The global existence of the smoothing solution for the Navier-Stokes equations

Wang Jianfeng
Department of Mathematics, Nanjing University, Nanjing, 210093,
P. R. China.

Abstract. This paper discussed the global existence of the smoothing solution for the Navier-Stokes equations. At first, we construct the theory of the linear equations which is about the unknown four variables functions with constant coefficients. Secondly, we use this theory to convert the Navier-Stokes equations into the simultaneous of the first order linear partial differential equations with constant coefficients and the quadratic equations. Thirdly, we use the Fourier transformation to convert the first order linear partial differential equations with constant coefficients into the linear equations, and we get the explicit general solution of it. At last, we convert the quadratic equations into the integral equations or the question to find the fixed-point of a continuous mapping. We use the theories about the Poisson's equation, the heat-conduct equation, the Schauder fixed-point theorem to prove that the fixed-point is exist, hence the smoothing solution for the Navier-Stokes equations is globally exist.

Keywords. smoothing solution, Poisson's equation, heat-conduct equation, the Schauder fixed-point theorem, globally exist.

---

1 Introduction

We consider the dynamical equations for an viscous and incompressible fluid just as follows.

$$divu = 0 \tag{1.1}$$

$$\frac{\partial u}{\partial t} - \mu\Delta u + \sum_{k=1}^{3} u_k \frac{\partial u}{\partial x_k} + \frac{1}{\rho}\, gradp = F \tag{1.2}$$

where $u = (u_1,\ u_2,\ u_3)$ is the velocity vector, it is the macro motion velocity of the material particle. μ is the dynamic viscosity coefficient, we assume it is a constant. ρ is the density of the material particle, it is the mass of per volume fluid. Because the fluid is incompressible, we can assume ρ is a constant, too. p is intensity of the pressure, it is the pressure on per area fluid, the direction is perpendicular to such area. $F = (F_1,\ F_2,\ F_3)$ is the density of the body force, it is the external force of per unit mass. $u = (u_1,\ u_2,\ u_3)$ and $p,\ F$ are all functions on the variables of the time t and position $x = (x_1,\ x_2,\ x_3)$. We assume $\tau = \dfrac{1}{\rho}$, τ is called the specific volume of the fluid, it is just the volume of per unit mass. These equations are the second order partial differential equations about four unknown functions: $u = (u_1,\ u_2,\ u_3)$, p , and they are completed. These equations are called the Navier-Stokes equations.

In order to discuss it more conveniently, we often rewrite the Navier-Stokes equations as follows.

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0 \tag{1.3}$$

$$\frac{\partial u_j}{\partial t}-\mu(\frac{\partial^2 u_j}{\partial x_1^2}+\frac{\partial^2 u_j}{\partial x_2^2}+\frac{\partial^2 u_j}{\partial x_3^2})+\sum_{k=1}^{3}u_k\frac{\partial u_j}{\partial x_k}+\tau\frac{\partial p}{\partial x_j}=F_j\ ,\ j=1\ ,\ 2\ ,\ 3\ . \tag{1.4}$$

And we assume as follows.

Assumption 1.1 *(1)We only discuss the Navier-Stokes equations on the region as follows,* $t\in[0,\ T]$, *T is given,* $(x_1,\ x_2,\ x_3)^T\in K_1$, *K_1 is a bounded and closed set in* R^3 , *and when t is not in* $[0,\ T]$, *or* $(x_1,\ x_2,\ x_3)^T$ *is not in* K_1 , $u_j\equiv 0$, $p\equiv 0$, $F_j\equiv 0$, $j=1\ ,\ 2\ ,\ 3$.
(2)In the region $K_1'=[0,\ T]\times K_1$, $u_j\in C^2$, $p\in C^1$, $F_j\in C^1$, $j=1\ ,\ 2\ ,\ 3$.
(3)The boundary of K_1 , ∂K_1 *satisfies the exterior ball condition,* $\forall\ (x_1,\ x_2,\ x_3)^T\in\partial K_1$, *there exists a ball* $B_\rho(y)\subset R^3\setminus K_1^0$, *such that* $B_\rho(y)\cap K_1=(x_1,\ x_2,\ x_3)^T$, *where* $y=(y_1,\ y_2,\ y_3)^T$, $B_\rho(y)=\{z\in R^3|\ |z-y|\le\rho\ ,\ \rho>0\}$, *K_1^0 is the interior of* K_1 .
(4) K_1 *and* F_j , $j=1\ ,\ 2\ ,\ 3$, *satisfy the following,*

$$\max_{1\le j\le 9}\{\ |F^{-1}(\alpha_{j1}^T Y_1)|\ ,|F^{-1}(\alpha_{j2}^T Y_1)|\ ,|F^{-1}(\alpha_{j1}^T Y_1)F^{-1}(\alpha_{j2}^T Y_1)|\}\le\frac{\theta(1-\theta)}{M_{T,\ 3}M(K_1)(2\theta+M_{T,\ 3}M(K_1))}\ ,$$

where $0<\theta<1$, *and* $F^{-1}(\alpha_{j1}^T Y_1)$, $F^{-1}(\alpha_{j2}^T Y_1)$, $M_{T,\ 3}$, $M(K_1)$ *will be defined in the section 3.*

After that we will show that the smoothing solution for the Navier-Stokes equations is globally exist in the region K_1' .

2 Main conclusion

Theorem 2.1 *We consider the linear equations as follows.*

$$AX=\beta\ ,$$

where $A=(a_{ij})_{m\times s}\in R^{m\times s}$ *is a constant matrix,*

$$X=(X_1(x_1,\ x_2,\ x_3,\ t),\ X_2(x_1,\ x_2,\ x_3,\ t),\ \cdots\ ,\ X_s(x_1,\ x_2,\ x_3,\ t))^T$$

is the unknown s dimensional four variables functional vector,

$$\beta=(b_1(x_1,\ x_2,\ x_3,\ t),\ b_2(x_1,\ x_2,\ x_3,\ t),\ \cdots\ ,\ b_m(x_1,\ x_2,\ x_3,\ t))^T$$

is the known m dimensional four variables functional vector.
We can get the following conclusions.
(1)A necessary and sufficient condition for the existence of the solution of this equations is

$$\forall\ (x_1,\ x_2,\ x_3,\ t)^T\in R^4\ ,\ rank(A,\ \beta(x_1,\ x_2,\ x_3,\ t))=rank(A)\ ,$$

where $\beta(x_1,\ x_2,\ x_3,\ t)$ *is the value of* β , *when* $(x_1,\ x_2,\ x_3,\ t)^T$ *is given.*
(2)If the solution of this equations is exist, $rank(A)=r,\ r<s$, X_0 *is a particular solution of this equations, and the constant linearly independent solutions of the equations* $AX=0$ *are* $\eta_1\ ,\ \eta_2\ ,\cdots\ ,\ \eta_{s-r}$, *we denote* $A(\eta)=[\eta_1\ ,\ \eta_2\ ,\cdots\ ,\ \eta_{s-r}]$, *then its general solution can be expressed as*

$$X=X_0+A(\eta)Z_1\ ,$$

where Z_1 *is the arbitrary* $s-r$ *dimensional four variables functional vector.*

Theorem 2.2 *If $u_1,\ u_2,\ u_3 \in C^2$, $p \in C^1$, $F_1,\ F_2,\ F_3 \in C^1$, the Navier-Stokes equations can be converted into the simultaneous of the first order linear partial differential equations with constant coefficients and the quadratic equations, just as the following.*

$$\begin{cases} \dfrac{\partial(HZ+h)}{\partial t} = H_0 Z + h_0 \\ \dfrac{\partial(HZ+h)}{\partial x_i} = H_i Z\ ,\ i=1,\ 2,\ 3. \end{cases} \tag{2.1}$$

$$\begin{cases} e_{47}^T Z = (e_3^T Z)(-\alpha_1^T Z),\ e_{48}^T Z = (e_7^T Z)(e_1^T Z),\ e_{49}^T Z = (e_{11}^T Z)(e_2^T Z), \\ e_{50}^T Z = (e_3^T Z)(e_4^T Z),\ e_{51}^T Z = (e_7^T Z)(e_5^T Z),\ e_{52}^T Z = (e_{11}^T Z)(e_6^T Z), \\ e_{53}^T Z = (e_3^T Z)(e_8^T Z),\ e_{54}^T Z = (e_7^T Z)(e_9^T Z),\ e_{55}^T Z = (e_{11}^T Z)(e_{10}^T Z)\ , \end{cases} \tag{2.2}$$

where

$$\begin{aligned}
&H = (-\alpha_1,\ -\alpha_2,\ -\alpha_3,\ -\alpha_4,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7,\ e_8,\ e_9,\ e_{10},\ e_{11},\ e_{16})^T\ , \\
&H_0 = (e_{21},\ e_{17},\ e_{27},\ e_{37},\ e_{22},\ e_{23},\ -\alpha_2,\ e_{31},\ e_{32},\ e_{33},\ -\alpha_3,\ e_{41},\ e_{42},\ e_{43},\ -\alpha_4,\ e_{12})^T\ , \\
&H_1 = (e_{18},\ e_{21},\ e_{31},\ e_{41},\ e_{24},\ e_{25},\ -\alpha_1,\ e_{28},\ e_{34},\ e_{35},\ e_4,\ e_{38},\ e_{44},\ e_{45},\ e_8,\ e_{13})^T\ , \\
&H_2 = (e_{24},\ e_{22},\ e_{32},\ e_{42},\ e_{19},\ e_{26},\ e_1,\ e_{34},\ e_{29},\ e_{36},\ e_5,\ e_{44},\ e_{39},\ e_{46},\ e_9,\ e_{14})^T\ , \\
&H_3 = (e_{25},\ e_{23},\ e_{33},\ e_{43},\ e_{26},\ e_{20},\ e_2,\ e_{35},\ e_{36},\ e_{30},\ e_6,\ e_{45},\ e_{46},\ e_{40},\ e_{10},\ e_{15})^T\ , \\
&\alpha_1 = (0_4,\ 1,\ 0_4,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0_{39})^T, \\
&\alpha_2 = (0_{12},\ \tau,\ 0_4,\ -\mu,\ -\mu,\ -\mu,\ 0_{26},\ 1,\ 1,\ 1,\ 0_6)^T, \\
&\alpha_3 = (0_{13},\ \tau,\ 0_{13},\ -\mu,\ -\mu,\ -\mu,\ 0_{19},\ 1,\ 1,\ 1,\ 0_3)^T, \\
&\alpha_4 = (0_{14},\ \tau,\ 0_{22},\ -\mu,\ -\mu,\ -\mu,\ 0_{11},\ 0,\ 1,\ 1,\ 1)^T\ ,
\end{aligned}$$

and e_i is the ith 55 dimensional unit coordinate vector, $1 \le i \le 55$, $h_0 = (0_6,\ F_1,\ 0_3,\ F_2,\ 0_3,\ F_3,\ 0)^T$, $h = (0,\ F_1,\ F_2,\ F_3,\ 0_{12})^T$, Z is an unknown 55 dimensional functional vector. Moreover

$$(e_3,\ e_7,\ e_{11},\ e_{16})^T Z = (u_1,\ u_2,\ u_3,\ p)^T$$

is the solution of the Navier-Stokes equations.

Theorem 2.3 *We consider the first order linear partial differential equations with constant coefficients and the quadratic equations in the theorem 2.2, moreover when t is not in $[0,\ T]$, or $(x_1,\ x_2,\ x_3)^T$ is not in K_1 , $Z \equiv 0$, then we can get the following conclusions.*
(1)We can use the Fourier transformation to convert (2.1) into the linear equations as follows.

$$\begin{pmatrix} i\xi_0 H - H_0 \\ i\xi_1 H - H_1 \\ i\xi_2 H - H_2 \\ i\xi_3 H - H_3 \end{pmatrix} F(Z) = \begin{pmatrix} F(h_0) - i\xi_0 F(h) \\ -i\xi_1 F(h) \\ -i\xi_2 F(h) \\ -i\xi_3 F(h) \end{pmatrix}\ , \tag{2.3}$$

where

$$\begin{aligned}
F(Z) &= \int_{R^4} Z e^{-i\xi_0 t - i\sum_{j=1}^3 \xi_j x_j} dt dx_1 dx_2 dx_3\ , \\
F(h) &= \int_{R^4} h e^{-i\xi_0 t - i\sum_{j=1}^3 \xi_j x_j} dt dx_1 dx_2 dx_3\ , \\
F(h_0) &= \int_{R^4} h_0 e^{-i\xi_0 t - i\sum_{j=1}^3 \xi_j x_j} dt dx_1 dx_2 dx_3\ .
\end{aligned}$$

We can get the explicit general solution of it,

$$F(Z) = Y_1 + A_1(\eta) Z_1\ ,\ or\ Z = F^{-1}(Y_1 + A_1(\eta) Z_1)\ ,$$

where

$$\begin{aligned} Y_1 \;=\; & (i\xi_2 y_3,\ i\xi_3 y_3,\ y_3,\ i\xi_1 y_7,\ i\xi_2 y_7,\ i\xi_3 y_7,\ y_7,\ i\xi_1 y_{11},\ i\xi_2 y_{11},\ i\xi_3 y_{11},\ y_{11}, \\ & i\xi_0 y_{16},\ i\xi_1 y_{16},\ i\xi_2 y_{16},\ i\xi_3 y_{16},\ y_{16},\ (i\xi_0)^2 y_3,\ (i\xi_1)^2 y_3,\ (i\xi_2)^2 y_3,\ (i\xi_3)^2 y_3,\ i\xi_0 i\xi_1 y_3, \\ & i\xi_0 i\xi_2 y_3,\ i\xi_0 i\xi_3 y_3,\ i\xi_1 i\xi_2 y_3,\ i\xi_1 i\xi_3 y_3,\ i\xi_2 i\xi_3 y_3,\ (i\xi_0)^2 y_7,\ (i\xi_1)^2 y_7,\ (i\xi_2)^2 y_7,\ (i\xi_3)^2 y_7, \\ & i\xi_0 i\xi_1 y_7,\ i\xi_0 i\xi_2 y_7,\ i\xi_0 i\xi_3 y_7,\ i\xi_1 i\xi_2 y_7,\ i\xi_1 i\xi_3 y_7,\ i\xi_2 i\xi_3 y_7,\ (i\xi_0)^2 y_{11},\ (i\xi_1)^2 y_{11},\ (i\xi_2)^2 y_{11}, \\ & (i\xi_3)^2 y_{11},\ i\xi_0 i\xi_1 y_{11},\ i\xi_0 i\xi_2 y_{11},\ i\xi_0 i\xi_3 y_{11},\ i\xi_1 i\xi_2 y_{11},\ i\xi_1 i\xi_3 y_{11},\ i\xi_2 i\xi_3 y_{11},\ 0_{9\times1}^T)^T\ , \end{aligned}$$

and

$$\begin{aligned} y_3 &= \frac{i\xi_1(i\xi_1 F(F_1) + i\xi_2 F(F_2) + i\xi_3 F(F_3))}{a^2 b\tau} - \frac{F(F_1)}{a\tau}\ , \\ y_7 &= \frac{i\xi_2(i\xi_1 F(F_1) + i\xi_2 F(F_2) + i\xi_3 F(F_3))}{a^2 b\tau} - \frac{F(F_2)}{a\tau}\ , \\ y_{11} &= \frac{i\xi_3(i\xi_1 F(F_1) + i\xi_2 F(F_2) + i\xi_3 F(F_3))}{a^2 b\tau} - \frac{F(F_3)}{a\tau}\ , \\ y_{16} &= \frac{i\xi_1 F(F_1) + i\xi_2 F(F_2) + i\xi_3 F(F_3)}{ab\tau}\ , \end{aligned}$$

where

$$a = \frac{\mu[(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2] - i\xi_0}{\tau} \neq 0\ ,\ b = \frac{(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2}{a} \neq 0\ ,$$

when $(\xi_1,\ \xi_2,\ \xi_3) \neq 0$. $A_1(\eta) = (\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5,\ \eta_6,\ \eta_7,\ \eta_8,\ \eta_9)$, *and*

$$\begin{aligned} \eta_j \;=\; & (i\xi_2 y_3,\ i\xi_3 y_3,\ y_3,\ i\xi_1 y_7,\ i\xi_2 y_7,\ i\xi_3 y_7,\ y_7,\ i\xi_1 y_{11},\ i\xi_2 y_{11},\ i\xi_3 y_{11},\ y_{11}, \\ & i\xi_0 y_{16},\ i\xi_1 y_{16},\ i\xi_2 y_{16},\ i\xi_3 y_{16},\ y_{16},\ (i\xi_0)^2 y_3,\ (i\xi_1)^2 y_3,\ (i\xi_2)^2 y_3,\ (i\xi_3)^2 y_3,\ i\xi_0 i\xi_1 y_3, \\ & i\xi_0 i\xi_2 y_3,\ i\xi_0 i\xi_3 y_3,\ i\xi_1 i\xi_2 y_3,\ i\xi_1 i\xi_3 y_3,\ i\xi_2 i\xi_3 y_3,\ (i\xi_0)^2 y_7,\ (i\xi_1)^2 y_7,\ (i\xi_2)^2 y_7,\ (i\xi_3)^2 y_7, \\ & i\xi_0 i\xi_1 y_7,\ i\xi_0 i\xi_2 y_7,\ i\xi_0 i\xi_3 y_7,\ i\xi_1 i\xi_2 y_7,\ i\xi_1 i\xi_3 y_7,\ i\xi_2 i\xi_3 y_7,\ (i\xi_0)^2 y_{11},\ (i\xi_1)^2 y_{11},\ (i\xi_2)^2 y_{11}, \\ & (i\xi_3)^2 y_{11},\ i\xi_0 i\xi_1 y_{11},\ i\xi_0 i\xi_2 y_{11},\ i\xi_0 i\xi_3 y_{11},\ i\xi_1 i\xi_2 y_{11},\ i\xi_1 i\xi_3 y_{11},\ i\xi_2 i\xi_3 y_{11},\ e_j^T)^T\ , \end{aligned}$$

here e_j *is the jth* 9 *dimensional unit coordinate vector,* $1 \leq j \leq 9$, *moreover when* $j = 1,\ 2,\ 3$,

$$y_3 = \frac{\xi_1^2}{a^2 b\tau} + \frac{1}{a\tau}\ ,\ y_7 = \frac{-\ i\xi_1 i\xi_2}{a^2 b\tau}\ ,\ y_{11} = \frac{-\ i\xi_1 i\xi_3}{a^2 b\tau}\ ,\ y_{16} = \frac{-\ i\xi_1}{ab\tau}\ ,$$

when $j = 4,\ 5,\ 6$,

$$y_7 = \frac{\xi_2^2}{a^2 b\tau} + \frac{1}{a\tau}\ ,\ y_3 = \frac{-\ i\xi_1 i\xi_2}{a^2 b\tau}\ ,\ y_{11} = \frac{-\ i\xi_2 i\xi_3}{a^2 b\tau}\ ,\ y_{16} = \frac{-\ i\xi_2}{ab\tau}\ ,$$

when $j = 7,\ 8,\ 9$,

$$y_{11} = \frac{\xi_3^2}{a^2 b\tau} + \frac{1}{a\tau}\ ,\ y_3 = \frac{-\ i\xi_1 i\xi_3}{a^2 b\tau}\ ,\ y_7 = \frac{-\ i\xi_2 i\xi_3}{a^2 b\tau}\ ,\ y_{16} = \frac{-\ i\xi_3}{ab\tau}\ .$$

$Z_1 = (Z_{1j})_{9\times1}$, *moreover we assume*

$$F^{-1}(Y_1 + A_1(\eta) Z_1 I_{\{(\xi_1,\ \xi_2,\ \xi_3)\neq 0\}}) = F^{-1}(Y_1 + A_1(\eta) Z_1)\ ,$$

and $Z_1 \in\ \Omega_1$, *where*

$$\Omega_1 = \{Z_1 | H[F^{-1}(Y_1 + A_1(\eta) Z_1)] = H[F^{-1}(Y_1 + A_1(\eta) Z_1)] I_{K_1'} \in C^1(K_1').\}\ .$$

(2)We can convert (2.2) into the integral equations as follows,

$$\begin{cases} Z_{11} = F[F^{-1}(e_3^T(Y_1 + A_1(\eta)Z_1))F^{-1}(-\alpha_1^T(Y_1 + A_1(\eta)Z_1))] = f_1(Z_1) \ , \\ Z_{12} = F[F^{-1}(e_7^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_1^T(Y_1 + A_1(\eta)Z_1))] = f_2(Z_1) \ , \\ Z_{13} = F[F^{-1}(e_{11}^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_2^T(Y_1 + A_1(\eta)Z_1))] = f_3(Z_1) \ , \\ Z_{14} = F[F^{-1}(e_3^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_4^T(Y_1 + A_1(\eta)Z_1))] = f_4(Z_1) \ , \\ Z_{15} = F[F^{-1}(e_7^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_5^T(Y_1 + A_1(\eta)Z_1))] = f_5(Z_1) \ , \\ Z_{16} = F[F^{-1}(e_{11}^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_6^T(Y_1 + A_1(\eta)Z_1))] = f_6(Z_1) \ , \\ Z_{17} = F[F^{-1}(e_3^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_8^T(Y_1 + A_1(\eta)Z_1))] = f_7(Z_1) \ , \\ Z_{18} = F[F^{-1}(e_7^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_9^T(Y_1 + A_1(\eta)Z_1))] = f_8(Z_1) \ , \\ Z_{19} = F[F^{-1}(e_{11}^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_{10}^T(Y_1 + A_1(\eta)Z_1))] = f_9(Z_1) \ . \end{cases} \tag{2.4}$$

This is also the question to find the fixed-points of $f(Z_1)$, where $f(Z_1) = (f_j(Z_1))_{9\times 1}$. We can use the theories about the Poisson's equation, the heat-conduct equation, the Schauder fixed-point theorem to prove that the fixed-point is exist in the region K_1' , and the smoothing solution for the Navier-Stokes equations is globally exist in the region K_1' .

3 Proof

Proof of theorem2-1. (1)Necessity. Because the solution of $AX = \beta$ is exist, we assume the particular solution is X_0 , then $\forall\ (x_1,\ x_2,\ x_3,\ t)^T \in R^4$, we know that

$$AX_0(x_1,\ x_2,\ x_3,\ t) = \beta(x_1,\ x_2,\ x_3,\ t)$$

satisfies,

$$X_0(x_1,\ x_2,\ x_3,\ t) \text{ and } \beta(x_1,\ x_2,\ x_3,\ t)$$

are values of X_0 and β , when $(x_1,\ x_2,\ x_3,\ t)^T$ is given. Hence

$$rank(A,\ \beta(x_1,\ x_2,\ x_3,\ t)) = rank(A),\ \forall\ (x_1,\ x_2,\ x_3,\ t)^T \in R^4\ .$$

Sufficiency. Because $\forall\ (x_1,\ x_2,\ x_3,\ t)^T \in R^4$, we can get

$$rank(A,\ \beta(x_1,\ x_2,\ x_3,\ t)) = rank(A)\ ,$$

then the solution of

$$AX = \beta(x_1,\ x_2,\ x_3,\ t)$$

is exist. We assume the particular solution is $X_0(x_1,\ x_2,\ x_3,\ t)$, then $\forall\ (x_1,\ x_2,\ x_3,\ t)^T \in R^4$, we let

$$X = (X_1(x_1,\ x_2,\ x_3,\ t),\ X_2(x_1,\ x_2,\ x_3,\ t),\ \cdots,\ X_s(x_1,\ x_2,\ x_3,\ t))^T = X_0(x_1,\ x_2,\ x_3,\ t)\ ,$$

and we can get X satisfies $AX = \beta$.

(2)When the solution of $AX = \beta$ is exist, we assume X_0 is a particular solution, X is the arbitrary solution of this equations, then $\forall\ (x_1,\ x_2,\ x_3,\ t)^T \in R^4$, we can get

$$A(X(x_1,\ x_2,\ x_3,\ t) - X_0(x_1,\ x_2,\ x_3,\ t)) = 0\ ,$$

according to the linear equations theory, there exists a unique team of real numbers

$$C_1,\ C_2,\ \cdots,\ C_{s-r}$$

such that

$$X(x_1,\ x_2,\ x_3,\ t) - X_0(x_1,\ x_2,\ x_3,\ t) = C_1\eta_1 + C_2\eta_2 + \cdots + C_{s-r}\eta_{s-r}\ ,$$

we let

$$Z_1(x_1,\ x_2,\ x_3,\ t) = (C_1,\ C_2,\ \cdots,\ C_{s-r})^T\ ,$$

then

$$X(x_1,\ x_2,\ x_3,\ t) = X_0(x_1,\ x_2,\ x_3,\ t) + A(\eta)Z_1(x_1,\ x_2,\ x_3,\ t)\ .$$

Because $(x_1,\ x_2,\ x_3,\ t)^T$ is arbitrary, we can learn that

$$X = X_0 + A(\eta)Z_1\ .$$

On the other hand

$$A[X_0 + A(\eta)Z_1] = \beta + 0Z_1 = \beta\ .$$

This is to say

$$\forall Z_1\ ,\ X_0 + A(\eta)Z_1$$

is the solution of the equation $AX = \beta$. Hence

$$X_0 + A(\eta)Z_1$$

is the general solution of the equation $AX = \beta$.

Proof of theorem2-2. We can rewrite the Navier-Stokes equations as following.

$$AX = \beta\ ,$$

where

$$A = \begin{pmatrix} 1, & 0, & 0, & 0, & 0_4, & 1, & 0_4, & 1, & 0, & 0, & 0, & 0, & 0, & 0, & 0_{39} \\ 0, & 1, & 0, & 0, & 0_{12}, & \tau, & 0_4, & -\mu, & -\mu, & -\mu, & 0_{26}, & 1, & 1, & 1, & 0_6 \\ 0, & 0, & 1, & 0, & 0_{13}, & \tau, & 0_{13}, & -\mu, & -\mu, & -\mu, & 0_{19}, & 1, & 1, & 1, & 0_3 \\ 0, & 0, & 0, & 1, & 0_{14}, & \tau, & 0_{22}, & -\mu, & -\mu, & -\mu, & 0_{11}, & 0, & 1, & 1, & 1 \end{pmatrix}_{4\times 59}$$

0_n is the row vector which is made up of n zeroes, X includes $u_1,\ u_2,\ u_3,\ p$ and all their first order partial derivative, and all the second order partial derivative of $u_1,\ u_2,\ u_3$, and all the products which they are in the Navier-Stokes equations,

$$X = \ (\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, \frac{\partial u_3}{\partial t}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_1}{\partial x_3}, u_1, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_2}{\partial x_3}, u_2, \frac{\partial u_3}{\partial x_1}, \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_3}, u_3,$$
$$\frac{\partial p}{\partial t}, \frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}, p, \frac{\partial^2 u_1}{\partial t^2}, \frac{\partial^2 u_1}{\partial x_1^2}, \frac{\partial^2 u_1}{\partial x_2^2}, \frac{\partial^2 u_1}{\partial x_3^2}, \frac{\partial^2 u_1}{\partial t\partial x_1}, \frac{\partial^2 u_1}{\partial t\partial x_2}, \frac{\partial^2 u_1}{\partial t\partial x_3},$$

$$\frac{\partial^2 u_1}{\partial x_1 \partial x_2},\ \frac{\partial^2 u_1}{\partial x_1 \partial x_3},\ \frac{\partial^2 u_1}{\partial x_2 \partial x_3},\ \frac{\partial^2 u_2}{\partial t^2},\ \frac{\partial^2 u_2}{\partial x_1^2},\ \frac{\partial^2 u_2}{\partial x_2^2},\ \frac{\partial^2 u_2}{\partial x_3^2},\ \frac{\partial^2 u_2}{\partial t \partial x_1},\ \frac{\partial^2 u_2}{\partial t \partial x_2},\ \frac{\partial^2 u_2}{\partial t \partial x_3},$$

$$\frac{\partial^2 u_2}{\partial x_1 \partial x_2},\ \frac{\partial^2 u_2}{\partial x_1 \partial x_3},\ \frac{\partial^2 u_2}{\partial x_2 \partial x_3},\ \frac{\partial^2 u_3}{\partial t^2},\ \frac{\partial^2 u_3}{\partial x_1^2},\ \frac{\partial^2 u_3}{\partial x_2^2},\ \frac{\partial^2 u_3}{\partial x_3^2},\ \frac{\partial^2 u_3}{\partial t \partial x_1},\ \frac{\partial^2 u_3}{\partial t \partial x_2},\ \frac{\partial^2 u_3}{\partial t \partial x_3},$$

$$\frac{\partial^2 u_3}{\partial x_1 \partial x_2},\ \frac{\partial^2 u_3}{\partial x_1 \partial x_3},\ \frac{\partial^2 u_3}{\partial x_2 \partial x_3},\ u_1\frac{\partial u_1}{\partial x_1},\ u_2\frac{\partial u_1}{\partial x_2},\ u_3\frac{\partial u_1}{\partial x_3},\ u_1\frac{\partial u_2}{\partial x_1},\ u_2\frac{\partial u_2}{\partial x_2},\ u_3\frac{\partial u_2}{\partial x_3},$$

$$u_1\frac{\partial u_3}{\partial x_1},\ u_2\frac{\partial u_3}{\partial x_2},\ u_3\frac{\partial u_3}{\partial x_3})^T,$$

$$\beta = (0,\ F_1,\ F_2,\ F_3)\ .$$

We can see that $A = (E_4, A_1)$, A_1 is a 4×55 matrix. We assume $A_1 = (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)^T$, where

$$\begin{aligned}
\alpha_1 &= (0_4,\ 1,\ 0_4,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0_{39})^T,\\
\alpha_2 &= (0_{12},\ \tau,\ 0_4,\ -\mu,\ -\mu,\ -\mu,\ 0_{26},\ 1,\ 1,\ 1,\ 0_6)^T,\\
\alpha_3 &= (0_{13},\ \tau,\ 0_{13},\ -\mu,\ -\mu,\ -\mu,\ 0_{19},\ 1,\ 1,\ 1,\ 0_3)^T,\\
\alpha_4 &= (0_{14},\ \tau,\ 0_{22},\ -\mu,\ -\mu,\ -\mu,\ 0_{11},\ 0,\ 1,\ 1,\ 1)^T\ .
\end{aligned}$$

We let $X_0 = (0,\ F_1,\ F_2,\ F_3,\ 0_{55})^T$, $A(\eta) = \begin{pmatrix} -A_1 \\ E_{55} \end{pmatrix}_{59\times 55}$, then $AX_0 = \beta$, $AA(\eta) = 0$, and the columns of $A(\eta)$ are linear independent, so according to (2) in the theorem 2.2, we can get the general solution of $AX = \beta$ as follows,

$$X = X_0 + A(\eta)Z\ .$$

We write the 59 components of X in details.

$$\frac{\partial u_1}{\partial x_1} = -\alpha_1^T Z\ ,\ \frac{\partial u_1}{\partial t} = F_1 - \alpha_2^T Z\ ,\ \frac{\partial u_2}{\partial t} = F_2 - \alpha_3^T Z\ ,\ \frac{\partial u_3}{\partial t} = F_3 - \alpha_4^T Z\ ,\ \frac{\partial u_1}{\partial x_2} = e_1^T Z\ ,$$

$$\frac{\partial u_1}{\partial x_3} = e_2^T Z\ ,\ u_1 = e_3^T Z\ ,\ \frac{\partial u_2}{\partial x_1} = e_4^T Z\ ,\ \frac{\partial u_2}{\partial x_2} = e_5^T Z\ ,\ \frac{\partial u_2}{\partial x_3} = e_6^T Z\ ,\ u_2 = e_7^T Z\ ,$$

$$\frac{\partial u_3}{\partial x_1} = e_8^T Z\ ,\ \frac{\partial u_3}{\partial x_2} = e_9^T Z\ ,\ \frac{\partial u_3}{\partial x_3} = e_{10}^T Z\ ,\ u_3 = e_{11}^T Z\ ,\ \frac{\partial p}{\partial t} = e_{12}^T Z\ ,\ \frac{\partial p}{\partial x_1} = e_{13}^T Z\ ,$$

$$\frac{\partial p}{\partial x_2} = e_{14}^T Z\ ,\ \frac{\partial p}{\partial x_3} = e_{15}^T Z\ ,\ p = e_{16}^T Z\ ,\ \frac{\partial^2 u_1}{\partial t^2} = e_{17}^T Z\ ,\ \frac{\partial^2 u_1}{\partial x_1^2} = e_{18}^T Z\ ,\ \frac{\partial^2 u_1}{\partial x_2^2} = e_{19}^T Z\ ,$$

$$\frac{\partial^2 u_1}{\partial x_3^2} = e_{20}^T Z\ ,\ \frac{\partial^2 u_1}{\partial t \partial x_1} = e_{21}^T Z\ ,\ \frac{\partial^2 u_1}{\partial t \partial x_2} = e_{22}^T Z\ ,\ \frac{\partial^2 u_1}{\partial t \partial x_3} = e_{23}^T Z\ ,\ \frac{\partial^2 u_1}{\partial x_1 \partial x_2} = e_{24}^T Z\ ,$$

$$\frac{\partial^2 u_1}{\partial x_1 \partial x_3} = e_{25}^T Z\ ,\ \frac{\partial^2 u_1}{\partial x_2 \partial x_3} = e_{26}^T Z\ ,\ \frac{\partial^2 u_2}{\partial t^2} = e_{27}^T Z\ ,\ \frac{\partial^2 u_2}{\partial x_1^2} = e_{28}^T Z\ ,\ \frac{\partial^2 u_2}{\partial x_2^2} = e_{29}^T Z\ ,$$

$$\frac{\partial^2 u_2}{\partial x_3^2} = e_{30}^T Z\ ,\ \frac{\partial^2 u_2}{\partial t \partial x_1} = e_{31}^T Z\ ,\ \frac{\partial^2 u_2}{\partial t \partial x_2} = e_{32}^T Z\ ,\ \frac{\partial^2 u_2}{\partial t \partial x_3} = e_{33}^T Z\ ,\ \frac{\partial^2 u_2}{\partial x_1 \partial x_2} = e_{34}^T Z\ ,$$

$$\frac{\partial^2 u_2}{\partial x_1 \partial x_3} = e_{35}^T Z\ ,\ \frac{\partial^2 u_2}{\partial x_2 \partial x_3} = e_{36}^T Z\ ,\ \frac{\partial^2 u_3}{\partial t^2} = e_{37}^T Z\ ,\ \frac{\partial^2 u_3}{\partial x_1^2} = e_{38}^T Z\ ,\ \frac{\partial^2 u_3}{\partial x_2^2} = e_{39}^T Z\ ,$$

$$\frac{\partial^2 u_3}{\partial x_3^2} = e_{40}^T Z\ ,\ \frac{\partial^2 u_3}{\partial t \partial x_1} = e_{41}^T Z\ ,\ \frac{\partial^2 u_3}{\partial t \partial x_2} = e_{42}^T Z\ ,\ \frac{\partial^2 u_3}{\partial t \partial x_3} = e_{43}^T Z\ ,\ \frac{\partial^2 u_3}{\partial x_1 \partial x_2} = e_{44}^T Z\ ,$$
$$\frac{\partial^2 u_3}{\partial x_1 \partial x_3} = e_{45}^T Z\ ,\ \frac{\partial^2 u_3}{\partial x_2 \partial x_3} = e_{46}^T Z\ ,\ u_1 \frac{\partial u_1}{\partial x_1} = e_{47}^T Z\ ,\ u_2 \frac{\partial u_1}{\partial x_2} = e_{48}^T Z\ ,\ u_3 \frac{\partial u_1}{\partial x_3} = e_{49}^T Z\ ,$$
$$u_1 \frac{\partial u_2}{\partial x_1} = e_{50}^T Z\ ,\ u_2 \frac{\partial u_2}{\partial x_2} = e_{51}^T Z\ ,\ u_3 \frac{\partial u_2}{\partial x_3} = e_{52}^T Z\ ,\ u_1 \frac{\partial u_3}{\partial x_1} = e_{53}^T Z\ ,\ u_2 \frac{\partial u_3}{\partial x_2} = e_{54}^T Z\ ,$$
$$u_3 \frac{\partial u_3}{\partial x_3} = e_{55}^T Z\ ,$$

here e_i is the ith 55 dimensional unit coordinate vector, $1 \leq i \leq 55$. There are 9 quadratic items, hence we can get 9 quadratic constraints which Z need to satisfy just as following:

$$\begin{cases} e_{47}^T Z = (e_3^T Z)(-\alpha_1^T Z),\ e_{48}^T Z = (e_7^T Z)(e_1^T Z),\ e_{49}^T Z = (e_{11}^T Z)(e_2^T Z), \\ e_{50}^T Z = (e_3^T Z)(e_4^T Z),\ e_{51}^T Z = (e_7^T Z)(e_5^T Z),\ e_{52}^T Z = (e_{11}^T Z)(e_6^T Z), \\ e_{53}^T Z = (e_3^T Z)(e_8^T Z),\ e_{54}^T Z = (e_7^T Z)(e_9^T Z),\ e_{55}^T Z = (e_{11}^T Z)(e_{10}^T Z). \end{cases} \tag{3.1}$$

Because $u_1,\ u_2,\ u_3 \in C^2$, $p \in C^1$, we can get 46 differential constraints which Z need to satisfy just as following:

$$\frac{\partial(e_3^T Z)}{\partial x_1} = -\alpha_1^T Z\ ,\ \frac{\partial(e_3^T Z)}{\partial t} = F_1 - \alpha_2^T Z\ ,\ \frac{\partial(e_7^T Z)}{\partial t} = F_2 - \alpha_3^T Z\ ,\ \frac{\partial(e_{11}^T Z)}{\partial t} = F_3 - \alpha_4^T Z\ ,$$
$$\frac{\partial(e_3^T Z)}{\partial x_2} = e_1^T Z\ ,\ \frac{\partial(e_3^T Z)}{\partial x_3} = e_2^T Z\ ,\ \frac{\partial(e_7^T Z)}{\partial x_1} = e_4^T Z\ ,\ \frac{\partial(e_7^T Z)}{\partial x_2} = e_5^T Z\ ,\frac{\partial(e_7^T Z)}{\partial x_3} = e_6^T Z\ ,$$
$$\frac{\partial(e_{11}^T Z)}{\partial x_1} = e_8^T Z\ ,\ \frac{\partial(e_{11}^T Z)}{\partial x_2} = e_9^T Z\ ,\ \frac{\partial(e_{11}^T Z)}{\partial x_3} = e_{10}^T Z\ ,\ \frac{\partial(e_{16}^T Z)}{\partial t} = e_{12}^T Z\ ,\ \frac{\partial(e_{16}^T Z)}{\partial x_1} = e_{13}^T Z\ ,$$
$$\frac{\partial(e_{16}^T Z)}{\partial x_2} = e_{14}^T Z\ ,\ \frac{\partial(e_{16}^T Z)}{\partial x_3} = e_{15}^T Z\ ,\ \frac{\partial^2(e_3^T Z)}{\partial t^2} = e_{17}^T Z\ ,\ \frac{\partial^2(e_3^T Z)}{\partial x_1^2} = e_{18}^T Z\ ,\ \frac{\partial^2(e_3^T Z)}{\partial x_2^2} = e_{19}^T Z\ ,$$
$$\frac{\partial^2(e_3^T Z)}{\partial x_3^2} = e_{20}^T Z\ ,\ \frac{\partial^2(e_3^T Z)}{\partial t \partial x_1} = e_{21}^T Z\ ,\ \frac{\partial^2(e_3^T Z)}{\partial t \partial x_2} = e_{22}^T Z\ ,\ \frac{\partial^2(e_3^T Z)}{\partial t \partial x_3} = e_{23}^T Z\ ,\ \frac{\partial^2(e_3^T Z)}{\partial x_1 \partial x_2} = e_{24}^T Z\ ,$$
$$\frac{\partial^2(e_3^T Z)}{\partial x_1 \partial x_3} = e_{25}^T Z\ ,\ \frac{\partial^2(e_3^T Z)}{\partial x_2 \partial x_3} = e_{26}^T Z\ ,\ \frac{\partial^2(e_7^T Z)}{\partial t^2} = e_{27}^T Z\ ,\ \frac{\partial^2(e_7^T Z)}{\partial x_1^2} = e_{28}^T Z\ ,\ \frac{\partial^2(e_7^T Z)}{\partial x_2^2} = e_{29}^T Z\ ,$$
$$\frac{\partial^2(e_7^T Z)}{\partial x_3^2} = e_{30}^T Z\ ,\ \frac{\partial^2(e_7^T Z)}{\partial t \partial x_1} = e_{31}^T Z\ ,\ \frac{\partial^2(e_7^T Z)}{\partial t \partial x_2} = e_{32}^T Z\ ,\ \frac{\partial^2(e_7^T Z)}{\partial t \partial x_3} = e_{33}^T Z\ ,\ \frac{\partial^2(e_7^T Z)}{\partial x_1 \partial x_2} = e_{34}^T Z\ ,$$
$$\frac{\partial^2(e_7^T Z)}{\partial x_1 \partial x_3} = e_{35}^T Z\ ,\ \frac{\partial^2(e_7^T Z)}{\partial x_2 \partial x_3} = e_{36}^T Z\ ,\ \frac{\partial^2(e_{11}^T Z)}{\partial t^2} = e_{37}^T Z\ ,\ \frac{\partial^2(e_{11}^T Z)}{\partial x_1^2} = e_{38}^T Z\ ,\frac{\partial^2(e_{11}^T Z)}{\partial x_2^2} = e_{39}^T Z\ ,$$
$$\frac{\partial^2(e_{11}^T Z)}{\partial x_3^2} = e_{40}^T Z\ ,\ \frac{\partial^2(e_{11}^T Z)}{\partial t \partial x_1} = e_{41}^T Z\ ,\ \frac{\partial^2(e_{11}^T Z)}{\partial t \partial x_2} = e_{42}^T Z\ ,\ \frac{\partial^2(e_{11}^T Z)}{\partial t \partial x_3} = e_{43}^T Z\ ,\ \frac{\partial^2(e_{11}^T Z)}{\partial x_1 \partial x_2} = e_{44}^T Z\ ,$$
$$\frac{\partial^2(e_{11}^T Z)}{\partial x_1 \partial x_3} = e_{45}^T Z\ ,\ \frac{\partial^2(e_{11}^T Z)}{\partial x_2 \partial x_3} = e_{46}^T Z\ .$$

Because the second order partial derivative can be taken as the partial derivative of the first order partial derivative, and $u_1,\ u_2,\ u_3 \in C^2$, we can learn that their second order mixed partial derivatives are equal, the above differential constraints are equivalent to 64 first order differential constraints as follows:

$$\frac{\partial(e_3^TZ)}{\partial x_1}=-\alpha_1^TZ\ ,\ \frac{\partial(e_3^TZ)}{\partial t}=F_1-\alpha_2^TZ\ ,\ \frac{\partial(e_7^TZ)}{\partial t}=F_2-\alpha_3^TZ\ ,\ \frac{\partial(e_{11}^TZ)}{\partial t}=F_3-\alpha_4^TZ\ ,$$

$$\frac{\partial(e_3^TZ)}{\partial x_2}=e_1^TZ\ ,\ \frac{\partial(e_3^TZ)}{\partial x_3}=e_2^TZ\ ,\ \frac{\partial(e_7^TZ)}{\partial x_1}=e_4^TZ\ ,\ \frac{\partial(e_7^TZ)}{\partial x_2}=e_5^TZ\ ,\frac{\partial(e_7^TZ)}{\partial x_3}=e_6^TZ\ ,$$

$$\frac{\partial(e_{11}^TZ)}{\partial x_1}=e_8^TZ\ ,\ \frac{\partial(e_{11}^TZ)}{\partial x_2}=e_9^TZ\ ,\ \frac{\partial(e_{11}^TZ)}{\partial x_3}=e_{10}^TZ\ ,\ \frac{\partial(e_{16}^TZ)}{\partial t}=e_{12}^TZ\ ,\ \frac{\partial(e_{16}^TZ)}{\partial x_1}=e_{13}^TZ\ ,$$

$$\frac{\partial(e_{16}^TZ)}{\partial x_2}=e_{14}^TZ\ ,\ \frac{\partial(e_{16}^TZ)}{\partial x_3}=e_{15}^TZ\ ,\ \frac{\partial(F_1-\alpha_2^TZ)}{\partial t}=e_{17}^TZ\ ,\ \frac{\partial(-\alpha_1^TZ)}{\partial x_1}=e_{18}^TZ\ ,$$

$$\frac{\partial(e_1^TZ)}{\partial x_2}=e_{19}^TZ\ ,\ \frac{\partial(e_2^TZ)}{\partial x_3}=e_{20}^TZ\ ,\ \frac{\partial(F_1-\alpha_2^TZ)}{\partial x_1}=\frac{\partial(-\alpha_1^TZ)}{\partial t}=e_{21}^TZ\ ,$$

$$\frac{\partial(F_1-\alpha_2^TZ)}{\partial x_2}=\frac{\partial(e_1^TZ)}{\partial t}=e_{22}^TZ\ ,\ \frac{\partial(F_1-\alpha_2^TZ)}{\partial x_3}=\frac{\partial(e_2^TZ)}{\partial t}=e_{23}^TZ\ ,$$

$$\frac{\partial(-\alpha_1^TZ)}{\partial x_2}=\frac{\partial(e_1^TZ)}{\partial x_1}=e_{24}^TZ\ ,\ \frac{\partial(-\alpha_1^TZ)}{\partial x_3}=\frac{\partial(e_2^TZ)}{\partial x_1}=e_{25}^TZ\ ,\ \frac{\partial(e_1^TZ)}{\partial x_3}=\frac{\partial(e_2^TZ)}{\partial x_2}=e_{26}^TZ\ ,$$

$$\frac{\partial(F_2-\alpha_3^TZ)}{\partial t}=e_{27}^TZ\ ,\ \frac{\partial(e_4^TZ)}{\partial x_1}=e_{28}^TZ\ ,\ \frac{\partial(e_5^TZ)}{\partial x_2}=e_{29}^TZ\ ,\ \frac{\partial(e_6^TZ)}{\partial x_3}=e_{30}^TZ\ ,$$

$$\frac{\partial(F_2-\alpha_3^TZ)}{\partial x_1}=\frac{\partial(e_4^TZ)}{\partial t}=e_{31}^TZ\ ,\ \frac{\partial(F_2-\alpha_3^TZ)}{\partial x_2}=\frac{\partial(e_5^TZ)}{\partial t}=e_{32}^TZ\ ,$$

$$\frac{\partial(F_2-\alpha_3^TZ)}{\partial x_3}=\frac{\partial(e_6^TZ)}{\partial t}=e_{33}^TZ\ ,\ \frac{\partial(e_4^TZ)}{\partial x_2}=\frac{\partial(e_5^TZ)}{\partial x_1}=e_{34}^TZ\ ,\ \frac{\partial(e_4^TZ)}{\partial x_3}=\frac{\partial(e_6^TZ)}{\partial x_1}=e_{35}^TZ\ ,$$

$$\frac{\partial(e_5^TZ)}{\partial x_3}=\frac{\partial(e_6^TZ)}{\partial x_2}=e_{36}^TZ\ ,\ \frac{\partial(F_3-\alpha_4^TZ)}{\partial t}=e_{37}^TZ\ ,\ \frac{\partial(e_8^TZ)}{\partial x_1}=e_{38}^TZ\ ,\frac{\partial(e_9^TZ)}{\partial x_2}=e_{39}^TZ\ ,$$

$$\frac{\partial(e_{10}^TZ)}{\partial x_3}=e_{40}^TZ\ ,\ \frac{\partial(F_3-\alpha_4^TZ)}{\partial x_1}=\frac{\partial(e_8^TZ)}{\partial t}=e_{41}^TZ\ ,\ \frac{\partial(F_3-\alpha_4^TZ)}{\partial x_2}=\frac{\partial(e_9^TZ)}{\partial t}=e_{42}^TZ\ ,$$

$$\frac{\partial(F_3-\alpha_4^TZ)}{\partial x_3}=\frac{\partial(e_{10}^TZ)}{\partial t}=e_{43}^TZ\ ,\ \frac{\partial(e_8^TZ)}{\partial x_2}=\frac{\partial(e_9^TZ)}{\partial x_1}=e_{44}^TZ\ ,\ \frac{\partial(e_8^TZ)}{\partial x_3}=\frac{\partial(e_{10}^TZ)}{\partial x_1}=e_{45}^TZ\ ,$$

$$\frac{\partial(e_9^TZ)}{\partial x_3}=\frac{\partial(e_{10}^TZ)}{\partial x_2}=e_{46}^TZ\ .$$

We write them into the equations, we assume

$$U=(\frac{\partial u_1}{\partial x_1},\ \frac{\partial u_1}{\partial t},\ \frac{\partial u_2}{\partial t},\ \frac{\partial u_3}{\partial t},\ \frac{\partial u_1}{\partial x_2},\ \frac{\partial u_1}{\partial x_3},\ u_1\ ,\ \frac{\partial u_2}{\partial x_1},\ \frac{\partial u_2}{\partial x_2},\ \frac{\partial u_2}{\partial x_3},\ u_2\ ,\ \frac{\partial u_3}{\partial x_1},\ \frac{\partial u_3}{\partial x_2},\ \frac{\partial u_3}{\partial x_3},\ u_3\ ,\ p\ )^T\ ,$$

$$H=(-\alpha_1,\ -\alpha_2,\ -\alpha_3,\ -\alpha_4,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7,\ e_8,\ e_9,\ e_{10},\ e_{11},\ e_{16})^T\ ,$$

$h=(0,\ F_1,\ F_2,\ F_3,\ 0_{12})^T$, then we can get $U=HZ+h$, moreover we assume

$$H_0=(e_{21},\ e_{17},\ e_{27},\ e_{37},\ e_{22},\ e_{23},\ -\alpha_2,\ e_{31},\ e_{32},\ e_{33},\ -\alpha_3,\ e_{41},\ e_{42},\ e_{43},\ -\alpha_4,\ e_{12})^T\ ,$$

$$H_1=(e_{18},\ e_{21},\ e_{31},\ e_{41},\ e_{24},\ e_{25},\ -\alpha_1,\ e_{28},\ e_{34},\ e_{35},\ e_4,\ e_{38},\ e_{44},\ e_{45},\ e_8,\ e_{13})^T\ ,$$

$$H_2=(e_{24},\ e_{22},\ e_{32},\ e_{42},\ e_{19},\ e_{26},\ e_1,\ e_{34},\ e_{29},\ e_{36},\ e_5,\ e_{44},\ e_{39},\ e_{46},\ e_9,\ e_{14})^T\ ,$$

$$H_3=(e_{25},\ e_{23},\ e_{33},\ e_{43},\ e_{26},\ e_{20},\ e_2,\ e_{35},\ e_{36},\ e_{30},\ e_6,\ e_{45},\ e_{46},\ e_{40},\ e_{10},\ e_{15})^T\ ,$$

$h_0=(0_6,\ F_1,\ 0_3,\ F_2,\ 0_3,\ F_3,\ 0)^T$, then we can get

$$\begin{cases}\dfrac{\partial U}{\partial t}=H_0Z+h_0\\ \dfrac{\partial U}{\partial x_i}=H_iZ\ ,\ i=1,\ 2,\ 3,\end{cases}\tag{3.2}$$

or

$$\begin{cases}\dfrac{\partial(HZ+h)}{\partial t}=H_0Z+h_0\\ \dfrac{\partial(HZ+h)}{\partial x_i}=H_iZ\ ,\ i=1,\ 2,\ 3.\end{cases}\tag{3.3}$$

Now we have converted the Navier-Stokes equations into the simultaneous of the first order linear partial differential equations with constant coefficients (3.3) and 9 quadratic polynomial equations (3.1).

And we get a necessary condition for the existence of the solution of the Navier-Stokes equations as follows, $\exists$ 55 dimensional four variables functional vector Z , such that Z satisfies the first order linear partial differential equations with constant coefficients (3.3) and 9 quadratic polynomial equations (3.1).

In fact, this condition is also sufficient. If Z satisfies the above first order linear partial differential equations with constant coefficients (3.3) and 9 quadratic polynomial equations (3.1), we let

$$(u_1,\ u_2,\ u_3,\ p)^T=(e_3,\ e_7,\ e_{11},\ e_{16})^TZ\ ,$$

then from (3.3) we can get $U=HZ+h$, where

$$U=(\frac{\partial u_1}{\partial x_1},\frac{\partial u_1}{\partial t},\frac{\partial u_2}{\partial t},\frac{\partial u_3}{\partial t},\frac{\partial u_1}{\partial x_2},\frac{\partial u_1}{\partial x_3},\ u_1\ ,\frac{\partial u_2}{\partial x_1},\frac{\partial u_2}{\partial x_2},\frac{\partial u_2}{\partial x_3},\ u_2\ ,\frac{\partial u_3}{\partial x_1},\frac{\partial u_3}{\partial x_2},\frac{\partial u_3}{\partial x_3},\ u_3\ ,\ p\ )^T\ ,$$

again from (3.3) we can learn that $\dfrac{\partial U}{\partial t}=H_0Z+h_0$, $\dfrac{\partial U}{\partial x_i}=H_iZ$, $i=1,\ 2,\ 3$, and from (3.1) we can get $X=X_0+A(\eta)Z$, where X includes $u_1,\ u_2,\ u_3,\ p$ and all their first order partial derivative, and all the second order partial derivative of $u_1,\ u_2,\ u_3$, and all the products which they are in the Navier-Stokes equations, then we can get $AX=\beta$, this is just the Navier-Stokes equations.

Hence we can get the corollary as follows.

Corollary 3.1 *If we assume $u_1,\ u_2,\ u_3\in C^2$, $p\in C^1$, $F_1,\ F_2,\ F_3\in C^1$, then a necessary and sufficient condition for the existence of the solution for the Navier-Stokes equations is that $\exists$ 55 dimensional four variables functional vector Z , such that Z satisfies the first order linear partial differential equations with constant coefficients (3.3) and 9 quadratic equations (3.1).*

Under this circumstance, $(u_1,\ u_2,\ u_3,\ p)^T=(e_3,\ e_7,\ e_{11},\ e_{16})^TZ$ is the solution of the Navier-Stokes equations, here e_i is the ith 55 dimensional unit coordinate vector, $1\le i\le 55$.

Proof of theorem2-3. (1) Under the assumption (1.1) and from the theorem 2.2, we can learn that when t is not in $[0\ ,\ T]$, or $(x_1,\ x_2,\ x_3)$ is not in K_1 , $Z\equiv 0$, hence Z can do the Fourier transformation with t , x_1 , x_2 , x_3 , we use the Fourier transformation to convert (2.1) into the linear equations as follows.

$$\begin{pmatrix} i\xi_0 H - H_0 \\ i\xi_1 H - H_1 \\ i\xi_2 H - H_2 \\ i\xi_3 H - H_3 \end{pmatrix} F(Z) = \begin{pmatrix} F(h_0) - i\xi_0 F(h) \\ -i\xi_1 F(h) \\ -i\xi_2 F(h) \\ -i\xi_3 F(h) \end{pmatrix} , \tag{3.4}$$

where

$$F(Z) = \int_{R^4} Z e^{-i\xi_0 t - i\sum_{j=1}^{3}\xi_j x_j} dt dx_1 dx_2 dx_3 \ ,$$
$$F(h) = \int_{R^4} h e^{-i\xi_0 t - i\sum_{j=1}^{3}\xi_j x_j} dt dx_1 dx_2 dx_3 \ ,$$
$$F(h_0) = \int_{R^4} h_0 e^{-i\xi_0 t - i\sum_{j=1}^{3}\xi_j x_j} dt dx_1 dx_2 dx_3 \ .$$

We assume that

$$B = \begin{pmatrix} i\xi_0 H - H_0 \\ i\xi_1 H - H_1 \\ i\xi_2 H - H_2 \\ i\xi_3 H - H_3 \end{pmatrix}_{64\times 55} , \ G = \begin{pmatrix} F(h_0) - i\xi_0 F(h) \\ -i\xi_1 F(h) \\ -i\xi_2 F(h) \\ -i\xi_3 F(h) \end{pmatrix}_{64\times 1} ,$$

next we solve the linear equations $BY = G$, where $Y = (y_j)_{55\times 1}$. We can get $rank(B) = 46$, when $(\xi_1 \ , \ \xi_2 \ , \ \xi_3) \neq 0$, moreover the first 46 columns are linear independent.
We write out all the rows of the matrix B.

$$\begin{aligned}
&-i\xi_0\alpha_1^T - e_{21}^T \ , \ -i\xi_1\alpha_1^T - e_{18}^T \ , \ -i\xi_2\alpha_1^T - e_{24}^T \ , \ -i\xi_3\alpha_1^T - e_{25}^T \ , \\
&-i\xi_0\alpha_2^T - e_{17}^T \ , \ -i\xi_1\alpha_2^T - e_{21}^T \ , \ -i\xi_2\alpha_2^T - e_{22}^T \ , \ -i\xi_3\alpha_2^T - e_{23}^T \ , \\
&-i\xi_0\alpha_3^T - e_{27}^T \ , \ -i\xi_1\alpha_3^T - e_{31}^T \ , \ -i\xi_2\alpha_3^T - e_{32}^T \ , \ -i\xi_3\alpha_3^T - e_{33}^T \ , \\
&-i\xi_0\alpha_4^T - e_{37}^T \ , \ -i\xi_1\alpha_4^T - e_{41}^T \ , \ -i\xi_2\alpha_4^T - e_{42}^T \ , \ -i\xi_3\alpha_4^T - e_{43}^T \ , \\
&i\xi_0 e_1^T - e_{22}^T \ , \ i\xi_1 e_1^T - e_{24}^T \ , \ i\xi_2 e_1^T - e_{19}^T \ , \ i\xi_3 e_1^T - e_{26}^T \ , \\
&i\xi_0 e_2^T - e_{23}^T \ , \ i\xi_1 e_2^T - e_{25}^T \ , \ i\xi_2 e_2^T - e_{26}^T \ , \ i\xi_3 e_2^T - e_{20}^T \ , \\
&i\xi_0 e_3^T + \alpha_2^T \ , \ i\xi_1 e_3^T + \alpha_1^T \ , \ i\xi_2 e_3^T - e_1^T \ , \ i\xi_3 e_3^T - e_2^T \ , \\
&i\xi_0 e_4^T - e_{31}^T \ , \ i\xi_1 e_4^T - e_{28}^T \ , \ i\xi_2 e_4^T - e_{34}^T \ , \ i\xi_3 e_4^T - e_{35}^T \ , \\
&i\xi_0 e_5^T - e_{32}^T \ , \ i\xi_1 e_5^T - e_{34}^T \ , \ i\xi_2 e_5^T - e_{29}^T \ , \ i\xi_3 e_5^T - e_{36}^T \ , \\
&i\xi_0 e_6^T - e_{33}^T \ , \ i\xi_1 e_6^T - e_{35}^T \ , \ i\xi_2 e_6^T - e_{36}^T \ , \ i\xi_3 e_6^T - e_{30}^T \ , \\
&i\xi_0 e_7^T + \alpha_3^T \ , \ i\xi_1 e_7^T - e_4^T \ , \ i\xi_2 e_7^T - e_5^T \ , \ i\xi_3 e_7^T - e_6^T \ , \\
&i\xi_0 e_8^T - e_{41}^T \ , \ i\xi_1 e_8^T - e_{38}^T \ , \ i\xi_2 e_8^T - e_{44}^T \ , \ i\xi_3 e_8^T - e_{45}^T \ , \\
&i\xi_0 e_9^T - e_{42}^T \ , \ i\xi_1 e_9^T - e_{44}^T \ , \ i\xi_2 e_9^T - e_{39}^T \ , \ i\xi_3 e_9^T - e_{46}^T \ , \\
&i\xi_0 e_{10}^T - e_{43}^T \ , \ i\xi_1 e_{10}^T - e_{45}^T \ , \ i\xi_2 e_{10}^T - e_{46}^T \ , \ i\xi_3 e_{10}^T - e_{40}^T \ , \\
&i\xi_0 e_{11}^T + \alpha_4^T \ , \ i\xi_1 e_{11}^T - e_8^T \ , \ i\xi_2 e_{11}^T - e_9^T \ , \ i\xi_3 e_{11}^T - e_{10}^T \ , \\
&i\xi_0 e_{16}^T - e_{12}^T \ , \ i\xi_1 e_{16}^T - e_{13}^T \ , \ i\xi_2 e_{16}^T - e_{14}^T \ , \ i\xi_3 e_{16}^T - e_{15}^T \ ,
\end{aligned}$$

where e_i is the ith 55 dimensional unit coordinate vector, $1 \le i \le 55$, and

$$\alpha_1 = (0_4,\ 1,\ 0_4,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0_{39})^T,$$
$$\alpha_2 = (0_{12},\ \tau,\ 0_4,\ -\mu,\ -\mu,\ -\mu,\ 0_{26},\ 1,\ 1,\ 1,\ 0_6)^T,$$
$$\alpha_3 = (0_{13},\ \tau,\ 0_{13},\ -\mu,\ -\mu,\ -\mu,\ 0_{19},\ 1,\ 1,\ 1,\ 0_3)^T,$$
$$\alpha_4 = (0_{14},\ \tau,\ 0_{22},\ -\mu,\ -\mu,\ -\mu,\ 0_{11},\ 0,\ 1,\ 1,\ 1)^T .$$

First we let $y_{47+j-1} = 0$, $1 \le j \le 9$, we will show that the solution of $BY = 0$ is only 0 , when $(\xi_1,\ \xi_2,\ \xi_3) \ne 0$, hence the first 46 columns are linear independent, and $rank(B) \ge 46$.
From the 7th row we can get $y_1 = i\xi_2 y_3,\ y_2 = i\xi_3 y_3,\ y_5 + y_{10} = -i\xi_1 y_3$, and from the first, the second, the 5th , the 6th rows, we can get $y_{18} = (i\xi_1)^2 y_3,\ y_{19} = (i\xi_2)^2 y_3,\ y_{20} = (i\xi_3)^2 y_3,\ y_{13} = a y_3$, where

$$a = \frac{\mu[(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2] - i\xi_0}{\tau} \ne 0 ,$$

when $(\xi_1,\ \xi_2,\ \xi_3) \ne 0$, moreover $y_{17} = (i\xi_0)^2 y_3$, $y_{21} = i\xi_0 i\xi_1 y_3,\ y_{22} = i\xi_0 i\xi_2 y_3,\ y_{23} = i\xi_0 i\xi_3 y_3,$
$y_{24} = i\xi_1 i\xi_2 y_3,\ y_{25} = i\xi_1 i\xi_3 y_3,\ y_{26} = i\xi_2 i\xi_3 y_3$.
From the 11th row we can get $y_4 = i\xi_1 y_7,\ y_5 = i\xi_2 y_7,\ y_6 = i\xi_3 y_7$, and from the 8th, the third, the 9th , the 10th rows, we can get $y_{28} = (i\xi_1)^2 y_7,\ y_{29} = (i\xi_2)^2 y_7,\ y_{30} = (i\xi_3)^2 y_7,\ y_{14} = a y_7$, moreover $y_{27} = (i\xi_0)^2 y_7$, $y_{31} = i\xi_0 i\xi_1 y_7,\ y_{32} = i\xi_0 i\xi_2 y_7,\ y_{33} = i\xi_0 i\xi_3 y_7,\ y_{34} = i\xi_1 i\xi_2 y_7,\ y_{35} = i\xi_1 i\xi_3 y_7,$
$y_{36} = i\xi_2 i\xi_3 y_7$.
From the 15th row we can get $y_8 = i\xi_1 y_{11},\ y_9 = i\xi_2 y_{11},\ y_{10} = i\xi_3 y_{11}$, and from the 12th, the 13th, the 14th , the 4th rows, we can get $y_{38} = (i\xi_1)^2 y_{11},\ y_{39} = (i\xi_2)^2 y_{11},\ y_{40} = (i\xi_3)^2 y_{11},\ y_{15} = a y_{11}$, moreover $y_{37} = (i\xi_0)^2 y_{11}$, $y_{41} = i\xi_0 i\xi_1 y_{11},\ y_{42} = i\xi_0 i\xi_2 y_{11},\ y_{43} = i\xi_0 i\xi_3 y_{11},\ y_{44} = i\xi_1 i\xi_2 y_{11},$
$y_{45} = i\xi_1 i\xi_3 y_{11},\ y_{46} = i\xi_2 i\xi_3 y_{11}$.
And from the last row, we can get $y_{12} = i\xi_0 y_{16},\ y_{13} = i\xi_1 y_{16},\ y_{14} = i\xi_2 y_{16},\ y_{15} = i\xi_3 y_{16}$.
Because $y_{13} = a y_3,\ y_{14} = a y_7,\ y_{15} = a y_{11}$, we can get

$$y_3 = \frac{i\xi_1}{a} y_{16},\ y_7 = \frac{i\xi_2}{a} y_{16},\ y_{11} = \frac{i\xi_3}{a} y_{16} .$$

From $(y_5 + y_{10}) = y_5 + y_{10}$, we can get $-i\xi_1 y_3 = i\xi_2 y_7 + i\xi_3 y_{11}$. This is equal to the following.

$$[\frac{(i\xi_1)^2}{a} + \frac{(i\xi_2)^2}{a} + \frac{(i\xi_3)^2}{a}] y_{16} = 0 .$$

Hence $y_{16} = 0$, when $(\xi_1,\ \xi_2,\ \xi_3) \ne 0$, and we can get $y_3 = 0,\ y_7 = 0,\ y_{11} = 0$, the solution of $BY = 0$ is only 0 , the first 46 columns are linear independent, and $rank(B) \ge 46$.
Next we will in turn let $y_{47+j-1} = 1,\ y_k = 0,\ 47 \le k \le 55,\ k \ne 47 + j - 1,\ 1 \le j \le 9$, we can work out 9 linear independent solutions of $BY = 0$, we assume they are $\eta_j,\ 1 \le j \le 9$. This means that $rank(B) \le 46$, hence $rank(B) = 46$, when $(\xi_1,\ \xi_2,\ \xi_3) \ne 0$.
If we let $y_{47+j-1} = 1,\ y_k = 0,\ 47 \le k \le 55,\ k \ne 47 + j - 1,\ 1 \le j \le 3$, we only need to notice that $\alpha_2^T Y$ will change into $\alpha_2^T Y + 1$, and $y_{13} = i\xi_1 y_{16} = a y_3 - \frac{1}{\tau}$, then we can get $y_3 = \frac{i\xi_1}{a} y_{16} + \frac{1}{a\tau}$, and $y_7 = \frac{i\xi_2}{a} y_{16}$, $y_{11} = \frac{i\xi_3}{a} y_{16}$. Hence we can get the following,

$$b y_{16} + \frac{i\xi_1}{a\tau} = 0 ,\ \text{or}\ y_{16} = -\frac{i\xi_1}{ab\tau} .$$

Moreover we can work out y_3 , y_7 , y_{11} and η_j , $1 \le j \le 3$.
If we let $y_{47+j-1} = 1,\ y_k = 0,\ 47 \le k \le 55,\ k \ne 47+j-1,\ 4 \le j \le 6$, we only need to notice that $\alpha_3^T Y$ will change into $\alpha_3^T Y + 1$, and $y_{14} = i\xi_2 y_{16} = a y_7 - \dfrac{1}{\tau}$, then we can get $y_7 = \dfrac{i\xi_2}{a} y_{16} + \dfrac{1}{a\tau}$, and $y_3 = \dfrac{i\xi_1}{a} y_{16}$, $y_{11} = \dfrac{i\xi_3}{a} y_{16}$. Hence we can get the following,

$$b y_{16} + \frac{i\xi_2}{a\tau} = 0\ ,\ \text{or}\ y_{16} = -\frac{i\xi_2}{ab\tau}\ .$$

Moreover we can work out y_3 , y_7 , y_{11} and η_j , $4 \le j \le 6$.
If we let $y_{47+j-1} = 1,\ y_k = 0,\ 47 \le k \le 55,\ k \ne 47+j-1,\ 7 \le j \le 9$, we only need to notice that $\alpha_4^T Y$ will change into $\alpha_4^T Y + 1$, and $y_{15} = i\xi_3 y_{16} = a y_{11} - \dfrac{1}{\tau}$, then we can get $y_{11} = \dfrac{i\xi_3}{a} y_{16} + \dfrac{1}{a\tau}$, and $y_3 = \dfrac{i\xi_1}{a} y_{16}$, $y_7 = \dfrac{i\xi_2}{a} y_{16}$. Hence we can get the following,

$$b y_{16} + \frac{i\xi_3}{a\tau} = 0\ ,\ \text{or}\ y_{16} = -\frac{i\xi_3}{ab\tau}\ .$$

Moreover we can work out y_3 , y_7 , y_{11} and η_j , $7 \le j \le 9$.
We write η_j , $1 \le j \le 9$, as follows.

$$\begin{aligned} \eta_j \ =\ & (i\xi_2 y_3,\ i\xi_3 y_3,\ y_3,\ i\xi_1 y_7,\ i\xi_2 y_7,\ i\xi_3 y_7,\ y_7,\ i\xi_1 y_{11},\ i\xi_2 y_{11},\ i\xi_3 y_{11},\ y_{11}, \\ & i\xi_0 y_{16},\ i\xi_1 y_{16},\ i\xi_2 y_{16},\ i\xi_3 y_{16},\ y_{16},\ (i\xi_0)^2 y_3,\ (i\xi_1)^2 y_3,\ (i\xi_2)^2 y_3,\ (i\xi_3)^2 y_3,\ i\xi_0 i\xi_1 y_3, \\ & i\xi_0 i\xi_2 y_3,\ i\xi_0 i\xi_3 y_3,\ i\xi_1 i\xi_2 y_3,\ i\xi_1 i\xi_3 y_3,\ i\xi_2 i\xi_3 y_3,\ (i\xi_0)^2 y_7,\ (i\xi_1)^2 y_7,\ (i\xi_2)^2 y_7,\ (i\xi_3)^2 y_7, \\ & i\xi_0 i\xi_1 y_7,\ i\xi_0 i\xi_2 y_7,\ i\xi_0 i\xi_3 y_7,\ i\xi_1 i\xi_2 y_7,\ i\xi_1 i\xi_3 y_7,\ i\xi_2 i\xi_3 y_7,\ (i\xi_0)^2 y_{11},\ (i\xi_1)^2 y_{11},\ (i\xi_2)^2 y_{11}, \\ & (i\xi_3)^2 y_{11},\ i\xi_0 i\xi_1 y_{11},\ i\xi_0 i\xi_2 y_{11},\ i\xi_0 i\xi_3 y_{11},\ i\xi_1 i\xi_2 y_{11},\ i\xi_1 i\xi_3 y_{11},\ i\xi_2 i\xi_3 y_{11},\ e_j^T)^T\ , \end{aligned}$$

here e_j is the jth 9 dimensional unit coordinate vector, $1 \le j \le 9$, moreover when $j = 1,\ 2,\ 3$,

$$y_3 = \frac{\xi_1^2}{a^2 b\tau} + \frac{1}{a\tau}\ ,\ y_7 = \frac{-\ i\xi_1 i\xi_2}{a^2 b\tau}\ ,\ y_{11} = \frac{-\ i\xi_1 i\xi_3}{a^2 b\tau}\ ,\ y_{16} = \frac{-\ i\xi_1}{ab\tau}\ ,$$

where $b = \dfrac{(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2}{a} \ne 0$, when $(\xi_1,\ \xi_2,\ \xi_3) \ne 0$, when $j = 4,\ 5,\ 6$,

$$y_7 = \frac{\xi_2^2}{a^2 b\tau} + \frac{1}{a\tau}\ ,\ y_3 = \frac{-\ i\xi_1 i\xi_2}{a^2 b\tau}\ ,\ y_{11} = \frac{-\ i\xi_2 i\xi_3}{a^2 b\tau}\ ,\ y_{16} = \frac{-\ i\xi_2}{ab\tau}\ ,$$

when $j = 7,\ 8,\ 9$,

$$y_{11} = \frac{\xi_3^2}{a^2 b\tau} + \frac{1}{a\tau}\ ,\ y_3 = \frac{-\ i\xi_1 i\xi_3}{a^2 b\tau}\ ,\ y_7 = \frac{-\ i\xi_2 i\xi_3}{a^2 b\tau}\ ,\ y_{16} = \frac{-\ i\xi_3}{ab\tau}\ .$$

Finally we work out a particular solution Y_1 of $BY = G$. We can see that

$$\begin{aligned} G \ =\ & (0,\ -i\xi_0 F(F_1),\ -i\xi_0 F(F_2),\ -i\xi_0 F(F_3),\ 0_2,\ F(F_1),\ 0_3,\ F(F_2),\ 0_3,\ F(F_3),\ 0,\ 0, \\ & i\xi_1 F(F_1),\ -i\xi_1 F(F_2),\ -i\xi_1 F(F_3),\ 0_{12},\ 0,\ i\xi_2 F(F_1),\ -i\xi_2 F(F_2),\ -i\xi_2 F(F_3),\ 0_{12}, \\ & 0,\ i\xi_3 F(F_1),\ -i\xi_3 F(F_2),\ -i\xi_3 F(F_3),\ 0_{12})^T\ , \end{aligned}$$

If we let $y_{47+j-1}=0,\ 1\le j\le 9$, then we can get $y_{13}=i\xi_1 y_{16}=ay_3+\dfrac{F(F_1)}{\tau}$, $y_{14}=i\xi_2 y_{16}=ay_7+\dfrac{F(F_2)}{\tau}$, and $y_{15}=i\xi_3 y_{16}=ay_{11}+\dfrac{F(F_3)}{\tau}$. Hence we can get the following,

$$by_{16}=\frac{i\xi_1 F(F_1)+i\xi_2 F(F_2)+i\xi_3 F(F_3)}{a\tau}\ ,\ \text{or}\ y_{16}=\frac{i\xi_1 F(F_1)+i\xi_2 F(F_2)+i\xi_3 F(F_3)}{ab\tau}\ .$$

Moreover we can work out y_3 , y_7 , y_{11} and Y_1 .
We write Y_1 as follows.

$$\begin{aligned} Y_1 \ =\ & (i\xi_2 y_3,\ i\xi_3 y_3,\ y_3,\ i\xi_1 y_7,\ i\xi_2 y_7,\ i\xi_3 y_7,\ y_7,\ i\xi_1 y_{11},\ i\xi_2 y_{11},\ i\xi_3 y_{11},\ y_{11},\\ & i\xi_0 y_{16},\ i\xi_1 y_{16},\ i\xi_2 y_{16},\ i\xi_3 y_{16},\ y_{16},\ (i\xi_0)^2 y_3,\ (i\xi_1)^2 y_3,\ (i\xi_2)^2 y_3,\ (i\xi_3)^2 y_3,\ i\xi_0 i\xi_1 y_3,\\ & i\xi_0 i\xi_2 y_3,\ i\xi_0 i\xi_3 y_3,\ i\xi_1 i\xi_2 y_3,\ i\xi_1 i\xi_3 y_3,\ i\xi_2 i\xi_3 y_3,\ (i\xi_0)^2 y_7,\ (i\xi_1)^2 y_7,\ (i\xi_2)^2 y_7,\ (i\xi_3)^2 y_7,\\ & i\xi_0 i\xi_1 y_7,\ i\xi_0 i\xi_2 y_7,\ i\xi_0 i\xi_3 y_7,\ i\xi_1 i\xi_2 y_7,\ i\xi_1 i\xi_3 y_7,\ i\xi_2 i\xi_3 y_7,\ (i\xi_0)^2 y_{11},\ (i\xi_1)^2 y_{11},\ (i\xi_2)^2 y_{11},\\ & (i\xi_3)^2 y_{11},\ i\xi_0 i\xi_1 y_{11},\ i\xi_0 i\xi_2 y_{11},\ i\xi_0 i\xi_3 y_{11},\ i\xi_1 i\xi_2 y_{11},\ i\xi_1 i\xi_3 y_{11},\ i\xi_2 i\xi_3 y_{11},\ 0_{9\times 1}^T)^T\ , \end{aligned}$$

where

$$\begin{aligned} y_3 &= \frac{i\xi_1(i\xi_1 F(F_1)+i\xi_2 F(F_2)+i\xi_3 F(F_3))}{a^2 b\tau}-\frac{F(F_1)}{a\tau}\ ,\\ y_7 &= \frac{i\xi_2(i\xi_1 F(F_1)+i\xi_2 F(F_2)+i\xi_3 F(F_3))}{a^2 b\tau}-\frac{F(F_2)}{a\tau}\ ,\\ y_{11} &= \frac{i\xi_3(i\xi_1 F(F_1)+i\xi_2 F(F_2)+i\xi_3 F(F_3))}{a^2 b\tau}-\frac{F(F_3)}{a\tau}\ ,\\ y_{16} &= \frac{i\xi_1 F(F_1)+i\xi_2 F(F_2)+i\xi_3 F(F_3)}{ab\tau}\ . \end{aligned}$$

And we can test such Y_1 satisfies $BY_1=G$.
If we assume $A_1(\eta)=(\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5,\ \eta_6,\ \eta_7,\ \eta_8,\ \eta_9)$, then we can get the explicit general solutions of (3.4) as follows,

$$F(Z)=Y_1+A_1(\eta)Z_1\ ,\text{or}\ Z=F^{-1}(Y_1+A_1(\eta)Z_1)\ ,$$

where $Z_1=(Z_{1j})_{9\times 1}$, moreover we assume

$$F^{-1}(Y_1+A_1(\eta)Z_1 I_{\{(\xi_1,\ \xi_2,\ \xi_3)\neq 0\}})=F^{-1}(Y_1+A_1(\eta)Z_1)\ ,$$

and $Z_1\in\ \Omega_1$. Because Z need to satisfy

$$HZ=HZI_{K_1'}\in C^1(K_1')\ ,$$

we can get

$$\begin{aligned} &\Omega_1=\{Z_1|H[F^{-1}(Y_1+A_1(\eta)Z_1)]=H[F^{-1}(Y_1+A_1(\eta)Z_1)]I_{K_1'}\in C^1(K_1').\}\ ,\text{where}\\ &H=(-\alpha_1,\ -\alpha_2,\ -\alpha_3,\ -\alpha_4,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7,\ e_8,\ e_9,\ e_{10},\ e_{11},\ e_{16})^T\ . \end{aligned}$$

(2)We can learn from (1) that $F(e_{47+j-1}^T Z) = Z_{1j}$, $1 \leq j \leq 9$, hence we can convert (2.2) into the integral equations as follows,

$$\begin{cases} Z_{11} = F[F^{-1}(e_3^T(Y_1 + A_1(\eta)Z_1))F^{-1}(-\alpha_1^T(Y_1 + A_1(\eta)Z_1))] = f_1(Z_1) \ , \\ Z_{12} = F[F^{-1}(e_7^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_1^T(Y_1 + A_1(\eta)Z_1))] = f_2(Z_1) \ , \\ Z_{13} = F[F^{-1}(e_{11}^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_2^T(Y_1 + A_1(\eta)Z_1))] = f_3(Z_1) \ , \\ Z_{14} = F[F^{-1}(e_3^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_4^T(Y_1 + A_1(\eta)Z_1))] = f_4(Z_1) \ , \\ Z_{15} = F[F^{-1}(e_7^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_5^T(Y_1 + A_1(\eta)Z_1))] = f_5(Z_1) \ , \\ Z_{16} = F[F^{-1}(e_{11}^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_6^T(Y_1 + A_1(\eta)Z_1))] = f_6(Z_1) \ , \\ Z_{17} = F[F^{-1}(e_3^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_8^T(Y_1 + A_1(\eta)Z_1))] = f_7(Z_1) \ , \\ Z_{18} = F[F^{-1}(e_7^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_9^T(Y_1 + A_1(\eta)Z_1))] = f_8(Z_1) \ , \\ Z_{19} = F[F^{-1}(e_{11}^T(Y_1 + A_1(\eta)Z_1))F^{-1}(e_{10}^T(Y_1 + A_1(\eta)Z_1))] = f_9(Z_1) \ . \end{cases} \tag{3.5}$$

This is also the question to find the fixed-points of $f(Z_1)$, where $f(Z_1) = (f_j(Z_1))_{9\times 1}$. But we can not see immediately $f(Z_1) \in \Omega_1$, even if $Z_1 \in \Omega_1$. We need to attain this point first. We assume Ω_2 as follows.

$$\Omega_2 = \{Z_1 | \exists\ h_1 = h_1 I_{K_1'} \in C^1(K_1') \ , \text{ such that } Z_1 = F(h_1)\ .\} \ ,$$

where

$$h_1 = (h_{1j})_{9\times 1} \ , \ F(h_1) = \int_{R^4} h_1 e^{-i\xi_0 t - i\sum_{j=1}^3 \xi_j x_j} dt dx_1 dx_2 dx_3 \ .$$

We will prove that $\Omega_2 \subset \Omega_1$ and $f(Z_1) \in \Omega_2$, if $Z_1 \in \Omega_2$. We look at a lemma as follows.

Lemma 3.1 *$\forall\ h_1 = h_1 I_{K_1'} \in\ C(K_1')$, and h_1 satisfies the Hölder condition, $\exists\ h_2 = h_2 I_{K_1'} \in C^1[0,\ T] \bigcap C^\infty(K_1)$, such that $F(h_1) = a^2 b\tau F(h_2)$, where*

$$a = \frac{\mu[(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2] - i\xi_0}{\tau} \ , \ b = \frac{(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2}{a} \ ,$$

and h_1 satisfies the Hölder condition means that $\exists\ C > 0$,$0 < \alpha < 1$, such that

$$|h_1(x_t) - h_1(x_t')| = \max_{1\leq j\leq 9} |h_{1j}(x_t) - h_{1j}(x_t')| \leq C|x_t - x_t'|^\alpha \ , \ \forall\ x_t \ , \ x_t' \ \in\ K_1' \ .$$

Proof of lemma3-1. We see that $a^2 b\tau = (\mu[(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2] - i\xi_0)((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)$, hence this lemma is equivalent to $\forall\ h_1 = h_1 I_{K_1'} \in\ C(K_1')$, and h_1 satisfies the Hölder condition, $\exists\ h_2 = h_2 I_{K_1'} \in C^1[0,\ T] \bigcap C^\infty(K_1)$, such that

$$\mu \sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial^4 h_2}{\partial x_j^2 \partial x_k^2} - \sum_{j=1}^3 \frac{\partial^3 h_2}{\partial t \partial x_j^2} = h_1 \ , \ a.e. \tag{3.6}$$

If we let $v = \mu \triangle h_2 - \partial h_2/\partial t$, then we convert (3.6) into follows,

$$\begin{cases} \triangle v = h_1 \ , \\ \mu \triangle h_2 - \dfrac{\partial h_2}{\partial t} = v \ . \end{cases} \tag{3.7}$$

We see that (3.7) are the simultaneous of Poisson's equation and the heat-conduct equation, they are all classical mathematical-physics equations and h_1 satisfies the Hölder condition, the boundary of K_1 , ∂K_1 satisfies the exterior ball condition, we know their solutions are all exist. We can write h_2 as follows.

$$v(t, M_0) = -\frac{1}{4\pi}\int_{K_1}\frac{h_1(t,M)}{r_{MM_0}}dx_1dx_2dx_3\ ,\ \text{where } M=(x_1,x_2,x_3)\ ,\ M_0=(x_{10},x_{20},x_{30})\ ,$$
$$r_{MM_0}=\sqrt{(x_1-x_{10})^2+(x_2-x_{20})^2+(x_3-x_{30})^2}\ ,$$
$$h_2(t,M)=(\frac{1}{2\sqrt{\pi\mu}})^3\int_0^t\int_{R^3}\frac{v(\tau_1,y_1,y_2,y_3)}{(\sqrt{t-\tau_1})^3}\ e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}}dy_1dy_2dy_3d\tau_1\ .$$

We assume the measure of the boundary of K_1 is 0 , then we can get that h_2 satisfies (3.6) . Next we will prove $h_2=h_2I_{K_1'}\in C^1[0,\ T]\bigcap C^\infty(K_1)$.
First we can see v is continuous on the region K_1' , and from

$$\frac{\partial^k h_2(t,M)}{\partial x_1^{a_1}\partial x_2^{a_2}\partial x_3^{a_3}}=(\frac{1}{2\sqrt{\pi\mu}})^3\int_0^t\int_{R^3}\frac{v(\tau_1,y_1,y_2,y_3)}{(\sqrt{t-\tau_1})^3}\ \frac{\partial^k e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}}}{\partial x_1^{a_1}\partial x_2^{a_2}\partial x_3^{a_3}}dy_1dy_2dy_3d\tau_1\ ,$$

where $k=a_1+a_2+a_3$, a_1 , a_2 , a_3 are all nonnegative integral numbers, we know $h_2=h_2I_{K_1'}\in C^\infty(K_1)$. And because

$$\frac{\partial h_2}{\partial t}=\mu\triangle h_2-v\ ,$$

we know $h_2=h_2I_{K_1'}\in C^1[0,\ T]\bigcap C^\infty(K_1)$.
We know h_1 satisfies the Hölder condition if $h_1=h_1I_{K_1'}\in C^1(K_1')$, hence we can get the corollary as follows.

Corollary 3.2 *$\forall\ h_1=h_1I_{K_1'}\in C^1(K_1')$, $\exists\ h_2=h_2I_{K_1'}\in C^2[0,\ T]\bigcap C^\infty(K_1)$, such that $F(h_1)=a^2b\tau F(h_2)$, where*

$$a=\frac{\mu[(i\xi_1)^2+(i\xi_2)^2+(i\xi_3)^2]-i\xi_0}{\tau}\ ,\ b=\frac{(i\xi_1)^2+(i\xi_2)^2+(i\xi_3)^2}{a}\ .$$

From the corollary 3.2 and the assumption 1.1 , we can get $\exists\ F_{j1}=F_{j1}I_{K_1'}\in C^2[0,\ T]\bigcap C^\infty(K_1)$, such that $F(F_j)=a^2b\tau F(F_{j1})$, $1\le j\le 3$. And we can get the following.

$$F_{j1}(t,M)=(\frac{1}{2\sqrt{\pi\mu}})^3\int_0^t\int_{R^3}\frac{F_{jv}(\tau_1,y_1,y_2,y_3)}{(\sqrt{t-\tau_1})^3}\ e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}}dy_1dy_2dy_3d\tau_1\ ,$$
$$F_{jv}(t,M_0)=-\frac{1}{4\pi}\int_{K_1}\frac{F_j(t,M)}{r_{MM_0}}dx_1dx_2dx_3\ ,\ 1\le\ j\le 3\ .$$

Next we see $H[F^{-1}(Y_1)]$, where

$$H=(-\alpha_1,\ -\alpha_2,\ -\alpha_3,\ -\alpha_4,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7,\ e_8,\ e_9,\ e_{10},\ e_{11},\ e_{16})^T\ ,$$
$$\alpha_1=(0_4,\ 1,\ 0_4,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0_{39})^T,$$
$$\alpha_2=(0_{12},\ \tau,\ 0_4,\ -\mu,\ -\mu,\ -\mu,\ 0_{26},\ 1,\ 1,\ 1,\ 0_6)^T,$$
$$\alpha_3=(0_{13},\ \tau,\ 0_{13},\ -\mu,\ -\mu,\ -\mu,\ 0_{19},\ 1,\ 1,\ 1,\ 0_3)^T,$$
$$\alpha_4=(0_{14},\ \tau,\ 0_{22},\ -\mu,\ -\mu,\ -\mu,\ 0_{11},\ 0,\ 1,\ 1,\ 1)^T\ ,$$

and e_i is the ith 55 dimensional unit coordinate vector, $1 \le i \le 55$,

$$\begin{aligned} Y_1 \;=\; & (i\xi_2 y_3,\ i\xi_3 y_3,\ y_3,\ i\xi_1 y_7,\ i\xi_2 y_7,\ i\xi_3 y_7,\ y_7,\ i\xi_1 y_{11},\ i\xi_2 y_{11},\ i\xi_3 y_{11},\ y_{11}, \\ & i\xi_0 y_{16},\ i\xi_1 y_{16},\ i\xi_2 y_{16},\ i\xi_3 y_{16},\ y_{16},\ (i\xi_0)^2 y_3,\ (i\xi_1)^2 y_3,\ (i\xi_2)^2 y_3,\ (i\xi_3)^2 y_3,\ i\xi_0 i\xi_1 y_3, \\ & i\xi_0 i\xi_2 y_3,\ i\xi_0 i\xi_3 y_3,\ i\xi_1 i\xi_2 y_3,\ i\xi_1 i\xi_3 y_3,\ i\xi_2 i\xi_3 y_3,\ (i\xi_0)^2 y_7,\ (i\xi_1)^2 y_7,\ (i\xi_2)^2 y_7,\ (i\xi_3)^2 y_7, \\ & i\xi_0 i\xi_1 y_7,\ i\xi_0 i\xi_2 y_7,\ i\xi_0 i\xi_3 y_7,\ i\xi_1 i\xi_2 y_7,\ i\xi_1 i\xi_3 y_7,\ i\xi_2 i\xi_3 y_7,\ (i\xi_0)^2 y_{11},\ (i\xi_1)^2 y_{11},\ (i\xi_2)^2 y_{11}, \\ & (i\xi_3)^2 y_{11},\ i\xi_0 i\xi_1 y_{11},\ i\xi_0 i\xi_2 y_{11},\ i\xi_0 i\xi_3 y_{11},\ i\xi_1 i\xi_2 y_{11},\ i\xi_1 i\xi_3 y_{11},\ i\xi_2 i\xi_3 y_{11},\ 0^T_{9\times 1})^T\ , \end{aligned}$$

and

$$\begin{aligned} y_3 &= \frac{i\xi_1(i\xi_1 F(F_1) + i\xi_2 F(F_2) + i\xi_3 F(F_3))}{a^2 b\tau} - \frac{F(F_1)}{a\tau} \\ &= i\xi_1(i\xi_1 F(F_{11}) + i\xi_2 F(F_{21}) + i\xi_3 F(F_{31})) - abF(F_{11})\ , \\ y_7 &= \frac{i\xi_2(i\xi_1 F(F_1) + i\xi_2 F(F_2) + i\xi_3 F(F_3))}{a^2 b\tau} - \frac{F(F_2)}{a\tau} \\ &= i\xi_2(i\xi_1 F(F_{11}) + i\xi_2 F(F_{21}) + i\xi_3 F(F_{31})) - abF(F_{21})\ , \\ y_{11} &= \frac{i\xi_3(i\xi_1 F(F_1) + i\xi_2 F(F_2) + i\xi_3 F(F_3))}{a^2 b\tau} - \frac{F(F_3)}{a\tau} \\ &= i\xi_3(i\xi_1 F(F_{11}) + i\xi_2 F(F_{21}) + i\xi_3 F(F_{31})) - abF(F_{31})\ , \\ y_{16} &= \frac{i\xi_1 F(F_1) + i\xi_2 F(F_2) + i\xi_3 F(F_3)}{ab\tau} \\ &= a(i\xi_1 F(F_{11}) + i\xi_2 F(F_{21}) + i\xi_3 F(F_{31}))\ . \end{aligned}$$

We assume $H[F^{-1}(Y_1)] = W_1(F_{11},\ F_{21},\ F_{31})$, where $W_1(F_{11},\ F_{21},\ F_{31}) = (w_{1j})_{16\times 1}$, and we can get the following.

$$\begin{aligned} & w_{11} = -(e_5^T + e_{10}^T)F^{-1}(Y_1) = -F^{-1}(i\xi_2 y_7 + i\xi_3 y_{11})\ , \\ & w_{12} = -[\tau e_{13}^T - \mu(e_{18}^T + e_{19}^T + e_{20}^T) + e_{47}^T + e_{48}^T + e_{49}^T]F^{-1}(Y_1) \\ & = -F^{-1}\{\tau i\xi_1 y_{16} - \mu[(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2]y_3\}\ , \\ & w_{13} = -[\tau e_{14}^T - \mu(e_{28}^T + e_{29}^T + e_{30}^T) + e_{50}^T + e_{51}^T + e_{52}^T]F^{-1}(Y_1) \\ & = -F^{-1}\{\tau i\xi_2 y_{16} - \mu[(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2]y_7\}\ , \\ & w_{14} = -[\tau e_{15}^T - \mu(e_{38}^T + e_{39}^T + e_{40}^T) + e_{53}^T + e_{54}^T + e_{55}^T]F^{-1}(Y_1) \\ & = -F^{-1}\{\tau i\xi_1 y_{16} - \mu[(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2]y_{11}\}\ , \\ & w_{15} = e_1^T F^{-1}(Y_1) = F^{-1}(i\xi_2 y_3)\ ,\ w_{16} = e_2^T F^{-1}(Y_1) = F^{-1}(i\xi_3 y_3)\ , \\ & w_{17} = e_3^T F^{-1}(Y_1) = F^{-1}(y_3)\ ,\ w_{18} = e_4^T F^{-1}(Y_1) = F^{-1}(i\xi_1 y_7)\ , \\ & w_{19} = e_5^T F^{-1}(Y_1) = F^{-1}(i\xi_2 y_7)\ ,\ w_{110} = e_6^T F^{-1}(Y_1) = F^{-1}(i\xi_3 y_7)\ , \\ & w_{111} = e_7^T F^{-1}(Y_1) = F^{-1}(y_7)\ ,\ w_{112} = e_8^T F^{-1}(Y_1) = F^{-1}(i\xi_1 y_{11})\ , \\ & w_{113} = e_9^T F^{-1}(Y_1) = F^{-1}(i\xi_2 y_{11})\ ,\ w_{114} = e_{10}^T F^{-1}(Y_1) = F^{-1}(i\xi_3 y_{11})\ , \\ & w_{115} = e_{11}^T F^{-1}(Y_1) = F^{-1}(y_{11})\ ,\ w_{116} = e_{16}^T F^{-1}(Y_1) = F^{-1}(y_{16})\ . \end{aligned}$$

Because $F_{j1} = F_{j1} I_{K_1'} \in C^2[0,\ T]\bigcap C^\infty(K_1)$, $1 \le j \le 3$, we can get as follows,

$$F^{-1}(i\xi_0 F(F_{j1})) = F^{-1}(F(\frac{\partial F_{j1}}{\partial t})) = \frac{\partial F_{j1}}{\partial t}\ ,\ 1 \le j \le 3\ ,$$

for the same reason we can get the following,

$$F^{-1}(i\xi_1 F(F_{j1})) = \frac{\partial F_{j1}}{\partial x_1}\ ,\ F^{-1}(i\xi_2 F(F_{j1})) = \frac{\partial F_{j1}}{\partial x_2}\ ,\ F^{-1}(i\xi_3 F(F_{j1})) = \frac{\partial F_{j1}}{\partial x_3}\ ,\ 1 \le j \le 3\ .$$

Hence we can get the following,

$$\begin{aligned}
F^{-1}(y_3) &= \frac{\partial^2 F_{21}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_1 \partial x_3} - \frac{\partial^2 F_{11}}{\partial x_2^2} - \frac{\partial^2 F_{11}}{\partial x_3^2}\ ,\\
F^{-1}(y_7) &= \frac{\partial^2 F_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{21}}{\partial x_1^2} - \frac{\partial^2 F_{21}}{\partial x_3^2}\ ,\\
F^{-1}(y_{11}) &= \frac{\partial^2 F_{11}}{\partial x_1 \partial x_3} + \frac{\partial^2 F_{21}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{31}}{\partial x_1^2} - \frac{\partial^2 F_{31}}{\partial x_2^2}\ ,\\
F^{-1}(y_{16}) &= \frac{1}{\tau}\ [\ \mu\triangle(\frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{21}}{\partial x_2} + \frac{\partial F_{31}}{\partial x_3}) - \frac{\partial^2 F_{11}}{\partial t \partial x_1} - \frac{\partial^2 F_{21}}{\partial t \partial x_2} - \frac{\partial^2 F_{31}}{\partial t \partial x_3}\ ]\ .
\end{aligned}$$

And we can get the following,

$$w_{11} = -\frac{\partial F^{-1}(y_7)}{\partial x_2} - \frac{\partial F^{-1}(y_{11})}{\partial x_3} = \frac{\partial^3 F_{21}}{\partial x_1^2 \partial x_2} + \frac{\partial^3 F_{31}}{\partial x_1^2 \partial x_3} - \frac{\partial^3 F_{11}}{\partial x_1 \partial x_2^2} - \frac{\partial^3 F_{11}}{\partial x_1 \partial x_3^2}\ ,$$

$$w_{12} = -\frac{\partial \tau F^{-1}(y_{16})}{\partial x_1} + \mu\triangle F^{-1}(y_3)\ ,\ w_{13} = -\frac{\partial \tau F^{-1}(y_{16})}{\partial x_2} + \mu\triangle F^{-1}(y_7)\ ,$$

$$w_{14} = -\frac{\partial \tau F^{-1}(y_{16})}{\partial x_3} + \mu\triangle F^{-1}(y_{11})\ ,$$

$$w_{15} = \frac{\partial F^{-1}(y_3)}{\partial x_2} = \frac{\partial^3 F_{21}}{\partial x_1 \partial x_2^2} + \frac{\partial^3 F_{31}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 F_{11}}{\partial x_2^3} - \frac{\partial^3 F_{11}}{\partial x_2 \partial x_3^2}\ ,$$

$$w_{16} = \frac{\partial F^{-1}(y_3)}{\partial x_3} = \frac{\partial^3 F_{21}}{\partial x_1 \partial x_2 \partial x_3} + \frac{\partial^3 F_{31}}{\partial x_1 \partial x_3^2} - \frac{\partial^3 F_{11}}{\partial x_2^2 \partial x_3} - \frac{\partial^3 F_{11}}{\partial x_3^3}\ ,$$

$$w_{17} = F^{-1}(y_3) = \frac{\partial^2 F_{21}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_1 \partial x_3} - \frac{\partial^2 F_{11}}{\partial x_2^2} - \frac{\partial^2 F_{11}}{\partial x_3^2}\ ,$$

$$w_{18} = \frac{\partial F^{-1}(y_7)}{\partial x_1} = \frac{\partial^3 F_{11}}{\partial x_1^2 \partial x_2} + \frac{\partial^3 F_{31}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 F_{21}}{\partial x_1^3} - \frac{\partial^3 F_{21}}{\partial x_1 \partial x_3^2}\ ,$$

$$w_{19} = \frac{\partial F^{-1}(y_7)}{\partial x_2} = \frac{\partial^3 F_{11}}{\partial x_1 \partial x_2^2} + \frac{\partial^3 F_{31}}{\partial x_2^2 \partial x_3} - \frac{\partial^3 F_{21}}{\partial x_1^2 \partial x_2} - \frac{\partial^3 F_{21}}{\partial x_2 \partial x_3^2}\ ,$$

$$w_{110} = \frac{\partial F^{-1}(y_7)}{\partial x_3} = \frac{\partial^3 F_{11}}{\partial x_1 \partial x_2 \partial x_3} + \frac{\partial^3 F_{31}}{\partial x_2 \partial x_3^2} - \frac{\partial^3 F_{21}}{\partial x_1^2 \partial x_3} - \frac{\partial^3 F_{21}}{\partial x_3^3}\ ,$$

$$w_{111} = F^{-1}(y_7) = \frac{\partial^2 F_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{21}}{\partial x_1^2} - \frac{\partial^2 F_{21}}{\partial x_3^2}\ ,$$

$$w_{112} = \frac{\partial F^{-1}(y_{11})}{\partial x_1} = \frac{\partial^3 F_{11}}{\partial x_1^2 \partial x_3} + \frac{\partial^3 F_{21}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 F_{31}}{\partial x_1^3} - \frac{\partial^3 F_{31}}{\partial x_1 \partial x_2^2}\ ,$$

$$w_{113} = \frac{\partial F^{-1}(y_{11})}{\partial x_2} = \frac{\partial^3 F_{11}}{\partial x_1 \partial x_2 \partial x_3} + \frac{\partial^3 F_{21}}{\partial x_2^2 \partial x_3} - \frac{\partial^3 F_{31}}{\partial x_1^2 \partial x_2} - \frac{\partial^3 F_{31}}{\partial x_2^3}\ ,$$

$$w_{114} = \frac{\partial F^{-1}(y_{11})}{\partial x_3} = \frac{\partial^3 F_{11}}{\partial x_1 \partial x_3^2} + \frac{\partial^3 F_{21}}{\partial x_2 \partial x_3^2} - \frac{\partial^3 F_{31}}{\partial x_1^2 \partial x_3} - \frac{\partial^3 F_{31}}{\partial x_2^2 \partial x_3}\ ,$$

$$w_{115} = F^{-1}(y_{11}) = \frac{\partial^2 F_{11}}{\partial x_1 \partial x_3} + \frac{\partial^2 F_{21}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{31}}{\partial x_1^2} - \frac{\partial^2 F_{31}}{\partial x_2^2} ,$$

$$w_{116} = F^{-1}(y_{16}) = \frac{1}{\tau} [\mu \triangle (\frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{21}}{\partial x_2} + \frac{\partial F_{31}}{\partial x_3}) - \frac{\partial^2 F_{11}}{\partial t \partial x_1} - \frac{\partial^2 F_{21}}{\partial t \partial x_2} - \frac{\partial^2 F_{31}}{\partial t \partial x_3}] .$$

And we can see that W_1 is the function to do the partial derivation with $F_{11},\ F_{21},\ F_{31}$ no more than the fourth order and their linear combination, moreover no more than the first order with the variable t . Hence $H[F^{-1}(Y_1)] = H[F^{-1}(Y_1)]I_{K_1'} \in C^1(K_1')$.

If $Z_1 \in \Omega_2$, then there exists $h_1 = h_1 I_{K_1'} \in C^1(K_1')$, such that $Z_1 = F(h_1)$. Again from the corollary 3.2 , we can get $\exists\ h_2 = h_2 I_{K_1'} \in C^2[0,\ T] \bigcap C^\infty(K_1)$, such that $F(h_1) = a^2 b\tau F(h_2)$. Next we see $H[F^{-1}(A_1(\eta) Z_1)]$, where $A_1(\eta) = (\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5,\ \eta_6,\ \eta_7,\ \eta_8,\ \eta_9)$,

$$\begin{aligned} \eta_j \ = \ & (i\xi_2 y_3,\ i\xi_3 y_3,\ y_3,\ i\xi_1 y_7,\ i\xi_2 y_7,\ i\xi_3 y_7,\ y_7,\ i\xi_1 y_{11},\ i\xi_2 y_{11},\ i\xi_3 y_{11},\ y_{11}, \\ & i\xi_0 y_{16},\ i\xi_1 y_{16},\ i\xi_2 y_{16},\ i\xi_3 y_{16},\ y_{16},\ (i\xi_0)^2 y_3,\ (i\xi_1)^2 y_3,\ (i\xi_2)^2 y_3,\ (i\xi_3)^2 y_3,\ i\xi_0 i\xi_1 y_3, \\ & i\xi_0 i\xi_2 y_3,\ i\xi_0 i\xi_3 y_3,\ i\xi_1 i\xi_2 y_3,\ i\xi_1 i\xi_3 y_3,\ i\xi_2 i\xi_3 y_3,\ (i\xi_0)^2 y_7,\ (i\xi_1)^2 y_7,\ (i\xi_2)^2 y_7,\ (i\xi_3)^2 y_7, \\ & i\xi_0 i\xi_1 y_7,\ i\xi_0 i\xi_2 y_7,\ i\xi_0 i\xi_3 y_7,\ i\xi_1 i\xi_2 y_7,\ i\xi_1 i\xi_3 y_7,\ i\xi_2 i\xi_3 y_7,\ (i\xi_0)^2 y_{11},\ (i\xi_1)^2 y_{11},\ (i\xi_2)^2 y_{11}, \\ & (i\xi_3)^2 y_{11},\ i\xi_0 i\xi_1 y_{11},\ i\xi_0 i\xi_2 y_{11},\ i\xi_0 i\xi_3 y_{11},\ i\xi_1 i\xi_2 y_{11},\ i\xi_1 i\xi_3 y_{11},\ i\xi_2 i\xi_3 y_{11},\ e_j^T)^T , \end{aligned}$$

here e_j is the jth 9 dimensional unit coordinate vector, $1 \le j \le 9$, moreover
when $j = 1,\ 2,\ 3$,

$$y_3 = \frac{1}{a^2 b\tau}(\xi_1^2 + ab) ,\ y_7 = \frac{1}{a^2 b\tau}(-i\xi_1 i\xi_2) ,\ y_{11} = \frac{1}{a^2 b\tau}(-i\xi_1 i\xi_3) ,\ y_{16} = \frac{1}{a^2 b\tau}(-i\xi_1 a) ,$$

where $b = \dfrac{(i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2}{a} \neq 0$, and $(\xi_1,\ \xi_2,\ \xi_3) \neq (0\ ,\ 0\ ,\ 0)$,
when $j = 4,\ 5,\ 6$,

$$y_7 = \frac{1}{a^2 b\tau}(\xi_2^2 + ab) ,\ y_3 = \frac{1}{a^2 b\tau}(-i\xi_1 i\xi_2) ,\ y_{11} = \frac{1}{a^2 b\tau}(-i\xi_2 i\xi_3) ,\ y_{16} = \frac{1}{a^2 b\tau}(-i\xi_2 a) ,$$

when $j = 7,\ 8,\ 9$,

$$y_{11} = \frac{1}{a^2 b\tau}(\xi_3^2 + ab) ,\ y_3 = \frac{1}{a^2 b\tau}(-i\xi_1 i\xi_3) ,\ y_7 = \frac{1}{a^2 b\tau}(-i\xi_2 i\xi_3) ,\ y_{16} = \frac{1}{a^2 b\tau}(-i\xi_3 a) .$$

Hence we can get

$$\begin{aligned} & H[F^{-1}(A_1(\eta) Z_1)] = H[F^{-1}(A_1(\eta) F(h_1))] \\ & = H[F^{-1}(A_1(\eta) a^2 b\tau F(h_2))] = H\{F^{-1}[(a^2 b\tau A_1(\eta)) F(h_2)]\} , \end{aligned}$$

and we can see that $a^2 b\tau A_1(\eta)$ is a polynomial matrix. We assume

$$H\{F^{-1}[(a^2 b\tau A_1(\eta)) F(h_2)]\} = W_2(h_2) ,$$

where $W_2(h_2) = (w_{2j})_{16\times 1}$, and we assume $h_2 = (h_{2j})_{9\times 1}$, then we can get the following.

$$\begin{aligned} w_{21} \ = \ & -(e_5^T + e_{10}^T) F^{-1}[(a^2 b\tau A_1(\eta)) F(h_2)] \\ = \ & -F^{-1}\{i\xi_2[-i\xi_1 i\xi_2 (F(h_{21}) + F(h_{22}) + F(h_{23})) + (\xi_2^2 + ab)(F(h_{24}) + F(h_{25}) + F(h_{26})) \\ & -i\xi_1 i\xi_2 (F(h_{27}) + F(h_{28}) + F(h_{29}))] + i\xi_3[-i\xi_1 i\xi_3 (F(h_{21}) + F(h_{22}) + F(h_{23})) \\ & -i\xi_2 i\xi_3 (F(h_{24}) + F(h_{25}) + F(h_{26})) + (\xi_3^2 + ab)(F(h_{27}) + F(h_{28}) + F(h_{29}))]\} , \end{aligned}$$

$$
\begin{aligned}
w_{22} &= -[\tau e_{13}^T - \mu(e_{18}^T + e_{19}^T + e_{20}^T) + e_{47}^T + e_{48}^T + e_{49}^T]F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] \\
&= -F^{-1}\{[\tau i\xi_1(-i\xi_1 a) - \mu((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)(\xi_1^2 + ab) + a^2 b\tau](F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad +[\tau i\xi_1(-i\xi_2 a) - \mu((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)(-i\xi_1 i\xi_2)](F(h_{24}) + F(h_{25}) + F(h_{26})) \\
&\quad +[\tau i\xi_1(-i\xi_3 a) - \mu((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)(-i\xi_1 i\xi_3)](F(h_{27}) + F(h_{28}) + F(h_{29}))\} \ , \\
w_{23} &= -[\tau e_{14}^T - \mu(e_{28}^T + e_{29}^T + e_{30}^T) + e_{50}^T + e_{51}^T + e_{52}^T]F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] \\
&= -F^{-1}\{[\tau i\xi_2(-i\xi_1 a) - \mu((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)(-i\xi_1 i\xi_2)](F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad +[\tau i\xi_2(-i\xi_2 a) - \mu((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)(\xi_2^2 + ab) + a^2 b\tau](F(h_{24}) + F(h_{25}) + F(h_{26})) \\
&\quad +[\tau i\xi_2(-i\xi_3 a) - \mu((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)(-i\xi_2 i\xi_3)](F(h_{27}) + F(h_{28}) + F(h_{29}))\} \ , \\
w_{24} &= -[\tau e_{15}^T - \mu(e_{38}^T + e_{39}^T + e_{40}^T) + e_{53}^T + e_{54}^T + e_{55}^T]F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] \\
&= -F^{-1}\{[\tau i\xi_3(-i\xi_1 a) - \mu((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)(-i\xi_1 i\xi_3)](F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad +[\tau i\xi_3(-i\xi_2 a) - \mu((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)(-i\xi_2 i\xi_3)](F(h_{24}) + F(h_{25}) + F(h_{26})) \\
&\quad +[\tau i\xi_3(-i\xi_3 a) - \mu((i\xi_1)^2 + (i\xi_2)^2 + (i\xi_3)^2)(\xi_3^2 + ab) + a^2 b\tau](F(h_{27}) + F(h_{28}) + F(h_{29}))\} \ , \\
w_{25} &= e_1^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{i\xi_2[(\xi_1^2 + ab)(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad -i\xi_1 i\xi_2(F(h_{24}) + F(h_{25}) + F(h_{26})) - i\xi_1 i\xi_3(F(h_{27}) + F(h_{28}) + F(h_{29}))]\} \ , \\
w_{26} &= e_2^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{i\xi_3[(\xi_1^2 + ab)(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad -i\xi_1 i\xi_2(F(h_{24}) + F(h_{25}) + F(h_{26})) - i\xi_1 i\xi_3(F(h_{27}) + F(h_{28}) + F(h_{29}))]\} \ , \\
w_{27} &= e_3^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{(\xi_1^2 + ab)(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad -i\xi_1 i\xi_2(F(h_{24}) + F(h_{25}) + F(h_{26})) - i\xi_1 i\xi_3(F(h_{27}) + F(h_{28}) + F(h_{29}))\} \ , \\
w_{28} &= e_4^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{i\xi_1[-i\xi_1 i\xi_2(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad +(\xi_2^2 + ab)(F(h_{24}) + F(h_{25}) + F(h_{26})) - i\xi_2 i\xi_3(F(h_{27}) + F(h_{28}) + F(h_{29}))]\} \ , \\
w_{29} &= e_5^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{i\xi_2[-i\xi_1 i\xi_2(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad +(\xi_2^2 + ab)(F(h_{24}) + F(h_{25}) + F(h_{26})) - i\xi_2 i\xi_3(F(h_{27}) + F(h_{28}) + F(h_{29}))]\} \ , \\
w_{210} &= e_6^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{i\xi_3[-i\xi_1 i\xi_2(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad +(\xi_2^2 + ab)(F(h_{24}) + F(h_{25}) + F(h_{26})) - i\xi_2 i\xi_3(F(h_{27}) + F(h_{28}) + F(h_{29}))]\} \ , \\
w_{211} &= e_7^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{-i\xi_1 i\xi_2(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad +(\xi_2^2 + ab)(F(h_{24}) + F(h_{25}) + F(h_{26})) - i\xi_2 i\xi_3(F(h_{27}) + F(h_{28}) + F(h_{29}))\} \ , \\
w_{212} &= e_8^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{i\xi_1[-i\xi_1 i\xi_3(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad -i\xi_2 i\xi_3(F(h_{24}) + F(h_{25}) + F(h_{26})) + (\xi_3^2 + ab)(F(h_{27}) + F(h_{28}) + F(h_{29}))]\} \ , \\
w_{213} &= e_9^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{i\xi_2[-i\xi_1 i\xi_3(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad -i\xi_2 i\xi_3(F(h_{24}) + F(h_{25}) + F(h_{26})) + (\xi_3^2 + ab)(F(h_{27}) + F(h_{28}) + F(h_{29}))]\} \ , \\
w_{214} &= e_{10}^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{i\xi_3[-i\xi_1 i\xi_3(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad -i\xi_2 i\xi_3(F(h_{24}) + F(h_{25}) + F(h_{26})) + (\xi_3^2 + ab)(F(h_{27}) + F(h_{28}) + F(h_{29}))]\} \ , \\
w_{215} &= e_{11}^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{-i\xi_1 i\xi_3(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad -i\xi_2 i\xi_3(F(h_{24}) + F(h_{25}) + F(h_{26})) + (\xi_3^2 + ab)(F(h_{27}) + F(h_{28}) + F(h_{29}))\} \ , \\
w_{216} &= e_{16}^T F^{-1}[(a^2 b\tau A_1(\eta))F(h_2)] = F^{-1}\{-i\xi_1 a(F(h_{21}) + F(h_{22}) + F(h_{23})) \\
&\quad -i\xi_2 a(F(h_{24}) + F(h_{25}) + F(h_{26})) - i\xi_3 a(F(h_{27}) + F(h_{28}) + F(h_{29}))\} \ .
\end{aligned}
$$

Because $h_2 = h_2 I_{K_1'} \in C^2[0,\ T] \bigcap C^\infty(K_1)$, we can get

$$F^{-1}(i\xi_0 F(h_2)) = F^{-1}(F(\frac{\partial h_2}{\partial t})) = \frac{\partial h_2}{\partial t} ,$$

for the same reason we can get

$$F^{-1}(i\xi_1 F(h_2)) = \frac{\partial h_2}{\partial x_1} ,\ F^{-1}(i\xi_2 F(h_2)) = \frac{\partial h_2}{\partial x_2} ,\ F^{-1}(i\xi_3 F(h_2)) = \frac{\partial h_2}{\partial x_3} .$$

If we assume $h_{31} = h_{21} + h_{22} + h_{23}$, $h_{32} = h_{24} + h_{25} + h_{26}$, $h_{33} = h_{27} + h_{28} + h_{29}$, then we can get the following.

$$w_{21} = \frac{\partial^3 h_{31}}{\partial x_1 \partial x_2^2} + \frac{\partial^3 h_{31}}{\partial x_1 \partial x_3^2} - \frac{\partial^3 h_{32}}{\partial x_1^2 \partial x_2} - \frac{\partial^3 h_{33}}{\partial x_1^2 \partial x_3} ,\ w_{22} = \frac{\partial^3 h_{31}}{\partial t \partial x_2^2} + \frac{\partial^3 h_{31}}{\partial t \partial x_3^2} - \frac{\partial^3 h_{32}}{\partial t \partial x_1 \partial x_2} - \frac{\partial^3 h_{33}}{\partial t \partial x_1 \partial x_3} ,$$

$$w_{23} = \frac{\partial^3 h_{32}}{\partial t \partial x_1^2} + \frac{\partial^3 h_{32}}{\partial t \partial x_3^2} - \frac{\partial^3 h_{31}}{\partial t \partial x_1 \partial x_2} - \frac{\partial^3 h_{33}}{\partial t \partial x_2 \partial x_3} ,\ w_{24} = \frac{\partial^3 h_{33}}{\partial t \partial x_1^2} + \frac{\partial^3 h_{33}}{\partial t \partial x_2^2} - \frac{\partial^3 h_{31}}{\partial t \partial x_1 \partial x_3} - \frac{\partial^3 h_{32}}{\partial t \partial x_2 \partial x_3} ,$$

$$w_{25} = \frac{\partial^3 h_{31}}{\partial x_2^3} + \frac{\partial^3 h_{31}}{\partial x_2 \partial x_3^2} - \frac{\partial^3 h_{32}}{\partial x_1 \partial x_2^2} - \frac{\partial^3 h_{33}}{\partial x_1 \partial x_2 \partial x_3} ,\ w_{26} = \frac{\partial^3 h_{31}}{\partial x_2^2 \partial x_3} + \frac{\partial^3 h_{31}}{\partial x_3^3} - \frac{\partial^3 h_{32}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 h_{33}}{\partial x_1 \partial x_3^2} ,$$

$$w_{27} = \frac{\partial^2 h_{31}}{\partial x_2^2} + \frac{\partial^2 h_{31}}{\partial x_3^2} - \frac{\partial^2 h_{32}}{\partial x_1 \partial x_2} - \frac{\partial^2 h_{33}}{\partial x_1 \partial x_3} ,\ w_{28} = \frac{\partial^3 h_{32}}{\partial x_1^3} + \frac{\partial^3 h_{32}}{\partial x_1 \partial x_3^2} - \frac{\partial^3 h_{31}}{\partial x_1^2 \partial x_2} - \frac{\partial^3 h_{33}}{\partial x_1 \partial x_2 \partial x_3} ,$$

$$w_{29} = \frac{\partial^3 h_{32}}{\partial x_1^2 \partial x_2} + \frac{\partial^3 h_{32}}{\partial x_2 \partial x_3^2} - \frac{\partial^3 h_{31}}{\partial x_1 \partial x_2^2} - \frac{\partial^3 h_{33}}{\partial x_2^2 \partial x_3} ,\ w_{210} = \frac{\partial^3 h_{32}}{\partial x_1^2 \partial x_3} + \frac{\partial^3 h_{32}}{\partial x_3^3} - \frac{\partial^3 h_{31}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 h_{33}}{\partial x_2 \partial x_3^2} ,$$

$$w_{211} = \frac{\partial^2 h_{32}}{\partial x_1^2} + \frac{\partial^2 h_{32}}{\partial x_3^2} - \frac{\partial^2 h_{31}}{\partial x_1 \partial x_2} - \frac{\partial^2 h_{33}}{\partial x_2 \partial x_3} ,\ w_{212} = \frac{\partial^3 h_{33}}{\partial x_1^3} + \frac{\partial^3 h_{33}}{\partial x_1 \partial x_2^2} - \frac{\partial^3 h_{31}}{\partial x_1^2 \partial x_3} - \frac{\partial^3 h_{32}}{\partial x_1 \partial x_2 \partial x_3} ,$$

$$w_{213} = \frac{\partial^3 h_{33}}{\partial x_1^2 \partial x_2} + \frac{\partial^3 h_{33}}{\partial x_2 \partial x_3^2} - \frac{\partial^3 h_{31}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 h_{32}}{\partial x_2^2 \partial x_3} ,\ w_{214} = \frac{\partial^3 h_{33}}{\partial x_1^2 \partial x_3} + \frac{\partial^3 h_{33}}{\partial x_3^3} - \frac{\partial^3 h_{31}}{\partial x_1 \partial x_3^2} - \frac{\partial^3 h_{32}}{\partial x_2 \partial x_3^2} ,$$

$$w_{215} = \frac{\partial^2 h_{33}}{\partial x_1^2} + \frac{\partial^2 h_{33}}{\partial x_2^2} - \frac{\partial^2 h_{31}}{\partial x_1 \partial x_3} - \frac{\partial^3 h_{32}}{\partial x_2 \partial x_3} ,$$

$$w_{216} = \frac{1}{\tau}(\frac{\partial^2 h_{31}}{\partial t \partial x_1} + \frac{\partial^2 h_{32}}{\partial t \partial x_2} + \frac{\partial^2 h_{33}}{\partial t \partial x_3}) - \frac{\mu}{\tau}(\frac{\partial \triangle h_{31}}{\partial x_1} + \frac{\partial \triangle h_{32}}{\partial x_2} + \frac{\partial \triangle h_{33}}{\partial x_3}) .$$

We can see that W_2 is the function to do the partial derivation with the components of h_2 no more than the third order and their linear combination, moreover no more than the first order with the variable t . Hence $H[F^{-1}(A_1(\eta)Z_1)] = H[F^{-1}(A_1(\eta)Z_1)]I_{K_1'} \in C^1(K_1')$.
Now we can get $H[F^{-1}(Y_1 + A_1(\eta)Z_1)] = H[F^{-1}(Y_1 + A_1(\eta)Z_1)]I_{K_1'} \in C^1(K_1')$, if $Z_1 \in \Omega_2$, this means that $\Omega_2 \subset \Omega_1$, moreover the components of $F^{-1}(f(Z_1))$ are all in the $H[F^{-1}(Y_1 + A_1(\eta)Z_1)]$, hence $f(Z_1) \in \Omega_2$, if $Z_1 \in \Omega_2$. And we can get if $Z_1 \in \Omega_1$ and $Z_1 = f(Z_1)$, then $Z_1 \in \Omega_2$.
Secondly we prove the fixed-point of $f(Z_1)$ is exist, where $Z_1 \in \Omega_2$. In order to discuss more conveniently, we assume $f_j(Z_1) = F[F^{-1}(\alpha_{j1}^T(Y_1 + A_1(\eta)Z_1))F^{-1}(\alpha_{j2}^T(Y_1 + A_1(\eta)Z_1))]$, $1 \le j \le 9$, where $\alpha_{11} = e_3$, $\alpha_{21} = e_7$, $\alpha_{31} = e_{11}$, $\alpha_{41} = e_3$, $\alpha_{51} = e_7$, $\alpha_{61} = e_{11}$, $\alpha_{71} = e_3$, $\alpha_{81} = e_7$, $\alpha_{91} = e_{11}$, $\alpha_{12} = -\alpha_1$, $\alpha_{22} = e_1$, $\alpha_{32} = e_2$, $\alpha_{42} = e_4$, $\alpha_{52} = e_5$, $\alpha_{62} = e_6$, $\alpha_{72} = e_8$, $\alpha_{82} = e_9$, $\alpha_{92} = e_{10}$, and $\alpha_1 = e_5 + e_{10}$, e_i is the *ith* 55 dimensional unit coordinate vector, $1 \le i \le 55$.

And we assume the set Ω_C as follows.

$$\begin{aligned}\Omega_C &= \{h_1|\ h_1 \in C(K_1')\ ,\ \exists\ M>0\ ,\ \text{such that } |h_1| \le M\ ,\ \text{moreover } \exists\ C>0\ ,\ 0<\alpha<1\ ,\\ &\quad \text{such that } \forall\ x_t\ ,\ x_t' \in K_1'\ ,\ |h_1(x_t)-h_1(x_t')| \le C|x_t-x_t'|^\alpha\ .\}\ ,\\ &\quad \text{where } h_1=(h_{1j})_{9\times 1}\ ,\ |h_1| = \max_{1\le j\le 9}|h_{1j}|\ ,\ |h_{1j}| = \max_{x_t\in K_1'}|h_{1j}(x_t)|\ ,\ x_t=(t,\ x_1,\ x_2,\ x_3)^T\ .\end{aligned}$$

From the Arzela-Ascoli theorem, we know Ω_C is a compact set. And we assume M in the Ω_C satisfies the following.

$$\begin{aligned}&\max_{1\le j\le 9}\{\ |F^{-1}(\alpha_{j1}^TY_1)|\ ,|F^{-1}(\alpha_{j2}^TY_1)|\ ,|F^{-1}(\alpha_{j1}^TY_1)F^{-1}(\alpha_{j2}^TY_1)|\} \le \theta M\ ,\ \text{where } 0<\theta<1\ ,\\ &|F^{-1}(\alpha_{j1}^TY_1)| = \max_{x_t\in K_1'}|F^{-1}(\alpha_{j1}^TY_1)(x_t)|\ ,\ |F^{-1}(\alpha_{j2}^TY_1)| = \max_{x_t\in K_1'}|F^{-1}(\alpha_{j2}^TY_1)(x_t)|\ ,\\ &|F^{-1}(\alpha_{j1}^TY_1)F^{-1}(\alpha_{j2}^TY_1)| = \max_{x_t\in K_1'}|F^{-1}(\alpha_{j1}^TY_1)(x_t)F^{-1}(\alpha_{j2}^TY_1)(x_t)|\ ,\ 1\le j\le 9\ .\end{aligned}$$

We assume $\forall\ x_t\ ,\ x_t' \in K_1'\ ,\ \forall\ j\ ,\ 1\le j\le 9\ ,\ \exists\ C_1>0$, such that

$$\begin{aligned}&|F^{-1}(\alpha_{j1}^TY_1)(x_t)-F^{-1}(\alpha_{j1}^TY_1)(x_t')| \le C_1|x_t-x_t'|^\alpha\ ,\\ &|F^{-1}(\alpha_{j2}^TY_1)(x_t)-F^{-1}(\alpha_{j2}^TY_1)(x_t')| \le C_1|x_t-x_t'|^\alpha\ ,\\ &|F^{-1}(\alpha_{j1}^TY_1)(x_t)F^{-1}(\alpha_{j2}^TY_1)(x_t)-F^{-1}(\alpha_{j1}^TY_1)(x_t')F^{-1}(\alpha_{j2}^TY_1)(x_t')| \le C_1|x_t-x_t'|^\alpha\ .\end{aligned}$$

Next we assume $g(h_1)=F^{-1}[f(F(h_1))]$, and $g_j(h_1)=F^{-1}[f_j(F(h_1))],\ 1\le j\le 9$, where $h_1\in\Omega_C$. We will prove that the fixed-point of $g(h_1)$ is exist. Moreover we can get $h_1\in C^1(K_1')$, if $h_1=g(h_1)$ and $h_1\in\Omega_C$.
We only need to show $|g_j(h_1)|\le M$, moreover $\forall\ x_t\ ,\ x_t'\in K_1'$, we can get $|g_j(x_t)-g_j(x_t')|\le C|x_t-x_t'|^\alpha\ ,\ 1\le j\le 9$. We can see that

$$\begin{aligned}g_j(h_1) &= F^{-1}[\alpha_{j1}^T(Y_1+A_1(\eta)F(h_1))]F^{-1}[\alpha_{j2}^T(Y_1+A_1(\eta)F(h_1))]\ ,\\ &= F^{-1}(\alpha_{j1}^TY_1)F^{-1}(\alpha_{j2}^TY_1)+F^{-1}(\alpha_{j1}^TY_1)F^{-1}(\alpha_{j2}^TA_1(\eta)F(h_1))+\\ &\quad F^{-1}(\alpha_{j2}^TY_1)F^{-1}(\alpha_{j1}^TA_1(\eta)F(h_1))+\\ &\quad F^{-1}(\alpha_{j1}^TA_1(\eta)F(h_1))F^{-1}(\alpha_{j2}^TA_1(\eta)F(h_1))\ ,\end{aligned}$$

and from the lemma 3.1 , we can get $\exists\ h_2=h_2I_{K_1'}\in C^1[0,\ T]\bigcap C^\infty(K_1)$, such that

$$\begin{aligned}g_j(h_1) &= F^{-1}(\alpha_{j1}^TY_1)F^{-1}(\alpha_{j2}^TY_1)+F^{-1}(\alpha_{j1}^TY_1)F^{-1}(\alpha_{j2}^TA_1(\eta)a^2b\tau F(h_2))+\\ &\quad F^{-1}(\alpha_{j2}^TY_1)F^{-1}(\alpha_{j1}^TA_1(\eta)a^2b\tau F(h_2))+\\ &\quad F^{-1}(\alpha_{j1}^TA_1(\eta)a^2b\tau F(h_2))F^{-1}(\alpha_{j2}^TA_1(\eta)a^2b\tau F(h_2))\\ &= F^{-1}(\alpha_{j1}^TY_1)F^{-1}(\alpha_{j2}^TY_1)+F^{-1}(\alpha_{j1}^TY_1)W_{j2}(h_2)+\\ &\quad F^{-1}(\alpha_{j2}^TY_1)W_{j1}(h_2)+W_{j1}(h_2)W_{j2}(h_2)\ ,\end{aligned}$$

where $W_{j1}(h_2)=F^{-1}(\alpha_{j1}^T(a^2b\tau A_1(\eta))F(h_2))\ ,\ W_{j2}(h_2)=F^{-1}(\alpha_{j2}^T(a^2b\tau A_1(\eta))F(h_2))\ ,\ 1\le j\le 9$, and $\alpha_{11}=e_3\ ,\ \alpha_{21}=e_7\ ,\ \alpha_{31}=e_{11}\ ,\ \alpha_{41}=e_3\ ,\ \alpha_{51}=e_7\ ,\ \alpha_{61}=e_{11}\ ,\ \alpha_{71}=e_3\ ,\ \alpha_{81}=e_7$, $\alpha_{91}=e_{11}$, $\alpha_{12}=-\alpha_1\ ,\ \alpha_{22}=e_1\ ,\ \alpha_{32}=e_2\ ,\ \alpha_{42}=e_4\ ,\ \alpha_{52}=e_5\ ,\ \alpha_{62}=e_6\ ,\ \alpha_{72}=e_8$,

$\alpha_{82} = e_9$, $\alpha_{92} = e_{10}$, and $\alpha_1 = e_5 + e_{10}$, e_i is the ith 55 dimensional unit coordinate vector, $1 \leq i \leq 55$. And we can get the following.

$$
\begin{aligned}
F^{-1}(\alpha_{11}^T Y_1) &= w_{17} = \frac{\partial^2 F_{21}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_1 \partial x_3} - \frac{\partial^2 F_{11}}{\partial x_2^2} - \frac{\partial^2 F_{11}}{\partial x_3^2} , \\
F^{-1}(\alpha_{12}^T Y_1) &= w_{11} = \frac{\partial^3 F_{21}}{\partial x_1^2 \partial x_2} + \frac{\partial^3 F_{31}}{\partial x_1^2 \partial x_3} - \frac{\partial^3 F_{11}}{\partial x_1 \partial x_2^2} - \frac{\partial^3 F_{11}}{\partial x_1 \partial x_3^2} , \\
F^{-1}(\alpha_{21}^T Y_1) &= w_{111} = \frac{\partial^2 F_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{21}}{\partial x_1^2} - \frac{\partial^2 F_{21}}{\partial x_3^2} , \\
F^{-1}(\alpha_{22}^T Y_1) &= w_{15} = \frac{\partial^3 F_{21}}{\partial x_1 \partial x_2^2} + \frac{\partial^3 F_{31}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 F_{11}}{\partial x_2^3} - \frac{\partial^3 F_{11}}{\partial x_2 \partial x_3^2} , \\
F^{-1}(\alpha_{31}^T Y_1) &= w_{115} = \frac{\partial^2 F_{11}}{\partial x_1 \partial x_3} + \frac{\partial^2 F_{21}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{31}}{\partial x_1^2} - \frac{\partial^2 F_{31}}{\partial x_2^2} , \\
F^{-1}(\alpha_{32}^T Y_1) &= w_{16} = \frac{\partial^3 F_{21}}{\partial x_1 \partial x_2 \partial x_3} + \frac{\partial^3 F_{31}}{\partial x_1 \partial x_3^2} - \frac{\partial^3 F_{11}}{\partial x_2^2 \partial x_3} - \frac{\partial^3 F_{11}}{\partial x_3^3} , \\
F^{-1}(\alpha_{41}^T Y_1) &= w_{17} = \frac{\partial^2 F_{21}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_1 \partial x_3} - \frac{\partial^2 F_{11}}{\partial x_2^2} - \frac{\partial^2 F_{11}}{\partial x_3^2} , \\
F^{-1}(\alpha_{42}^T Y_1) &= w_{18} = \frac{\partial^3 F_{11}}{\partial x_1^2 \partial x_2} + \frac{\partial^3 F_{31}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 F_{21}}{\partial x_1^3} - \frac{\partial^3 F_{21}}{\partial x_1 \partial x_3^2} , \\
F^{-1}(\alpha_{51}^T Y_1) &= w_{111} = \frac{\partial^2 F_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{21}}{\partial x_1^2} - \frac{\partial^2 F_{21}}{\partial x_3^2} , \\
F^{-1}(\alpha_{52}^T Y_1) &= w_{19} = \frac{\partial^3 F_{11}}{\partial x_1 \partial x_2^2} + \frac{\partial^3 F_{31}}{\partial x_2^2 \partial x_3} - \frac{\partial^3 F_{21}}{\partial x_1^2 \partial x_2} - \frac{\partial^3 F_{21}}{\partial x_2 \partial x_3^2} , \\
F^{-1}(\alpha_{61}^T Y_1) &= w_{115} = \frac{\partial^2 F_{11}}{\partial x_1 \partial x_3} + \frac{\partial^2 F_{21}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{31}}{\partial x_1^2} - \frac{\partial^2 F_{31}}{\partial x_2^2} , \\
F^{-1}(\alpha_{62}^T Y_1) &= w_{110} = \frac{\partial^3 F_{11}}{\partial x_1 \partial x_2 \partial x_3} + \frac{\partial^3 F_{31}}{\partial x_2 \partial x_3^2} - \frac{\partial^3 F_{21}}{\partial x_1^2 \partial x_3} - \frac{\partial^3 F_{21}}{\partial x_3^3} , \\
F^{-1}(\alpha_{71}^T Y_1) &= w_{17} = \frac{\partial^2 F_{21}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_1 \partial x_3} - \frac{\partial^2 F_{11}}{\partial x_2^2} - \frac{\partial^2 F_{11}}{\partial x_3^2} , \\
F^{-1}(\alpha_{72}^T Y_1) &= w_{112} = \frac{\partial^3 F_{11}}{\partial x_1^2 \partial x_3} + \frac{\partial^3 F_{21}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 F_{31}}{\partial x_1^3} - \frac{\partial^3 F_{31}}{\partial x_1 \partial x_2^2} , \\
F^{-1}(\alpha_{81}^T Y_1) &= w_{111} = \frac{\partial^2 F_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{21}}{\partial x_1^2} - \frac{\partial^2 F_{21}}{\partial x_3^2} , \\
F^{-1}(\alpha_{82}^T Y_1) &= w_{113} = \frac{\partial^3 F_{11}}{\partial x_1 \partial x_2 \partial x_3} + \frac{\partial^3 F_{21}}{\partial x_2^2 \partial x_3} - \frac{\partial^3 F_{31}}{\partial x_1^2 \partial x_2} - \frac{\partial^3 F_{31}}{\partial x_2^3} , \\
F^{-1}(\alpha_{91}^T Y_1) &= w_{115} = \frac{\partial^2 F_{11}}{\partial x_1 \partial x_3} + \frac{\partial^2 F_{21}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{31}}{\partial x_1^2} - \frac{\partial^2 F_{31}}{\partial x_2^2} , \\
F^{-1}(\alpha_{92}^T Y_1) &= w_{114} = \frac{\partial^3 F_{11}}{\partial x_1 \partial x_3^2} + \frac{\partial^3 F_{21}}{\partial x_2 \partial x_3^2} - \frac{\partial^3 F_{31}}{\partial x_1^2 \partial x_3} - \frac{\partial^3 F_{31}}{\partial x_2^2 \partial x_3} ,
\end{aligned}
$$

$$
\begin{aligned}
F_{j1}(t,M) &= (\frac{1}{2\sqrt{\pi\mu}})^3\int_0^t\int_{R^3}\frac{F_{jv}(\tau_1,y_1,y_2,y_3)}{(\sqrt{t-\tau_1})^3}\ e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}}dy_1dy_2dy_3d\tau_1\ ,\\
F_{jv}(t,M_0) &= -\frac{1}{4\pi}\int_{K_1}\frac{F_j(t,M)}{r_{MM_0}}dx_1dx_2dx_3\ ,\ 1\le\ j\le 3\ ,\\
W_{11}(h_2) &= w_{27}=\frac{\partial^2h_{31}}{\partial x_2^2}+\frac{\partial^2h_{31}}{\partial x_3^2}-\frac{\partial^2h_{32}}{\partial x_1\partial x_2}-\frac{\partial^2h_{33}}{\partial x_1\partial x_3}\ ,\\
W_{12}(h_2) &= w_{21}=\frac{\partial^3h_{31}}{\partial x_1\partial x_2^2}+\frac{\partial^3h_{31}}{\partial x_1\partial x_3^2}-\frac{\partial^3h_{32}}{\partial x_1^2\partial x_2}-\frac{\partial^3h_{33}}{\partial x_1^2\partial x_3}\ ,\\
W_{21}(h_2) &= w_{211}=\frac{\partial^2h_{32}}{\partial x_1^2}+\frac{\partial^2h_{32}}{\partial x_3^2}-\frac{\partial^2h_{31}}{\partial x_1\partial x_2}-\frac{\partial^2h_{33}}{\partial x_2\partial x_3}\ ,\\
W_{22}(h_2) &= w_{25}=\frac{\partial^3h_{31}}{\partial x_2^3}+\frac{\partial^3h_{31}}{\partial x_2\partial x_3^2}-\frac{\partial^3h_{32}}{\partial x_1\partial x_2^2}-\frac{\partial^3h_{33}}{\partial x_1\partial x_2\partial x_3}\ ,\\
W_{31}(h_2) &= w_{215}=\frac{\partial^2h_{33}}{\partial x_1^2}+\frac{\partial^2h_{33}}{\partial x_2^2}-\frac{\partial^2h_{31}}{\partial x_1\partial x_3}-\frac{\partial^3h_{32}}{\partial x_2\partial x_3}\ ,\\
W_{32}(h_2) &= w_{26}=\frac{\partial^3h_{31}}{\partial x_2^2\partial x_3}+\frac{\partial^3h_{31}}{\partial x_3^3}-\frac{\partial^3h_{32}}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3h_{33}}{\partial x_1\partial x_3^2}\ ,\\
W_{41}(h_2) &= w_{27}=\frac{\partial^2h_{31}}{\partial x_2^2}+\frac{\partial^2h_{31}}{\partial x_3^2}-\frac{\partial^2h_{32}}{\partial x_1\partial x_2}-\frac{\partial^2h_{33}}{\partial x_1\partial x_3}\ ,\\
W_{42}(h_2) &= w_{28}=\frac{\partial^3h_{32}}{\partial x_1^3}+\frac{\partial^3h_{32}}{\partial x_1\partial x_3^2}-\frac{\partial^3h_{31}}{\partial x_1^2\partial x_2}-\frac{\partial^3h_{33}}{\partial x_1\partial x_2\partial x_3}\ ,\\
W_{51}(h_2) &= w_{211}=\frac{\partial^2h_{32}}{\partial x_1^2}+\frac{\partial^2h_{32}}{\partial x_3^2}-\frac{\partial^2h_{31}}{\partial x_1\partial x_2}-\frac{\partial^2h_{33}}{\partial x_2\partial x_3}\ ,\\
W_{52}(h_2) &= w_{29}=\frac{\partial^3h_{32}}{\partial x_1^2\partial x_2}+\frac{\partial^3h_{32}}{\partial x_2\partial x_3^2}-\frac{\partial^3h_{31}}{\partial x_1\partial x_2^2}-\frac{\partial^3h_{33}}{\partial x_2^2\partial x_3}\ ,\\
W_{61}(h_2) &= w_{215}=\frac{\partial^2h_{33}}{\partial x_1^2}+\frac{\partial^2h_{33}}{\partial x_2^2}-\frac{\partial^2h_{31}}{\partial x_1\partial x_3}-\frac{\partial^3h_{32}}{\partial x_2\partial x_3}\ ,\\
W_{62}(h_2) &= w_{210}=\frac{\partial^3h_{32}}{\partial x_1^2\partial x_3}+\frac{\partial^3h_{32}}{\partial x_3^3}-\frac{\partial^3h_{31}}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3h_{33}}{\partial x_2\partial x_3^2}\ ,\\
W_{71}(h_2) &= w_{27}=\frac{\partial^2h_{31}}{\partial x_2^2}+\frac{\partial^2h_{31}}{\partial x_3^2}-\frac{\partial^2h_{32}}{\partial x_1\partial x_2}-\frac{\partial^2h_{33}}{\partial x_1\partial x_3}\ ,\\
W_{72}(h_2) &= w_{212}=\frac{\partial^3h_{33}}{\partial x_1^3}+\frac{\partial^3h_{33}}{\partial x_1\partial x_2^2}-\frac{\partial^3h_{31}}{\partial x_1^2\partial x_3}-\frac{\partial^3h_{32}}{\partial x_1\partial x_2\partial x_3}\ ,\\
W_{81}(h_2) &= w_{211}=\frac{\partial^2h_{32}}{\partial x_1^2}+\frac{\partial^2h_{32}}{\partial x_3^2}-\frac{\partial^2h_{31}}{\partial x_1\partial x_2}-\frac{\partial^2h_{33}}{\partial x_2\partial x_3}\ ,\\
W_{82}(h_2) &= w_{213}=\frac{\partial^3h_{33}}{\partial x_1^2\partial x_2}+\frac{\partial^3h_{33}}{\partial x_2\partial x_3^2}-\frac{\partial^3h_{31}}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3h_{32}}{\partial x_2^2\partial x_3}\ ,\\
W_{91}(h_2) &= w_{215}=\frac{\partial^2h_{33}}{\partial x_1^2}+\frac{\partial^2h_{33}}{\partial x_2^2}-\frac{\partial^2h_{31}}{\partial x_1\partial x_3}-\frac{\partial^3h_{32}}{\partial x_2\partial x_3}\ ,\\
W_{92}(h_2) &= w_{214}=\frac{\partial^3h_{33}}{\partial x_1^2\partial x_3}+\frac{\partial^3h_{33}}{\partial x_3^3}-\frac{\partial^3h_{31}}{\partial x_1\partial x_3^2}-\frac{\partial^3h_{32}}{\partial x_2\partial x_3^2}\ ,
\end{aligned}
$$

where $h_2 = (h_{2j})_{9\times 1}$, and $h_{31} = h_{21} + h_{22} + h_{23}$, $h_{32} = h_{24} + h_{25} + h_{26}$, $h_{33} = h_{27} + h_{28} + h_{29}$. Hence we can get $W_{j1}(h_2)$, $W_{j2}(h_2)$, $1 \le j \le 9$, are the functions to do the partial derivation with the components of h_2 no more than the third order and their linear combination, moreover only do the partial derivation with the variables x_1 , x_2 , x_3 . From the lemma 3.1 , we know

$$v(t, M_0) = -\frac{1}{4\pi}\int_{K_1} \frac{h_1(t,M)}{r_{MM_0}} dx_1dx_2dx_3 \text{ , where } M = (x_1, x_2, x_3) \text{ , } M_0 = (x_{10}, x_{20}, x_{30}) \text{ ,}$$
$$r_{MM_0} = \sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + (x_3 - x_{30})^2} \text{ ,}$$
$$h_2(t, M) = (\frac{1}{2\sqrt{\pi\mu}})^3 \int_0^t \int_{R^3} \frac{v(\tau_1, y_1, y_2, y_3)}{(\sqrt{t-\tau_1})^3} \; e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}} dy_1dy_2dy_3d\tau_1 \text{ .}$$

And we can get the lemma as follows.

Lemma 3.2 $\exists\ M_{T,\ 1} > 0$, *and* $M_{T,\ 1}$ *is independent with* M , K_1 , $\forall\ h_1 \in\ \Omega_C$, *we can get the following,*

$$\max_{1\le i\le 3}\{|h_2| \text{ , } |\frac{\partial h_2}{\partial t}| \text{ , } |\frac{\partial h_2}{\partial x_i}| \text{ , } |\frac{\partial \frac{\partial h_2}{\partial x_i}}{\partial t}|\} \le\ MM_{T,\ 1}M(K_1) \text{ ,}$$

where

$$h_1 = (h_{1m})_{9\times 1} \text{ , } h_2 = h_2(t,\ M) = (h_{2m}(t,\ M))_{9\times 1} \text{ , } |h_2| = \max_{1\le m\le 9} \max_{(t,\ M)\in\ K_1'} |h_{2m}(t,\ M)| \text{ ,}$$
$$|\frac{\partial h_2}{\partial t}| = \max_{1\le m\le 9} \max_{(t,\ M)\in\ K_1'} |\frac{\partial h_{2m}(t,\ M)}{\partial t}| \text{ , } |\frac{\partial h_2}{\partial x_i}| = \max_{1\le m\le 9} \max_{(t,\ M)\in\ K_1'} |\frac{\partial h_{2m}(t,\ M)}{\partial x_i}| \text{ ,}$$
$$|\frac{\partial \frac{\partial h_2}{\partial x_i}}{\partial t}| = \max_{1\le m\le 9} \max_{(t,\ M)\in\ K_1'} |\frac{\partial \frac{\partial h_{2m}(t,\ M)}{\partial x_i}}{\partial t}| \text{ , } M(K_1) = \max_{M_0\in K_1} \frac{1}{4\pi}\int_{K_1} \frac{1}{r_{MM_0}} dx_1dx_2dx_3 \text{ .}$$

Proof of lemma3-2. We denote $h_{2m} = h_{2m}(t,\ M)$, $1 \le m \le 9$, and we assume $v(t, M) = (v_m)_{9\times 1}$, where $v_m = v_m(t, M)$, then from the lemma 3.1 , we can get $\forall\ m$, $1 \le m \le 9$,

$$v_m(t, M_0) = -\frac{1}{4\pi}\int_{K_1} \frac{h_{1m}(t,M)}{r_{MM_0}} dx_1dx_2dx_3 \text{ , where } M = (x_1, x_2, x_3) \text{ , } M_0 = (x_{10}, x_{20}, x_{30}) \text{ ,}$$
$$r_{MM_0} = \sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + (x_3 - x_{30})^2} \text{ ,}$$
$$h_{2m}(t, M) = (\frac{1}{2\sqrt{\pi\mu}})^3 \int_0^t \int_{R^3} \frac{v_m(\tau_1, y_1, y_2, y_3)}{(\sqrt{t-\tau_1})^3} \; e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}} dy_1dy_2dy_3d\tau_1 \text{ .}$$

if $h_1 \in\ \Omega_C$, then we can get

$$|v_m| \le |h_{1m}| \; \frac{1}{4\pi}\int_{K_1} \frac{1}{r_{MM_0}} dx_1dx_2dx_3 \le MM(K_1) \text{ ,}$$

moreover

$$|h_{2m}| \le (\frac{1}{2\sqrt{\pi\mu}})^3 \int_0^t \int_{R^3} \frac{MM(K_1)}{(\sqrt{t-\tau_1})^3} \; e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}} dy_1dy_2dy_3d\tau_1 \le TMM(K_1) \text{ ,}$$

where

$$M(K_1) = \max_{M_0\in K_1} \frac{1}{4\pi}\int_{K_1} \frac{1}{r_{MM_0}} dx_1dx_2dx_3 \text{ .}$$

And we know

$$\frac{\partial h_{2m}}{\partial t} = \lim_{\tau_1\to t}(\frac{1}{2\sqrt{\pi\mu}})^3\int_{R^3}\frac{v_m(\tau_1,y_1,y_2,y_3)}{(\sqrt{t-\tau_1})^3}\ e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}}dy_1dy_2dy_3 +$$

$$(\frac{1}{2\sqrt{\pi\mu}})^3\int_0^t\int_{R^3}\frac{\partial(\frac{v_m(\tau_1,y_1,y_2,y_3)}{(\sqrt{t-\tau_1})^3}\ e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}})}{\partial t}dy_1dy_2dy_3d\tau_1\ .$$

If we let $y_j = x_j + 2\sqrt{\mu(t-\tau_1)}\theta_j\ ,\ j=1\ ,\ 2\ ,\ 3$, then we can get as follows,

$$\begin{aligned}\frac{\partial h_{2m}}{\partial t} &= \lim_{\tau_1\to t}(\frac{1}{\sqrt{\pi}})^3\int_{R^3}v_m(\tau_1,x_j+2(\sqrt{\mu(t-\tau_1)})\theta_j,\ j=1,2,3,)\ e^{-(\theta_1^2+\theta_2^2+\theta_3^2)}d\theta_1d\theta_2d\theta_3 + \\ &(\frac{1}{2\sqrt{\pi\mu}})^3\int_0^t\int_{R^3}\frac{-3}{2}\frac{v_m(\tau_1,y_1,y_2,y_3)}{(\sqrt{t-\tau_1})^5}\ e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}}dy_1dy_2dy_3d\tau_1 + \\ &(\frac{1}{2\sqrt{\pi\mu}})^3\int_0^t\int_{R^3}\frac{v_m(\tau_1,y_1,y_2,y_3)}{4\mu(\sqrt{t-\tau_1})^7}\ \sum_{i=1}^3(x_i-y_i)^2\ e^{-\frac{\sum_{i=1}^3(x_i-y_i)^2}{4\mu(t-\tau_1)}}dy_1dy_2dy_3d\tau_1\ .\end{aligned}$$

If we let $y_j = x_j + 2\sqrt{\mu}(\sqrt{(t-\tau_1)})^5\theta_j\ ,\ j=1\ ,\ 2\ ,\ 3$, to the second integral, and we let $y_j = x_j + 2\sqrt{\mu}(\sqrt{(t-\tau_1)})^7\theta_j\ ,\ j=1\ ,\ 2\ ,\ 3$, to the third integral, then we can get as follows,

$$\begin{aligned}\frac{\partial h_{2m}}{\partial t} &= v_m(t,x_1,x_2,x_3) + (\frac{1}{\sqrt{\pi}})^3\int_0^t\int_{R^3}\frac{-3}{2}\ v_m^{(1)}(t-\tau_1)^5\ e^{-(t-\tau_1)^4(\theta_1^2+\theta_2^2+\theta_3^2)}d\theta_1d\theta_2d\theta_3d\tau_1 + \\ &(\frac{1}{\sqrt{\pi}})^3\int_0^t\int_{R^3}v_m^{(2)}(t-\tau_1)^{14}(\theta_1^2+\theta_2^2+\theta_3^2)\ e^{-(t-\tau_1)^6(\theta_1^2+\theta_2^2+\theta_3^2)}d\theta_1d\theta_2d\theta_3d\tau_1\ ,\end{aligned}$$

where

$$v_m^{(1)} = v_m(\tau_1,x_j+2\sqrt{\mu}(\sqrt{(t-\tau_1)})^5\theta_j,j=1,2,3,)\ ,\ v_m^{(2)} = v_m(\tau_1,x_j+2\sqrt{\mu}(\sqrt{(t-\tau_1)})^7\theta_j,j=1,2,3,)\ .$$

Hence we can get

$$\begin{aligned}|\frac{\partial h_{2m}}{\partial t}| &\leq MM(K_1) + (\frac{1}{\sqrt{\pi}})^3\int_0^t\int_{R^3}\frac{3}{2}MM(K_1)(t-\tau_1)^5\ e^{-(t-\tau_1)^4(\theta_1^2+\theta_2^2+\theta_3^2)}d\theta_1d\theta_2d\theta_3d\tau_1 + \\ &(\frac{1}{\sqrt{\pi}})^3\int_0^t\int_{R^3}MM(K_1)(t-\tau_1)^{14}(\theta_1^2+\theta_2^2+\theta_3^2)\ e^{-(t-\tau_1)^6(\theta_1^2+\theta_2^2+\theta_3^2)}d\theta_1d\theta_2d\theta_3d\tau_1\ .\end{aligned}$$

We assume $\varphi_1(t,\ \tau_1)$, $\varphi_2(t,\ \tau_1)$ as follows,

$$\varphi_1(t,\ \tau_1) = \int_{R^3}(t-\tau_1)^5\ e^{-(t-\tau_1)^4(\theta_1^2+\theta_2^2+\theta_3^2)}d\theta_1d\theta_2d\theta_3\ ,$$

$$\varphi_2(t,\ \tau_1) = \int_{R^3}(t-\tau_1)^{14}(\theta_1^2+\theta_2^2+\theta_3^2)\ e^{-(t-\tau_1)^6(\theta_1^2+\theta_2^2+\theta_3^2)}d\theta_1d\theta_2d\theta_3\ .$$

We can see that $\varphi_1(t,\ \tau_1),\ \varphi_2(t,\ \tau_1)$ are continuous about $t,\ \tau_1$ on the region $0\leq\tau_1\leq t\ ,\ 0\leq t\leq T$, hence they are bounded on the region $0\leq\tau_1\leq t\ ,\ 0\leq t\leq T$. We assume there exist $M'>0$, such that $|\varphi_1(t,\ \tau_1)|\leq M'\ ,\ |\varphi_2(t,\ \tau_1)|\leq M'$, where $0\leq\tau_1\leq t\ ,\ 0\leq t\leq T$. Hence we can get

$$|\frac{\partial h_{2m}}{\partial t}| \leq MM(K_1) + (\frac{1}{\sqrt{\pi}})^3\frac{5}{2}TMM(K_1)M'\ .$$

And we know

$$\frac{\partial h_{2m}}{\partial x_i} = (\frac{1}{2\sqrt{\pi\mu}})^3 \int_0^t \int_{R^3} \frac{v_m(\tau_1, y_1, y_2, y_3)}{4\mu(\sqrt{t-\tau_1})^5}\ (-2)(x_i - y_i)\ e^{-\frac{\sum_{j=1}^3 (x_j - y_j)^2}{4\mu(t-\tau_1)}} dy_1 dy_2 dy_3 d\tau_1\ .$$

If we let $y_j = x_j + 2\sqrt{\mu}(\sqrt{(t-\tau_1)})^5 \theta_j\ ,\ j = 1\ ,\ 2\ ,\ 3$, we can get the follows,

$$\frac{\partial h_{2m}}{\partial x_i} \quad = \quad (\frac{1}{\sqrt{\pi}})^3 \int_0^t \int_{R^3} \frac{1}{\sqrt{\mu}}\ v_m^{(1)}\ (\sqrt{t-\tau_1})^{15} \theta_i\ e^{-(t-\tau_1)^4(\theta_1^2+\theta_2^2+\theta_3^2)} d\theta_1 d\theta_2 d\theta_3 d\tau_1\ .$$

We assume $\varphi_3(t,\ \tau_1)$ as follows,

$$\varphi_3(t,\ \tau_1) = \int_{R^3} (\sqrt{t-\tau_1})^{15} \theta_i\ e^{-(t-\tau_1)^4(\theta_1^2+\theta_2^2+\theta_3^2)} d\theta_1 d\theta_2 d\theta_3\ ,$$

We can also see that $\varphi_3(t,\ \tau_1)$ is continuous about $t,\ \tau_1$ on the region $0 \leq \tau_1 \leq t\ ,\ 0 \leq t \leq T$, hence it is bounded on the region $0 \leq \tau_1 \leq t\ ,\ 0 \leq t \leq T$. We assume there exist $M'' > 0$, such that $|\varphi_3(t,\ \tau_1)| \leq M''$, where $0 \leq \tau_1 \leq t\ ,\ 0 \leq t \leq T$. Hence we can get

$$|\frac{\partial h_{2m}}{\partial x_i}| \leq (\frac{1}{\sqrt{\pi}})^3 \frac{T}{\sqrt{\mu}}\ MM(K_1)M''\ .$$

At last we see from

$$\begin{aligned}\frac{\partial \frac{\partial h_{2m}}{\partial x_i}}{\partial t} \quad = \quad & \lim_{\tau_1 \to t} (\frac{1}{2\sqrt{\pi\mu}})^3 \int_{R^3} \frac{v_m(\tau_1, y_1, y_2, y_3)}{4\mu(\sqrt{t-\tau_1})^5}\ (-2)(x_i - y_i)\ e^{-\frac{\sum_{j=1}^3 (x_j - y_j)^2}{4\mu(t-\tau_1)}} dy_1 dy_2 dy_3 + \\ & (\frac{1}{2\sqrt{\pi\mu}})^3 [\int_0^t \int_{R^3} \frac{v_m(\tau_1, y_1, y_2, y_3)}{4\mu(\sqrt{t-\tau_1})^7}\ (5)(x_i - y_i)\ e^{-\frac{\sum_{j=1}^3 (x_j - y_j)^2}{4\mu(t-\tau_1)}} dy_1 dy_2 dy_3 d\tau_1 + \\ & \int_0^t \int_{R^3} \frac{v_m(\tau_1, y_1, y_2, y_3)}{16\mu^2(\sqrt{t-\tau_1})^9}\ (-2)(x_i - y_i) \sum_{j=1}^3 (x_j - y_j)^2\ e^{-\frac{\sum_{i=1}^3 (x_i - y_i)^2}{4\mu(t-\tau_1)}} dy_1 dy_2 dy_3 d\tau_1]\ .\end{aligned}$$

If we let $y_j = x_j + 2\sqrt{\mu}(\sqrt{(t-\tau_1)})^5 \theta_j\ ,\ j = 1\ ,\ 2\ ,\ 3$, to the first integral, and we let $y_j = x_j + 2\sqrt{\mu}(\sqrt{(t-\tau_1)})^7 \theta_j\ ,\ j = 1\ ,\ 2\ ,\ 3$, to the second integral, $y_j = x_j + 2\sqrt{\mu}(\sqrt{(t-\tau_1)})^9 \theta_j$, $j = 1\ ,\ 2\ ,\ 3$, to the third integral, then we can get as follows,

$$\begin{aligned}\frac{\partial \frac{\partial h_{2m}}{\partial x_i}}{\partial t} \quad = \quad & \lim_{\tau_1 \to t} (\frac{1}{\sqrt{\pi}})^3 \int_{R^3} \frac{1}{\sqrt{\mu}}\ v_m^{(1)}\ (\sqrt{t-\tau_1})^{15} \theta_i\ e^{-(t-\tau_1)^4(\theta_1^2+\theta_2^2+\theta_3^2)} d\theta_1 d\theta_2 d\theta_3 + \\ & (\frac{1}{\sqrt{\pi}})^3 \int_0^t \int_{R^3} \frac{v_m^{(2)}}{2\sqrt{\mu}} (\sqrt{t-\tau_1})^{21}\ (5)\theta_i\ e^{-(t-\tau_1)^6 \sum_{j=1}^3 \theta_j^2} dy_1 dy_2 dy_3 d\tau_1 + \\ & (\frac{1}{\sqrt{\pi}})^3 \int_0^t \int_{R^3} \frac{v_m^{(3)}}{\sqrt{\mu}} (\sqrt{t-\tau_1})^{45} \theta_i (\theta_1^2 + \theta_2^2 + \theta_3^2) e^{-(t-\tau_1)^8 \sum_{j=1}^3 \theta_j^2} dy_1 dy_2 dy_3 d\tau_1,\end{aligned}$$

where

$$v_m^{(3)} = v_m(\tau_1, x_j + 2\sqrt{\mu}(\sqrt{(t-\tau_1)})^9 \theta_j, j = 1, 2, 3,)\ .$$

We assume $\varphi_4(t,\ \tau_1)$, $\varphi_5(t,\ \tau_1)$ as follows,

$$\varphi_4(t,\ \tau_1)=\int_{R^3}(\sqrt{t-\tau_1})^{21}\theta_i\ e^{-(t-\tau_1)^6(\theta_1^2+\theta_2^2+\theta_3^2)}d\theta_1 d\theta_2 d\theta_3\ ,$$
$$\varphi_5(t,\ \tau_1)=\int_{R^3}(\sqrt{t-\tau_1})^{45}\theta_i(\theta_1^2+\theta_2^2+\theta_3^2)\ e^{-(t-\tau_1)^8(\theta_1^2+\theta_2^2+\theta_3^2)}d\theta_1 d\theta_2 d\theta_3\ .$$

We can see that $\varphi_4(t,\ \tau_1)$, $\varphi_5(t,\ \tau_1)$ are continuous about $t,\ \tau_1$ on the region $0\le\tau_1\le t$, $0\le t\le T$, hence they are bounded on the region $0\le\tau_1\le t$, $0\le t\le T$. We assume there exist $M'''>0$, such that $|\varphi_4(t,\ \tau_1)|\le M'''$, $|\varphi_5(t,\ \tau_1)|\le M'''$, where $0\le\tau_1\le t$, $0\le t\le T$. Hence we can get

$$|\frac{\partial\frac{\partial h_{2m}}{\partial x_i}}{\partial t}|\le(\frac{1}{\sqrt{\pi}})^3\frac{MM(K_1)}{\sqrt{\mu}}\ M''+(\frac{1}{\sqrt{\pi}})^3\frac{7}{2\sqrt{\mu}}\ TMM(K_1)M'''\ .$$

If we let

$$M_{T,\ 1}=\max\{T\ ,\ 1+(\frac{1}{\sqrt{\pi}})^3\frac{5}{2}\ TM'\ ,\ (\frac{1}{\sqrt{\pi}})^3\frac{T}{\sqrt{\mu}}\ M''\ ,\ (\frac{1}{\sqrt{\pi}})^3\frac{1}{\sqrt{\mu}}\ M''+(\frac{1}{\sqrt{\pi}})^3\frac{7}{2\sqrt{\mu}}\ TM'''\}\ ,$$

then we can get

$$\max_{1\le i\le 3}\{|h_2|\ ,\ |\frac{\partial h_2}{\partial t}|\ ,\ |\frac{\partial h_2}{\partial x_i}|\ ,\ |\frac{\partial\frac{\partial h_2}{\partial x_i}}{\partial t}|\}\le\ MM_{T,\ 1}M(K_1)\ .$$

Hence the lemma is true.

Corollary 3.3 *$\exists\ M_{T,\ 2}>0$, and $M_{T,\ 2}$ is independent with M , K_1 ,$\forall\ h_1\in\ \Omega_C$, we can get the following,*

$$\max_{i,\ j,\ k,\ l}\{\ |\frac{\partial^2 h_2}{\partial x_i\partial x_j}|\ ,\ |\frac{\partial\frac{\partial^2 h_2}{\partial x_i\partial x_j}}{\partial t}|\ ,\ |\frac{\partial^3 h_2}{\partial x_i\partial x_j\partial x_k}|\ ,\ |\frac{\partial\frac{\partial^3 h_2}{\partial x_i\partial x_j\partial x_k}}{\partial t}|\ ,\ |\frac{\partial^4 h_2}{\partial x_i\partial x_j\partial x_k\partial x_l}|\}\le\ MM_{T,\ 2}M(K_1)\ ,$$

where

$$h_1=(h_{1m})_{9\times 1}\ ,\ h_2=h_2(t,\ M)=(h_{2m}(t,\ M))_{9\times 1}\ ,\ |\frac{\partial^2 h_2}{\partial x_i\partial x_j}|=\max_{1\le m\le 9}\max_{(t,\ M)\in\ K_1'}|\frac{\partial^2 h_{2m}(t,\ M)}{\partial x_i\partial x_j}|\ ,$$
$$|\frac{\partial\frac{\partial^2 h_2}{\partial x_i\partial x_j}}{\partial t}|=\max_{1\le m\le 9}\max_{(t,\ M)\in\ K_1'}|\frac{\partial\frac{\partial^2 h_{2m}(t,\ M)}{\partial x_i\partial x_j}}{\partial t}|\ ,\ |\frac{\partial^3 h_2}{\partial x_i\partial x_j\partial x_k}|=\max_{1\le m\le 9}\max_{(t,\ M)\in\ K_1'}|\frac{\partial^3 h_{2m}(t,\ M)}{\partial x_i\partial x_j\partial x_k}|\ ,$$
$$|\frac{\partial\frac{\partial^3 h_2}{\partial x_i\partial x_j\partial x_k}}{\partial t}|=\max_{1\le m\le 9}\max_{(t,\ M)\in\ K_1'}|\frac{\partial\frac{\partial^3 h_{2m}(t,\ M)}{\partial x_i\partial x_j\partial x_k}}{\partial t}|\ ,\ M(K_1)=\max_{M_0\in K_1}\frac{1}{4\pi}\int_{K_1}\frac{1}{r_{MM_0}}dx_1dx_2dx_3\ ,$$
$$|\frac{\partial^4 h_2}{\partial x_i\partial x_j\partial x_k\partial x_l}|=\max_{1\le m\le 9}\max_{(t,\ M)\in\ K_1'}|\frac{\partial^4 h_{2m}(t,\ M)}{\partial x_i\partial x_j\partial x_k\partial x_l}|\ ,\ 1\le i,\ j,\ k,\ l\le 3\ .$$

Corollary 3.4 *$\exists\ M_{T,\ 3}>0$, and $M_{T,\ 3}$ is independent with M , K_1 ,$\forall\ h_1\in\ \Omega_C$, we can get the following,*

$$\max\{\ |W_{jl}(h_2)|\ ,\ |\frac{\partial W_{jl}(h_2)}{\partial t}|\ ,\ |\frac{\partial W_{jl}(h_2)}{\partial x_i}|\ ,\ 1\le j\le 9\ ,\ 1\le l\le 2\ ,\ 1\le i\le 3\ .\}\le\ MM_{T,\ 3}M(K_1)\ .$$

From the corollary 3.4 , we can get

$$|W_{jl}(h_2)(x_t) - W_{jl}(h_2)(x'_t)| \le 4MM_{T,\ 3}M(K_1)|x_t - x'_t| \le MM_{T,\ 4}M(K_1)|x_t - x'_t|^{\alpha}\ ,\ \forall\ x_t\ ,\ x'_t \in K'_1\ ,$$

where

$$1 \le j \le 9\ ,\ 1 \le l \le 2\ ,\ M_{T,\ 4} = 4M_{T,\ 3} \max_{x_t,\ x'_t \in K'_1} |x_t - x'_t|^{1-\alpha}\ .$$

Now we can get

$$\begin{aligned} |g_j(h_1)| &= |F^{-1}(\alpha_{j1}^T Y_1)F^{-1}(\alpha_{j2}^T Y_1) + F^{-1}(\alpha_{j1}^T Y_1)W_{j2}(h_2) + \\ &\quad F^{-1}(\alpha_{j2}^T Y_1)W_{j1}(h_2) + W_{j1}(h_2)W_{j2}(h_2)| \\ &\le \theta M + 2\theta M^2\ M_{T,\ 3}M(K_1) + [MM_{T,\ 3}M(K_1)]^2\ ,\ 1 \le j \le 9\ . \end{aligned}$$

And from

$$\begin{aligned} & g_j(h_1)(x_t) - g_j(h_1)(x'_t) \\ = \ & F^{-1}(\alpha_{j1}^T Y_1)(x_t)F^{-1}(\alpha_{j2}^T Y_1)(x_t) + F^{-1}(\alpha_{j1}^T Y_1)(x_t)W_{j2}(h_2)(x_t) + \\ & F^{-1}(\alpha_{j2}^T Y_1)(x_t)W_{j1}(h_2)(x_t) + W_{j1}(h_2)(x_t)W_{j2}(h_2)(x_t) - \\ & [F^{-1}(\alpha_{j1}^T Y_1)(x'_t)F^{-1}(\alpha_{j2}^T Y_1)(x'_t) + F^{-1}(\alpha_{j1}^T Y_1)(x'_t)W_{j2}(h_2)(x'_t) + \\ & F^{-1}(\alpha_{j2}^T Y_1)(x'_t)W_{j1}(h_2)(x'_t) + W_{j1}(h_2)(x'_t)W_{j2}(h_2)(x'_t)] \end{aligned}$$

$$\begin{aligned} = \ & F^{-1}(\alpha_{j1}^T Y_1)(x_t)F^{-1}(\alpha_{j2}^T Y_1)(x_t) - F^{-1}(\alpha_{j1}^T Y_1)(x'_t)F^{-1}(\alpha_{j2}^T Y_1)(x'_t) + \\ & F^{-1}(\alpha_{j1}^T Y_1)(x_t)(W_{j2}(h_2)(x_t) - W_{j2}(h_2)(x'_t)) + (F^{-1}(\alpha_{j1}^T Y_1)(x_t) - F^{-1}(\alpha_{j1}^T Y_1)(x'_t))W_{j2}(h_2)(x'_t) + \\ & F^{-1}(\alpha_{j2}^T Y_1)(x_t)(W_{j1}(h_2)(x_t) - W_{j1}(h_2)(x'_t)) + (F^{-1}(\alpha_{j2}^T Y_1)(x_t) - F^{-1}(\alpha_{j2}^T Y_1)(x'_t))W_{j1}(h_2)(x'_t) + \\ & W_{j1}(h_2)(x_t)(W_{j2}(h_2)(x_t) - W_{j2}(h_2)(x'_t)) + (W_{j1}(h_2)(x_t) - W_{j1}(h_2)(x'_t))W_{j2}(h_2)(x'_t)\ , \end{aligned}$$

we can get

$$\begin{aligned} & |g_j(h_1)(x_t) - g_j(h_1)(x'_t)| \le \\ & \{C_1 + 2[\theta M^2\ M_{T,\ 4}M(K_1) + C_1MM_{T,\ 3}M(K_1) + M^2M_{T,\ 3}M_{T,\ 4}M(K_1)^2]\}|x_t - x'_t|^{\alpha}\ ,\ 1 \le j \le 9\ . \end{aligned}$$

Hence $g(h_1) = (g_j(h_1))_{9\times 1} \in \Omega_C$, if the followings are true.

$$\begin{cases} \theta M + 2\theta M^2\ M_{T,\ 3}M(K_1) + [MM_{T,\ 3}M(K_1)]^2 \le M\ , \\ C_1 + 2[\theta M^2\ M_{T,\ 4}M(K_1) + C_1MM_{T,\ 3}M(K_1) + M^2M_{T,\ 3}M_{T,\ 4}M(K_1)^2] \le C\ . \end{cases} \tag{3.8}$$

If we let C is big enough, and from the assumption 1.1, if we let

$$M = \frac{1-\theta}{M_{T,\ 3}M(K_1)(2\theta + M_{T,\ 3}M(K_1))}\ ,$$

then (3.8) will be true.
At last we need to prove the mapping $T:\ h_1 \to\ g(h_1)$ is continuous about h_1 in the Ω_C .
We assume $|h_1^{(n)} - h_1^*| \to 0$, when $n \to +\infty$, and $h_1^{(n)}\ ,\ h_1^* \in \Omega_C\ ,\ n \ge 1$, from the lemma 3.1 we

know that there exist $h_2^{(n)} = h_2^{(n)} I_{K_1'} \in C^1[0,\ T] \bigcap C^{\infty}(K_1)$, $h_2^* = h_2^* I_{K_1'} \in C^1[0,\ T] \bigcap C^{\infty}(K_1)$, such that $F(h_1^{(n)}) = a^2 b\tau F(h_2^{(n)})$, $F(h_1^*) = a^2 b\tau F(h_2^*)$, where

$$v^{(n)}(t, M_0) = -\frac{1}{4\pi} \int_{K_1} \frac{h_1^{(n)}(t, M)}{r_{MM_0}} dx_1 dx_2 dx_3 \ , \text{ and } M = (x_1, x_2, x_3) \ , \ M_0 = (x_{10}, x_{20}, x_{30}) \ ,$$

$$h_2^{(n)}(t, M) = (\frac{1}{2\sqrt{\pi\mu}})^3 \int_0^t \int_{R^3} \frac{v^{(n)}(\tau_1, y_1, y_2, y_3)}{(\sqrt{t-\tau_1})^3}\ e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}} dy_1 dy_2 dy_3 d\tau_1 \ ,$$

$$v^*(t, M_0) = -\frac{1}{4\pi} \int_{K_1} \frac{h_1^*(t, M)}{r_{MM_0}} dx_1 dx_2 dx_3 \ , \text{ and } M = (x_1, x_2, x_3) \ , \ M_0 = (x_{10}, x_{20}, x_{30}) \ ,$$

$$h_2^*(t, M) = (\frac{1}{2\sqrt{\pi\mu}})^3 \int_0^t \int_{R^3} \frac{v^*(\tau_1, y_1, y_2, y_3)}{(\sqrt{t-\tau_1})^3}\ e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{4\mu(t-\tau_1)}} dy_1 dy_2 dy_3 d\tau_1 \ .$$

From the Lebesgue dominated convergence theorem, we can get $|v^{(n)} - v^*| \to 0$, when $n \to +\infty$. And we know the partial derivation of $h_2^{(n)}$ which is no more than the third order, only do the partial derivation with the variables x_1 , x_2 , x_3 , can pass through the integral. Again from the Lebesgue dominated convergence theorem, we can get when $n \to +\infty$,

$$|h_2^{(n)} - h_2^*| \to 0 \ , \ |\frac{\partial h_2^{(n)}}{\partial x_i} - \frac{\partial h_2^*}{\partial x_i}| \to 0 \ , \ |\frac{\partial^2 h_2^{(n)}}{\partial x_i \partial x_j} - \frac{\partial^2 h_2^*}{\partial x_i \partial x_j}| \to 0 \ ,$$

$$|\frac{\partial^3 h_2^{(n)}}{\partial x_i \partial x_j \partial x_k} - \frac{\partial^3 h_2^*}{\partial x_i \partial x_j \partial x_k}| \to 0 \ , \ 1 \le i \ , \ j \ , \ k \le 3 \ .$$

Hence we can get when $n \to +\infty$,

$$|W_{j1}(h_2^{(n)}) - W_{j1}(h_2^*)| \to 0 \ , \ |W_{j1}(h_2^{(n)}) - W_{j1}(h_2^*)| \to 0 \ , \ 1 \le j \le 9 \ .$$

Now we can get

$$\begin{aligned}
& |g_j(h_1^{(n)}) - g_j(h_1^*)| \\
= \ & |F^{-1}(\alpha_{j1}^T Y_1) F^{-1}(\alpha_{j2}^T Y_1) + F^{-1}(\alpha_{j1}^T Y_1) W_{j2}(h_2^{(n)}) + F^{-1}(\alpha_{j2}^T Y_1) W_{j1}(h_2^{(n)}) + W_{j1}(h_2^{(n)}) W_{j2}(h_2^{(n)}) - \\
& [F^{-1}(\alpha_{j1}^T Y_1) F^{-1}(\alpha_{j2}^T Y_1) + F^{-1}(\alpha_{j1}^T Y_1) W_{j2}(h_2^*) + F^{-1}(\alpha_{j2}^T Y_1) W_{j1}(h_2^*) + W_{j1}(h_2^*) W_{j2}(h_2^*)]| \\
\le \ & |F^{-1}(\alpha_{j1}^T Y_1)||W_{j2}(h_2^{(n)}) - W_{j2}(h_2^*)| + |F^{-1}(\alpha_{j2}^T Y_1)||W_{j1}(h_2^{(n)}) - W_{j1}(h_2^*)| + \\
& |W_{j1}(h_2^{(n)}) - W_{j1}(h_2^*)||W_{j2}(h_2^{(n)})| + |W_{j1}(h_2^*)||W_{j2}(h_2^{(n)}) - W_{j2}(h_2^*)| \\
\le \ & [\theta M + M M_{T,\ 3} M(K_1)]\ [\ |W_{j1}(h_2^{(n)}) - W_{j1}(h_2^*)| + |W_{j2}(h_2^{(n)}) - W_{j2}(h_2^*)|\ ] \ ,
\end{aligned}$$

this means that $|g_j(h_1^{(n)}) - g_j(h_1^*)| \to 0$, $1 \le j \le 9$, when $n \to +\infty$.
Hence T is continuous about h_1 in the Ω_C . According to the Schauder fixed-point theorem we can learn that there exists $h_1 \in \Omega_C$, such that $h_1 = g(h_1)$. And $\forall\ j$, $1 \le j \le 9$, we know that

$$g_j(h_1) = F^{-1}(\alpha_{j1}^T Y_1) F^{-1}(\alpha_{j2}^T Y_1) + F^{-1}(\alpha_{j1}^T Y_1) W_{j2}(h_2) + F^{-1}(\alpha_{j2}^T Y_1) W_{j1}(h_2) + W_{j1}(h_2) W_{j2}(h_2) \ ,$$

where $h_2 = h_2 I_{K_1'} \in C^1[0,\ T] \bigcap C^{\infty}(K_1)$, and $W_{j1}(h_2)$, $W_{j2}(h_2)$, $1 \le j \le 9$, are the functions to do the partial derivation with the components of h_2 no more than the third order and their linear combination, moreover only do the partial derivation with the variables x_1 , x_2 , x_3 . And the components of $F^{-1}(\alpha_{j1}^T Y_1)$, $F^{-1}(\alpha_{j2}^T Y_1)$, $1 \le j \le 9$, are all in the $H[F^{-1}(Y_1)]$, moreover

$$H[F^{-1}(Y_1)] = H[F^{-1}(Y_1)] I_{K_1'} \in C^1(K_1') \ .$$

This means that $g_j(h_1) \in C^1(K_1')$, $1 \le j \le 9$, hence the fixed-point

$$h_1 = g(h_1) = (g_j(h_1))_{9\times 1} \in C^1(K_1') \ .$$

At last if we let $Z_1 = F(h_1)$, where h_1 is the fixed-point of $g(h_1)$, then $Z_1 \in \Omega_2$, and Z_1 is the fixed point of $f(Z_1)$. This is also to say that the smoothing solution for the Navier-Stokes equations is globally exist, and we can get the following.

$$\begin{aligned} u_1 &= e_3^T Z = e_3^T[F^{-1}(Y_1 + A_1(\eta)Z_1)] = w_{17} + w_{27} \\ &= \frac{\partial^2 F_{21}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_1 \partial x_3} - \frac{\partial^2 F_{11}}{\partial x_2^2} - \frac{\partial^2 F_{11}}{\partial x_3^2} + \frac{\partial^2 h_{31}}{\partial x_2^2} + \frac{\partial^2 h_{31}}{\partial x_3^2} - \frac{\partial^2 h_{32}}{\partial x_1 \partial x_2} - \frac{\partial^2 h_{33}}{\partial x_1 \partial x_3} \ , \end{aligned}$$

$$\begin{aligned} u_2 &= e_7^T Z = e_7^T[F^{-1}(Y_1 + A_1(\eta)Z_1)] = w_{111} + w_{211} \\ &= \frac{\partial^2 F_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 F_{31}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{21}}{\partial x_1^2} - \frac{\partial^2 F_{21}}{\partial x_3^2} + \frac{\partial^2 h_{32}}{\partial x_1^2} + \frac{\partial^2 h_{32}}{\partial x_3^2} - \frac{\partial^2 h_{31}}{\partial x_1 \partial x_2} - \frac{\partial^2 h_{33}}{\partial x_2 \partial x_3} \ , \\ u_3 &= e_{11}^T Z = e_{11}^T[F^{-1}(Y_1 + A_1(\eta)Z_1)] = w_{115} + w_{215} \\ &= \frac{\partial^2 F_{11}}{\partial x_1 \partial x_3} + \frac{\partial^2 F_{21}}{\partial x_2 \partial x_3} - \frac{\partial^2 F_{31}}{\partial x_1^2} - \frac{\partial^2 F_{31}}{\partial x_2^2} + \frac{\partial^2 h_{33}}{\partial x_1^2} + \frac{\partial^2 h_{33}}{\partial x_2^2} - \frac{\partial^2 h_{31}}{\partial x_1 \partial x_3} - \frac{\partial^3 h_{32}}{\partial x_2 \partial x_3} \ , \\ p &= e_{16}^T Z = e_{16}^T[F^{-1}(Y_1 + A_1(\eta)Z_1)] = w_{116} + w_{216} \\ &= \frac{1}{\tau}\,[\ \mu\triangle(\frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{21}}{\partial x_2} + \frac{\partial F_{31}}{\partial x_3}) - \frac{\partial^2 F_{11}}{\partial t \partial x_1} - \frac{\partial^2 F_{21}}{\partial t \partial x_2} - \frac{\partial^2 F_{31}}{\partial t \partial x_3}\ ] + \\ &\quad \frac{1}{\tau}\,(\frac{\partial^2 h_{31}}{\partial t \partial x_1} + \frac{\partial^2 h_{32}}{\partial t \partial x_2} + \frac{\partial^2 h_{33}}{\partial t \partial x_3}) - \frac{\mu}{\tau}\,(\frac{\partial \triangle h_{31}}{\partial x_1} + \frac{\partial \triangle h_{32}}{\partial x_2} + \frac{\partial \triangle h_{33}}{\partial x_3}) \ , \end{aligned}$$

where

$$\begin{aligned} &v(t, M_0) = -\frac{1}{4\pi}\int_{K_1} \frac{h_1(t, M)}{r_{MM_0}} dx_1 dx_2 dx_3 \ , \text{ and } M = (x_1, x_2, x_3) \ , \ M_0 = (x_{10}, x_{20}, x_{30}) \ , \\ &h_2(t, M) = (\frac{1}{2\sqrt{\pi\mu}})^3 \int_0^t \int_{R^3} \frac{v(\tau_1, y_1, y_2, y_3)}{(\sqrt{t-\tau_1})^3}\ e^{-\frac{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}{4\mu(t-\tau_1)}} dy_1 dy_2 dy_3 d\tau_1 \ , \\ &h_{31} = h_{21} + h_{22} + h_{23} \ , \ h_{32} = h_{24} + h_{25} + h_{26} \ , \ h_{33} = h_{27} + h_{28} + h_{29} \ , \\ &F_{j1}(t, M) = (\frac{1}{2\sqrt{\pi\mu}})^3 \int_0^t \int_{R^3} \frac{F_{jv}(\tau_1, y_1, y_2, y_3)}{(\sqrt{t-\tau_1})^3}\ e^{-\frac{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}{4\mu(t-\tau_1)}} dy_1 dy_2 dy_3 d\tau_1 \ , \\ &F_{jv}(t, M_0) = -\frac{1}{4\pi}\int_{K_1} \frac{F_j(t, M)}{r_{MM_0}} dx_1 dx_2 dx_3 \ , \ 1 \le\ j \le 3 \ , \end{aligned}$$

and h_1 is the fixed-point of $g(h_1)$, where $g(h_1) = (g_j(h_1))_{9\times 1}$, and

$$g_j(h_1) = F^{-1}(\alpha_{j1}^T Y_1) F^{-1}(\alpha_{j2}^T Y_1) + F^{-1}(\alpha_{j1}^T Y_1) W_{j2}(h_2) + F^{-1}(\alpha_{j2}^T Y_1) W_{j1}(h_2) + W_{j1}(h_2) W_{j2}(h_2) \ ,$$

$1 \le\ j \le\ 9$. And because $Z_1 = F(h_1) \in \Omega_2$, we can get

$$u_j \in C^2[0,\ T] \bigcap C^\infty(K_1) \ , \ 1 \le j \le 3 \ , \ p \in C^1[0,\ T] \bigcap C^\infty(K_1) \ .$$

Finally we say our sincerely thanks to Schauder according to at least two points as following.

(1)The Schauder fixed-point theorem.
(2)If h_1 is Hölder continuous, ∂K_1 satisfies the exterior ball condition, then the smoothing solution of Poisson's equation exists. This is also proved by Schauder, and it leads to that the Newtonian potential of h_1 ,

$$\frac{-1}{4\pi}\int_{K_1}\frac{h_1}{r_{MM_0}}dx_1dx_2dx_3$$

is a solution of Poisson's equation.
If there is no important help from Schauder, we can not see so far with the Navier-Stokes equations. Maybe you will ask why not to do the Laplace transformation with the variable t just as usual. It is very difficult to discuss the fixed-point of $f(Z_1)$ if we do that. We have tried to do the Laplace transformation with the variable t , but we can not discuss the inverse Laplace transformation in $f(Z_1)$. This is the reason why we also do the Fourier transformation with the variable t .

References

[1] David Gilbarg, Neil S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Heidelberg, New York, 1977.

[2] Chaohao Gu, Daqian Li, Shuxing Chen, Songmu Zheng, Yongji Tian, The mathematical-physics equations, (second edition), Higher education press, 2005.

[3] Gongqing Zhang, Yuanqu Lin, Functional analysis, Peking university press, 2010.

[4] Pieter Wesseling, Principles of computational fluid dynamics, Springer-Verlag Berlin Heidelberg 2001.

[5] Shuxing Chen, The general theory of the partial differential equation, Higher education press, 1981.

[6] Daqian Li, Tiehu Qing, Physics and partial differential equations, (second edition), Higher education press, 2005.

[7] Caisheng Chen, Gang Li, Jidong Chou, Wenchu Wang, The mathematical-physics equations, Science education press, 2008.

[8] Jingpu Lu, Zhi Guan, The general theory of the partial differential equation, Higher education press, 2004.