(us) \leq \ell(u)$ and $\ell(vs) \leq \ell(v)$, then

$$W_L u \leq W_L v \iff W_L us \leq W_L v \iff W_L us \leq W_L vs.$$

Furthermore,

Proposition 4.5. *Bruhat order is the unique partial order on $W_L \backslash W_G$ such that*

(1) *for all $W_L w \in W_L \backslash W_G$, $W_L w \leq W_L 1 \iff W_L w = W_L 1$;*
(2) *$\leq$ has property $Z(s, W_L u, W_L v)$.*

Again, Bruhat order for $P\backslash G/B$ may be described using only simple relations and we focus on reformulations those relations.

4.2. **Weyl Group and Roots.** The goal is to understand simple relations while avoiding the introduction of additional machinery. Using the intuition gained from $P\backslash G/B$, consider:

Theorem 4.6. *Let $w \in W_G$ and let $\alpha \in \Delta^+$. Then*

$$\begin{array}{lcll} W_L w & = & W_L w s_\alpha & \text{if } w\alpha \in \Delta(\mathfrak{l}, \mathfrak{t}) \\ W_L w & \xrightarrow{s_\alpha} & W_L w s_\alpha & \text{if } w\alpha \in \Delta(\mathfrak{n}, \mathfrak{t}) \\ W_L w & \xleftarrow{s_\alpha} & W_L w s_\alpha & \text{if } w\alpha \in \Delta(\mathfrak{n}^-, \mathfrak{t}) \end{array}$$

Note that this is analogous to Bruhat order relations for $B\backslash G/B$ with $\mathfrak{l}$ analogous to $\mathfrak{t}$, $\Delta(\mathfrak{t}, \mathfrak{t}) = \{\}$.

Proof. We will find conjugation to be a very convenient tool in studying Bruhat order:

$$W_L w s_\alpha = W_L w s_\alpha w^{-1} w = W_L s_{w\alpha} w.$$

It now follows that $W_L w s_\alpha = W_L w \iff w\alpha \in \Delta(\mathfrak{l}, \mathfrak{t})$. The second and third statements then follow from the formula above and from Theorem 3.5. □

Remark 4.7. Compare this characterization to John Stembridge's characterization of parabolic Bruhat order in [**?**] in which W_L-cosets are associated with W_G-orbits in the dual space of the Cartan subalgebra with stabilizer W_L. Bruhat order then corresponds to the partial order on the root lattice.

4.3. **Roots and Pullbacks.** Since $\alpha_{\dot{w}} = w\alpha$, we may reformulate Bruhat order as follows:

Theorem 4.8. *Let α be a positive root and $w \in W_G$. Then*

$$\begin{array}{lcl} P\dot{w}B & \xrightarrow{\alpha} & P\dot{w}\dot{s}_\alpha B \text{ if } \alpha_{\dot{w}} \in \Delta(\mathfrak{n}, \mathfrak{t}) \\ P\dot{w}B & \xleftarrow{\alpha} & P\dot{w}\dot{s}_\alpha B \text{ if } \alpha_{\dot{w}} \in \Delta(\mathfrak{n}^-, \mathfrak{t}). \end{array}$$

4.4. **Closure order.**

Definition 4.9. The closure order on $P\backslash G/B$ is defined by $\mathcal{O}_1 \preceq^P \mathcal{O}_2$ if $\mathcal{O}_1 \subset \overline{\mathcal{O}}_2$.

Definition 4.10. Let $\alpha \in \Pi$ and $w \in W$. If $P\dot{w}B \preceq^P P\dot{w}\dot{s}_\alpha^{-1}B$, then write

$$P\dot{w}B \preceq^P_\alpha P\dot{w}\dot{s}_\alpha^{-1}B.$$

Again, we will see that closure order and Bruhat order on $P\backslash G/B$ are the same.

Lemma 4.11. *Let $\alpha \in \Pi$ and $w \in W$. Then:*

$$\pi_\alpha^{-1}\pi_\alpha P\dot{w}B = P\dot{w}B \cup P\dot{w}\dot{s}_\alpha B = \begin{cases} P\dot{w}B & \text{if } w\alpha \in \Delta(\mathfrak{l},\mathfrak{t}) \\ P\dot{w}B \cup P\dot{w}\dot{s}_\alpha B & \text{if } w\alpha \notin \Delta(\mathfrak{l},\mathfrak{t}). \end{cases}$$

Proof. This follows from Theorem 4.6 and the correspondence $P\backslash G/B \leftrightarrow W_L\backslash W_G$. □

Theorem 4.12. *Let $w \in W_G$ and let $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t})$. Then $\mathcal{O}_w \preceq_\alpha^P \mathcal{O}_{ws_\alpha} \iff W_L w \le W_L ws_\alpha$:*

$$\begin{array}{rcl} P\dot{w}B & = & P\dot{w}\dot{s}_\alpha B \text{ if } w\alpha \in \Delta(\mathfrak{l},\mathfrak{t}) \\ P\dot{w}B & \preceq_\alpha^P & P\dot{w}\dot{s}_\alpha B \text{ if } w\alpha \in \Delta(\mathfrak{n},\mathfrak{t}) \\ P\dot{w}B & \succeq_\alpha^P & P\dot{w}\dot{s}_\alpha B \text{ if } w\alpha \in \Delta(\mathfrak{n}^-,\mathfrak{t}) \end{array}$$

Furthermore, closure order and Bruhat order on $P\backslash G/B$ are the same:

$$\mathcal{O}_u \preceq^P \mathcal{O}_v \iff W_L u \le W_L v.$$

Proof. We only need to prove the last three statements. Since $B\dot{w}B \preceq_\alpha^B B\dot{w}\dot{s}_\alpha B$ if $w\alpha > 0$ and since $P\dot{w}\dot{s}_\alpha B = P\dot{w}\dot{s}_\alpha B$ if $w\alpha \in \Delta(\mathfrak{l},\mathfrak{t})$, therefore $P\dot{w}B \preceq_\alpha P\dot{w}\dot{s}_\alpha B$ if $w\alpha \in \Delta(\mathfrak{n},\mathfrak{t})$. Similarly, $P\dot{w}B \succeq_\alpha P\dot{w}\dot{s}_\alpha B$ if $w\alpha \in \Delta(\mathfrak{n}^-,\mathfrak{t})$. Thus $\xrightarrow{\alpha}$ and $\preceq_\alpha^P$ are equivalent. Since Bruhat order on $P\backslash G/B$ depends only on simple relations, it follows that if $W_L u \le W_L v$, then $\mathcal{O}_u \preceq^P \mathcal{O}_v$.

Heuristically, Bruhat order and closure order on $P\backslash G/B$ are the same since:

(1) Bruhat order is induced by Bruhat order on $B\backslash G/B$ and closure order and Bruhat order for $B\backslash G/B$ are the same;
(2) the topology on G/B is the quotient topology.

For details, see the proof of the equivalence of Bruhat order and closure order for $K\backslash G/P$, Theorem 8.2. Here, we provide an alternate proof using minimal length coset representatives.

To prove the converse, we wish to show that if $u, v \in W_G$, then $P\dot{u}B \subset \overline{P\dot{v}B} \Rightarrow W_L u \le W_L v$. We may assume that u, v are chosen to be minimal length representatives.

$$\begin{array}{rcl} P\dot{u}B & \subset & \overline{P\dot{v}B} \\ \iff \cup_{w\in W_L} B\dot{w}\dot{u}B & \subset & \overline{\cup_{w\in W_L} B\dot{w}\dot{v}B} = \cup_{w\in W_L}\overline{B\dot{w}\dot{v}B} \\ \iff \text{for every } w \in W_L,\ B\dot{w}\dot{u}B & \subset & \overline{B\dot{x}\dot{v}B} \text{ for some } x \in W_L \\ \Rightarrow u & \le & xv \text{ for some } x \in W_L \\ \Rightarrow u & \le & v \text{ by Lemma 3.5 of [?].} \end{array}$$

□

4.5. **Cross Actions.**

Definition 4.13. The cross action of W on $P\backslash G/B$ is the action generated by

$$s_\alpha \times P\dot{w}B := P\dot{w}\dot{s}_\alpha^{-1}B$$

where α is a positive root.

It follows immediately:

Theorem 4.14. *Let α be a positive root and $w \in W$. Then the cross action corresponding to α satisfies:*

$$\begin{array}{rcl} P\dot{w}B & = & s_\alpha \times P\dot{w}B = P\dot{w}\dot{s}_\alpha^{-1}B \text{ if } w\alpha \in \Delta(\mathfrak{l},\mathfrak{t}) \\ P\dot{w}B & \xrightarrow{\alpha} & s_\alpha \times P\dot{w}B = P\dot{w}\dot{s}_\alpha^{-1}B \text{ if } w\alpha \in \Delta(\mathfrak{n},\mathfrak{t}) \\ P\dot{w}B & \xleftarrow{\alpha} & s_\alpha \times B\dot{w}B = P\dot{w}\dot{s}_\alpha^{-1}B \text{ if } w\alpha \in \Delta(\mathfrak{n}^-,\mathfrak{t}). \end{array}$$

Proof. Again, if $w\alpha \in \Delta(\mathfrak{l},\mathfrak{t})$, then $\dot{s}_{w\alpha} \in L \subset P$. The remainder of the proof resembles previous arguments. □

5. Reduced Expressions for $B\backslash G/B$ and $P\backslash G/B$

Throughout this section, wherever $w = s_{i_1}s_{i_2}\cdots s_{i_k}$, let $w_1 = s_{i_1}$, $w_2 = s_{i_1}s_{i_2}, \ldots, w_k = s_{i_1}s_{i_2}\cdots s_{i_k} = w$.

5.1. $B\backslash G/B$.

An important aspect of Bruhat order is understanding decompositions of Weyl group elements into products of simple reflections.

Definition 5.1. Let $w \in W_G$. Then $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ is a *reduced expression* for w (or B*-reduced expression*) if k is minimal.

The following result is standard:

Proposition 5.2. *Let $w \in W_G$ where $w = s_{i_1}s_{i_2}\cdots s_{i_k}$. Then $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced expression if and only if $w_j\alpha_{i_{j+1}} > 0$ for $j = 1, 2, \ldots, k-1$.*

This is the equivalent definition for reduced expression that generalizes nicely to $P\backslash G/B$.

5.2. $P\backslash G/B$.

Again, we wish to simplify the existing literature and limit the introduction of complex machinery as much as possible.

Definition 5.3. An element $w \in W_G$ is P*-minimal* if it is a minimal length coset representative for W_Lw. Equivalently, $w\alpha \in \Delta(\mathfrak{n},\mathfrak{t})$ for every $\alpha \in I$.

Definition 5.4. Let $w \in W_G$ be P-minimal. A P*-reduced expression* for w is $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ where $w_j\alpha_{i_{j+1}} \in \Delta(\mathfrak{n},\mathfrak{t})$ for $j = 1, 2, \ldots, k-1$. We define $\ell^P(w) = k$, the P*-length* of w.

Proposition 5.5. *Every coset in $W_L\backslash W_G$ has a unique P-minimal representative and every P-minimal representative has a P-reduced expression.*

Proof. The first statement follows immediately from the definition of P-minimal element. The second statement follows from the equations

$$\begin{array}{rcll} W_Lw & = & W_Ls_\alpha & \text{if } w\alpha \in \Delta(\mathfrak{l},\mathfrak{t}) \\ W_Lw & \xrightarrow{\alpha} & W_Ls_\alpha & \text{if } w\alpha \in \Delta(\mathfrak{n},\mathfrak{t}) \\ W_Lw & \xleftarrow{\alpha} & W_Ls_\alpha & \text{if } w\alpha \in \Delta(\mathfrak{n}^-,\mathfrak{t}). \end{array}$$

□

6. Exchange Property for $B\backslash G/B$ and $P\backslash G/B$

6.1. $B\backslash G/B$**.** The Exchange Property for W_G is the following:

Theorem 6.1. *Let $w = s_{i_1}s_{i_2}\dots s_{i_k} \in W_G$ be a reduced expression and let $\alpha \in \Pi$. If $w\alpha < 0$, then $\ell(w) > \ell(ws_\alpha)$. Furthermore, the Exchange Property states that there exists some j such that $ws_\alpha = s_{i_1}s_{i_2}\cdots\hat{s}_{i_j}\cdots s_{i_k}$ is a reduced expression for ws_α.*

6.2. $P\backslash G/B$**.** The Exchange Property for $W_L\backslash W_G$ may be described similarly:

Theorem 6.2. *Let $w = s_{i_1}s_{i_2}\cdots s_{i_k} \in W_G$ be a P-reduced expression and let $\alpha \in \Pi$. If $w\alpha \in \Delta(\mathfrak{n}^-, \mathfrak{t})$, then there exists some j such that $ws_\alpha = s_{i_1}s_{i_2}\cdots\hat{s}_{i_j}\cdots s_{i_k}$ and it is a P-reduced expression for ws_α.*

Proof. We know that there exists j such that $s_{i_1}s_{i_2}\cdots\hat{s}_{i_j}\cdots s_{i_k}$ is a B-reduced expression for ws_α. Since $w\alpha \in \Delta(\mathfrak{n}^-, \mathfrak{t})$, therefore $PwB \neq Pws_\alpha B$, $\dim Pws_\alpha B = \dim Pws_\alpha B - 1$, and $\ell(ws_\alpha) = k - 1$. Since, as we analyze each location in any expression, roots in $\mathfrak{l}$ fix the orbit, roots in $\mathfrak{n}$ increase the orbit dimension, while roots in $\mathfrak{n}^-$ decrease the orbit dimension, therefore each simple reflection in our expression must increase dimension, whence $ws_\alpha = s_{i_1}s_{i_2}\cdots\hat{s}_{i_j}\cdots s_{i_k}$ must be a P-reduced expression as well. $\square$

7. Bruhat Order on $K\backslash G/B$

Bruhat order for $K\backslash G/B$ may differ in "direction" in the literature due to a preference to associate the minimal length reduced expression with the open dense orbit since the open orbit is unique while the closed minimal dimension orbits generally are not.

7.1. **Parametrizing** $K\backslash G/B$**.** We use Richardson-Springer's parametrization of $K\backslash G/B$ from their seminal paper [?]. A good reference is [?]. Recall that $K = G^\theta$ and $B = TU$ is a θ-stable Borel subgroup.

Notation 7.1. Modifying Richardson-Springer's parametrization for $B\backslash G/K$, we set:

- $\mathscr{V} := \{x \in G | x^{-1}\theta(x) \in N(T)\}$
- $V := K \times T$ -orbits on $\mathscr{V}$: $(k,t)\cdot x = kxt^{-1}$.
- $\dot{v} \in \mathscr{V}$ is a representative of v.

Proposition 7.2. [?] *V is in bijection with $K\backslash G/B$.*

7.2. **Closure Order.** Here, we change our approach and define:

Definition 7.3. Bruhat order on $K\backslash G/B$ is defined to be closure order on $K\backslash G/B$. For $v_1, v_2 \in V$,

$$\mathcal{O}_1 \preceq^K \mathcal{O}_2 \quad \text{if} \quad \mathcal{O}_1 \subset \overline{\mathcal{O}}_2$$
$$\text{and } v_1 \preceq^K v_2 \iff \mathcal{O}_{v_1} \preceq^K \mathcal{O}_{v_2}.$$

We will then show various definitions to be equivalent, as we did for $B\backslash G/B$ and for $P\backslash G/B$, by understanding simple relations and how Bruhat order reduces to understanding simple relations.

We will need the following notation:

Notation 7.4. Let $\alpha \in \Pi$ and $v, v' \in V$. If $\mathcal{O}_v \subset \pi_\alpha^{-1}\pi_\alpha\mathcal{O}_{v'}$ and $\dim\mathcal{O}_{v'} = \dim\mathcal{O}_v + 1$, then write $\mathcal{O}_v \preceq_\alpha^K \mathcal{O}_{v'}$ or $v \preceq_\alpha^K v'$.

7.3. **Real Forms and Root Types.** In order to discuss Bruhat order in detail, we must discuss real forms and root types. A real form of the complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_0$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. A less obvious way to specify a real form is to select a Cartan involution θ. (Use the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and the fact that the Killing form is positive definite on $i\mathfrak{k}_0 \oplus \mathfrak{s}_0$.)

We begin by studying the Cartan subalgebra.

Lemma 7.5. *([?]) Given any θ-stable Cartan subalgebra $\mathfrak{t}$ and Cartan decomposition $\mathfrak{t}_0 = \mathfrak{t}_0^c \oplus \mathfrak{t}_0^n$ of its real form, it is known that roots $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$ are real-valued on $i\mathfrak{t}_0^c \oplus \mathfrak{t}_0^n$. Thus $\theta\alpha = -\bar{\alpha}$.*

Remark 7.6. Recall that $\bar{\alpha}(X) = \overline{\alpha(\bar{X})}$.

Definition 7.7. Given $\mathfrak{t}$ a θ-stable CSA, *relative to θ*, $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$ is:

(1) real if $\theta\alpha = -\alpha$
(2) imaginary if $\theta\alpha = \alpha$
(3) complex if $\theta\alpha \neq \pm\alpha$.

Definition 7.8. Given $g \in G$, recall $\alpha_g := \mathrm{Ad}^*_{g^{-1}}\,\alpha : \mathfrak{t}_g = \mathrm{Ad}_g\,\mathfrak{t} \to \mathbb{C}$. *Relative to g, α is:*

(1) *real if $\bar{\alpha}_g = -\alpha_g$*
(2) *imaginary if $\bar{\alpha}_g = \alpha_g$*
(3) *complex if $\bar{\alpha}_g \neq \pm\alpha_g$.*

Notation 7.9. Let $v \in V$ and let $n = \dot{v}^{-1}\theta(\dot{v})$, and $w = nT$. Since $v \in V$, it follows that $\theta(w) = w^{-1}$.

We study the particular case where $g = \dot{v} \in \mathscr{V}$.

Lemma 7.10. *([?]) If $v \in V$, then $\dot{v}T\dot{v}^{-1}$ is a θ-stable Cartan subgroup.*

This allows us to describe root types using θ.

Definition 7.11. If $v \in V$, then *relative to v* α is:

(1) real if $\theta\alpha_v = -\alpha_v$
(2) imaginary if $\theta\alpha_v = \alpha_v$
(3) complex if $\theta\alpha_v \neq \pm\alpha_v$.

Equivalently,

Definition 7.12. ([?]) *Relative to v (or w)* α is:

(1) real if $w\theta\alpha = -\alpha$
(2) imaginary if $w\theta\alpha = \alpha$
(3) complex if $w\theta\alpha \neq \pm\alpha$.

Proposition 7.13. *The previous two definitions for real, complex, imaginary are consistent.*

Proof. Let $T_1 = vTv^{-1}$, which is θ-stable because $v \in \mathscr{V}$. Then $-\bar{\alpha}_v = \theta\alpha_v$. Given $t_1 \in T_1$,

$$\theta\alpha_v(t_1) = \alpha_v(\theta^{-1}(t_1)) = \alpha(v^{-1}\theta^{-1}(t_1)v)$$

whereas for $t = v^{-1}t_1v \in T$,

$$\begin{aligned} w\theta\alpha(t) = \theta\alpha(n^{-1}tn) &= \alpha(\theta^{-1}(\theta(v)^{-1}v)\theta^{-1}(t)\theta^{-1}(v^{-1}\theta(v))) \\ &= \alpha(v^{-1}\theta^{-1}(vtv^{-1})v) \\ &= \alpha(v^{-1}\theta^{-1}(t_1)v). \end{aligned}$$

Since $\alpha_v(t_1) = \alpha(t)$, we see therefore that

$$\begin{aligned} \theta\alpha_v(t_1) = \alpha_v(t_1) &\iff w\theta\alpha(t) = \alpha(t) \\ \theta\alpha_v(t_1) = -\alpha_v(t_1) &\iff w\theta\alpha(t) = -\alpha(t) \\ \theta\alpha_v(t_1) \neq \pm\alpha_v(t_1) &\iff w\theta\alpha(t) \neq \pm\alpha(t). \end{aligned}$$

□

It follows from these computations that:

Corollary 7.14. *For $\alpha \in \Pi$, $v \in V$, $w = \dot{v}^{-1}\theta(\dot{v})T \in W$,*

$$w\theta\alpha = (\theta\alpha_v)_{v^{-1}}.$$

We may further distinguish imaginary roots as compact or noncompact.

Definition 7.15. Let α be an imaginary root. Normalizing the one-parameter subgroup x_α appropriately,

$$\theta(x_\alpha(\xi)) = x_{\theta\alpha}(c_\alpha\xi) = x_\alpha(c_\alpha\xi) \quad \text{where } c_\alpha = \pm 1.$$

The root α is said to be *compact imaginary* if $c_\alpha = 1$ and *noncompact imaginary* if $c_\alpha = -1$.

Definition 7.16. Suppose the root α is imaginary relative to $v \in V$. Then, normalizing x_{α_v} appropriately,

$$\theta(x_{\alpha_v}(\xi)) = x_{\theta\alpha_v}(c_{\alpha_v}\xi) = x_{\alpha_v}(c_{\alpha_v}\xi) \quad \text{where } c_{\alpha_v} = \pm 1.$$

We say that α is *compact relative to v* if $c_{\alpha_v} = 1$ and *noncompact* if $c_{\alpha_v} = -1$.

Lemma 7.17. *([?], p. 527) Recall $\dot{s}_\alpha$ was defined using one-parameter subgroups. Then*

i) *if α is real: $\theta(\dot{s}_\alpha) = \dot{s}_{-\alpha}$*
ii) *if α is compact imaginary: $\theta(\dot{s}_\alpha) = \dot{s}_\alpha$*
iii) *if α is complex $\theta(\dot{s}_\alpha) = \dot{s}_{\theta\alpha}$*
iv) *if α is noncompact imaginary: $\theta(\dot{s}_\alpha) = \dot{s}_{-\alpha}$.*

Recall that $\dot{s}_\alpha = \phi_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order 4, therefore either $\dot{s}_\alpha^2 = 1$ or $\dot{s}_\alpha^4 = 1$. Let $m_\alpha = \dot{s}_\alpha^2$. Lemma 14.11 of [?] allows us to make the definition:

Definition 7.18. A simple root α is said to be of type II if $m_\alpha = 1$ and of type I otherwise.

Remark 7.19.
(1) The above condition is conjugation invariant since the order of an element is conjugation invariant. That is, α is type I (resp. type II) if and only if α_g is type I (resp. type II).
(2) Furthermore, the definition makes no reference to the real form.
(3) All the roots in each Weyl group orbit must be of the same type.
(4) If a simple Lie algebra is simply laced, then its roots are either all type I or all type II.

Proposition 7.20. *If α is type II, then*

$$\dot{s}_{-\alpha} = \dot{s}_\alpha = \dot{s}_\alpha^{-1}.$$

If α is type I, then

$$\dot{s}_{-\alpha} = \dot{s}_\alpha^{-1} \neq \dot{s}_\alpha.$$

Proof. Since $\dot{s}_\alpha = \phi_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ while $\dot{s}_{-\alpha} = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we see that $\dot{s}_{-\alpha} = \dot{s}_\alpha^{-1}$. Considering order, we obtain the proposition. □

7.4. **Cross Actions and Cayley Transforms.** We will see that simple relations for Bruhat order on $K\backslash G/B$ may be described by cross actions and Cayley transforms.

Springer showed that if $v \in \mathscr{V}$ and $n \in N(T)$, then $vn^{-1} \in \mathscr{V}$, so there is a left W_G-action on $\mathscr{V}$ and also on V [**?**].

Definition 7.21. The *cross action* on $K\backslash G/B$ corresponds to Springer's W_G-action on V. That is,

$$s_\alpha \times K\dot{v}B = K\dot{v}\dot{s}_\alpha^{-1}B.$$

In section 4.3 of [**?**], one finds that:

Case	Type of α wrt v	$s_\alpha \times v$
i.I)	real type I	$s_\alpha \times v \neq v$
i.II)	real type II	$s_\alpha \times v = v$
ii)	compact imaginary	$s_\alpha \times v = v$
iii)	complex	$s_\alpha \times v \neq v$
iv.I)	noncompact imaginary type I	$s_\alpha \times v \neq v$
iv.II)	noncompact imaginary type II	$s_\alpha \times v = v$

This is easy to see using Definition 7.18 and Lemma 7.17. For example, if α is type II imaginary relative to v, then $s_\alpha \times KvB = Kv\dot{s_\alpha}^{-1}B = K\dot{s}_{\alpha_v}^{-1}vB = KvB$ since $\theta(\dot{s}_{\alpha_v}) = \dot{s}_{-\alpha_v} = \dot{s}_{\alpha_v}^{-1} = \dot{s}_{\alpha_v}$.

Suppose $\alpha \in \Pi$ is noncompact imaginary relative to $v \in V$. In section 6.7 of [**?**], Springer defines the automorphism $\psi(g) = \dot{v}^{-1}\theta(\dot{v})\theta(g)\theta(\dot{v})^{-1}\dot{v}$. We observe that this is simply $\theta_n(g) := int(n) \circ \theta(g)$. Since $\theta(n) = n^{-1}$, this is an involutive automorphism. It is now easy to see, as Springer pointed out, that ψ descends to an involutive automorphism of G_α, the subgroup corresponding to $\pm\alpha$, since α is imaginary relative to θ_n. Springer shows that $\psi(x_\alpha(m)) = x_\alpha(-m)$, $\psi(x_{-\alpha}(m)) = x_{-\alpha}(m)$, and $\psi(\dot{s}_\alpha) = \dot{s}_\alpha^{-1}$. Springer claims that there is $z_\alpha \in G_\alpha$ such that $z_\alpha\psi(z_\alpha)^{-1} = \dot{s}_\alpha$. We see we may choose $z_\alpha = x_\alpha(-1)x_{-\alpha}(1/2)$ since $z_\alpha\psi(z_\alpha)^{-1} = x_\alpha(-1)x_{-\alpha}(1/2)x_{-\alpha}(1/2)x_\alpha(-1) = \dot{s}_\alpha$. Then:

Definition 7.22. Given $v \in V$, $\alpha \in \Pi$ noncompact imaginary relative to v, the *Cayley transform* of v through α is

$$c^\alpha(K\dot{v}B) = K\dot{v}z_\alpha^{-1}B.$$

The Cayley transform is known to increase orbit dimension by 1.

In order to relate cross actions and Cayley transforms to simple relations in the closure order, we must understand how projection onto the variety of parabolics of type α followed by taking the preimage affects dimensions. Either $\dim \pi_\alpha^{-1}\pi_\alpha(\mathcal{O}_v) = \dim \mathcal{O}_v$ or $\dim \pi_\alpha^{-1}\pi_\alpha(\mathcal{O}_v) = \dim \mathcal{O}_v + 1$. Whether the dimension increases or not depends on the type of the root α relative to $v \in V$ parametrizing the orbit.

We label the possible cases which may be found in [?]:

Case	Root Type of α_v	$\dim \pi_\alpha^{-1}\pi_\alpha \mathcal{O}_v$
i.I)	real type I	same
i.II)	real type II	same
ii)	compact imaginary	same
iii.a)	complex	same
iii.b)	complex	+1
iv.I)	noncompact imaginary type I	+1
iv.II)	noncompact imaginary type II	+1

Case	Root Type of α_v	$\pi_\alpha^{-1}\pi_\alpha \mathcal{O}$	Other Types	Bruhat Relation
iii.b)	complex	$\mathcal{O} \cup s_\alpha \mathcal{O}$	α complex wrt. $s_\alpha \mathcal{O}$	$\mathcal{O} \preceq_\alpha^K s_\alpha \mathcal{O}$
iv.I)	noncpt type I	$\mathcal{O} \cup s_\alpha \mathcal{O} \cup c^\alpha \mathcal{O}$	α real type I wrt. $c^\alpha \mathcal{O}$	$\mathcal{O} \preceq_\alpha^K c^\alpha \mathcal{O}$
			α noncpt type I wrt. $s_\alpha \mathcal{O}$	$s_\alpha \mathcal{O} \preceq_\alpha^K c^\alpha \mathcal{O}$
iv.II)	noncpt type II	$\mathcal{O} \cup c^\alpha \mathcal{O}$	α real type II wrt. $c^\alpha \mathcal{O}$	$\mathcal{O} \preceq_\alpha^K c^\alpha \mathcal{O}$

Theorem 7.23. *If $v \preceq_\alpha^K v'$, then either:*

- *$v' = s_\alpha \times v$ where α is type iii.b) relative to v, or*
- *$v' = c^\alpha \times v$ where α is type iv) relative to v.*

$\mathcal{O}_v$ type	$\preceq_\alpha^K$	*$\mathcal{O}'_v$ type*	*Relationship*
iv.I)	$\preceq_\alpha^K$	*i.I)*	$\mathcal{O}_{v'} = c^\alpha \mathcal{O}_v$
iv.II)	$\preceq_\alpha^K$	*i.II)*	$\mathcal{O}_{v'} = c^\alpha \mathcal{O}_v$
iii.b)	$\preceq_\alpha^K$	*iii.a)*	$\mathcal{O}_{v'} = s_\alpha \mathcal{O}_v$

7.5. **Weyl Group and Roots.**

Theorem 7.24. *Let $v \in V$ and $\alpha \in \Pi$. Simple relations for Bruhat order on $K\backslash G/B$ may be formulated by the existence of $v' \in V$ such that:*

$\mathcal{O}_v \preceq_\alpha^K \mathcal{O}_{v'}$	*iff $w\theta\alpha > 0$ and $\mathfrak{g}_{\alpha_v} \not\subset \mathfrak{k}$*
$\mathcal{O}_v \succeq_\alpha^K \mathcal{O}_{v'}$	*iff $w\theta\alpha < 0$ and $\mathfrak{g}_{\alpha_v} \not\subset \mathfrak{k}$.*

We note that this description of Bruhat order is analogous to the descriptions for $B\backslash G/B$ and $P\backslash G/B$ as follows. The reductive subalgebra $\mathfrak{k}$ plays an analogous role to $\mathfrak{l}$ in $P\backslash G/B$ and to $\mathfrak{t}$ in $B\backslash G/B$. Furthermore, in the cases $B\backslash G/B$ and $P\backslash G/B$, $\theta = \mathrm{Id}$.

Proof. Consider the following table:

Case	Root Type of α_v	$\dim \pi_\alpha^{-1}\pi_\alpha \mathcal{O}$	Combinatorial Description
i.I)	real	same	$w\theta\alpha = -\alpha < 0$
i.II)	real	same	$w\theta\alpha = -\alpha < 0$
ii)	compact imaginary	same	$w\theta\alpha = \alpha > 0$ but $\mathfrak{g}_{\alpha_v} \subset \mathfrak{k}$
iii.a)	complex	same	$w\theta\alpha < 0$
iii.b)	complex	+1	$w\theta\alpha > 0$, $\mathfrak{g}_{\alpha_v} \not\subset \mathfrak{k}$
iv.I)	noncompact imaginary type I	+1	$w\theta\alpha = \alpha > 0$, $\mathfrak{g}_{\alpha_v} \not\subset \mathfrak{k}$
iv.II)	noncompact imaginary type II	+1	$w\theta\alpha = \alpha > 0$, $\mathfrak{g}_{\alpha_v} \not\subset \mathfrak{k}$

The real and imaginary cases follow immediately from definitions.

In the complex case, either $w\theta\alpha > 0$ or $w\theta\alpha < 0$. Since B is θ-stable, therefore $\theta\Delta^+ = \Delta^+$. If $w\theta\alpha > 0$, then $\theta(w\theta\alpha) > 0$. Since $\theta(w) = w^{-1}$, it may be shown that $\theta(w\theta\alpha) = w^{-1}\alpha$. From $w^{-1}\alpha > 0$, we conclude that $\ell(s_\alpha w) = \ell(w) + 1$. From $\theta\Delta^+ = \Delta^+$, we also conclude that $\theta\Pi = \Pi$, whence $\theta\alpha$ is a simple root. Since α is complex relative to v so that $w\theta\alpha \neq \pm\alpha$, therefore $s_\alpha w\theta\alpha > 0$, whence $\ell(s_\alpha w s_{\theta\alpha}) = \ell(s_\alpha w) + 1 = \ell(w) + 2$. Similarly, if $w\theta\alpha < 0$, then $\theta(w\theta\alpha) = w^{-1}\alpha < 0$ as well and $\ell(s_\alpha w s_{\theta\alpha}) = \ell(s_\alpha w) - 1 = \ell(w) - 2$. The complex case now follows from the case analysis in 4.3 of [**?**]. □

7.6. **Roots and Pullbacks.**

Theorem 7.25. *Let $v \in V$ and $\alpha \in \Pi$. Simple relations for Bruhat order on $K\backslash G/B$ may be formulated by the existence of $v' \in V$ such that:*

$\mathcal{O}_v \preceq_\alpha^K \mathcal{O}_{v'}$	*if $\theta\alpha_v > 0$ (i.e. $\in \Delta^+(\mathfrak{g},\mathfrak{t})_v$) and $\mathfrak{g}_{\alpha_v} \not\subset \mathfrak{k}$*
$\mathcal{O}_v \succeq_\alpha^K \mathcal{O}_{v'}$	*if $\theta\alpha_v < 0$ and $\mathfrak{g}_{\alpha_v} \not\subset \mathfrak{k}$.*

Proof. This follows from the previous theorem and from Corollary 7.14. □

7.7. **Reducing to Simple Relations and $K\backslash G/B$ in More Depth.** In [**?**], Richardson and Springer define standard order on V (5.2) which they then show to be equivalent to Bruhat order (section 6). Standard order is defined only using simple relations; therefore the scope of our results was not reduced by considering only simple relations. We review the definitions leading to the definition of standard order and discuss some basic results since we will need them in future discussions.

Definition 7.26. ([**?**], 3.10) Given the Coxeter group (W, S), the monoid $M(W)$ has generators $m(s)$ $(s \in S)$ and the relations:

(1) $m(s)^2 = m(s) \quad s \in S$;
(2) braid relations: if $s, t \in S$ are distinct, then
  (a) $o(st) = 2k$: $\quad (m(s)m(t))^k = (m(t)m(s))^k$
  (b) $o(st) = 2k+1$: $\quad (m(s)m(t))^k m(s) = (m(t)m(s))^k m(t)$.

Proposition 7.27. *([**?**], 3.10)*

(1) *If $w = s_1 s_2 \dots s_\ell$ is a reduced decomposition of w, then $m(w) := m(s_1)m(s_2)\cdots m(s_\ell) \in M(W)$ is independent of the reduced decomposition chosen.*
(2) *$M(W) = \{m(w) : w \in W\}$.*
(3) $m(w)m(s) = \begin{cases} m(ws) & \textit{if } ws > w \\ m(w) & \textit{if } ws < w. \end{cases}$

Definition 7.28. ([**?**], 4.7) There is an action of the monoid $M(W)$ on V: if $\mathcal{O}_{v'}$ is the unique dense orbit in KvP_α, then $m(s_\alpha)v = v'$. Thus:

$$\begin{aligned} \text{If } \mathcal{O}_v \preceq_\alpha^K \mathcal{O}_{v'} \text{ then} \quad m(s_\alpha)v &= v'. \\ \text{Otherwise,} \quad m(s_\alpha)v &= v. \end{aligned}$$

The monoidal action should be thought of in the following way. When considering Weyl group actions, $s \in S$ is self-inverse, so acting twice by s should return the original element. The action of s can both raise and lower dimensions. In contrast, the monoidal action of $s \in S$ on $v \in V$ only changes v if a cross action or Cayley transform corresponding to s raises the dimension. Thus repeated monoidal actions of s are the same as acting once. This agrees with $m(s)^2 = m(s)$. Considering a string of simple monoidal actions, we may always remove the simple elements which do not raise dimension. As for the Weyl group action on $B\backslash G/B$ and on $P\backslash G/B$, any element of V can be obtained by $M(W)$ acting on the closed orbits in V.

Notation 7.29. Let V_0 be the set of closed orbits in V.

Definition 7.30. ([?], 4.1) The *length* of an element of V is defined as follows:

(1) If $v \in V_0$, then $\ell(v) = 0$.
(2) If $v = m(s)u$ where $v \neq u$, then $\ell(v) = \ell(u) + 1$.

Definition 7.31. ([?]) A *sequence* in S is $\mathbf{s} = (s_1, \ldots, s_k)$. The *length* of $\mathbf{s}$ is k and $m(\mathbf{s}) = m(s_k)\cdots m(s_1)$.

Definition 7.32. ([?], 5.2) Given $u, v \in V$, write $u \xrightarrow{\alpha} v$ or $u \xrightarrow{s_\alpha} v$ if there exists $x \in V$, $t \in S$, and a sequence $\mathbf{s} \in S$ such that

(1) $u = m(\mathbf{s})x$ and $\ell(u) = \ell(x) + \ell(\mathbf{s})$;
(2) $v = m(\mathbf{s})m(t)x$ and $\ell(v) = \ell(x) + \ell(\mathbf{s}) + 1$.

The relation defined by

$$u \leq v \text{ if there exists a sequence } u = v_0 \xrightarrow{\alpha_1} v_1 \cdots \xrightarrow{\alpha_k} v_k = v$$

is the *standard order* on V.

Again, Richardson and Springer show in [?] that Bruhat order on $K\backslash G/B$ and standard order are the same.

The inverse of a cross action is single valued. The inverse of a type II Cayley transform is single valued while the inverse of a type I Cayley transform is double valued. We wish to understand how elements of $K\backslash G/B$ may be identified using sequences in S.

Proposition 7.33. *Given a sequence in V*

$$v_0 \xrightarrow{s_1} v_1 \xrightarrow{s_2} v_2 \cdots \xrightarrow{s_k} v_k$$

where α_k is noncompact type I relative to v_{k-1}, there is a sequence

$$u_0 \xrightarrow{s_1} u_1 \xrightarrow{s_2} u_2 \cdots \xrightarrow{s_{k-1}} u_{k-1} = s_{\alpha_k} \times v_{k-1} \xrightarrow{s_k} v_k = c^{\alpha_k} v_{k-1}$$

with each α_j the same types relative to v_{j-1} and to u_{j-1} (eg. α_k is noncompact type I relative to both u_{k-1} and v_{k-1}).

Proof. Begin by letting $w_j = v_j^{-1}\theta(v_j)T \in W_G$. Since α_k is noncompact imaginary relative to v_{k-1}, therefore $w_{k-1}\theta\alpha_k = \alpha_k$. Therefore $(v_{k-1}\dot{s}_{\alpha_k}^{-1})^{-1}\theta(v_{k-1}\dot{s}_{\alpha_k}^{-1})T = s_{\alpha_k} w_{k-1}\theta(s_{\alpha_k}) = s_{\alpha_k} s_{w_{k-1}\theta\alpha_k} w_{k-1} = s_{\alpha_k}^2 w_{k-1} = w_{k-1}$. Recall Definition 7.12. Thus if β is real, imaginary, or complex relative to v and α is non-compact relative to v, then β is real, imaginary, or complex, respectively, relative to $s_\alpha \times v$. Remark 7.19 shows that if β is type I relative to v, then it is type I relative to $s_\alpha \times v$. If β is compact relative to v (that is, $d(\theta int(v))X_\beta = d\, int(v)X_\beta$ for $X_\beta \in \mathfrak{g}_\beta$) and α is non-compact type I or type II relative to v, then we see that β is compact relative to $s_\alpha \times v$:

$$\begin{aligned} d(\theta int(v\dot{s}_\alpha^{-1}))X_\beta &= d(\theta(int(v\dot{s}_\alpha^{-1}v^{-1})int(v))X_\beta \\ &= d(int(\theta(\dot{s}_{\alpha_v})\theta int(v))X_\beta \\ &= d(int(\dot{s}_{\alpha_v})int(v))X_\beta \\ &= d(int(v\dot{s}_\alpha))X_\beta \\ &= d(int(v\dot{s}_\alpha^{-1}))X_\beta. \end{aligned}$$

We see that whatever type some simple root β is relative to v_{k-1}, it is precisely the same type relative to $s_{\alpha_k} \times v_{k-1}$. Recall that $c^{\alpha_k}(s_{\alpha_k} \times v_{k-1}) = v_k$ as well. Since only noncompact type I roots cause ambiguity in taking inverses of cross actions and Cayley transforms, therefore by induction, the proposition holds. $\square$

Remark 7.34. There are two general methods of specifying any element $u \in V$ up to braid relations:

(1) There is $u_0 \in V_0$ and a sequence

$$u_0 \xrightarrow{s_1} u_1 \xrightarrow{s_2} u_2 \cdots \xrightarrow{u_k} u_k = u.$$

This specifies u unambiguously.

(2) Let the unique open dense orbit in $K\backslash G/B$ be KvB. There is a sequence

$$v = u_\ell \xleftarrow{s_\ell} u_{\ell-1} \xleftarrow{s_{\ell-1}} u_{\ell-2} \cdots \xleftarrow{s_{k+1}} u_k = u.$$

A sequence moving downwards from the open orbit does not necessarily uniquely identify the orbit u since the inverse Cayley transform is double valued for type I roots. To uniquely identify u, specify a choice for each type I inverse Cayley transform.

8. Bruhat Order on $K\backslash G/P$

8.1. **Bruhat and Closure Orders.**

Definition 8.1. Bruhat order on $K\backslash G/P$ is the partial order induced from Bruhat order on $K\backslash G/B$. That is, $KuP \leq KvP$ if there are orbit representatives u_0 and v_0, respectively, such that $Ku_0B \leq Kv_0B$.

Proposition 8.2. *Bruhat order on $K\backslash G/P$ is the same as closure order.*

Proof. Recall that $\pi_I : G/B \to G/P$ is the natural projection from the variety of Borel subgroups to the variety of parabolic subgroups of type I. Then π_I is continuous. We will show that $\overline{\pi_I(U)} = \pi_I(\overline{U})$ for all $U \subset G/B$.

First, consider the commutative diagram of natural projections

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[d, "q"'] & G/P \\ G/B. \arrow[ur, "\pi_I"'] & \end{tikzcd}$$

Since the topologies on G/B and on G/P are the quotient topologies, for $U \subset G/B$:

$$\begin{aligned}
\overline{\pi_I(U)} &= \overline{p(q^{-1}(U))} \\
&= \bigcap_{V \supset q^{-1}(U) \text{ closed}} p(V) \qquad \text{by definition of quotient topology} \\
&= \bigcap_{V \supset q^{-1}(U) \text{ closed}} \pi_I \circ q(V) \\
&= \bigcap_{X \supset U \text{ closed}} \pi_I(X) \\
&= \pi_I(\overline{U}).
\end{aligned}$$

We apply $\overline{\pi_I(U)} = \pi_I(\overline{U})$ to $KvP = \pi_I(KvB)$.

$$\overline{KvP} = \overline{\pi_I(KvB)} = \pi_I(\overline{KvB}) = \bigcup_{u \leq v} \pi_I(KuB) = \bigcup_{KuP \leq KvP} KuP.$$

The proposition follows. □

8.2. **Understanding KvP: I-Equivalence.** We wish to parametrize $K\backslash G/P$. As we will see, the key to parametrizing $K\backslash G/P$ is understanding the Bruhat order of $K\backslash G/B$ restricted to the B-orbits in a P-orbit.

Since $P \supset B$, therefore each P-orbit KvP can be expressed as a union of B-orbits KuB. This is an example of an I-equivalence class, defined in [**?**]:

Definition 8.3. Recall the map $\pi_I : G/B \to G/P$, the natural projection from the flag variety to the partial flag variety of parabolic subgroups of type I. Two orbits $\mathcal{O}, \mathcal{O}'$ in $K\backslash G/B$ are *I-equivalent* (write $\mathcal{O} \sim_I \mathcal{O}'$) if they project to the same K-orbit on G/P; i.e. $\pi_I(\mathcal{O}) = \pi_I(\mathcal{O}')$. The I-equivalence class of $\mathcal{O}$ is

$$[\mathcal{O}]_{\sim_I} = K\backslash \pi_I^{-1}(\pi_I(\mathcal{O})).$$

In [**?**], each I-equivalence class $KvP = \cup_{\mathcal{O} \sim_I \mathcal{O}_v} \mathcal{O}$ is shown to be in bijection with some double coset space $M_v\backslash L/B \cap L$. Note that these are M_v-orbits on $L/B \cap L$, the flag variety of L. The subgroup M_v is a spherical subgroup of L and thus there is a unique open dense orbit in $M_v\backslash L/B \cap L$. The bijection of I-equivalence classes with double coset spaces respects Bruhat order. Therefore, each I-equivalence class has a unique maximal element since $M_v\backslash L/B \cap L$ has a unique maximal element. These maximal elements are easy to specify combinatorially, giving us a succinct parametrization of $K\backslash G/P$. We now proceed to provide more details. Because we study both K-orbis on G/B and M_{v_0}-orbits on $L/L \cap B$, we use superscripts to indicate orbit type.

Notation 8.4. Let:

(1) $v_0 =$ a representative for $[\mathcal{O}^K]_{\sim_I}$ of minimal dimension
(2) $\tilde{\theta} = int(v_0^{-1}) \circ \theta \circ int(v_0)$
(3) $J = \{\alpha \in S : \alpha_{v_0} \text{ is real or imaginary}\} \cup \{\alpha \in S : \alpha_{v_0} \text{ is complex and } \theta(\alpha_{v_0}) \in S_{v_0}\}$.
(4) $P_J^I = L_J N_J^I$ the T-stable Levi decomposition of the parabolic subgroup of L corresponding to J
(5) $M_{v_0} = L_J^{\tilde{\theta}} N_J^I$.

We chose J so that L_J is $\tilde{\theta}$-stable, making M_{v_0} a mixed subgroup (see [?] for the definition and for more details). Cross actions and Cayley transforms for mixed subgroup orbits on the flag variety are defined by multiplying orbit representatives by $\dot{s}_\alpha^{-1}$ and by z_α^{-1}, respectively, as before.

Proposition 8.5. *([?]) There is a bijection*

$$M_{v_0}\backslash L/B \cap L \xrightarrow{\psi} K\backslash v_0 P/B = [\mathcal{O}^K_{v_0}]_{\sim_I}$$

such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r] \arrow[dr] & M_{v_0}\backslash L/B \cap L \arrow[d, "\psi"] \\ & K\backslash v_0 P/B. \end{tikzcd}$$

The unlabelled maps are the natural maps arising by choosing orbit representatives from L and from $v_0 L$. Furthermore,

$$\begin{array}{rrcl} & \psi\left(s_\alpha \times \mathcal{O}^{M_{v_0}}_\ell\right) & = & s_\alpha \times \psi\left(\mathcal{O}^{M_{v_0}}_\ell\right) \\ \textit{and} & \psi\left(c^\alpha\left(\mathcal{O}^{M_{v_0}}_\ell\right)\right) & = & c^\alpha\left(\psi\left(\mathcal{O}^{M_{v_0}}_\ell\right)\right). \end{array}$$

Remark 8.6.
1. Compare this with Brion and Helminck's parametrization of an I-equivalence class in section 1.5 of [?]. Additional results in [?] show that $M_{v_0}\backslash L/L\cap B$ is in bijection with $L_J^{\tilde{\theta}}\backslash L_J/B \cap L_J \times W_J\backslash W_I$ (a smaller $K\backslash G/B$ cross a Weyl group quotient), connecting the parametrization above to Brion and Helminck's.
2. In comparison, $P\backslash G/B$ is in bijection with $W^I := \{w \in W_G : w\alpha > 0$ for every $\alpha \in I\}$. Clearly $P\backslash G/B$ is in bijection with the unique maximal length coset representatives as well, giving us a parametrization analogous to our parametrization of $K\backslash G/P$.

Corollary 8.7. *Each I-equivalence class of orbits has a unique maximal orbit. Thus each P-orbit KvP contains a unique dense B-orbit.*

Remark 8.8. This is equivalent to Brion-Helminck's result in section 1.5 of [?].

8.3. **Parametrizing $K\backslash G/P$.**

Theorem 8.9. *Let $I \subset \Pi$ correspond to the standard parabolic P. Then the double coset space $K\backslash G/P$ is in bijection with V_P where*

$$\begin{array}{rcl} V_P & := & \{v \in V : \textit{ for every } \alpha \in I, w\theta\alpha < 0 \textit{ where } w = v^{-1}\theta(v)T\} \\ & = & \{v \in V : \textit{ for every } \alpha \in I, m(s_\alpha)v = v\}. \end{array}$$

In other words, $K\backslash G/P$ is in bijection with the I-maximal elements of V.

Proof. This follows immediately from the proposition and the corollary above and our characterization of Bruhat order for $K\backslash G/B$. ☐

Remark 8.10. Compare this to Brion and Helminck's parametrization in section 1.2 of [?].

8.4. **Behaviour of Simple Relations: Descent of the Monoidal Action.** Since Bruhat order for $K\backslash G/P$ is induced from Bruhat order on $K\backslash G/B$, which can be described using simple relations, one concludes that Bruhat order for $K\backslash G/P$ can be described using simple relations as well. However, the absence of a Borel subgroup among the two subgroups with respect to which we take double cosets complicates matters somewhat, obstructing the possibility of making a natural definition for $\xrightarrow{\alpha}$ consistent among all coset representatives.

Proposition 8.11. (1) *If $\alpha \in I$, then $\dot{s}_\alpha, z_\alpha \in L \subset P$; thus*

$$\pi_I(v) = \pi_I(m(s_\alpha)v) \quad \text{for all } v \in V.$$

(2) *If $\alpha \in \Pi \setminus I$,*

$$\pi_I(v) \neq \pi_I(m(s_\alpha)v) \iff v \neq m(s_\alpha)v.$$

Proof. This follows immediately from Proposition 8.5. □

Thus we may restrict our attention to simple relations in $K\backslash G/B$ for $\alpha \in \Pi \setminus I$.

Consider defining cross action to be $s_\alpha \times KvP = Kv\dot{s}_\alpha^{-1}P$. Since for $w \in W_L$,

$$\begin{aligned} KvB &\xrightarrow{s_\alpha \times} Kv\dot{s}_\alpha^{-1}B \subset Kv\dot{s}_\alpha^{-1}P \\ KvwB &\xrightarrow{s_\alpha \times} Kvw\dot{s}_\alpha^{-1}B \\ KvwB &\xrightarrow{s_{w^{-1}\alpha} \times} Kv\dot{s}_\alpha^{-1}wB \subset Kv\dot{s}_\alpha^{-1}P, \end{aligned}$$

we see that the cross action does not descend naturally from $K\backslash G/B$ to $K\backslash G/P$.

Lemma 8.12. *For $\alpha \in \Pi \setminus I$, L normalizes N, so $W_L\alpha \subset \Delta(\mathfrak{n}, \mathfrak{t})$.*

(1) *For any $w \in W_L$, the coefficient of α in the expression of $w\alpha$ as a linear combination of simple roots is* 1.
(2) *If $\beta \in W_L\alpha$ is a simple root, then $\beta = \alpha$.*

Proposition 8.13. *If $w \in W_L$, $\alpha \in \Pi \setminus I$, and $v \in V_P$, then*

$$\mathcal{O}^P_{m(s_{w\alpha})v} = \mathcal{O}^P_{m(s_\alpha)v}.$$

Proof. By Lemma 8.12 and Lemma 5.3.3 of [**?**], there is a reduced expression of $s_{w\alpha}$ of the form $s_1 \cdots s_k s_\alpha s_k \cdots s_1$ where the $s_i \in W_L$. Since v is maximal, $m(s_k \cdots s_1)v = v$. Then $m(s_{w\alpha})v = m(s_1 \cdots s_k)m(s_\alpha)m(s_k \cdots s_1)v = m(s_1 \cdots s_k)m(s_\alpha)v$. Since the monoidal action by W_L preserves P-orbits, the lemma follows. □

Proposition 8.14. *Let $v \in V_P$ and $u \in V$ with $u \sim_I v$. Let $w \in W_L$ be of minimal length such that $v = m(w)u$. If $\alpha \in \Pi \setminus I$, then $\mathcal{O}^P_{m(s_\alpha)v} = \mathcal{O}^P_{m(s_{w^{-1}\alpha})v} = \mathcal{O}^P_{m(s_{w^{-1}\alpha})u}$.*

Remark 8.15. It is tempting at this point, but incorrect, to conclude that the monoidal action of W_G on $K\backslash G/B$ descends naturally to a monoidal action on $K\backslash G/P$ as follows:

- $m(s_{w\alpha})v = m(ws_\alpha w^{-1})m(w)u \stackrel{?}{=} m(w)m(s_\alpha)u$.
- $\mathcal{O}^P_{m(s_{w\alpha})v} = \mathcal{O}^P_{m(s_\alpha)v}$ and $\mathcal{O}^P_{m(w)m(s_\alpha)u}\mathcal{O}^P_{m(s_\alpha)u}$ so $\mathcal{O}^P_{m(s_\alpha)v} = \mathcal{O}^P_{m(s_\alpha)u}$.

However, we cannot cancel inverses in $M(W_G)$, so the above argument is incorrect. It is easy to find a rank two counterexample.

Proposition 8.16. *If* $v \in V_P$, α *and* $\beta \in \Pi \backslash I$, *and* $\alpha \neq \beta$ *with* $v \xrightarrow{\alpha} m(s_\alpha)v$ *and* $v \xrightarrow{\beta} m(s_\beta)v$, *then*

$$\mathcal{O}^P_{m(s_\alpha)v} \neq \mathcal{O}^P_{m(s_\beta)v}.$$

Proof. Assume by contradiction that $m(s_\alpha)v$ and $m(s_\beta)v$ belong to the same P-orbit. Then there exist minimal length elements $w_\alpha, w_\beta \in W_L$ such that $m(w_\alpha)m(s_\alpha)v = m(w_\beta)m(s_\beta)v \in V^P$. Then $w_\alpha s_\alpha = w_\beta s_\beta \Rightarrow s_\alpha s_\beta = w_\alpha^{-1} w_\beta \in W_L \Rightarrow s_\alpha = s_\beta$–contradiction. □

9. Conclusion

It would be interesting to apply the simplifications of Bruhat order to the study of Kazhdan-Lusztig-Vogan polynomials. The theory of parabolic Kazhdan-Lusztig polynomials appears the most likely to benefit from the simplifications.

Another topic for future consideration is to further explore the philosophy of proving results for $P\backslash G/B$ and for $K\backslash G/B$ by reducing to $B\backslash G/B$ using our analogies for simple relations. For example, can it be applied to develop a better understanding of the exchange property and the deletion condition for $K\backslash G/B$?

Can the theory for $K\backslash G/B$ be simplified by using the Tits group?

Can the theories for $K\backslash G/P$ and $P\backslash G/B$ be made more similar by recasting results for $P\backslash G/B$ using maximal length representatives rather than minimal length representatives?

The reader will find more material on Bruhat order in [**?**]. In particular, it contains a description of Bruhat order for mixed subgroups (for which parabolic subgroups and symmetric subgroups are a special case) and for situations where one of the subgroups with respect to which we take double cosets is twisted by conjugation. The descriptions of Bruhat order through pullbacks of roots in particular carries over to the twisted case very naturally.

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