ERGODICITY OF POISSON PRODUCTS AND APPLICATIONS

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Abstract. For every σ-finite measure-preserving transformation T acting on a space X there is an associated probability preserving transformation T_* which acts on discrete countable subsets of X. This is the Poisson suspension of T. We prove ergodicity of the *Poisson-product* $T \times T_*$ under the assumption that T is ergodic and conservative. From this we deduce some probabilistic results: The ergodicity of the "first return of left-most transformation" associated with a measure preserving transformation on $\mathbb{R}_+$, and the non-existence of a T-invariant Poisson-thinning. We discuss ergodicity for the Poisson-product of measure preserving group actions, and related spectral properties.

1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and let $T : X \to X$ be a measure-preserving transforation. On the probability space $(X^*, \mathcal{B}^*, \mu^*)$ of countable subset of X, with the probability measure μ^* defined by the poisson-law with intensity μ, there is a natural probability-preserving transformation $T_* : X^* \to X^*$ associated with T. This map is called the *Poisson-suspension* of T. Ergodic properties of Poisson suspensions and the connections with ergodic properties of the underlaying map have been studied in [7]. In this paper we study ergodicity of the map $T \times T_*$, which acts on the product space $(X \times X^*, \mathcal{B} \times B, \mu^* \times \mu)$. We refer to this system as the *Poisson-product* associated with $(X, \mathcal{B}, \mu, T)$.

Here we prove the following basic result:

Theorem 1.1. *Let $(X, \mathcal{B}, \mu, T)$ be a conservative, measure preserving transformation with $\mu(X) = \infty$. Then the Poisson-product $T \times T_*$ is ergodic if and only if T is ergodic.*

In section 2 we briefly provide the terminology and background result. Section 3 contains the proof of theorem 1.1 stated above. the first return of leftmost transformation is studied in section 4.Section 5 is a discussion of Poisson products of measure preserving group actions, particularly locally compact abelian groups. The (non) existence of an invariant Poisson thinning is discussed is section 6.

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2. Preliminaries

In this section we briefly recall some basic definitions and results from infinite ergodic theory. Throughout this, we assume that $(X, \mathcal{B}, \mu)$ is a standard σ-finite measure space. We assume $T : X \to X$ is a measure preserving transformation of this space, and denote the measurable sets of positive measure by

$$\mathcal{B}^+ := \{B \in \mathcal{B} \ : \ \mu(B) > 0\}.$$

2.1. **Conservative transformations, and induced transformations.** A set $W \in \mathcal{B}$ is called a *wondering set* if $\mu(T^{-n}W \cap W) = 0$ for all $n > 0$. The transformation T is called *conservative* if there are no wondering sets in $\mathcal{B}^+$. Whenever T preserves a finite measure, it is conservative.

For a conservative T and $A \in \mathcal{B}^+$, the *first return time function*, defined for $x \in A$ by $\varphi_A(x) = \min\{n \geq 1 \ : \ T^n(x) \in A\}$ is finite μ-a.e.

The induced transformation on A is defined by $T_A(x) = T^{\varphi_A(x)}(x)$. If T is conservative and ergodic and $A \in \mathcal{B}^+$, $T_A : A \to A$ is a conservative, ergodic transformation of $(A, \mathcal{B} \cap A, \mu\mid_A)$. For proofs and a general discussion of conservative transformations, see propositions 1.5.2 and 1.5.3 and the corresponding section of [1].

2.2. **Cartesian product transformations.** Suppose T is conservative, and $S : Y \to Y$ is a probability preserving transformation of $(\mathcal{Y}, \mathcal{C}, \nu)$ with $\nu(Y) = 1$. It follows (as in proposition 1.2.4 in [1]) that the *cartesian product transformation* $T \times S : X \times Y \to X \times Y$ is a conservative, measure-preserving transforation of the cartesian product measure-space $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \times \nu)$.

2.3. L^∞**-Eigenvalues of measure preserving transformations.** A function $f \in L^\infty(X, \mathcal{B}, \mu)$ is an L^∞*-eigenfunction* of T if $f \neq 0$ and $Tf = \lambda f$ for some $\lambda \in \mathbb{C}$. The corresponding λ is called an L^∞-eigenvalue of T. This is an eigenvalue which corresponds to the associated operator on $L^\infty(\mu)$ obtained from T.

We briefly recall some well known facts about the L^∞-eigenvalues of a non-singular transformations: If T is ergodic and f is an L^∞-eigenfunction, it follows that $|f|$ is constant almost-everywhere. The L^∞-eigenvalues of T are

$$e(T) := \{\lambda \in \mathbb{C} \ : \ \exists\ f \in L^\infty(X, \mathcal{B}, \mu)\ f \neq 0 \text{ and } Tf = \lambda f\}.$$

An eigenvalue λ with $\lambda \neq 1$ is impossible for T conservative. To see this, note that under the contrary assumption The set

$$\{x \in X \ : \ |f(x)| \in (|\lambda|^k, |\lambda|^{k+1}]\}$$

would be a non trivial wondering set for some $k \in \mathbb{Z}$ if $|\lambda| > 1$. Thus, for T conservative, $e(T) \subset S^1$.

$e(T)$ is a group with respect to complex multiplication, and carries a natural polish topology, with respect to which the natural embedding in

$$S^1 = \{x \in \mathbb{C} \ : \ |x| = 1\}$$

is continuous.

We now recall some known properties about $e(T)$: In general, $e(T)$ can be uncountable, and can be of arbitrary Hausdorff dimension $\alpha \in (0, 1)$. However, $e(T)$ is a *weak Dirichlet* set : $\liminf_{n\to\infty} \int |1 - \chi_n(s)| dp(s) = 0$ whenever p is a probability measure on S^1 with $p(e(T)) = 1$. As a consequence we know $e(T)$ is a set has measure zero with respect to Haar measure on S^1. For proofs and details, see section 2.6 in [1] and references within.

2.4. **The L^2-spectrum.** A measure preserving transformation T gives an associated unitary operator U_T on $L^2(\mu)$. With a unitary operator, there is an associated spectral measure on S^1, defined up to equivalence.

The *spectral type* of a unitary operator U on a Hilbert space H, denoted σ_U, is a positive measure on S^1 satisfying:

(a)
$$\langle U^n f, g\rangle = \int_{S^1} \chi_n(s) h(f,g)(s) d\sigma_U(s),$$
where $h : H \times H \to L^1(\sigma_U)$ is a sesquilinear map.

(b) σ_U is minimal with that property, in the sense that it satisfies $\sigma_U << \sigma$ for any measure σ on $e(T)$ satisfying (a).

The spectral type σ_U is defined up to measure class. Existence of σ_U is a formulation the scalar spectral theorem.

For a measure-preserving transformation T, The *spectral type* of T σ_T is the spectral type of the associated unitary operator U_T on $L^2(\mu)$. For a probability preserving transformation S , the *restricted spectral type* is the spectral type the unitary operator U_S restricted to L^2-functions with integral zero.

2.5. **The Poisson suspension.** For a σ-finite measure space $(X, \mathcal{B}, \mu)$, The associated *Poisson measure* $\mu^* \in \mathcal{P}(X^*, \mathcal{B}^*)$ is a probability measure on the standard measurable space $(X^*, \mathcal{B}^*)$ of countable subset, which is uniquely defined by requiring that $|\omega \cap A_i|$ are jointly independent random variables for pairwise disjoint $A_1, A_2, \ldots, A_n \in \mathcal{B}$, and that for $A \in \mathcal{B}$, the cardinally of the random set $|\omega \cap A|$ is distributed Poisson with expectancy $\mu(A)$:

$$\mu^* \left(|\omega \cap A| = k\right) = e^{-\mu(A)} \frac{\mu(A)^k}{k!}. \tag{1}$$

The *Poisson suspension* of a measure preserving map $T : X \to X$, denoted $T_* : X^* \to X^*$ defined by $T_*(\gamma) = \{T(x) \ : \ x \in \omega\}$. T_* is a probability-preserving transformation.

The well known *Fock representation* of $L^2(\mu^*)$ gives an isomorphism

$$L^2(\mu^*) \equiv \bigoplus_{n=0}^{\infty} L^2(\mu)^{\otimes n}.$$

This isomorphism directly gives the following (as in [7]):

Proposition 2.1. *If σ is the spectral-type of T. The restricted spectral type of T_* is given by:*

$$\sigma_{T_*} = \sum_{n \geq 1} \frac{1}{n!} \sigma^{\otimes n}.$$

Recall that a probability-preserving transformation is ergodic iff its restricted spectral type has no atom at $\lambda = 1$, and is *weak mixing* iff its restricted spectral type has no atoms in S^1. Thus, as in [7], it follows that T_* is ergodic iff T_* is weakly-mixing iff there are no T-invariant sets of finite measure in $\mathcal{B}^+$.

3. Ergodicity of Poisson product for conservative transformations

We now describe our proof of theorem 1.1. The argument we use is an adaptation of [2].

To proof our result, we invoke the following condition for ergodicity of cartesian product, due to M. Keane:

Theorem. ***(The Ergodic Multiplier Theorem)***

Let S be a probability preserving transformation and T a conservative, ergodic, non-singular transformation. $S \times T$ is ergodic iff $\sigma_S(e(T)) = 0$, where:

- *σ_S is the restricted spectral type of S,*
- *$e(T)$ is the group of L_∞-eigenvalues of T.*

A proof of this result, along with related discussions can be found in section 2.7 of [1].

By proposition 2.1, the restricted spectral-type of the Poisson suspension T_*, is a sum of the convolution powers of the spectral type of T.

We make use of the following basic lemma about convolution of measures and measure classes. A short proof is provided here for the sake of completeness:

Lemma 3.1. *Let $\mathbb{G}$ be a Polish group, and let $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{G})$ be σ-finite measures on $\mathcal{G}$ which are in the same equivalence class (have the same null-sets). For any probability measure ν on $\mathbb{G}$, the measures $\mu_1 * \nu$ and $\mu_2 * \nu$ are in the same equivalence class of measures.*

Proof. Let $\epsilon > 0$ and choose $M > 1$. There is some $\delta > 0$ such that $\mu_1(B) \geq \frac{\epsilon}{M}$ implies $\mu_2(B) \geq \delta$. Choose $A \in \mathcal{B}(\mathbb{G})$ with $(\mu_1 * \nu)(A) = \epsilon > 0$. Thus ,

$$\nu\left((\{g \in \mathbb{G} \ : \ \mu_1(Ag) \geq \frac{\epsilon}{M}\}\right) \geq \epsilon - \frac{\epsilon}{M}.$$

Thus,

$$\nu\left((\{g \in \mathbb{G} \ : \ \mu_2(Ag) \geq \delta\}\right) \geq \epsilon - \frac{\epsilon}{M}.$$

It follows that $\mu_2 * \nu(A) \geq \delta\epsilon(1 - \frac{1}{M})$. We have proved that μ_1 is absolutely continuous with respect to μ_2. Replacing the roles of μ_1 and μ_2, we conclude that μ_1 and μ_2 are in the same equivalence class.

□

From this we deduce the following lemma:

Lemma 3.2. *Let T be a conservative, measure-preserving transformation. Let σ_{T_*} be a measure from the maximal the spectral type of T_*. The group $e(T)$ acts non-singularly on σ_{T_*}.*

Proof. The claim of this lemma is that for any $t \in e(T)$, $\exists g_t \in L^1(\sigma_{T_*})$ such that for all $f \in L^\infty(\sigma_{T_*})$:

$$\int f(s) d\sigma_{T_*}(s) = \int f(s+t) g_t(s) d\sigma_{T_*}(s).$$

In other words,

$$\forall t \in e(T), \ \sigma_{T_*} \sim R_t \sigma_{T_*},$$

where R_t denotes convolution with dirac measure at t.

Since the restricted spectral-type σ_T^* of the Poisson suspension is a sum of convolution powers of σ_T, it is sufficient to prove that for all $n \geq 1$,

$$(2) \qquad \forall t \in e(T),\ \sigma_T^{\otimes n} \sim R_t \sigma_T^{\otimes n}.$$

For $n = 1$, this follows from the relation

$$\mu_g * \delta_\lambda = \mu_{g \cdot f_\lambda},$$

where f_λ is an L^∞-eigenfunction with eigenvalue λ (see [2]).

As equation (2) holds for $n = 1$, it follows for $n > 1$ by lemma 3.1, with $t \in e(T)$, σ_T and $\sigma_T * \delta_t$ taking the roles of μ_1 and μ_2 and $\sigma_T^{\otimes(n-1)}$ as ν.

□

Completing the proof of theorem 1.1:

By the ergodic multiplier theorem, proving ergodicity of the Poisson-product amounts to proving $\sigma_{T_*}(e(T)) = 0$. Since $\sigma_{T_*} = \sum_{n \geq 1} \frac{1}{n!} \sigma_T^{\otimes n}$, it is sufficient to prove that $\sigma_T^{\otimes n}(e(T)) = 0$ for all $n \geq 1$.

Suppose the contrary: $\sigma_T^{\otimes n}(e(T)) > 0$ for some n. From lemma 3.2, it follow that $\sigma_T^{\otimes n} |_{e(T)}$ is a quasi-invariant measure on $e(T)$. Thus, $e(T)$ can be furnished with a locally-compact second-countable topology, respecting the Borel structure inherited from S^1, with Haar measure which is equivalent to $\sigma_T^{\otimes n} |_{e(T)}$.

With the above topology, we have that $e(T)$ is a locally compact group, continuously embedded in S^1, where the topological embedding is also a group embedding. In this situation, it follows as in [2] that $e(T)$ is either discrete or $e(T) = S^1$.

Suppose $e(T)$ is discrete. It follows that $\sigma_T^{\otimes n}$ has atoms. As any convolution power of an atom-free measure is itself atom-free, this would imply σ_T has atoms, which means T has $L^2(\mu)$ eigenfunctions. But this is impossible as we assumed T is ergodic and infinite-measure-preserving. The alternative is that $e(T) = S^1$. This is impossible by the remark that $e(T)$ must be a null set with respect to Haar measure on S^1.

This completes the proof of theorem 1.1.

4. The First Return of Leftmost transformation

In this section we specialize to the case where $X = \mathbb{R}_+$ is the set of positive real numbers, $\mathcal{B}$ is the Borel σ-algebra of X, and μ is Lebesgue measure on X. $T : X \to X$ is a conservative, ergodic, Lebesgue-measure-preserving map of the positive real numbers.

Example 4.1. *The unsigned version of Bool's transformation $T(x) = |x - \frac{1}{x}|$ is a classical example of an ergodic, Lebesgue-measure-preserving map of the positive real numbers. See* [3] *for a proof of ergodicity and discussions of this transformation.*

We define the following function:

$$(3) \qquad t_1 : X^* \to X \text{ by } t_1(\omega) = \inf \omega.$$

This map is defined μ^*-almost everywhere: μ^*-almost surely, $t_1(\omega)$ is the leftmost point of ω, as ω is a discrete countable subset of $\mathbb{R}$.

Define the *first return of leftmost time* $\kappa_T : (\mathbb{R}_+)^* \to \mathbb{N} \cup \{+\infty\}$ by:

$$(4) \qquad \kappa(\omega) = \inf\{k \geq 1 \ : \ t_1(T_*^k(\omega)) = T^k(t_1(\omega))\}.$$

μ^*-Almost surely, $\kappa_T(\omega)$ is the smallest positive number of iterations of T_* which must be applied to ω in order for the left-most point to return to the leftmost location. A priory, κ_T is could be infinite. Nevertheless, we will soon show that when

T is conservative and measure preserving, κ_T is finite μ^*-almost surely. Finally, the *first return of leftmost transformation* associated with T, $T_*^\kappa : \omega \to \omega$, is defined by $T_*^\kappa(\omega) := T_*^{\kappa(\omega)}(\omega)$. It is the map of X_* obtained by reapplying T_* till once again there are no points to the left of the point which was originally leftmost.

Let

$$X_0 = \{(x, \omega) \in X \times X^* \ : \ \omega \cap [0, x] = \emptyset\} \tag{5}$$

Proposition 4.1. *Let $T : \mathbb{R}_+ \to \mathbb{R}_+$ be conservative and Lebesgue-measure-preserving. Then the first return of leftmost transformation associated with T is isomorphic to the induced map of the Poisson product on the set X_0 defined by equation (5):*

$$(X^*, \mathcal{B}^*, \mu_*, T_*^\kappa) \cong (X_0, \mathcal{B}_0, \mu_0, (T \times T_*)_{X_0})$$

Where $\mu_0 = (\mu \times \mu_*) \mid_{X_0}$ is the restriction of the measure product $\mu \times \mu_*$ to the set X_0, and $\mathcal{B}_0 = (\mathcal{B} \otimes \mathcal{B}^*) \cap X_0$ is the restriction of the σ-algebra on the product space to subset of X_0.

In particular, it follows that $\mu_0(X_0) = 1$, so $(X_0, \mathcal{B}_0, \mu_0)$ is a probability space.

Proof. Consider the map

$$\Phi : X_0 \to X^* \text{ given by } \Phi(x, \omega) = \{x\} \cup \omega.$$

For a non-empty, discrete $\omega \in X^*$ we have:

$$\Phi^{-1}(\omega) = (t_1(\omega), \omega \setminus t_1(\omega)).$$

Thus Φ is invertible on a set of full μ^*-measure in X^*.

As T is conservative and T_* is a probability preserving transformation, the Poisson product $T \times T_*$ is also conservative. We will see in a moment that $\mu \times \mu^*(X_0) > 0$. Therefore, the return time φ_{X_0} is finite almost everywhere on X_0.

Since $\kappa \circ \Phi = \Phi \circ \varphi_{X_0}$, it follows that κ is finite μ^*-a.e.

We also have

$$\Phi(T^n x, T_*^n \omega) = T_*^n(\Phi(x, \omega))$$

whenever (x, ω) and $(T^n x, T_*^n \omega)$ are in X_0. Thus,

$$\Phi \circ (T \times T_*)_{X_0} = T_*^\kappa \circ \Phi.$$

It remains to check that $\Phi^{-1}\mu^* = \mu_0$. it is sufficient to verify that $\mu^*(A) = \mu_0(\Phi^{-1}(A))$ for sets $A \in \mathcal{B}^*$ of the form

$$A = \bigcap_{k=0}^{N} \mathcal{N}(a_k, a_{k+1}, n_k),$$

Where:

where $0 = a_0 < a_1 < a_2 < \ldots < a_N$, $n_k \geq 0$ for $k = 1, \ldots N$ and

$$\mathcal{N}(x, y, n) := \{\omega \in X^* \ : \ |\omega \cap [x, y)| = n\},$$

with $0 < x < y$ positive real numbers , and $n \geq 0$ an integer.

Finite intersections of such sets are a basis for the σ-algebra $\mathcal{B}^*$.

Verifying this is a direct calculation involving elementary integration, which we include here for the sake of completeness. By definition of μ^*:

$$\mu^*(A) = \prod_{k=0}^{N} \frac{\mu([a_k, a_{k+1}))^{n_k}}{n_k!} \exp\left(-\mu([a_k, a_{k+1}))\right)$$

which simplifies to:

$$\mu^*(A) = \exp(-a_N) \prod_{k=0}^{N} \frac{(a_{k+1}-a_k)^{n_k}}{n_k!} \tag{6}$$

Assuming the n_k's are not all zero,let k_0 the smallest k such that $n_k \geq 1$, and let $I_0 = [a_{k_0}, a_{k_0+1})$. We have:

$$\Phi^{-1}(A) = \bigcap_{k \neq k_0} (X \times \mathcal{N}(a_k, a_{k+1}, n_k)) \cap \bigcup_{x \in I_0} \{x\} \times (\mathcal{N}(a_{k_0}, x, 0) \cap \mathcal{N}(x, a_{k_0+1}, n_{k_0} - 1)).$$

Thus,

$$\mu_0(\Phi^{-1}(A)) = T_0 \int_{I_0} \exp(-(x - a_{k_0})) \exp(-(a_{k_0} - x)) \frac{(a_{k_0+1} - x)^{n_{k_0}-1}}{(n_{k_0}-1)!} dx$$

where

$$T_0 = \prod_{k \neq k_0} \frac{(a_{k+1}-a_k)^{n_k}}{n_k!} \exp(a_{k+1} - a_k)$$

By integrating this rational function of a single variable, it is easily verified that the last expression is equal to the expression on right hand side of the formula (6).

The equality of the measures for the case $n_k = 0$ for all $k = 1, \dots N$ is verified by a very similar simple computation.

In particular, it follows that $\mu_0(X_0) = 1$.

□

It would be interesting to establish other ergodic properties of T^κ. For example, what conditions on T are required for T_*^κ to be weakly mixing?

5. Poisson-products for measure-preserving group actions

We now consider the following setup: Let $\mathbb{G}$ be a locally compact, non-compact, second countable abelian group acting on a σ-finite measure space $(X, \mathcal{B}, \mu)$. A measure-preserving $\mathbb{G}$-action, or a measure preserving $\mathbb{G}$-*flow* is

$$T : \mathbb{G} \times X \to X,$$

Such that $g \in \mathbb{G} \mapsto T(g, \cdot) \in Aut(X, \mathcal{B}, \mu)$ is a group representation of $\mathbb{G}$ in the group of measure preserving automorphisms of $(X, \mathcal{B}, \mu)$.

We say a $\mathbb{G}$-action is *ergodic* if whenever $A \in \mathcal{B}$ satisfies $T(g, A) = A$ for all $g \in \mathbb{G}$ it implies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

The L^∞-*spectra* $Sp(T)$ of the $\mathbb{G}$-action T is the set of elements $\chi \in \widehat{\mathbb{G}}$, the character group $\mathbb{G}$, such that $f(T_g x) = \chi(g) f(x)$ for some non zero $f \in L^\infty(X, \mu)$. In case $\mathbb{G} = Z$, the spectra is simply the group L^∞-eigenvalues. As in the case $\mathbb{G} = \mathbb{Z}$, $Sp(T_g)$ is a weak-Dirichlet set in $\widehat{\mathbb{G}}$. For more details, see K. Schmidt's paper [8] which addresses L^∞-*spectra* of non-singular abelian group actions.

The L^2-*spectral type* of T is an equivalence class of Borel measures σ_T on the character group $\widehat{\mathbb{G}}$. For any non-zero $f \in L^2(\mu)$ $\sigma_f << \sigma_T$, where the measure σ_f is given by:

$$\hat{\sigma}_f(g) = \int f(T_g(x)) \overline{f(x)} d\mu(x).$$

The spectral type of σ_T is the minimal measure class with that property. Analogously, the restricted spectral type of a probability-preserving $\mathbb{G}$-action is defined.

With these definitions, the ergodic multiplier theorem generalizes to $\mathbb{G}$-actions of locally compact abelian group: The product of an ergodic measure preserving $\mathbb{G}$-action T and a probability preserving $\mathbb{G}$-action S is ergodic iff $Sp(T)$ is null with respect to the restricted spectral type of σ_T.

When $\mathbb{G} \neq \mathbb{Z}$, for example $\mathbb{G} = \mathbb{Z}^2$, there are obstructions to ergodicity of $T \times T_*$ under the assumption that T is an ergodic, infinite measure preserving $\mathbb{Z}^2$-action.

To see this we consider the following simple example:

Let $a, b \in \mathbb{R}$ be two real numbers, linearly independent over the rational numbers $\mathbb{Q}$. Define $T : \mathbb{Z}^2 \times \mathbb{R} \to \mathbb{R}$ by

$$T((m,n),x) = x + am + bn.$$

Evidently, T an ergodic $\mathbb{Z}^2$ action on $\mathbb{R}$, preserving Lebesgue measure. Note that for any $\tau \in \mathbb{R}$, the function $f_\tau \in L^\infty(\mathbb{R})$ defined by

$$f_\tau(x) = \exp(i\tau x),$$

is satisfies

$$f_\tau(T((m,n),x)) = \exp(i\tau(x + am + bn)) = \chi_{ta,tb}(m,n)\exp(i\tau x),$$

where $\chi(\tau a, \tau b) = \exp(i\tau am + \tau bn)$. The map $t \to \chi(ta, tb)$ is a continuous embedding of $\mathbb{R}$ in $\widehat{\mathbb{Z}^2}$.

It is elementary to check that $T \times T_*$ is not ergodic: For example,

$$\{(x,\omega) \in \mathbb{R} \times \mathbb{R}^* \ : \ [x+1, x-1] \cap \omega = \emptyset\}$$

is a non-trivial $T \times T_*$-invariant set.

Say $W \in \mathbb{B}$ is a *wandering set* with resect to the action T of a locally-compact group $\mathbb{G}$ if $\mu(T(g,W) \cap W) = 0$ for all g in the complement of some compact $K \subset \mathbb{G}$. Call a $\mathbb{G}$ action *conservative* if there are no-non trivial wandering sets. By this definition, the $\mathbb{Z}^2$ action T in the example above is conservative.

6. Existence of T-Invariant Poisson thinning

A (deterministic) *Poisson thinning* is a $\mathcal{B}^*$-measurable map $\Psi : X^* \to X^*$, such that μ^*-almost surely $\Psi(\omega) \subset \omega$ and $\Psi\mu^* = (\theta\mu)^*$ for some $\theta \in (0,1)$. A Poisson thinning Ψ is called *T-invariant* if $\Psi T_* = T_*\Psi$.

Proposition 6.1. *If $T \times T_*$ is ergodic, there does not exist a T-invariant Poisson thinning.*

Proof. For $f \in L^1(\mu)$, define $f^* \in L^1(\mu^*)$ by $f^*(\omega) = \int f(x)d\omega(x)$ where we identify $\omega \in X^*$ with a purely atomic measure assigning mass 1 to each $x \in \omega$.

From the definitions of f^* and μ^*, it follows that

$$\int f(x)d\mu(x) = \int f^*(\omega)d\mu^*(\omega).$$

Suppose by contradiction that Ψ is a Poisson thinning with parameter θ. Define $\hat{\Psi} : X \times X^* \to \{0,1\}$ by

$$\hat{\Psi}(x,\omega) = \begin{cases} 1 & x \in \Psi(\omega \cup \{x\}) \\ 0 & \text{otherwise} \end{cases}$$

From T_*-invariance of Ψ it follows that $\hat{\Psi}(Tx, T_*\omega) = \hat{\Psi}(x,\omega)$. Since we assume $T \times T_*$ is ergodic, $\hat{\Psi}$ is constant $\mu \times \mu^*$ almost everywhere, so we can write $\hat{\Psi}(x,\omega) = \alpha$.

For $\omega \in X^*$ and $f \in L^1(\mu)$, we have:

$$f^*(\Psi(\omega)) = \int f(x) d\Psi(\omega) = \int f(x)\hat{\Psi}(\omega, x) d\omega = \alpha f^*(\omega).$$

In particular,

$$\int f^*(\Psi(\omega)) d\mu^*(\omega) = \alpha \int f^* d\mu^*.$$

On the other hand, since Ψ is a Poisson-thinning,

$$\int f^*(\Psi(\omega)) d\mu^*(\omega) = \theta \int f^* d\mu^*$$

It thus follows that $\alpha = \theta$.

Since $\hat{\Psi}$ takes only values in $\{0,1\}$, we must have $\theta = 0$ or $\theta = 1$, which is impossible.

□

Combining the above result with theorem 1.1, we have the following:

Corollary 6.2. *If T is a conservative ergodic infinite-measure preserving transformation, there does not exit a T-invariant Poisson thinning.*

The proof of proposition 6.1 generalizes directly, and gives the following:

Proposition 6.3. *Suppose a group $\mathbb{G}$ acts on $(X, \mathcal{B}, \mu)$ by measure preserving transformations $T : \mathbb{G} \times X \to X$. Further, suppose the Poisson-product action of $\mathbb{G}$ on $(X \times X^*, \mathcal{B} \otimes \mathcal{B}^*, \mu \times \mu^*)$ is ergodic.*

Then there does not exist a Poisson-Thinning which is jointly $T(g,\cdot)$-invariant for all $g \in \mathbb{G}$.

However, as discussed in section 5, ergodicity of the Poisson-product of a measure-preserving group action is not guaranteed. Indeed, For the $\mathbb{R}$-action on $\mathbb{R}$ by translations there exists a T-invariant Poisson-thinning. Such translation invariant thinning was constructed in [4]. This result has been extended by Holroyd Lyons and Soo to construct a translation-invariant $\mathbb{R}^d$ Poisson-splitting in [6]. As far as results about non-existence of a Poisson-thinning, Evans has shown in [5] that with respect to any non-compact group of linear transformations there is no invariant Poisson-thinning on $\mathbb{R}^d$.

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