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CO-QUASI-INVARIANT SPACES FOR FINITE COMPLEX REFLECTION GROUPS

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Abstract. We study, in a global uniform manner, the quotient of the ring of polynomials in ℓ sets of n variables, by the ideal generated by diagonal quasi-invariant polynomials for general permutation groups $W = G(r, n)$. We show that, for each such group W, there is an explicit universal symmetric function that gives the $\mathbb{N}^\ell$-graded Hilbert series for these spaces. This function is universal in that its dependance on ℓ only involves the number of variables it is calculated with. We also discuss the combinatorial implications of the observed fact that it affords an expansion as a positive coefficient polynomial in the complete homogeneous symmetric functions.

Contents

1. Introduction

For rank n classical families of finite complex reflection groups W, we contribute to the description of the diagonal co-quasi-invariant space $\mathcal{Q}_W$ for W, in several (say ℓ) sets of n variables. Here, the use of the term diagonal refers to the fact that W is considered as a diagonal subgroup of W^ℓ, acting on the ℓ^{th}-tensor power $\mathcal{R}_n^{(\ell)}$ of the symmetric algebra of the defining representation of W. Instead of the usual one, the action considered here is the so-called Hivert-action. Invariant polynomials under this action are known as quasi-invariants[1] (or quasi-symmetric for the symmetric group). Our space $\mathcal{Q}_W^{(\ell)}$ is simply the quotient of $\mathcal{R}_n^{(\ell)}$ by the ideal generated by constant-term-free quasi-invariants for W.

[1]Under the same name, an entirely different notion has been considered in [7, 8]. However, the terminology of quasi-symmetric polynomials being well ingrained, it seems awkward to call their generalization to other reflection groups by any other name than quasi-invariant.

We show that the associated multigraded Hilbert series, denoted $\mathcal{Q}_W^{(\ell)}(q_1,\dots,q_\ell)$ (which is symmetric in the q_i), can be described in an uniform manner as a positive coefficient linear combination of Schur polynomials

$$\mathcal{Q}_W^{(\ell)}(q_1,\dots,q_\ell)=\sum_\mu c_\mu\, s_\mu(q_1,\dots,q_\ell), \tag{1.1}$$

with the c_μ independent of ℓ, and μ running through a finite set of integer partitions that depend only on the group W. This is a typical phenomena in many similar situations such as considered in [6]. It has the striking feature that we can give explicit formulas for the dimension of $\mathcal{Q}_W^{(\ell)}$ for all ℓ. To see why this is so striking, it may be worthwhile to recall that, for the entirely analogous context of diagonal co-invariant spaces (i.e. the one corresponding to the usual diagonal action of W on $\mathcal{R}_n^{(\ell)}$), a large body of work has only recently settled the special case $\ell=2$, but that we know almost nothing yet for $\ell\geq 3$.

2. Our context

A down to earth description of our context may be given as follows. Consider a $\ell\times n$ matrix of variables $X:=(x_{ij})$. For any fixed i (a row of X), we say that the variables $x_{i1},x_{i2},\dots,x_{in}$ form the i^{th} *set of variables*. In some instances it is worth simplifying this notation, and write

$$X=\begin{pmatrix} x_1 & x_2 & \cdots & x_n\\ y_1 & y_2 & \cdots & y_n\\ \vdots & \vdots & \ddots & \vdots\\ z_1 & z_2 & \cdots & z_n\end{pmatrix}.$$

Thus, $x=x_1,\dots,x_n$ stands for the first set of variables, $y=y_1,\dots,y_n$ for the second set, …, and $z=z_1,\dots,z_n$ for the last set.

With the aim of certain describing polynomials in the variables X, we choose to denote by X_j the j^{th} column of X, for $1\leq j\leq n$. We assume the same convention for any $\ell\times n$ matrix of non-negative integers A. Moreover, if the A_i are the columns of A, we write

$$A=A_1A_2\cdots A_n.$$

We then consider the monomials

$$X_j^{A_j}:=\prod_{i=1}^{\ell} x_{ij}^{a_{ij}},\qquad \text{as well as}\qquad X^A:=\prod_{j=1}^{n} X_j^{A_j}.$$

For the monomials X^A, who clearly form a basis of the space of polynomials $\mathcal{R}_n^{(\ell)}:=\mathbb{Q}[X]$, the corresponding *degree vector*:

$$\deg(X^A):=\sum_j A_j,$$

lies in $\mathbb{N}^\ell$.

Given $r, n \in \mathbb{N}^+$, recall that the *generalized symmetric group* $W = G(r, n)$ may be described as the group of $n \times n$ matrices having exactly one non zero coefficient in each row and each column, which is an r^{th} root of unity. One usually considers W as acting on polynomials in $\mathcal{R}_n^{(\ell)} = \mathbb{Q}[X]$ by replacement of the variables by the matrix $X\,w$. With this point of view, we may consider that W is generated by the *transpositions* s_j, which exchange columns j and $j+1$ in X, together with s_0 which multiplies the first column of X by a (chosen) primitive r^{th} root of unity. These generators s_j satisfy the usual Coxeter relations for $j \geq 1$:

$$\begin{aligned} s_j^2 &= \text{Id}, \qquad (s_j\, s_{j+1})^3 = \text{Id}, \quad \text{and} \\ s_j\, s_k &= s_k\, s_j, \quad \text{when} \quad |j-k| > 1. \end{aligned}$$

For the special pseudo-reflection s_0, we have $s_0^r = \text{Id}$ and $(s_0\, s_1)^{2\,r} = \text{Id}$. This is the diagonal action which is considered for the "usual" definition of the diagonal co-invariant space for W (see [5]). Rather than this space, we consider a variant below.

Our point of departure from the "classical" situation is to consider rather the diagonal *Hivert-action* of W on polynomials. As introduced in [9], this action is described in terms of its effect on monomials. In a first step, the effect of simple transpositions s_j is described, and then one checks the compatibility with the above Coxeter relations. Only the s_j, for $j \geq 1$, are affected. One sets

$$s_j \cdot (\cdots X_j^{A_j} X_{j+1}^{A_{j+1}} \cdots) := \begin{cases} (\cdots X_{j+1}^{A_{j+1}} X_j^{A_j} \cdots) & \text{if} \quad A_j = 0 \quad \text{or} \quad A_{j+1} = 0, \\ (\cdots X_j^{A_j} X_{j+1}^{A_{j+1}} \cdots) & \text{otherwise.} \end{cases} \tag{2.1}$$

In other words, s_j exchanges variables x_{*j} and x_{*j+1}, if and only if there is no "collision" of exponents. For example, we have

$$s_1 \cdot x_1^2\, y_2^3\, y_3^4 = x_1^2\, y_2^3\, y_3^4, \qquad \text{whereas} \qquad s_1 \cdot x_1^2\, y_3^3\, y_4^4 = x_2^2\, y_3^3\, y_3^4.$$

Checking that this is compatible with the Coxeter relations allows one to extend this action to the whole group W.

A polynomial is said to be *diagonally quasi-invariant* for W if

$$w \cdot f(X) = f(X), \qquad \text{for all} \quad w \in W.$$

For example, taking $W = G(1, 3)$ (the symmetric group $\mathbb{S}_3$) and $\ell = 1$, we have the quasi-invariant (or quasi-symmetric) polynomials:

$$\begin{array}{lll} x_1 + x_2 + x_3, & {x_1}^2 + {x_2}^2 + {x_3}^2, & {x_1}^3 + {x_2}^3 + {x_3}^3, \\ x_1x_2 + x_1x_3 + x_2x_3, & x_1{x_2}^2 + x_1{x_3}^2 + x_2{x_3}^2, & {x_1}^2x_2 + {x_1}^2x_3 + {x_2}^2x_3. \end{array}$$

For $\ell = 2$, another $\mathbb{S}_3$-quasi-invariant is the polynomial

$$y_1x_2y_2 + y_1x_3y_3 + y_2x_3y_3.$$

The vector space of diagonally quasi-invariant for $W = G(r, n)$ is spanned by the *monomial basis* $\{M_A\}_{A \in \mathcal{B}_{r,n}}$, which are indexed by *r-composition-matrices*. These are the positive integer entries $\ell \times k$ matrices, $1 \leq k \leq n$, having all column sums congruent to 0 mod r, with no column sum actually vanishing. We say that we have an *r-matrix* if this last condition is dropped. The monomial quasi-invariant associated to such an r-composition-matrix is simply defined as

$$M_A := \sum_{Y \subseteq X} Y^A,$$

with Y running over all matrices obtained by selecting (in the order that they appear) k columns of X. We sometimes write $M[A]$ for $M_A(X)$. It is easy to check directly that this is indeed a basis. For example, we have

$$M\left[\begin{smallmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{smallmatrix}\right] = \sum_{a<b} x_a\, z_a^2\, x_b^3\, y_b.$$

We simply denote by J_W (or often simply by J) the ideal generated by constant-term-free diagonally W-quasi-invariant polynomials.

We may now define our main object of study, which is the *co-quasi-invariant* space:

$$\mathcal{Q}_W^{(\ell)} := \mathcal{R}_n^{(\ell)} / J_W. \tag{2.2}$$

To better analyze the structure of this space, we need to consider the action of the general linear group GL_ℓ on $\mathcal{R}_n^{(\ell)}$, defined by

$$(f \cdot \tau)(X) := f(\tau\, X), \qquad \text{for} \quad \tau \in GL_\ell. \tag{2.3}$$

Observing that the ideal $J = J_W$ is invariant under this action, we conclude that $\mathcal{Q}_W^{(\ell)}$ inherits a GL_ℓ-module structure. Since the ideal J is homogeneous for the vector-degree, $\mathcal{Q}_W^{(\ell)}$ may be graded by this same vector-degree, i.e.:

$$\mathcal{Q}_W^{(\ell)} = \bigoplus_{d \in \mathbb{N}^\ell} \mathcal{Q}_{W,d},$$

with $\mathcal{Q}_{W,d}$ denoting the homogeneous component of degree d of $\mathcal{Q}_W^{(\ell)}$. It follows that the associated *Hilbert series*, $\mathcal{Q}_W^{(\ell)}(\mathbf{q})$, coincides with the character of $\mathcal{Q}_W^{(\ell)}$ as a GL_ℓ-module. To help the reader parse this statement, let us assume that $\mathcal{B}$ is a basis consisting of homogeneous elements of $\mathcal{Q}_W^{(\ell)}$. This means that, for $f(X)$ in $\mathcal{B}$, we have

$$f(\mathbf{q}\, X) = \mathbf{q}^d\, f(X), \tag{2.4}$$

where $\mathbf{q}$ stands for the diagonal matrix

$$\mathbf{q} = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_\ell \end{pmatrix},$$

and $\mathbf{q}^d := q_1^{d_1} \cdots q_\ell^{d_\ell}$. Thus, an homogeneous $f(X)$ is an eigenvector of the linear transform $\mathbf{q}^*$ sending $f(x)$ to $f(\mathbf{q}\, X)$. Recall here that, by definition, the trace of $\mathbf{q}^*$, as a function of the q_i, is the character of $\mathcal{Q}_W^{(\ell)}$. Summing up, the Hilbert series of the space, defined by the expression

$$\mathcal{Q}_W^{(\ell)}(\mathbf{q}) := \sum_{d \in \mathbb{N}^\ell} \mathbf{q}^d\, \dim(\mathcal{Q}_{W,d}), \tag{2.5}$$

coincides with the (also usual) definition of the character of the corresponding (polynomial) representation of GL_ℓ.

The point of this last observation is that $\mathcal{Q}_W^{(\ell)}(\mathbf{q})$ is *Schur-positive*, since Schur functions $s_\mu(\mathbf{q})$ appear as characters of irreducible representations of GL_ℓ. Indeed, the decomposition into irreducibles of the polynomial GL_ℓ-representation $\mathcal{Q}_W^{(\ell)}(\mathbf{q})$ gives a formula of the form (1.1), with μ running through all partitions for which the homogeneous component $\mathcal{Q}_{W,\mu}$ is non-vanishing. For example, for the symmetric group, one finds the following expressions for $\mathcal{Q}_n := \mathcal{Q}_{\mathbb{S}_n}^{(\ell)}(\mathbf{q})$

$$\begin{aligned}
\mathcal{Q}_1 &= 1,\\
\mathcal{Q}_2 &= 1 + s_1(\mathbf{q}),\\
\mathcal{Q}_3 &= 1 + 2\,s_1(\mathbf{q}) + 2\,s_2(\mathbf{q}),\\
\mathcal{Q}_4 &= 1 + 3\,s_1(\mathbf{q}) + 5\,s_2(\mathbf{q}) + 2\,s_{11}(\mathbf{q}) + 5\,s_3(\mathbf{q}),\\
\mathcal{Q}_5 &= 1 + 4\,s_1(\mathbf{q}) + 9\,s_2(\mathbf{q}) + 5\,s_{11}(\mathbf{q}) + 14\,s_3(\mathbf{q})\\
&\qquad\qquad\qquad\qquad +10\,s_{21}(\mathbf{q}) + 14\,s_4(\mathbf{q}).
\end{aligned}$$

These examples exhibit the announced striking "independence" with respect to ℓ.

Before going on with our discussion, let us introduce another GL_ℓ-module which is isomorphic (both as a GL_ℓ-module and a W-module) to the space $\mathcal{Q}_W^{(\ell)}$. For each of the variables $x_{ij} \in X$, consider the partial derivation denoted by $\partial_{x_{ij}}$, or ∂_{ij} for short. For a polynomial $f(X)$, we then denote by $f(\partial_X)$ the differential operator obtained by replacing the variables in X by the corresponding derivation in ∂_X. The space $\mathcal{S}_W^{(\ell)}$ of *diagonally super-harmonic polynomials* with respect to W-quasi-invariants is simply defined to be the set of polynomial solutions $g(X)$ of the system of partial differential equations

$$f(\partial_X)(g(X)) = 0, \qquad \text{for} \quad f(X) \in J\,. \tag{2.6}$$

Evidently, we need only consider a generating set of J for these equations to characterize all solutions. The elementary proof (see [5]) that $\mathcal{Q}_W^{(\ell)}$ and $\mathcal{S}_W^{(\ell)}$ are isomorphic relies on the fact that there is a scalar product for which $\mathcal{S}_W^{(\ell)}$ appears as the orthogonal complement of J_W.

Following our established conventions, $\mathcal{S}_W^{(\ell)}(\mathbf{q})$ stands for the *Hilbert series* of the graded space $\mathcal{S}_W^{(\ell)}$. From the above discussion, this is equal to the Hilbert series $\mathcal{Q}_W^{(\ell)}(\mathbf{q})$. The advantage of working with $\mathcal{S}_W^{(\ell)}$ is that we may present a basis in terms of explicit polynomials (which give canonical representatives for equivalence classes in $\mathcal{Q}_W^{(\ell)}$).

To get a better feeling of how things work out, let us first consider the case $W = \mathbb{S}_3$ and $\ell = 2$. We may then check that we have the following bases $\mathcal{B}_d$ for the various homogeneous components $\mathcal{H}^{(2)}_{3,d}$ of the space $\mathcal{H}^{(2)}_3 = \mathcal{S}^{(2)}_W$.

$$\begin{aligned}
\mathcal{B}^{(2)}_{00} &= \{1\},\\
\mathcal{B}^{(2)}_{10} &= \{-x_1+x_2, -x_1+x_3\},\\
\mathcal{B}^{(2)}_{01} &= \{-y_1+y_2, -y_1+y_3\},\\
\mathcal{B}^{(2)}_{20} &= \{-(x_1-x_2)(x_1-2\,x_3+x_2),\\
&\qquad -(x_1-x_3)(x_1-2\,x_2+x_3)\},\\
\mathcal{B}^{(2)}_{11} &= \{x_2y_2-x_1y_2-x_3y_3+x_1y_3-y_1x_2+y_1x_3,\\
&\qquad x_2y_3-x_3y_3+x_1y_1-y_1x_2-x_1y_2+y_2x_3\},\\
\mathcal{B}^{(2)}_{02} &= \{-(y_1-y_2)(y_1-2\,y_3+y_2),\\
&\qquad -(y_1-y_3)(y_1-2\,y_2+y_3)\}.
\end{aligned}$$

Observe that we can calculate $\mathcal{B}^{(2)}_{0k}$ from $\mathcal{B}^{(2)}_{k0}$ by exchanging all the variables x_i by the corresponding y_i.

By contrast, for $\ell = 3$, the space $\mathcal{S}^{(3)}_3 = \mathcal{S}^{(3)}_W$ affords the following bases. For all d of the form $jk0$, we may choose $\mathcal{B}^{(3)}_{jk0} := \mathcal{B}^{(2)}_{jk}$. To get the bases for the other non-vanishing components of $\mathcal{S}^{(3)}_W$, we set $\mathcal{B}^{(3)}_{j0k}$ equal to the set of polynomials obtained by exchanging the y_i by the corresponding z_i for all elements of $\mathcal{B}^{(3)}_{jk0}$. In turn, we get $\mathcal{B}^{(3)}_{0jk}$ from $\mathcal{B}^{(3)}_{j0k}$, now exchanging the x-variables for the y-variables. It can then be checked that there are no other non-vanishing component in $\mathcal{H}^{(3)}_3$. One may also use Theorem 3.1 to see this. Our point here is that we get the two Hilbert series

$$\begin{aligned}
\mathcal{H}^{(2)}_3(q_1,q_2) &= 1+2\,(q_1+q_2)+2\,(q_1^2+q_2^2+q_1q_2)\\
\mathcal{H}^{(3)}_3(q_1,q_2,q_3) &= 1+2\,(q_1+q_2+q_3)+2\,(q_1^2+q_2^2+q_3^2+q_1q_2+q_1q_3+q_2q_3)
\end{aligned}$$

both taking the form $\mathcal{H}_3(\mathbf{q}) = 1+2\,s_1(\mathbf{q})+2\,s_2(\mathbf{q})$, as announced.

3. General results

Theorem 3.1. *For any given complex reflection group $W = G(r,n)$, the Hilbert series $\mathcal{Q}^{(\ell)}_W(\mathbf{q})$ affords an expansion in terms of Schur functions, with positive integer coefficients that are independent of ℓ, the sum being over the set of partitions of integers d:*

$$0 \le d \le 2\,r\,n - r - n, \tag{3.1}$$

and having at most n parts.

To better underline one of the most important feature of this statement, we may consider that the symmetric function involved in these expressions are written in terms of infinitely many variables

$$\mathbf{q} = q_1, q_2, q_3, \ \ldots$$

This makes formula (1.1) entirely independent of ℓ. To get the Hilbert series in the special case of ℓ sets of n variables, we simply specialize this "universal" formula by setting all variables q_k, for $k > \ell$, equal to zero. This process is made even more transparent by "removing the variables", writing s_μ (or h_μ) instead of $s_\mu(\mathbf{q})$ (or $h_\mu(\mathbf{q})$)[2] in (1.1). In other words, we consider $f(q_1, \ldots, q_\ell)$ as the evaluation, of a (variable free) symmetric functions f, in the set of ℓ variables $q_1, \ldots, q_\ell$. We may also drop the ℓ in $\mathcal{Q}_W^{(\ell)}$.

The following formula, for the case $\ell = 1$, is shown to hold in [1]. Namely, for the group $W = G(r, n)$, we have

$$\mathcal{Q}_W(q, 0, 0, \ldots) = \left(\frac{1-q^r}{1-q}\right)^n \cdot \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} q^{r\,k}. \tag{3.2}$$

In particular, for $n = 2$, we get

$$\mathcal{Q}_W(q, 0, 0, \ldots) = (1 + q + \ldots + q^{r-1})^2 \, (1 + q^r).$$

Since $h_k(q, 0, 0, \ldots) = q^k$, this is readily seen to be the specialization at

$$\mathbf{q} = q, 0, 0, \ldots$$

of the following "universal" formula (see [6]), for the groups $W = G(r, 2)$:

$$\left(\sum_{k=0}^{r-1} h_k\right)^2 + \sum_{k=0}^{r-1} (k+1)\, h_{r+k} + \sum_{k=1}^{r-1} (r-k)\, h_{2\,r-1+k}. \tag{3.3}$$

As such, it holds for the diagonal co-invariant space (under the classical action) which, in this very specific case, coincides with the space of diagonal co-quasi-invariant space.

A nice feature of the expression given in (3.3) is its *h-positivity*:

$$\sum_\mu a_\mu \, h_\mu, \qquad \text{with} \quad a_\mu \geq 0.$$

This appears to hold for many other reflection groups, in particular when W is a symmetric group, leading us to state the following.

Conjecture 3.2. *For the symmetric group, the Hilbert series $\mathcal{Q}_n(\mathbf{q})$ is h-positive.*

An immediate consequence of this conjecture is that $\mathcal{Q}_n(\mathbf{q})$ has to have a very specific form, since

$$\begin{aligned} \mathcal{Q}_n(q, 0, 0, \ldots) &= \sum_\mu a_\mu \, h_\mu(q, 0, 0, \ldots) \\ &= \sum_\mu a_\mu \, q^{|\mu|} \\ &= \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} q^k, \qquad (3.4) \\ &= \sum_\beta q^{\chi(\beta)}, \qquad (3.5) \end{aligned}$$

[2]As in Macdonald [10], we write h_k for the complete homogeneous symmetric functions.

with β running over the set of Dyck paths[3] of height n, and $\chi(\beta)$ taking as value the x-coordinate of the first point of the path at height n. The passage from (3.4) to (3.5) is classical. It follows that

Proposition 3.3. *Conjecture 3.2 implies that $\mathcal{Q}_n(\mathbf{q})$ affords an expression of the form*

$$\mathcal{Q}_n(\mathbf{q}) = \sum_{\beta} h_{\mu(\beta)}(\mathbf{q}), \tag{3.6}$$

with β running over the set of all Dyck paths of height n, and $\mu(\beta)$ some partition of the integer $\chi(\beta)$.

As of this writing, we do not have a rule for producing the partition $\mu(\beta)$ associated to β, which would have to be compatible with the actual values given in (4.4).

4. Low degree components

We discuss now how to get explicit polynomial formulas in the variable n for the coefficient of h_μ, when μ is a partition of a small enough integer. We restrict the discussion to the case $W = \mathbb{S}_n$, but much of it holds in generality. We exploit here the fact that low degree homogeneous components of the spaces $\mathcal{R}_n^{(\ell)}$ and $\mathcal{Q}_n^{(\ell)} \otimes J_n$ are isomorphic. This immediately implies that we have explicit formulas for the relevant homogeneous components of $\mathcal{Q}_n$, since we have the explicit expressions

$$\mathcal{R}_n^{(\ell)}(\mathbf{q}) = (1 + H(\mathbf{q}))^n, \qquad \text{and} \qquad J_n(\mathbf{q}) = \frac{1}{1 - H(\mathbf{q})}, \tag{4.1}$$

where $H(\mathbf{q}) := \sum_{k\geq 1} h_k(\mathbf{q})$. It follows that we may calculate the low degree terms of the Hilbert series $\mathcal{Q}_n(\mathbf{q})$, via the expansion

$$\begin{aligned}(1 + H(\mathbf{q}))^n(1 - H(\mathbf{q})) \;&=\; 1 + (n-1)h_1 + (n-1)h_2 + \tfrac{1}{2}\, n(n-3)h_1^2 + (n-1)h_3 \\ &\quad + \tfrac{1}{6}\, n(n-1)(n-5)h_1^3 + n(n-3)h_1h_2 + \ldots\end{aligned} \tag{4.2}$$

Observe that the coefficients for the various h_μ in $\mathcal{Q}_n(\mathbf{q})$ agree with those in the right-hand side of (4.2), whenever $n \geq 5$. This phenomenon seems to hold for n larger than twice the order of terms calculated.

Let us write $k(\mu)$ for the number of parts of a partition μ, and denote by

$$d_\mu := d_1! d_2! \cdots d_n!$$

the product of the factorials of multiplicities of parts in μ. Here d_i is the multiplicity of the part i. We then easily calculate that the coefficient of $h_\mu(\mathbf{q})$, in the right hand side of (4.2), can be written in the form

$$\frac{(n)_{k(\mu)}}{d_\mu} - \sum_{\nu} \frac{(n)_{k(\nu)}}{d_\nu}, \tag{4.3}$$

[3] See section 5 for more details.

where the summation is over the set of partitions that can be obtained by removing one part of μ. As usual, we denote by $(n)_k$ the product

$$(n)_k := n(n-1)\cdots(n-k+1).$$

Explicit values. Explicit calculations give the following h-positive expressions, in the case of the symmetric group.

$$\begin{aligned}
\mathcal{Q}_1 &= 1\\
\mathcal{Q}_2 &= 1+h_1,\\
\mathcal{Q}_3 &= 1+2\,h_1+2\,h_2,\\
\mathcal{Q}_4 &= 1+3\,h_1+3\,h_2+2\,{h_1}^2+5\,h_3,\\
\mathcal{Q}_5 &= 1+4\,h_1+4\,h_2+5\,{h_1}^2+4\,h_3+10\,h_1h_2+14\,h_4,\\
\mathcal{Q}_6 &= 1+5\,h_1+5\,h_2+9\,{h_1}^2+5\,h_3+18\,h_1h_2+5\,{h_1}^3\\
&\qquad +28\,h_3h_1+14\,{h_2}^2+42\,h_5.
\end{aligned}\tag{4.4}$$

For the groups $W=G(2,n)$, which is the hyperoctahedral group B_n, we have the values

$$\begin{aligned}
\mathcal{Q}_{G(2,2)} &= 1+2\,h_1+h_2+{h_1}^2+2\,h_3+h_4,\\
\mathcal{Q}_{G(2,3)} &= 1+3\,h_1+2\,h_2+3\,{h_1}^2+3\,h_3+{h_1}^3+3\,h_1h_2\\
&\qquad +6\,h_3h_1+2\,h_4+3\,h_1h_4+5\,h_5+6\,h_6+2\,h_7.
\end{aligned}$$

5. Colored quasi-symmetric polynomials

In light of the results and conjecture considered above, we think it worthwhile to reformulate results obtained in [2] from this new perspective. Indeed, the relevant formulas take a new and much nicer format which gives indirect support to our conjecture, since the Hilbert series considered happen to be provably h-positive. This was not noticed at the time of the writing of [2].

Let us consider the subspace of *colored*[4] quasi-symmetric polynomials of the space of diagonal $\mathbb{S}_n$-quasi-invariants (in the context of X being $\ell\times n$ matrix of variables). Contrary to our previous presentation, colored quasi-symmetric polynomials are not defined as invariants. They are rather described in terms of a basis, indexed by "colored composition". Recall that *colored compositions* of length p are $\ell\times p$-matrices

$$C=\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p}\\ c_{21} & c_{22} & \cdots & c_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ c_{\ell 1} & c_{\ell 2} & \cdots & c_{\ell p}\end{pmatrix}$$

with non negative entries, and such that the associated *entries reading word* (obtained by reading column by column from left to right, and each column from top to bottom) avoids the pattern of ℓ consecutive zeros.

[4]These where coined to be the $G(\ell,n)$-quasi-symmetric polynomials (or even B-quasi-symmetric when $\ell=2$) in [2, 4], but this terminology leads to confusion in the present context.

To each colored composition C, we associate a *monomial colored quasi-symmetric functions* by setting:

$$\overline{M}_C := \sum \prod_{1\leq i\leq \ell} \prod_{1\leq k\leq p} x_{i,a_{ik}}^{c_{ik}} \tag{5.1}$$

where the sum is over all choices of a_{ik} such that

$$\begin{aligned} a_{ik} &\leq a_{i+1,k}, &\quad \text{when } 1\leq i<\ell, \quad \text{and}\\ a_{\ell k} &< a_{1,k+1}, &\quad \text{for } 1\leq k<p. \end{aligned}$$

For example, we have

$$\overline{M}\begin{bmatrix} 1 & 3\\ 0 & 1\\ 2 & 0\end{bmatrix} = \sum_{a\leq b<c\leq d} x_a\, z_b^2\, x_c^3\, y_d.$$

An example helps us point out the difference between diagonal $\mathbb{S}_n$-quasi-invariants and colored quasi-symmetric polynomials. For $n=3$ and $\ell=2$, we have the three independent diagonal quasi-symmetric polynomials of degree 2:

$$\begin{aligned} &x_1y_1+x_2y_2+x_3y_3,\\ &x_1y_2+x_1y_3+x_2y_3, \qquad \text{and}\\ &x_2y_1+x_3y_1+x_3y_2, \end{aligned}$$

whereas we have only two

$$\begin{aligned} &x_1y_1+x_2y_2+x_3y_3+x_1y_2+x_1y_3+x_2y_3, \qquad \text{and}\\ &x_2y_1+x_3y_1+x_3y_2, \end{aligned}$$

colored quasi-symmetric polynomials in same degree.

We denote by K the ideal generated by constant-term-free colored quasi-symmetric polynomials assuming that the parameters n and ℓ are unambiguous from the context, and introduce the quotient of the ring of polynomials by the ideal K:

$$\mathcal{C}_n^{(\ell)} := \mathcal{R}_n^{(\ell)}/K. \tag{5.2}$$

The main result of [2] may be elegantly re-coined in terms of an h-positive expansion for the Hilbert series of $\mathcal{C}_n = \mathcal{C}_n^{(\ell)}$. This new formulation has the extra advantage that it has similarities with the formula that we would expect to find under the hypothesis that conjecture 3.2 holds. In order to state this reformulation, we need to recall some notions concerning *Dyck paths*. Recall that such a path is a sequence

$$\beta = p_0, p_1, \ldots, p_{2n}$$

of points $p_i=(x_i,y_i)$ in $\mathbb{N}\times\mathbb{N}$, with $x_i\leq y_i$, $p_0=(0,0)$, $p_{2n}=(n,n)$, and such that either

$$p_{i+1} = \begin{cases} p_i+(1,0) & \text{or,}\\ p_i+(0,1), \end{cases}$$

for all i. We say that n is the *height* of β, and that we have a *horizontal step*

$$s_i := (p_{i-1}, p_i),$$

at *level* $k\geq 1$, if $y_i=k=y_{i+1}$. The set of Dyck paths of height n is denoted by $\mathcal{D}_n$. To any given Dyck path β, we associate the composition $\nu(\beta)$ obtained by counting the number of level k horizontal steps (ignoring the situation when this number is zero), for $k < n$. An example is given in Figure 1, for the Dyck path

$$\begin{aligned}\beta \;=\; & (0,0),(0,1),(0,2),(1,2),(1,3),(1,4),(2,4),(3,4),(4,4),\\ & (4,5),(4,6),(5,6),(6,6),(6,7),(6,8),(7,8),(8,8)\end{aligned}$$

With these notions at hand, we may now give our new formula.

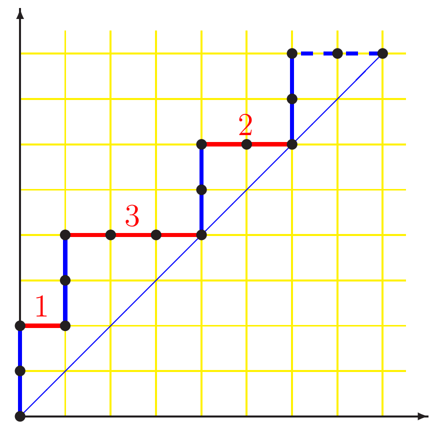

Figure 1. A Dyck path β with $\nu(\beta) = 132$.

Proposition 5.1. *The Hilbert series of the quotient $\mathcal{C}_n$ is given by the formula:*

$$\mathcal{C}_n(\mathbf{q}) = \sum_{\beta\in\mathcal{D}_n} h_{\nu(\beta)}(\mathbf{q}), \tag{5.3}$$

whose dependence on ℓ is entirely encapsulated in the number of variables in $\mathbf{q}$.

Observe that if we set all the q_i equal to 1, we get a combinatorial expression which is interesting on its own:

$$\sum_{\beta\in\mathcal{D}_n} \prod_{k\in\nu(\beta)} \binom{k+\ell-1}{k} = \frac{1}{\ell\, n+1}\binom{(\ell+1)\, n}{n}, \tag{5.4}$$

in which one may consider ℓ as a variable, hence we actually get a polynomial identity[5]. For integral values of ℓ, equation (5.4) may be proven via a simple bijection on paths. Observe also that both spaces $\mathcal{C}_n$ and $\mathcal{Q}_n$ coincide when $\ell = 1$.

6. Proofs

6.1. **Theorem 3.1.** Recall that we are asserting here that for any group $W = G(r,n)$ there exists a universal expression for the Hilbert series of $\mathcal{Q}_W$ of the form

$$\mathcal{Q}_W(\mathbf{q}) = \sum_{\mu} c_\mu\, s_\mu(\mathbf{q}), \qquad c_\mu \in \mathbb{N}, \tag{6.1}$$

[5]It is a well-known fact that the right-hand-side is actually a polynomial in ℓ.

with the sum running over partitions μ of integers $d \leq 2\,r\,n - r - n$, each such partitions having at most n parts. This restriction on the number of parts follows from the fact that this holds for the whole space $\mathcal{R}_n$, of which $\mathcal{Q}_W$ (or rather $\mathcal{S}_W$) can be considered as a subspace.

In order to prove inequality (3.1), we need to introduce some notations. The sum of all the entries an r-matrix A, divided by r, is an integer that we denote by $w_r(A)$. This is said to be the *r-size* of A. To avoid ambiguity, we restrict to r-matrices having a non-vanishing last column. The number of columns of such an r-matrix is its *length*.

Given an r-vector V (a single-column r-matrix), there is a lexicographically largest r-matrix $A(V)$ such that

- all of the columns of $A(V)$ are of r-size 1,
- the sum of the columns of $A(V)$ is V,
- the columns of $A(V)$ occur in decreasing lexicographic[6] order from left to right.

We denote $\theta(V)$ the *first column* of $A(V)$, and set $\Delta(V) := V - \theta(V)$. For $V^{\rm tr} = (2, 3, 4)$ ($A^{\rm tr}$ denotes the transpose of A), we get

$$A(V) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix},$$

hence $\theta(V)^{\rm tr} = (2, 1, 0)$ and $\Delta(V)^{\rm tr} = (0, 2, 4)$. We now associate to any r-vector V the smallest set, denoted by $S(V)$, of r-matrices that contains $A(V)$ and that is closed under the operation that consists in taking sum of consecutive columns. For example, still with $V^{\rm tr} = (2, 3, 4)$, we have

$$S(V) := \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 3 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}$$

For two ℓ-row matrices, $A = A_1 \cdots A_k$ and $B = B_1 \cdots B_j$, the *concatenation AB* is the matrix having columns

$$AB := A_1 \cdots A_k B_1 \cdots B_j.$$

For a general r-composition matrix $A = A_1\, A_2 \ldots A_k$, we define $S(A)$ to be the set obtained by all possible concatenation of matrices successively picked from each of the sets $S(A_i)$.

We now come to the definition of the polynomials $G[A] := G[A](X)$ that are used to prove (3.1). These are indexed by *trans r-matrices*, which is to say r-matrices $A = A_1\, A_2 \ldots A_k$ for which there exists $1 \leq j \leq k$ such that $w_r(A_1 \ldots A_j) \geq j$. It is clear that any r-composition matrix is trans.

[6]Considering entries from top to bottom.

Definition 6.1. *To a trans r-matrix A, we associate the W-quasi-invariant polynomial, $G[A] = G_A(X)$ recursively defined as follows.*

- *If A is an r-composition, we set*
$$G_A := \sum_{B \in S(A)} M_B.$$
- *If not, there is a unique column decomposition of A as a concatenation*
$$A = B\,\mathbf{0}\,V C,$$
where B an r-matrix (say of length j), V is a non-zero r-vector, and C is an r-composition. We then set
$$G[A] := G[B\,V C] - X_{j+1}^{\theta(V)}\, G[B\Delta(V)\, C].$$

It should be clear that when $A = B\,\mathbf{0}\,V C$ is trans, then so are both $B\,V\,C$ and $B\Delta(V)C$. Thus, the family $G[A]$ is well-defined by induction on the length of A.

It is helpful to consider an explicit an example. With $r = 2$ and $n = 4$, we compute that

$$\begin{aligned} G\left[\begin{smallmatrix} 0&2&0&1\\ 0&2&0&3 \end{smallmatrix}\right] &= G\left[\begin{smallmatrix} 0&2&1\\ 0&2&3 \end{smallmatrix}\right] - x_3\, y_3\, G\left[\begin{smallmatrix} 0&2&0\\ 0&2&0 \end{smallmatrix}\right] \\ &= G\left[\begin{smallmatrix} 2&1\\ 2&3 \end{smallmatrix}\right] - x_1^2 G\left[\begin{smallmatrix} 0&1\\ 2&3 \end{smallmatrix}\right] - x_3\, y_3 \left(G\left[\begin{smallmatrix} 2&0\\ 2&2 \end{smallmatrix}\right] - x_1^2\, G\left[\begin{smallmatrix} 0&0\\ 2&2 \end{smallmatrix}\right]\right). \end{aligned}$$

Since

$$\begin{aligned} G\left[\begin{smallmatrix} 2&1\\ 2&3 \end{smallmatrix}\right] &= M\left[\begin{smallmatrix} 2&1\\ 2&3 \end{smallmatrix}\right] + M\left[\begin{smallmatrix} 2&0&1\\ 2&0&3 \end{smallmatrix}\right] + M\left[\begin{smallmatrix} 2&1&0\\ 2&1&2 \end{smallmatrix}\right] + M\left[\begin{smallmatrix} 2&0&1&0\\ 0&2&1&2 \end{smallmatrix}\right], \\ G\left[\begin{smallmatrix} 0&1\\ 2&3 \end{smallmatrix}\right] &= M\left[\begin{smallmatrix} 0&1\\ 2&3 \end{smallmatrix}\right] + M\left[\begin{smallmatrix} 0&1&0\\ 2&1&2 \end{smallmatrix}\right], \\ G\left[\begin{smallmatrix} 2&0\\ 2&2 \end{smallmatrix}\right] &= M\left[\begin{smallmatrix} 2&0\\ 2&2 \end{smallmatrix}\right] + M\left[\begin{smallmatrix} 2&0&0\\ 0&2&2 \end{smallmatrix}\right], \\ G\left[\begin{smallmatrix} 0&0\\ 2&2 \end{smallmatrix}\right] &= M\left[\begin{smallmatrix} 0&0\\ 2&2 \end{smallmatrix}\right], \end{aligned}$$

we finally get

$$G\left[\begin{smallmatrix} 0&2&0&1\\ 0&2&0&3 \end{smallmatrix}\right] = x_2^2\, y_2^2\, x_4\, y_4^3 + x_2^2\, x_3\, y_3^3\, y_4^2 + x_2^2\, y_3^2\, x_4\, y_4^3 + x_3^2\, y_3^2\, x_4\, y_4^3 - x_3^3\, y_3^3\, y_4^2.$$

Observe that the lexicographic order leading monomial of $G\left[\begin{smallmatrix} 0&2&0&1\\ 0&2&0&3 \end{smallmatrix}\right]$ is precisely

$$X^{\left(\begin{smallmatrix} 0201\\ 0203 \end{smallmatrix}\right)} = x_2^2\, y_2^2\, x_4\, y_4^3.$$

This is shown to hold in full generality in the following proposition.

Proposition 6.2. *For any trans r-matrix A, the leading monomial of $G_A(X)$ is X^A.*

Proposition 6.2 is established through the next two lemmas.

Lemma 6.3. *For any r-composition matrix A, we have*
$$G_{\mathbf{0}A} = G_A(X_{-1}), \tag{6.2}$$
writing X_{-1} for the alphabet obtained from X by removing its first column of variables.

Proof. We write $A = VC$, with V the r-vector corresponding to the first column of A. In view of Definition 6.1, we have

$$G_{VC} = X^{\theta(V)} G_{\Delta(V)C} + G_{VC}(X_{-1}) \tag{6.3}$$

which implies (6.2). □

Lemma 6.4. *Let A be an r-matrix of length j, and D an r-composition matrix, then we have*

$$G_{AD}(X) = X^A G_{\mathbf{0}^j D}(X) + (\text{terms} <_{\text{lex}} X^A). \tag{6.4}$$

Proof. If A is an r-composition, (6.4) is a direct consequence of Definition 6.1. If not, equation (6.4) is shown to hold by induction on the length of A. Consider the unique factorization

$$A = B\mathbf{0}VC$$

with B is an r-matrix of length j, V a non-vanishing r-vector, and C an r-composition. We use Definition 6.1 and Lemma 6.3 to calculate that

$$\begin{aligned}
G_{B\mathbf{0}VCD} &= G_{BVCD} - X^{\mathbf{0}^j\theta(V)\mathbf{0}^{n-j-1}} G_{B\Delta(V)CD} \\
&= X^B\, G_{\mathbf{0}^j VCD} - X^{\mathbf{0}^{j-1}\theta(V)\mathbf{0}^{n-j}} X^B\, G_{\mathbf{0}^j \Delta(V)CD} + (\text{terms} <_{\text{lex}} X^B) \\
&= X^B\, G_{\mathbf{0}^{j+1} VCD} + (\text{terms} <_{\text{lex}} X^B) \\
&= X^B\, G_{\mathbf{0}^j VCD}(X_{-1}) + (\text{terms} <_{\text{lex}} X^B) \\
&= X^{B\mathbf{0}VC}\, G_{\mathbf{0}^{p-1} D}(X_{-1}) + (\text{terms} <_{\text{lex}} X^{B\mathbf{0}VC}) \\
&= X^{B\mathbf{0}VC}\, G_{\mathbf{0}^p D} + (\text{terms} <_{\text{lex}} X^{B\mathbf{0}VC}).
\end{aligned}$$

□

Proof of Condition (3.1). The proof is now an easy consequence of the following two observations.

- Any r-matrix A with $w_r(A) = n$ is trans. Thus any monomial X^A with A an r-matrix and $w_r(A) = n$ is the leading monomial of a polynomial in the ideal generated by quasi-invariant polynomials for the group $G(n, r)$.
- Any monomial of total degree strictly greater than $2rn - r - n$ is the multiple of a monomial X^A, with A an r-matrix and for which $w_r(A) = n$. Since it is true for any monomial of such degree, any monomial of degree strictly greater than $2rn - r - n$ lies in the ideal, whence (3.1).

□

6.2. **Colored polynomials.** The main aim of this subsection is to prove Proposition 5.1. Let us first show that formula (5.4) holds for all positive integer values of ℓ, hence it follows that we have a polynomial identity. Recall that an *ℓ-path* is a finite sequence of points $p_i = (x_i, y_i)$ in the plane such that $p_0 = (0,0)$ and

$$p_{i+1} = \begin{cases} p_i + (\ell, 0) & \text{or,} \\ p_i + (0,1). \end{cases}$$

We say that we have an *ℓ-Dyck path* if the path satisfies the further condition that $x_i \leq y_i$ for all i. Let us denote by $\mathcal{D}_n^{(\ell)}$ the set of ℓ-Dyck paths of height $n\,\ell$. It is well-known that ℓ-Dyck paths are enumerated by the *Fuss-Catalan numbers*:

$$\#\mathcal{D}_n^{(\ell)} = \frac{1}{\ell\, n+1}\binom{(\ell+1)n}{n}, \tag{6.5}$$

appearing as the right-hand side of (5.4). To relate this to the left-hand-side of (5.4), we consider *ℓ-colored Dyck paths* of height n, which are simply height n Dyck path β whose horizontal steps, at levels $k < n$, have been *colored* by elements of the set $\{1, 2, \ldots, \ell\}$. Here, we assume that the colors of steps on a same level are weakly increasing from left to right, according to the color order $1 < 2 < \ldots < \ell$. In other words, the coloring is a function γ, associating to each horizontal step s_i of β, a color $\gamma(s_i)$, in such a manner that

$$\gamma(s_i) \leq \gamma(s_{i+1}),$$

if both s_i and s_{i+1} are horizontal steps, hence inevitably at the same level. Thus ℓ-colored Dyck paths are pairs (β, γ), consisting of a path with its coloring.

We establish formula (5.4) by building a bijection between ℓ-colored Dyck paths of height n, and ℓ-Dyck paths of height $n\,\ell$. Given an ℓ-colored Dyck path (β, γ), we denote by a_{kj} the number of level k horizontal steps of color j, and we iteratively construct a path

$$\Phi(\beta, \gamma) = q_0, q_1 \ldots, \tag{6.6}$$

using this data. Starting with $q_0 = (0,0)$, and running through the a_{kj}'s as k goes from 1 to $n-1$ and j goes from 1 to ℓ, we successively add to π

- a_{kj} horizontal steps of length ℓ, followed by
- one vertical step $(0,1)$.

One easily checks that this indeed results in an ℓ-Dyck path of height $n\,\ell$. This transformation is illustrated in Figure 2 with $\{$red, green$\}$ as color set (i.e.: $\ell = 2$). It is easy to check that Φ is a bijection. □

With the intention of giving a proof of Proposition 5.1, let us recall the following result of [2] which generalizes the main result of [3].

Lemma 6.5. *A monomial basis of the quotient $\mathcal{C}_n^{(\ell)}$ is given by the monomials X^A such that $\pi(A)$ is an ℓ-Dyck path.*

This last statement uses the following "encoding" of monomials X^A in terms of lattice paths. The lattice path $\pi(A)$ is obtained by applying the following construction to

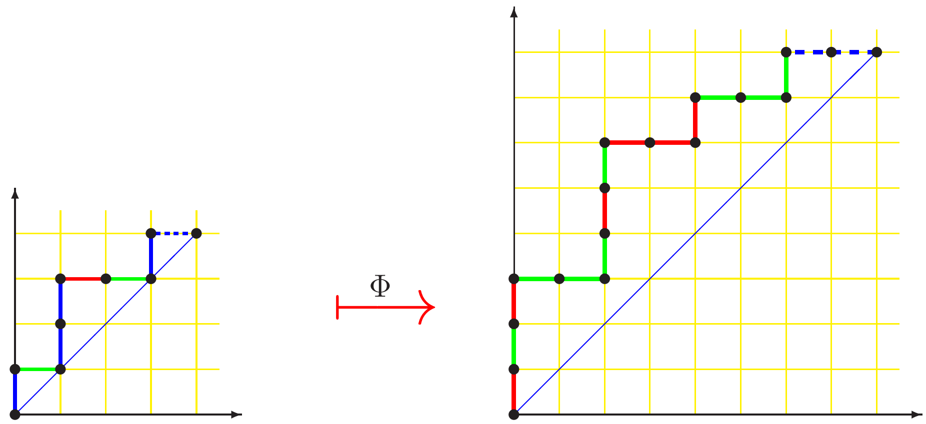

FIGURE 2. The transformation Φ.

entries-reading-word $w(A)$ of the $\ell \times n$ exponent matrix A. Starting with the point $(0,0)$, for each entry a of $w(A)$ we add a horizontal steps $(\ell, 0)$, followed by one vertical step $(0,1)$. For instance, Figure 3 represents the path $\pi(A)$ associated to the monomial

$$X^{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}} = x_1\, x_2\, y_2^2\, y_3,$$

whose exponent matrix has entries-reading-word 101201. Observe that each horizontal step is of length two. One associates variables to each level, namely x_i at even level $2\,(i-1)$, and y_i at odd level $2\,i-1$. We then read out the monomial from the path by the simple device of associating as exponent of the variable of a level, the number of horizontal steps on that level.

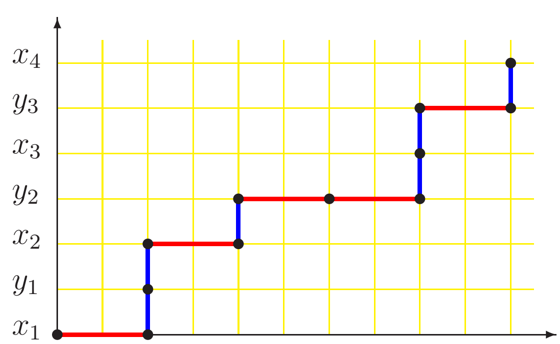

FIGURE 3. Example of $\pi(A)$ for $\ell = 2$.

Proof of Proposition 5.1. For a given Dyck path β, let us consider the set $\mathcal{C}(\beta)$ of ℓ-Dyck path π such that there exists a coloring γ with $\Phi(\beta,\gamma) = \pi$. In formula,

$$\mathcal{C}(\beta) = \{\ \pi \mid \exists \gamma \text{ such that } \Phi(\beta,\gamma) = \pi\},$$

with Φ as in (6.6). If a_k is the number of horizontal steps at level k in β, the choice of ℓ-coloring is equivalent to the choice of a monomial

$$x_{k1}^{a_{k1}} x_{k2}^{a_{k2}} \cdots x_{k\ell}^{a_{k\ell}},$$

with a_{kj} giving the number of steps getting to be colored j, hence $a_k = a_{k1}+\ldots+a_{k\ell}$. The Hilbert series of the resulting set of monomial is $h_{a_k}(\mathbf{q})$. Since there is independence in the choice of colorings at different levels, the Hilbert series of the monomials associated to ℓ-Dyck paths in $\mathcal{C}(\beta)$ is $h_{\nu(\beta)}(\mathbf{q})$. The fact that Φ is a bijection gives the proof of Proposition 5.1, in view of Lemma 6.5. □

7. Open problems

The main remaining open question in all the above considerations is to find a explicit (even conjectural) candidate for partitions $\mu(\beta)$, one for each Dyck path β, which would explain the h-positive expansion in Proposition 3.3. Naturally, similar questions may be stated whenever the universal Hilbert series $\mathcal{Q}_W(\mathbf{q})$, for a group W, happens to be h-positive. As discussed in the paper, the resulting entirely combinatorial description of the universal $\mathcal{Q}_W(\mathbf{q})$ would give, in one compact formula, the GL_ℓ-action characters for all the spaces $\mathcal{Q}_W^\ell$.

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