ON A CONJECTURE ON THE WEAK GLOBAL DIMENSION OF GAUSSIAN RINGS

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Abstract. In [2], Bazzoni and Glaz conjecture that the weak global dimension of a Gaussian ring is $0, 1$ or ∞. In this paper, we prove their conjecture in all cases except when R is a non-reduced local Gaussian ring with nilradical $\mathcal{N}$ satisfying $\mathcal{N}^2 = 0$.

1. Introduction

In her Thesis [10], H. Tsang, a student of Kaplansky introduced Gaussian rings. Noting that the content of a polynomial f over a commutative ring R is the ideal $c(f)$ generated by the coefficients of f, we now define a Gaussian ring.

Definition 1.1. A polynomial $f \in R[x]$ is called Gaussian if $c(f)c(g) = c(fg)$ for all $g \in R[x]$. The ring R is called Gaussian if each polynomial in $R[x]$ is Gaussian.

Among other things, H. Tsang ([10]) proved that an integral domain is Gaussian if and only if it is Prüfer (see Definition 2.3), a result also proved independently by R. Gilmer in [4]. Thus Gaussian rings provide another class of rings extending the class of Prüfer domains to rings with zero divisors. In [2], the authors consider five possible extensions of the Prüfer domain notion to the case of commutative rings with zero divisors, two among which are Gaussian rings and rings with weak global dimension (see Definition 2.2) at most one. The authors also consider the problem of determining the possible values for the weak global dimension of a Gaussian ring. At the end of their article, the authors make the following conjecture.

Conjecture (Bazzoni-Glaz, [2]). *The weak global dimension of a Gaussian ring is either* $0, 1$ *or* ∞.

In [6], the author shows that the weak global dimension of a coherent Gaussian ring is either ∞ or at most one. She also shows that the weak global dimension of a Gaussian ring is at most one if and only if it is reduced. So to prove the conjecture it is enough to show that $\operatorname{w.gl.dim} R = \infty$ for all non-reduced Gaussian rings R. Since $\operatorname{w.gl.dim} R = \sup\{\operatorname{w.gl.dim} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$, it is enough to prove the conjecture for non-reduced local Gaussian rings. For any non reduced local Gaussian ring R with nilradical $\mathcal{N}$, either (i) $\mathcal{N}$ is nilpotent or (ii) $\mathcal{N}$ is not nilpotent. Except when $\mathcal{N}^2 = 0$, the authors of [2] prove that if R satisfies (i), then $\operatorname{w.gl.dim} R = \infty$. In this paper we prove that if R satisfies

2010 *Mathematics Subject Classification.* Primary 13B25,13D05,13F05,16E30 Secondary 16E65.
Key words and phrases. Gaussian rings, weak dimension, content, non-noetherian rings.

(ii), then $\operatorname{w.gl.dim} R = \infty$ (cf. Theorem 5.4). In fact the authors of [2] do not exclude the case $\mathcal{N}^2 = 0$ ([2, Theorem 6.4]), but we exclude this case for correctness (cf. Section 5, Theorem 5.2). In this paper we solve the Bazzoni-Glaz conjecture in all cases except this case. With this in mind, we now state our Main theorem.

Main Theorem. *Let R be a non-reduced local Gaussian ring with nilradical $\mathcal{N}$. If $\mathcal{N}^2 \neq 0$, then* $\operatorname{w.gl.dim}(R) = \infty$.

In a recent paper [1], the authors have validated the Bazzoni-Glaz conjecture for the class of rings called fqp-rings. The class of fqp-rings fall strictly between the classes of arithmetical rings and Gaussian rings.

In Section 3, we consider some homological properties of local Gaussian rings. In particular we consider local Gaussian rings $(R, \mathfrak{m})$ which are not fields, with the property that each element of $\mathfrak{m}$ is a zero divisor. In this case we prove that $\operatorname{w.gl.dim} R \geq 3$.

In [2, Section 6], the authors consider local Gaussian rings $(R, \mathfrak{m})$ such that the maximal ideal $\mathfrak{m}$ coincides with the nilradical of R. With this set up in Section 4, we prove that if $\mathfrak{m}$ is not nilpotent, then $\operatorname{w.gl.dim} R = \infty$.

Finally in Section 5, we prove our main theorem. As a result of our Main Theorem, we reduce the Bazzoni-Glaz conjecture to the following **Conjecture**: Let R be a non-reduced local Gaussian ring with nilradical $\mathcal{N}$. If $\mathcal{N}^2 = 0$, then $\operatorname{w.gl.dim}(R) = \infty$.

Throughout this paper, R is a commutative ring with unit, $(R, \mathfrak{m})$ is a local ring(not necessarily Noetherian) with unique maximal ideal $\mathfrak{m}$. We denote the set of all prime ideals of R by $\operatorname{Spec}(R)$ and the set of all maximal ideals by $\operatorname{Max}(R)$.

2. Preliminary Results

In this section we will recall some definitions and results that we will need in later sections.

Definition 2.1. The flat dimension $fd(M)$ is the minimum integer (if it exists) such that there is a resolution of M by flat R modules $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$. If no finite resolution by flat R modules exists for M, then we set $fd(M) = \infty$.

Now we define the weak global dimension of a ring R denoted as $\operatorname{w.gl.dim}(R)$. It is also sometimes suggestively called as the Tor-dimension.

Definition 2.2. $\operatorname{w.gl.dim}(R)=\sup\{fd(M) \mid M \text{ is an } R\text{-module}\}$.

Recall that $\operatorname{w.gl.dim}(R)=\sup\{d \mid \operatorname{Tor}_d(M, N) \neq 0 \text{ for some } R\text{-modules } M, N\}$. The $\operatorname{w.gl.dim}(R) \leq 1$ if and only if every ideal of R is flat, or equivalently, if and only if every finitely generated ideal of R is flat.

We now define Prüfer domain. A Noetherian Prüfer domain is a Dedekind domain.

Definition 2.3. A domain is Prüfer if every non-zero finitely generated ideal is invertible.

L. Fuchs introduced the class of arithmetical rings in [3].

Definition 2.4. A ring R is arithmetical if the lattice of the ideals of R is distributive.

In [7], the author characterized arithmetical rings by the property that in every localization at a maximal ideal, the lattice of the ideals is linearly ordered by inclusion. Hence in a local arithmetical ring, the lattice of the ideals is linearly ordered by inclusion. Thus local arithmetical rings provide another class of rings extending the class of valuation domains to rings with zero-divisors.

The next theorem appears in Tsang's (see [10]) unpublished thesis.

Theorem 2.5 ([10])**.** *Let R be a Gaussian ring. If R is local, then*

(i) *R is Gaussian if and only if $R_{\mathfrak{m}}$ is Gaussian for all $\mathfrak{m} \in \mathrm{Max}(R)$;*
(ii) *R is Gaussian if and only if $R_{\mathfrak{p}}$ is Gaussian for all $\mathfrak{p} \in \mathrm{Spec}(R)$;*
(iii) *the prime ideals of R are linearly ordered under inclusion; and*
(iv) *the nilradical of R is the unique minimal prime ideal of R.*

We will need several equivalent characterizations of local Gaussian rings, which we now state.

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$. The following conditions are equivalent.*

(i) *R is a Gaussian ring;*
(ii) *If I is a finitely generated ideal of R and $(0 : I)$ is the annihilator of I, then $I/I \cap (0 : I)$ is a cyclic R-module;*
(iii) *Condition (ii) for two generated ideals;*
(iv) *For any two elements $a, b \in R$, the following two properties hold:*
  (a) *$(a, b)^2 = (a^2)$ or (b^2);*
  (b) *If $(a, b)^2 = (a^2)$ and $ab = 0$, then $b^2 = 0$.*
(v) *If $I = (a_1, a_2, \ldots, a_n)$ is a finitely generated ideal of R, then $I^2 = (a_i^2)$ for some $1 \leq i \leq n$.*

The implication $(iv) \Rightarrow (i)$ was noted by Lucas in [8] and the rest of Theorem 2.6 was proved by Tsang in [10]. The next two results can be found in [2].

Theorem 2.7 ([2])**.** *Let $(R, \mathfrak{m})$ be a local Gaussian ring and let $D = \{x \in R \mid x^2 = 0\}$. The following hold:*

(i) *D is an ideal of R, $D^2 = 0$, and R/D is an arithmetical ring;*
(ii) *For every $a \in R$, $(0 : a)$ and D are comparable and $D \subseteq Ra + (0 : a)$;*
(iii) *If $a \in m \setminus D$, then $(0 : a) \subseteq D$;*
(iv) *Let $\mathfrak{m}$ be the nilradical of R. If $\mathfrak{m}$ is not nilpotent, then $\mathfrak{m} = \mathfrak{m}^2 + D$ and $\mathfrak{m}^2 = \mathfrak{m}^3$.*

Proposition 2.8. *Let $(R, \mathfrak{m})$ be a local Gaussian ring. If $\mathfrak{m}$ is non-zero and nilpotent, then* $\mathrm{w.gl.dim}\, \mathfrak{m} = \infty$.

3. Some results on local Gaussian rings

It is well known that if the $\mathrm{w.gl.dim}\, R = n$, then there exists a cyclic R-module, say R/I such that $\mathrm{w.gl.dim}\, R/I = n$. In the next lemma we show that this cyclic module can be chosen with some additional properties.

Lemma 3.1. *Let R be local Gaussian ring with* $\operatorname{w.gl.dim}(R) = n$ *and let I be an ideal of R. If the* $\operatorname{w.gl.dim}_R(R/I) = n$*, then there exists an ideal $J \subset R$ such that* $\operatorname{w.gl.dim}_R(R/J) = n$ *and $J \supseteq I + D$.*

Proof. Let M be an R-module such that $\operatorname{Tor}_n(R/I, M) \neq 0$. Suppose the lemma is not true. Then we prove that the natural projection $R/I \to R/(I + x_1R + \cdots + x_mR)$ induces an inclusion

$$\operatorname{Tor}_n(R/I, M) \hookrightarrow \operatorname{Tor}_n(R/(I + x_1R + \cdots + x_mR), M) \tag{3.1.1}$$

for any finite subset $\{x_1, \ldots, x_m\} \subset D$. Set $I_0 = I$ and define I_p inductively as $I_p = I_{p-1} + x_pR$ for all $1 \leq p \leq m$. We have the following short exact sequence

$$0 \to (x_pR + I_{p-1})/I_{p-1} \to R/I_{p-1} \to R/I_p \to 0$$

for all $1 \leq p \leq m$. The homomorphism $f : R \to (x_pR + I_{p-1})/I_{p-1}$ defined by $f(r) = rx_p + I_{p-1}$ for all $r \in R$ induces an isomorphism $R/\operatorname{Ker} f \cong (x_pR + I_{p-1})/I_{p-1}$. Furthermore we have that $\operatorname{Ker} f \supset I_{p-1} + (0 : x_p) \supset I + D$. If $\operatorname{Tor}_n((x_pR + I_{p-1})/I_{p-1}, M) \neq 0$, then the lemma is true with $J = \operatorname{Ker} f$. So assume that $\operatorname{Tor}_n((x_pR + I_{p-1})/I_{p-1}, M) = 0$. In this case the natural projection $R/I_{p-1} \to R/I_p$ induces an inclusion $\operatorname{Tor}_n(R/I_{p-1}, M) \hookrightarrow \operatorname{Tor}_n(R/I_p, M)$ for all $1 \leq p \leq m$, proving (3.1.1).

Now let $\mathcal{X}$ denote the following class of ideals: $J \in \mathcal{X}$ iff $J \subset D$ and J is finitely generated. Then

$$\varinjlim_{J \in \mathcal{X}} \operatorname{Tor}_n(R/(I + J), M) = \operatorname{Tor}_n(R/(I + D), M) \tag{3.1.2}$$

Using (3.1.1) and (3.1.2), we obtain an inclusion $\operatorname{Tor}_n(R/I, M) \hookrightarrow \operatorname{Tor}_n(R/(I + D), M)$. Thus $\operatorname{Tor}_n(R/(I + D), M) \neq 0$ and the lemma is proved. □

The next lemma is an immediate consequence of the long exact sequence of Tor groups applied to the given short exact sequence. We note it here for the readers convenience.

Lemma 3.2. *Let R be a commutative (not necessarily local Gaussian) ring. Let M_1 and M_2 be R-modules and $f : M_1 \to M_2$ be an injective homomorphism. If the* $\operatorname{w.gl.dim}(R) = n$*, then $f_* : \operatorname{Tor}_n(M_1, -) \to \operatorname{Tor}_n(M_2, -)$ is also injective.*

Proof. The proof is a direct consequence of the long exact sequence of Tor groups applied to the given short exact sequence and the fact that $\operatorname{Tor}_{n+1}(M_2/M_1, -) = 0$. □

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a local Gaussian ring. If the* $\operatorname{w.gl.dim}(R) = n$*, then $\operatorname{Tor}_n(R/D, -) = 0$ for all $n \geq 1$.*

Proof. If the lemma is not true, then there exists a module M such that $\operatorname{Tor}_n(R/D, M) \neq 0$. Consider a free resolution of M: $\cdots \xrightarrow{\partial_{n+2}} R^{X_{n+1}} \xrightarrow{\partial_{n+1}} R^{X_n} \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} R^{X_0} \xrightarrow{\partial_0} M$, where X_i are sets. By assumption $\operatorname{Tor}_n(R/D, M) = \operatorname{Ker}(\overline{\partial_n})/\operatorname{Im}(\overline{\partial_{n+1}}) \neq 0$, where $\overline{\partial_i}$ is the natural homomorphism $\overline{\partial_i} : (R/D)^{X_i} \to (R/D)^{X_{i-1}}$ obtained after tensoring the above resolution by R/D for all $i \in \mathbb{N}$. Since $\operatorname{Tor}_n(R/D, M) \neq 0$, there exists a $\overline{w} \in \operatorname{Ker}(\overline{\partial_n})$ such that $\overline{w} \notin \operatorname{Im}(\overline{\partial_{n+1}})$. Let w be the representative of $\overline{w}$ in R^{X_n}. Hence $\partial_n(w) \in D^{X_{n-1}}$. Let $\lambda_1, \ldots, \lambda_m \in D$ be the finitely many non-zero entries of $\partial_n(w)$. Now we consider two cases.

Case 1. There exists an $a \in \mathfrak{m} \setminus D$ such that $a\lambda_j = 0$ for all $1 \leq j \leq m$.
Define a homomorphism $f : R/D \to R/aD$ which is multiplication by a. Using Theorem 2.7(iii), it follows that $(0 : a) \subset D$. This gives the injectivity of f. Therefore by Lemma 3.2, $f_* : \mathrm{Tor}_n(R/D, M) \to \mathrm{Tor}_n(R/(aD), M)$ is injective and hence $f_*(\overline{w}) \neq 0$. It is easy to verify that aw is a representative of $f_*(\overline{w})$ in R^{X_n}. Since $a \in (0 : \lambda_j)$ for all $1 \leq j \leq m$, we obtain that $\partial_n(aw) = a\partial_n(w) = 0$. This would imply that $f_*(\overline{w}) = 0$, a contradiction.

Case 2. For all $a \in R \setminus D$ at least one $a\lambda_j \neq 0$.
We have an injective homomorphism $g : R/D \to R^m$ defined by $g(1) = (\lambda_1, \ldots, \lambda_m)$. By Lemma 3.2, the induced homomorphism $g_* : \mathrm{Tor}_n(R/D, M) \to \mathrm{Tor}_n(R^m, M)$ is injective. This is a contradiction as $\mathrm{Tor}_n(R^m, M) = 0$. □

Lemma 3.4. *Let $(R, \mathfrak{m})$ be a local Gaussian ring such that each element of $\mathfrak{m}$ is a zero divisor. If $\lambda_1, \ldots, \lambda_n \in \mathfrak{m}$, then there exists a non-trivial element $a \in \mathfrak{m}$ such that $a\lambda_j = 0$ for all $1 \leq j \leq n$.*

Proof. We divide the proof into two cases.

Case 1: $\lambda_1, \ldots, \lambda_n \in D$.

First we claim that $D \neq 0$. Towards that end, let $0 \neq a, b \in \mathfrak{m}$ with $ab = 0$. By Theorem 2.6(iv), we have either $a^2 = 0$ or $b^2 = 0$. Thus $D \neq 0$. So take any $d \in D \setminus 0$. Using Theorem 2.7(i), we obtain that $d\lambda_i = 0$ for all $1 \leq i \leq n$.

Case 2: There exist $j \in \{1, 2, \ldots, n\}$ such that $\lambda_j \notin D$.

By Theorem 2.7(iii), it follows that $(0 : \lambda_j) \subseteq D$. Set $I = (\lambda_1, \ldots, \lambda_n)$. So $(0 : I) \subset (0 : \lambda_j) \subseteq D$. Using Theorem 2.6($ii$), we obtain that $I/I \cap (0 : I)$ is a cyclic R-module, say its generator is λ. Hence we can write $\lambda_i = r_i\lambda + d_i$ for all $1 \leq i \leq n$, where $d_i \in I \cap (0 : I)$ and $r_i \in R$. Observe that $\lambda \in \mathfrak{m} \setminus D$. Choose any $d \in (0 : \lambda) \setminus 0$. Using Theorem 2.7($iii$), it follows that $d \in D$. Multiplying the equation expressing λ_i in terms of λ with d, we obtain $d\lambda_i = dr_i\lambda + dd_i$ for all $1 \leq i \leq n$. Using Theorem 2.7(i), we obtain that $dd_i = 0$. Thus $d\lambda_i = 0$ for all $1 \leq i \leq n$.

□

Lemma 3.5. *Let $(R, \mathfrak{m})$ be a local Gaussian ring with* $\mathrm{w.gl.dim}(R) = n$. *If each element of $\mathfrak{m}$ is a zero divisor, then* $\mathrm{Tor}_n(R/\mathfrak{m}, -) = 0$ *for all $n \geq 1$.*

Proof. The proof of this Lemma follows by substituting $\mathfrak{m}$ for D in Lemma 3.3. As a result of Lemma 3.4, the proof of lemma 3.5 falls under Case 1 of Lemma 3.3. □

Proposition 3.6. *Let $(R, \mathfrak{m})$ be a local Gaussian ring. If $\mathfrak{m} \neq 0$ and each element of $\mathfrak{m}$ is a zero divisor, then* $\mathrm{w.gl.dim}(R) \geq 3$.

Proof. If $\mathfrak{m} = D$, then Proposition 2.8 implies that $\mathrm{w.gl.dim}(R) = \infty$. If $\mathfrak{m} \neq D$, then take any $x \in \mathfrak{m} \setminus D$ and consider the following resolution of R/xR:

$$0 \to (0 : x) \to R \xrightarrow{\sigma} R \xrightarrow{\tau} R/xR,$$

where τ is the natural projection and $\sigma(r) = xr$ for all $r \in R$. If $\mathrm{w.gl.dim}(R) < 3$, then $(0 : x)$ must be flat. Thus it suffices to show that $(0 : x)$ is not flat. We will use the fact that if M is a flat R module then $I \otimes M = IM$ for all ideals $I \subset R$. Set $I = xR$ and $M = (0 : x)$

and observe that $IM = 0$. Hence it suffices to show that $I \otimes (0 : x) \neq 0$. Since $x \notin D$, Theorem 2.7(iii) implies that $(0 : x) \subset D$. Define a homomorphism $\theta : I \otimes (0 : x) \to (0 : x)$ as follows: if $a \in I$ and $b \in (0 : x)$, then set $\theta(a \otimes b) = rb$, where $r \in R$ is such that $a = xr$. If there is another $r' \in R$ such that $a = xr'$, then $(r - r') \in (0 : x)$ which implies that $(r - r')b = 0$. Taking into account the last remark, it is easy to check that θ is well defined. Moreover the homomorphism $\theta' : (0 : x) \to I \otimes (0 : x)$ defined by $\theta'(c) = x \otimes c$ for all $c \in (0 : x)$ is an inverse of θ. Hence we have an isomorphism $\theta : I \otimes (0 : x) \cong (0 : x)$ which shows that $I \otimes (0 : x) \neq 0$, proving that $(0 : x)$ is not flat. □

4. Local Gaussian rings with nilradical being the maximal ideal

Let R be a local Gaussian ring which admits the following property:

Property 4.1. *For all $x \in D \setminus 0$, $(0 : x)$ is not cyclic modulo D. In other words there is no $a \in R \setminus D$ such that $(0 : x) = aR + D$.*

Lemma 4.2. *Let $(R, \mathfrak{m})$ be a local Gaussian ring such that each element of $\mathfrak{m}$ is a zero divisor. If R satisfies Property (4.1) and $\mathfrak{m} \neq D$, then*

(i) *$\mathfrak{m} = \mathfrak{m}^2 + D$;*
(ii) *for any finitely generated ideal $J \subset \mathfrak{m}$ there exist $x \in \mathfrak{m}$ such that $J^2 \subset x^2R$ and $x^2 \notin D$;*
(iii) *$\mathfrak{m}^2$ is flat.*

Proof. (i): Let $a \in \mathfrak{m} \setminus D$. Since every element of $\mathfrak{m}$ is a zero divisor, there exists $x \in D \setminus 0$ such that $ax = 0$. By Property (4.1), $(0 : x) \neq aR + D$. So there exists some $b \in \mathfrak{m}$ such that $b \in (0 : x)$ and $b \notin aR + D$. Theorem 2.7(i) implies that R/D is a local arithmetical ring. So $a \in bR + D$ and hence $a = br + d$ for some $r \in R$ and $d \in D$. Moreover $b \notin aR + D$ which implies that r is not a unit and hence $r \in \mathfrak{m}$. Thus $a \in \mathfrak{m}^2 + D$.

(ii): First we will show that if $x^2 \in D$ for all $x \in \mathfrak{m}$, then $\mathfrak{m}^2 \subset D$. Towards that end let $z \in \mathfrak{m}^2$. Such a z is of the form $z = \sum_{i=1}^{n} x_i y_i$, where $x_i, y_i \in m$ for all $1 \leq i \leq n$. Using Theorem 2.6(iv), if follows that $(x_i, y_i)^2 = (x_i^2)$ or (y_i^2) for all $1 \leq i \leq n$. This shows that $x_i y_i \in D$ for all $1 \leq i \leq n$. Recalling that D is an ideal of R, it follows that $z \in D$. Hence we have proved that $\mathfrak{m}^2 \subset D$. By (i) this would imply that $\mathfrak{m} = D$, a contradiction. Thus there exists an $x \in m$ such that $x^2 \notin D$. By Theorem 2.6(v), for any finitely generated ideal J we have $J^2 = y^2R$ for some $y \in J$. If $y^2 \notin D$ then we are done. If $y^2 \in D$, choose any $x \in \mathfrak{m}$ with $x^2 \notin D$ and observe that $y^2 \in x^2R$. Thus $J^2 \subset x^2R$.

(iii): To prove that $\mathfrak{m}^2$ is flat over R, we show that for any ideal $I \subset R$, the natural homomorphism $f : I \otimes \mathfrak{m}^2 \to \mathfrak{m}^2$ is injective. Assume that $w \in I \otimes \mathfrak{m}^2$ is such that $f(w) = 0$. Set $w = \sum_{i=1}^{k} z_i \otimes x_i y_i$, where $z_i \in I$ and $x_i, y_i \in \mathfrak{m}$. By (ii) there exist $x \in \mathfrak{m}$ such that $x^2 \notin D$ and $x_i y_i \in x^2R$ for all $1 \leq i \leq n$. Put $x_i y_i = x^2 r_i$, where $r_i \in R$. Then $w = z \otimes x^2$, where $z = \sum_{i=1}^{k} z_i r_i \in I$. Hence $f(z \otimes x^2) = 0 \Leftrightarrow zx^2 = 0$. If $z = 0$ then $w = 0$ and the proof is finished. So assume that $z \neq 0$. Using Theorem 2.7(iii), we obtain that $(0 : a) \subseteq D$ for all $a \in \mathfrak{m} \setminus D$. Since $x^2 \in \mathfrak{m} \setminus D$, it follows that $z \in D$ and either $zx = 0$ or $zx \neq 0$. If $zx = 0$, then $z \in D \setminus 0$ and $x \in (0 : z)$. It follows by Property (4.1) that $(0 : z) \neq xR + D$. So there exists $y \in \mathfrak{m}$ such that $y \in (0 : z)$ and $y \notin xR + D$.

By Theorem 2.7(i), we obtain that R/D is a local arithmetical ring. Hence $(y) \not\subset (x)$. So $x = cy + d'$, where $c \in \mathfrak{m}$ and $d' \in D$. Computing w, we obtain

$$w = z \otimes x^2 = z \otimes (cy + d')^2 = z \otimes (c^2y^2 + 2cyd' + d'^2) = zy^2 \otimes c^2 + zd' \otimes 2cy + z \otimes d'^2.$$

Noting that $d'^2, zd' \in D^2 = 0$ and that $zy^2 = 0$, we obtain $w = 0$. If $zx \neq 0$, we have $zx \in D \setminus 0$ and $x \in (0 : zx)$. By (4.1), there exists $h \in \mathfrak{m}$ such that $h \in (0 : zx)$ and $h \notin xR+D$. Using the same argument as above, there exists an $a \in \mathfrak{m}$ such that $x = ah+d''$. Observing that $zd'', d''^2 \in D^2$ we obtain that $w = z \otimes x^2 = z \otimes (ah + d'')^2 = z \otimes a^2h^2$. Furthermore, by (i) we can write $a = b + d$ where $b \in \mathfrak{m}^2$ and $d \in D$. Therefore

$$w = z \otimes (ah^2(b + d)) = z \otimes (ah^2b) + z \otimes (ah^2d) = (zah^2) \otimes b + (zd) \otimes (ah^2). \qquad (4.2.1)$$

Substituting $0 = zd \in D^2$ and $ah = x - d''$ in (4.2.1) and recalling that $h \in (0 : zx)$, we obtain $w = (zxh) \otimes b - zhd'' \otimes b = 0$. □

Lemma 4.3. *Let* $(R, \mathfrak{m})$ *be a local Gaussian ring with* $\mathrm{w.gl.dim}(R) = n$*. Let* $\mathfrak{m}$ *be the nilradical of* R*. If* R *satisfies Property (4.1) and* $\mathfrak{m}$ *is not nilpotent, then* $\mathrm{Tor}_{n-1}(R/\mathfrak{m}, -) = 0$ *for all* $n \geq 3$*.*

Proof. By applying the long exact sequence of Tor groups to the short exact sequence $0 \to \mathfrak{m}^2 \to R \to R/\mathfrak{m}^2 \to 0$ and using Lemma 4.2(iii), it follows that $\mathrm{Tor}_k(R/\mathfrak{m}^2, -) = 0$ for all $k \geq 2$. If $\mathfrak{m} = \mathfrak{m}^2$, then the lemma is proved. So assume that $\mathfrak{m} \neq \mathfrak{m}^2$. Observing that $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over $R/\mathfrak{m}$, we obtain that $\mathfrak{m}/\mathfrak{m}^2 = \bigoplus R/\mathfrak{m}$. Consider the short exact sequence $0 \to \mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}^2 \to R/\mathfrak{m} \to 0$. Consider the following segment of the corresponding long exact sequence of Tor groups

$$\mathrm{Tor}_n(R/\mathfrak{m}, -) \to \mathrm{Tor}_{n-1}(\mathfrak{m}/\mathfrak{m}^2, -) \to \mathrm{Tor}_{n-1}(R/\mathfrak{m}^2, -).$$

Using Lemma 3.5 and Lemma 4.2, we obtain that $\mathrm{Tor}_{n-1}(\mathfrak{m}/\mathfrak{m}^2, -) = 0$. This shows that $\mathrm{Tor}_{n-1}(\bigoplus R/\mathfrak{m}, -) = 0$. Recalling that $\mathrm{Tor}_{n-1}(\bigoplus R/\mathfrak{m}, -) = \bigoplus \mathrm{Tor}_{n-1}(R/\mathfrak{m}, -)$ proves our lemma. □

The idea of the next lemma is taken from [9], but we give a slightly different proof.

Lemma 4.4. *Let* $(R, \mathfrak{m})$ *be a local arithmetical ring with nilradical* $\mathfrak{m}$*. For any* $x \in \mathfrak{m} \setminus 0$*, if* $(0 : x) = I$ *then* $(0 : I) = (x)$*.*

Proof. Clearly $(x) \subseteq (0 : I)$. We want to show that $(0 : I) \subseteq (x)$. Towards that end assume that there exists a $z \in (0 : I)$ such that $z \notin (x)$. Recalling that the ideals in a local arithmetical ring are linearly ordered under inclusion, we obtain that $x = \lambda z$ where $\lambda \in \mathfrak{m}$. Hence $\lambda \notin I$ which implies that $I \subset (\lambda)$. By induction on k, we will show that $I \subset (\lambda^k)$ for all $k \in \mathbb{N}$. The case $k = 1$ is obvious. Let $b \in I$ be arbitrary. Since $I \subset (\lambda)$ there exists a $t \in \mathfrak{m}$ such that $b = \lambda t$. Notice that we have $0 = zb = z\lambda t = xt$. Hence $t \in I = (0 : x)$. By induction hypothesis $I \subset (\lambda^k)$. So $t = \lambda^k t_1$ where $t_1 \in \mathfrak{m}$. Hence $b = \lambda^{k+1}t_1$, where $t_1 \in \mathfrak{m}$. Thus $I \subset (\lambda^{k+1})$ for all $k \in \mathbb{N}$. Since λ is nilpotent, we obtain that $I = 0$, a contradiction. □

In what follows let $R' = R/D$ and $\mathfrak{m}' = \mathfrak{m}/D$. Recall that if R is a local Gaussian ring, then R' is a local arithmetical ring by Theorem 2.7(i).

Lemma 4.5. *Let $(R, \mathfrak{m})$ be a local Gaussian ring with nilradical $\mathfrak{m}$. If* $\mathrm{w.gl.dim}(R) = n \geq 1$, *then*

(i) *there is a non trivial element $x \in \mathfrak{m}'$ such that* $\mathrm{w.gl.dim}_R(R'/xR') = n$;
(ii) *for any non trivial element $z \in \mathfrak{m}'$ and ideal $J \subset R'$ such that $z \in J$, $zR' \neq J$, the natural projection $R'/zR' \to R'/J$ induces a trivial map* $0 : \mathrm{Tor}_n(R'/zR', -) \to \mathrm{Tor}_n(R'/J, -)$;
(iii) $\mathrm{w.gl.dim}_R(R'/zR') = n$ *for any non trivial element $z \in \mathfrak{m}'$.*

Proof. (i): There is an ideal $I \subset R$ such that $\mathrm{w.gl.dim}_R(R/I) = n$. Without loss of generality one can assume that $D \subset I$ (see Lemma 3.1). Hence $\mathrm{w.gl.dim}_R(R'/I') = n$, where $I' = I/D$. Using Lemma 3.3, we obtain that $I' \neq 0$. Let $\mathcal{X}$ denote the following class of ideals: $J \in \mathcal{X}$ iff $J \subset I'$ and J is finitely generated. Since $\mathrm{Tor}_n(R'/I', -) = \varinjlim_{J\in\mathcal{X}} \mathrm{Tor}_n(R'/J, -)$ and $\mathrm{w.gl.dim}_R(R'/I') = n$, there exist $x_1, \ldots, x_m \in I'$ such that $\mathrm{w.gl.dim}_R(R'/(x_1', \ldots, x_m)) = n$. Since R' is local arithmetical, $(x_1, \ldots, x_m) = (x_i)$ for some $1 \leq i \leq m$. Thus the first part of the lemma is proved.

(ii): Let $I = (0 : z) \subset R'$. Using Lemma 4.4, we obtain that $(0 : I) = zR'$. This implies that $(0 : I) \subset J$ and $(0 : I) \neq J$. Hence there exists $y \in I$ such that $(0 : y) \subset J$ and $(0 : y) \neq J$. Thus we have the inclusions $zR' \subset (0 : y) \subset J$ which give rise to the natural projections $R'/zR' \to R'/(0 : y) \to R'/J$. Using Lemma's 3.2 and 3.3, we obtain that $\mathrm{Tor}_n(R'/(0 : y), -) = 0$, since $R'/(0 : y) \cong yR' \subset R'$. Hence the composition of the following maps $\mathrm{Tor}_n(R'/zR', -) \to \mathrm{Tor}_n(R'/(0 : y), -) \to \mathrm{Tor}_n(R'/J, -)$ is trivial.

(iii): By (i) we have a non trivial element $x \in \mathcal{M}'$ such that $\mathrm{w.gl.dim}_R(R'/xR') = n$. For any non trivial element $z \in \mathfrak{m}'$, either $z \in xR'$ or $x \in zR'$. If $z \in xR'$ and $z \neq x$, then there exists $\lambda \in \mathfrak{m}'$ such that $z = \lambda x$. Define a map $\alpha : R'/xR' \to R'/zR'$ by $\alpha(r + xR') = \lambda r + zR'$ for all $r \in R'$. Since $x \notin (0 : \lambda)$, it follows that $(0 : \lambda) \subset xR'$. This shows that α is injective. Using Lemma 3.2, we obtain that α induces an inclusion $\mathrm{Tor}_n(R'/xR', -) \hookrightarrow \mathrm{Tor}_n(R'/zR', -)$. Thus $\mathrm{Tor}_n(R'/zR', -) \neq 0$.

In the case when $x \in zR'$ and $x \neq z$, there exists $\lambda' \in \mathfrak{m}'$ such that $x = \lambda' z$. Define a map $\sigma : R'/zR' \to R'/xR'$ by $\sigma(r + zR') = r + xR'$ for all $r \in R'$. Since $z \notin (0 : \lambda')$, we obtain that $(0 : \lambda') \subset zR'$. Thus σ is injective. Consider the short exact sequence

$$0 \to R'/zR' \xrightarrow{\sigma} R'/xR' \xrightarrow{\tau} R'/\lambda' R' \to 0\,,$$

where τ is the natural projection. Observe that $xR' \subset \lambda' R'$ and $xR' \neq \lambda' R'$. Using (ii), we see that τ induces the trivial map $0 : \mathrm{Tor}_n(R'/xR', -) \to \mathrm{Tor}_n(R'/\lambda' R', -)$. Therefore σ induces an epimorphism

$$\mathrm{Tor}_n(R'/zR', -) \twoheadrightarrow \mathrm{Tor}_n(R'/xR', -)\,,$$

which implies that $\mathrm{Tor}_n(R'/zR', -) \neq 0$. □

Let $\deg(r)$ denote the degree of nilpotency of $r \in R$. Noting that the nilpotency degree of an element $r \in R$ is the smallest $k \in \mathbb{N}$ such that $r^k = 0$, we state our next lemma.

Lemma 4.6. *Let $(R, \mathfrak{m})$ be a local Gaussian ring with nilradical $\mathfrak{m}$ and let $\lambda \in \mathfrak{m}$. If $\mathfrak{m}$ is not nilpotent, then there exists $z \in \mathfrak{m}$ such that* $\deg(z) > \deg(\lambda)$.

Proof. Let $\deg(\lambda) = n$. Suppose the lemma is not true, then $\deg(z) \leq \deg(\lambda)$ for all $z \in \mathfrak{m}$. Now we will show that $\mathfrak{m}^n = 0$. This will give us a contradiction, as $\mathfrak{m}$ is not nilpotent. Towards that end, let $z_1, \ldots, z_n \in \mathfrak{m}$ and consider $I = (z_1, \ldots, z_n)$. Using Theorem 2.6(ii), we can write $z_i = r_i z + d_i$ for some $z \in I$, $r_i \in R$ and $d_i \in I \cap (0 : I) \subset (0 : z_i)$ for all $1 \leq i \leq n$. So

$$z_1 z_2 \cdots z_n = \prod_{i=0}^{n} (z r_i + d_i) \tag{4.6.1}$$

After expanding the right hand side of (4.6.1), observe that every term of the expansion except the term $d_1 \cdots d_n$ contains a z and some d_i, where $1 \leq i \leq n$. Using Theorem 2.7(i), it follows that $d_1 \cdots d_n = 0$. Since $d_i z = 0$ for all $1 \leq i \leq n$, every term in the expansion is zero. Thus $\mathfrak{m}^n = 0$, a contradiction. □

Lemma 4.7. *Let $(R, \mathfrak{m})$ be a local Gaussian ring with nilradical $\mathfrak{m}$. If $\operatorname{w.gl.dim}(R) = n \geq 1$ and $\mathfrak{m}$ is not nilpotent, then $\operatorname{Tor}_n(R'/a\mathfrak{m}', -) = 0$ for all non trivial $a \in \mathfrak{m}'$.*

Proof. If $a\mathfrak{m}' = 0$, then Lemma 3.3 gives the desired result. So assume that $a\mathfrak{m}' \neq 0$. We claim that $a\mathfrak{m}'$ is not a finitely generated ideal. Suppose $a\mathfrak{m}'$ is a finitely generated ideal. Since R' is an arithmetical ring, there exists an element $\lambda \in \mathfrak{m}'$ such that $a\mathfrak{m}' = a\lambda R'$. Let $\deg(x)$ denote the degree of nilpotency of x for all $x \in \mathfrak{m}'$. Since $\mathfrak{m}$ is not nilpotent, $\mathfrak{m}'$ is not nilpotent. By Lemma 4.6, there exists $z \in \mathfrak{m}'$ such that $\deg(z) > \deg(\lambda)$. Observe that $\lambda \in z\mathfrak{m}'$, i.e. $\lambda = zh$ for some $h \in \mathfrak{m}'$. Hence $az \neq 0$. Furthermore $1 - hr$ is a unit for all $r \in R'$. This implies that $az - a\lambda r = a(z - zhr) = az(1 - hr) \neq 0$. Thus $az \notin a\lambda R'$, a contradiction.

Now let $\mathcal{X}$ be the following class of ideals: $J \in \mathcal{X}$ iff $J \subset a\mathfrak{m}'$ and J is finitely generated. Then $\operatorname{Tor}_n(R'/a\mathfrak{m}', -) = \varinjlim_{J \in \mathcal{X}} \operatorname{Tor}_n(R'/J, -)$. Since R' is a local arithmetical ring, there exists $c \in \mathfrak{m}'$ such that $I = cR'$ for all $I \in \mathcal{X}$. As $a\mathfrak{m}'$ is not finitely generated, $I \neq a\mathfrak{m}'$. Using Lemma 4.5(ii), we obtain that the natural projection $R'/I \to R'/a\mathfrak{m}'$ induces a trivial homomorphism $0 : \operatorname{Tor}_n(R'/I, -) \to \operatorname{Tor}_n(R'/a\mathfrak{m}', -)$. Thus the canonical homomorphism $\operatorname{Tor}_n(R'/I, -) \to \varinjlim_{J \in \mathcal{X}} \operatorname{Tor}_n(R'/J, -)$ is trivial for all $I \in \mathcal{X}$. This implies $\varinjlim_{J \in \mathcal{X}} \operatorname{Tor}_n(R'/J, -) = 0$. □

Theorem 4.8. *Let $(R, \mathfrak{m})$ be a local Gaussian ring with nilradical $\mathfrak{m}$. If $\mathfrak{m}$ is not nilpotent, then $\operatorname{w.gl.dim}(R) = \infty$.*

Proof. Suppose the theorem is not true, then the $\operatorname{w.gl.dim}(R) = n < \infty$. Using Proposition 3.6, we obtain that $n \geq 3$. We divide the proof into two cases.

Case 1. R does not satisfy Property (4.1).

Hence there exists $x \in D \setminus 0$ and $a \in R \setminus D$ such that $(0 : x) = aR + D$. Thus we have an isomorphism $R/(aR + D) \cong xR$. Using Lemma 3.2 and noting that $xR \subset R$, we obtain an inclusion $\operatorname{Tor}_n(R/(aR + D), -) \hookrightarrow \operatorname{Tor}_n(R, -)$. Hence $\operatorname{Tor}_n(R/(aR + D), -) = 0$. But using Lemma 4.5($iii$), we obtain that $\operatorname{w.gl.dim}_R(R/(aR + D)) = n$, a contradiction.

Case 2. R satisfies Property (4.1).

Consider the short exact sequence $0 \to aR'/a\mathfrak{m}' \to R'/a\mathfrak{m}' \to R'/aR' \to 0$. From the corresponding long exact sequence of Tor groups, consider the following segment

$$\mathrm{Tor}_n(R'/a\mathfrak{m}', -) \to \mathrm{Tor}_n(R'/aR', -) \to \mathrm{Tor}_{n-1}(aR'/a\mathfrak{m}', -)\,.$$

Applying Lemma 4.7, we obtain that $\mathrm{Tor}_n(R'/a\mathfrak{m}', -) = 0$. Observing that $aR'/a\mathfrak{m}' \cong R/\mathfrak{m}$ and using Lemma 4.3 yields $\mathrm{Tor}_{n-1}(aR'/a\mathfrak{m}', -) = 0$. Hence $\mathrm{Tor}_n(R'/aR', -) = 0$. But using Lemma 4.5 (*iii*), we obtain that $\mathrm{w.gl.dim}_R(R'/aR') = n$, a contradiction. □

5. Conjecture

In this section, we restate [2, Theorem 6.4] with an additional hypothesis and prove the theorem under this additional hypothesis. We also give an example to show that the proof of Theorem 6.4 as given in [2] is not complete. We need the next lemma to give a proof of the modification of [2, Theorem 6.4]. We can use the same idea to give a proof of our Main Theorem.

Lemma 5.1. *Let R be a local Gaussian ring with nilradical $\mathcal{N}$. If $\mathcal{N} \neq D$, then the maximal ideal of $R_{\mathcal{N}}$ is non-zero.*

Proof. Using Theorem 2.5, it follows that the nilradical $\mathcal{N}$ is the unique minimal prime ideal of R. Thus the maximal ideal and the nilradical of $R_{\mathcal{N}}$ coincide and let us denote it by $\mathcal{N}'$. We want to show that $\mathcal{N}' \neq 0$. Towards that end, let $x \in \mathcal{N} \setminus D$. We will show that $0 \neq \frac{x}{1} \in R_{\mathcal{N}}$. Suppose not, then there exists $y \in R \setminus \mathcal{N}$ such that $xy = 0$. Using Theorem 2.6(*iv*), it follows that $x^2 = 0$ or $y^2 = 0$, a contradiction. □

Noting that the nilpotency degree of an ideal I of R is the smallest $k \in \mathbb{N}$ such that $I^k = 0$, we now restate and prove Theorem 6.4 of [2].

Theorem 5.2. *Let R be a Gaussian ring admitting a maximal ideal $\mathfrak{m}$ such that the nilradical $\mathcal{N}$ of the localization $R_\mathfrak{m}$ is a non-zero nilpotent ideal. If the nilpotency degree of $\mathcal{N} \geq 3$, then* $\mathrm{w.gl.dim}(R) = \infty$.

Proof. Let $\mathfrak{m}$ be a maximal ideal of R such that $R_\mathfrak{m}$ has a non-zero nilpotent nilradical $\mathcal{N}$. Using Theorem 2.5, it follows that $\mathcal{N}$ is the unique minimal prime ideal. Recall that the Nilradical($S^{-1}R$)=S^{-1}(Nilradical(R)) for any multiplicative closed set $S \subset R$. Hence $\mathcal{N} = S^{-1}\mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of R and $S = R \setminus \mathfrak{m}$ is a multiplicatively closed set in R. Furthermore $\mathfrak{n}$ is a prime ideal of R. It is clear that the maximal ideal of $R_\mathfrak{n}$ is nilpotent. It follows from Lemma 5.1 that the maximal ideal of $R_\mathfrak{n}$ is non-zero. The rest of the proof follows [2, Theorem 6.4] mutatis mutandis. □

Remark. The hypotheses that the nilpotency degree of $\mathcal{N} \geq 3$ in the above Theorem ensures that $\mathcal{N} \neq D$.

We now give an example to show that the hypothesis on the nilpotency degree in Theorem 5.2 is necessary for the conclusion of Lemma 5.1 to hold.

Example 5.3. *let* $\mathbf{k}$ *be a field and* $\mathbf{k}[X,Y]$ *be a polynomial ring in two variables. Consider a set* $S \subset \mathbf{k}[X,Y]/(XY,Y^2)$ *defined by*

$$S = \{a + bY + a_1X + a_2X^2 + \cdots + a_nX^n,\ a, b, a_i \in \mathbf{k}, a \neq 0, n \geq 0\}.$$

Then S *is multiplicative set in* $\mathbf{k}[X,Y]/(XY,Y^2)$. *Define* $R = S^{-1}\big(\mathbf{k}[X,Y]/(XY,Y^2)\big)$. *It is easy to see that the unique maximal ideal of* R *is given by* $\mathfrak{m} = \{xf(x) + b_1y \mid f(x) \in \mathbf{k}[x]$ *and* $b_1 \in \mathbf{k}\}$, *where* x, y *are the images of* X, Y *in* R. *Any* $c, d \in \mathfrak{m}$ *has the form,* $c = \lambda_1 y + c_1x + c_2x^2 + \cdots c_nx^n$ *and* $d = \lambda_2 y + d_1x + d_2x^2 + \cdots + d_mx^m$, *where* $m, n \in \mathbb{N}$ *and* $c_i, d_j \in \mathbf{k}$ *for* $1 \leq i \leq n$ *and* $1 \leq j \leq m$. *Let* $i, j \in \mathbb{N}$ *be such that* $c_i \neq 0$ *and* $d_j \neq 0$ *and let* i, j *be the least natural numbers with this property. Then one can rewrite* c, d *as* $c = \lambda_1 y + x^i(c_i + c'_{i+1}x + \cdots + c'_nx^{n-i})$ *and* $d = \lambda_2 y + x^j(d_j + d'_{j+1}x + \cdots + d'_mx^{m-j})$, *where* $c'_t = c_{t+1}c_i^{-1}$ *and* $d'_s = d_{s+1}d_j^{-1}$ *for all* $i \leq t \leq n-i$ *and* $j \leq s \leq n-j$. *Observe that* $c_i + c'_{i+1}x + \cdots + c'_nx^{n-i}$ *and* $d_j + d'_{j+1}x + \cdots + d'_mx^{m-j}$ *are units in* R. *Furthermore* $(c,d)^2 = (c^2, cd, d^2)$, $c^2 = x^{2i}u_1^2$, $d^2 = x^{2j}u_2^2$ *and* $cd = x^{i+j}u_1u_2$, *where* $u_1, u_2 \notin \mathfrak{m}$. *Now using Theorem 2.6(iv), it can be verified that* R *is a local Gaussian ring. Its nilradical* $\mathfrak{n} = (y) \subset R$ *is not trivial, while the nilradical of* $R_{\mathfrak{n}}$ *is trivial as* $\frac{y}{1} = \frac{xy}{x} = 0$.

We now prove our Main Theorem.

Theorem 5.4 (Main Theorem)**.** *Let* R *be a non-reduced local Gaussian ring with nilradical* $\mathcal{N}$. *If* $\mathcal{N} \neq D$, *then* $\operatorname{w.gl.dim}(R) = \infty$.

Proof. By Theorem 2.5, the nilradical $\mathcal{N}$ is the unique minimal prime ideal. Thus the maximal ideal and the nilradical of $R_{\mathcal{N}}$ coincide and let us denote it by $\mathcal{N}'$. Since $\operatorname{w.gl.dim}(R) \geq \operatorname{w.gl.dim}(R_{\mathcal{N}})$, it suffices to show that $\operatorname{w.gl.dim}(R_{\mathcal{N}}) = \infty$. Using Lemma 5.1, it follows that $\mathcal{N}' \neq 0$. Hence we have a local Gaussian ring $(R_{\mathcal{N}}, \mathcal{N}')$ with $\mathcal{N}' \neq 0$. If $\mathcal{N}$ is nilpotent, then Theorem 5.2 implies that $\operatorname{w.gl.dim}(R_{\mathcal{N}}) = \infty$. If $\mathcal{N}$ is not nilpotent, then Theorem 4.8 implies that $\operatorname{w.gl.dim}(R_{\mathcal{N}}) = \infty$. □

We claim that to prove the Bazzoni-Glaz Conjecture, it remains to consider the case of a local Gaussian ring with nilradical $\mathfrak{n} = D \neq 0$. Let R be a Gaussian ring (not necessarily local) and let $\mathfrak{n}_{\mathfrak{p}}$ denote the nilradical of $R_{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Spec}(R)$. We have the following cases:

(i) $R_{\mathfrak{p}}$ is domain for all $\mathfrak{p} \in \operatorname{Spec}(R)$;
(ii) there exists a $\mathfrak{p} \in \operatorname{Spec}(R)$ such that the $\mathfrak{n}_{\mathfrak{p}} \neq 0$ and $\mathfrak{n}_{\mathfrak{p}}^2 \neq 0$;
(iii) there exists a $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{n}_{\mathfrak{p}} \neq 0$ and $\mathfrak{n}_{\mathfrak{p}}^2 = 0$.

We remind the reader that if $R_{\mathfrak{p}}$ is not a domain, then $\mathfrak{n}_{\mathfrak{p}} \neq 0$ and hence all possible cases are listed above. In case (i) $\operatorname{w.gl.dim}(R) \leq 1$, while in case (ii) $\operatorname{w.gl.dim}(R) = \infty$. Hence to prove the Bazzoni-Glaz Conjecture it remains to show the following conjecture.

Conjecture. *Let* R *be a non-reduced local Gaussian ring with nilradical* $\mathcal{N}$. *If* $\mathcal{N}^2 = 0$, *then* $\operatorname{w.gl.dim}(R) = \infty$.

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