Stratifications of derived categories from tilting modules over tame hereditary algebras

Hongxing Chen and **Changchang Xi***

School of Mathematical Sciences, Beijing Normal University,
Laboratory of Mathematics and Complex Systems,
100875 Beijing, People's Republic of China
E-mail: chx19830818@163.com(H.X.Chen) xicc@bnu.edu.cn(C.C.Xi)

Abstract

In this paper, we consider the endomorphism algebras of infinitely generated tilting modules of the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ over tame hereditary k-algebras R with k an arbitrary field, where $R_{\mathcal{U}}$ is the universal localization of R at an arbitrary set $\mathcal{U}$ of simple regular R-modules, and show that the derived module category of $\mathrm{End}_R(R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R)$ is a recollement of the derived module category $\mathscr{D}(R)$ of R and the derived module category $\mathscr{D}(\mathbb{A}_{\mathcal{U}})$ of the adèle ring $\mathbb{A}_{\mathcal{U}}$ associated with $\mathcal{U}$. When k is an algebraically closed field, the ring $\mathbb{A}_{\mathcal{U}}$ can be precisely described in terms of Laurent power series ring $k((x))$ over k. Moreover, if $\mathcal{U}$ is a union of finitely many cliques, we give two different stratifications of the derived category of $\mathrm{End}_R(R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R)$ by derived categories of rings, such that the two stratifications are of different finite lengths.

1 Introduction

Tilting modules over tame hereditary algebras have played a special role in the development of the representation theory of algebras: Finite-dimensional tilting modules provide a class of minimal representation-infinite algebras which can be used together with the covering techniques in [4] to judge whether an algebra is of finite representation type or not, while infinite-dimensional tilting modules involve the generic modules discovered by Ringel in [27], Prüfer modules and adic modules. Recently, Angeleri-Hügel and Sánchez classify all tilting modules over tame hereditary algebras up to equivalence in [3]. One of the main ingredients of their classification involves the universal localizations at simple regular modules, which were already studied by Crawley-Boevey in [13]. It is worthy to note that Krause and Stovicek show very recently in [21] that over hereditary rings universal localizations and ring epimorphisms coincide. For finite-dimensional tilting modules over tame hereditary algebras, their endomorphism algebras have been well understood from the view of torsion theory and derived categories (see [7], [18], [19], [28], and others). Especially, in this case, there are derived equivalences between the given tame hereditary algebras and the endomorphism algebras of titling modules. However, for infinite-dimensional tilting modules, one cannot get such derived equivalences any more (see [5]). Nevertheless, if they are good tilting modules, then the derived module categories of their endomorphism rings admit recollements by derived module categories of the given tame hereditary algebras themselves on the one side, and of certain universal localizations of their endomorphism rings on the other side, as shown by a general result in [8]. Here, not much is known about the precise structures of these universal localizations as well as the composition factors of these recollements. In fact, it seems to be very difficult to describe them in general.

* Corresponding author. Email: xicc@bnu.edu.cn; Fax: 0086 10 58802136; Tel.: 0086 10 58808877.
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In the present paper, we will study these new recollements arising from a class of good tilting modules over tame hereditary algebras more explicitly. In this special situation, we can describe precisely the universal localizations appearing in the recollements in terms of adèle rings which occur often in algebraic number theory (see [24, Chapter V], determine their derived composition factors, and provide two completely different stratifications of the derived module categories of the endomorphism rings of these tilting modules.

Let R be an indecomposable finite-dimensional tame hereditary algebra over an arbitrary field k. Of our interest are simple regular R-modules. Now, we fix a complete set $\mathscr{S}$ of all non-isomorphic simple regular R-modules, and consider the equivalence relation $\sim$ on $\mathscr{S}$ generated by

$$L_1 \sim L_2 \quad \text{for } L_1, L_2 \in \mathscr{S} \text{ if } \operatorname{Ext}^1_R(L_1, L_2) \neq 0.$$

The equivalence classes of this relation are called cliques (see [13]). It is well known that all cliques are finite, and all but at most three cliques consist of only one simple regular module. For a simple regular R-module L, we denote by $\mathscr{C}(L)$ the clique containing L. Similarly, for a subset $\mathcal{V}$ of $\mathscr{S}$, we denote by $\mathscr{C}(\mathcal{V})$ the union of all cliques $\mathscr{C}(L)$ with $L \in \mathcal{V}$.

Let $\mathcal{C}$ be a clique and $V \in \mathcal{C}$. Then there is a unique Prüfer R-module, denoted by $V[\infty]$, such that its regular socle is equal to V (see [27]). Moreover, for any two non-isomorphic simple regular modules in $\mathcal{C}$, the endomorphism rings of the Prüfer modules corresponding to them are isomorphic (see, for instance, Lemma 3.1(3)). Hence we define $D(\mathcal{C})$ to be $\operatorname{End}_R(V[\infty])$ for an arbitrary but fixed module $V \in \mathcal{C}$. It is shown that this ring is a (not necessarily commutative) discrete valuation ring. Therefore, the so-called division ring $Q(\mathcal{C})$ of fractions of $D(\mathcal{C})$ exists, which is the "smallest" division ring containing $D(\mathcal{C})$ as a subring up to isomorphism.

Let $\mathcal{U} \subseteq \mathscr{S}$ be a set of simple regular modules, and let $R_\mathcal{U}$ stand for the universal localization of R at $\mathcal{U}$ in the sense of Schofield and Crawley-Boevey. Then it is proved in [2] that the R-module $T_\mathcal{U} := R_\mathcal{U} \oplus R_\mathcal{U}/R$ is a tilting module. Following [3, Example 1.3], if $\mathcal{U}$ is a union of cliques, the R-module $T_\mathcal{U}$ is called the Reiten-Ringel tilting module associated with $\mathcal{U}$. This class of modules was studied first in [27] and generalized then in [25]. As a main objective of the present paper, we will concentrate us on the derived categories of the endomorphism rings of tilting modules $T_\mathcal{U}$ for arbitrary subsets $\mathcal{U}$ of $\mathscr{S}$.

Let $k[[x]]$ and $k((x))$ be the algebras of formal and Laurent power series over k in one variable x, respectively. For an index set I, we define the I-adèle ring of $k((x))$ by

$$\mathbb{A}_I := \left\{ \big(f_i\big)_{i\in I} \in \prod_{i\in I} k((x)) \;\middle|\; f_i \in k[[x]] \text{ for almost all } i \in I \right\},$$

where $\prod_{i\in I} k((x))$ stands for the direct product of I copies of $k((x))$. In particular, if I is a finite set, then $\mathbb{A}_I = k((x))^{|I|}$.

Our main result in this paper is the following theorem, which provides us a class of new recollements different from the one obtained by the structure of triangular matrix rings.

Theorem 1.1. *Let R be an indecomposable finite-dimensional tame hereditary algebra over an arbitrary field k. Let $\mathcal{U} = \mathcal{U}_0 \dot\cup \mathcal{U}_1$ be a non-empty set of simple regular R-modules, where $\mathcal{U}_0$ contains no cliques and $\mathcal{U}_1$ is the union of all cliques $\{\mathcal{C}_i\}_{i\in I}$ contained in $\mathcal{U}$, where I is an index set, and let B be the endomorphism ring of $R_\mathcal{U} \oplus R_\mathcal{U}/R$, where $R_\mathcal{U}$ stands for the universal localization of R at $\mathcal{U}$. Then there is the following recollement of derived module categories:*

$$\mathscr{D}(\mathbb{A}_\mathcal{U}) \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(R)\,,$$

where $\mathbb{A}_\mathcal{U}$ is the I-adèle ring with respect to the rings $Q(\mathcal{C}_i)$ for $i \in I$, that is,

$$\mathbb{A}_\mathcal{U} := \left\{ \big(f_i\big)_{i\in I} \in \prod_{i\in I} Q(\mathcal{C}_i) \;\middle|\; f_i \in D(\mathcal{C}_i) \textit{ for almost all } i \in I \right\}.$$

In particular, if k is algebraically closed, then $\mathbb{A}_{\mathcal{U}}$ is isomorphic to the I-adèle ring $\mathbb{A}_I$ of the Laurent power series ring $k((x))$.

Note that if the field k is algebraically closed then the set $\mathscr{S}$ of all non-isomorphic simple regular R-modules can be parameterized by the projective line $\mathbb{P}^1(k)$, and the adèle ring $\mathbb{A}_{\mathbb{P}^1(k)}$ is the same as the adèle ring $\mathbb{A}_{k(x)}$ of the rational function field $k(x)$ in global class field theory (see [24, Chapter VI] and [16, Theorem 2.1.4] for details). Thus, the adèle ring $\mathbb{A}_{k(x)}$ occurs in our recollement of Theorem 1.1 for the Reiten-Ringel tilting R-module $R_{\mathscr{S}} \oplus R_{\mathscr{S}}/R$.

As a consequence of Theorem 1.1, we can obtain new stratifications of the derived categories of the endomorphism rings of tilting modules arising from universal localizations at simple regular modules.

Corollary 1.2. *Let R be an indecomposable finite-dimensional tame hereditary algebra over an algebraically closed field k. Let r be the number of non-isomorphic simple R-modules. Suppose that $\mathcal{U}$ is a non-empty finite subset of $\mathscr{S}$ consisting of s cliques. Let B be the endomorphism ring of the Reiten-Ringel tilting R-module associated with $\mathcal{U}$. Then $\mathscr{D}(B)$ admits two stratifications by derived module categories, one is of length $r+s$ with the composition factors: r copies of the ring k and s copies of the ring $k((x))$, and the other is of length $r+s-1$ with the composition factors: $r-2$ copies of the ring k, s copies of the ring $k[[x]]$ and one copy of a Dedekind integral domain contained in the ring $k(x)$.*

Observe that if R is the Kronecker algebra and $\mathcal{U}$ consists of one simple regular module, then we re-obtain the stratifications, shown in the example of [8, Section 8], from Corollary 1.2.

Now, let us state the structure of this paper. In Section 2, we fix notations and recall some definitions and basic facts which will be used throughout the paper. In Section 3, we first prepare with a few lemmas, and then prove the main result, Theorem 1.1. In Section 4, we first consider the case of general tame hereditary algebras, and then turn to the special case of the Kronecker algebra. With these preparations in hand, together with a result in [20], we can determine the derived composition factors of the derived categories of the endomorphism rings of Reiten-Ringel tilting modules, and therefore get a proof of Corollary 1.2.

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2 Preliminaries

First, we recall some standard notations which will be used throughout this paper.

All rings considered are assumed to be associative and with identity, all ring homomorphisms preserve identity, and all full subcategories $\mathcal{D}$ of a given category $\mathcal{C}$ are closed under isomorphic images, that is, if X and Y are objects in $\mathcal{C}$, then $Y \in \mathcal{D}$ whenever $Y \simeq X$ with $X \in \mathcal{D}$.

Let R be a ring.

We denote by R-Mod the category of all unitary left R-modules, and by R-mod the category of finitely generated unitary left R-modules. Unless stated otherwise, by an R-module we mean a left R-module. For an R-module M, we denote by $\mathrm{add}(M)$ $\big($respectively, $\mathrm{Add}(M)\big)$ the full subcategory of R-Mod consisting of all direct summands of finite (respectively, arbitrary) direct sums of copies of M. If I is an index set, we denote by $M^{(I)}$ the direct sum of I copies of M.

If $f: M \to N$ is a homomorphism of R-modules, then the image of $x \in M$ under f is denoted by $(x)f$ instead of $f(x)$. Also, for any R-module X, the induced morphisms $\mathrm{Hom}_R(X,f): \mathrm{Hom}_R(X,M) \to \mathrm{Hom}_R(X,N)$ and $\mathrm{Hom}_R(f,X): \mathrm{Hom}_R(N,X) \to \mathrm{Hom}_R(M,X)$ are denoted by f^* and f_*, respectively.

Given a class $\mathcal{U}$ of R-modules, we denote by $\mathcal{F}(\mathcal{U})$ the full subcategory of R-Mod consisting of all those R-modules M which have a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that M_i/M_{i-1} is isomorphic to a module in $\mathcal{U}$ for each i. We say that M is a direct union of finite extensions of modules in $\mathcal{U}$ if M is the direct limit of a direct system of submodules of M belonging to $\mathcal{F}(\mathcal{U})$.

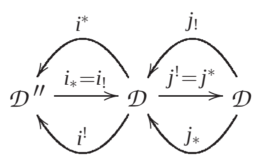
Let $\mathscr{D}(R)$ be the (unbounded) derived category of R-Mod, which is the localization of the homotopy category of R-Mod at all quasi-isomorphisms. Furthermore, we always identify R-Mod with the full subcategory of $\mathscr{D}(R)$ consisting of all stalk complexes concentrated on degree zero. It is well known that $\mathrm{Hom}_{\mathscr{D}(R)}(X, Y[n]) \simeq \mathrm{Ext}^n_R(X, Y)$ for any $X, Y \in R$-Mod and $n \in \mathbb{N}$, where $[n]$ stands for the n-th shift functor of $\mathscr{D}(R)$, and that the triangulated category $\mathscr{D}(R)$ has small coproducts, that is, coproducts indexed by sets exist in $\mathscr{D}(R)$.

If R is an Artin k-algebra over a commutative Artin ring k, we denote by D the usual duality, and by τ the Auslander-Reiten translation of R.

Now, let us recall the definition of recollements of triangulated categories. This notion was first introduced by Beilinson, Bernstein and Deligne in [6] to study the triangulated categories of perverse sheaves over singular spaces, and later was used by Cline, Parshall and Scott in [10] to stratify the derived categories of quasi-hereditary algebras arising from the representation theory of semisimple Lie algebras and algebraic groups.

Let $\mathcal{D}$ be a triangulated category with small coproducts. We denote by $[1]$ the shift functor of $\mathcal{D}$.

Definition 2.1. [6] Let $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$ if there are six triangle functors $i_*, i^*, i^!, j^!, j_*$ and $j_!$ as in the following diagram



such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs,

(2) i_*, j_* and $j_!$ are fully faithful,

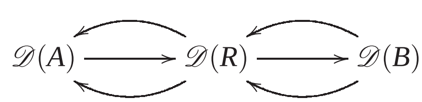
(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$),

(4) for each object $C \in \mathcal{D}$, there are two triangles in $\mathcal{D}$:

$$i_! i^!(C) \longrightarrow C \longrightarrow j_* j^*(C) \longrightarrow i_! i^!(C)[1] \quad \text{and} \quad j_! j^!(C) \longrightarrow C \longrightarrow i_* i^*(C) \longrightarrow j_! j^!(C)[1].$$

In the following, if $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$, we also say that there is a recollement among $\mathcal{D}'$, $\mathcal{D}$ and $\mathcal{D}''$, or very briefly, that $\mathcal{D}$ admits a recollement.

A well known example of recollements of derived categories of rings is given by triangular matrix rings: Suppose that A, B are rings, and that M is an A-B-bimodule. Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be the triangular matrix ring associated with A, B and M. Then there is a recollement of derived categories:



A generalization of this situation is the so-called stratifying ideals defined by Cline, Parshall and Scott, and can be found in [10].

Another type of examples of recollements of derived categories of rings appears in the tilting theory of infinitely generated tilting modules over arbitrary rings (see [8]). Before we state this kind of examples, we recall first the definition of tilting modules over arbitrary rings from [14], and then the notion of universal localizations which is closely related to constructing tilting modules.

Definition 2.2. An R-module T is called a tilting module (of projective dimension at most one) if the following conditions are satisfied:

$(T1)$ The projective dimension of T is at most 1, that is, there exists an exact sequence: $0 \to P_1 \to P_0 \to T \to 0$ with P_i projective for $i = 0,1$,

$(T2)$ $\mathrm{Ext}^i_R(T, T^{(\alpha)}) = 0$ for each $i \geq 1$ and each index set α, and

$(T3)$ there exists an exact sequence

$$0 \longrightarrow {}_RR \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

of R-modules such that $T_i \in \mathrm{Add}(T)$ for $i = 0,1$.

A tilting R-module T is called good if T_0 and T_1 in (T3) lie in $\mathrm{add}(T)$, and classical if T is good and finitely presented.

A special kind of good tilting modules can be constructed from injective ring epimorphisms, including particularly certain universal localizations. The following result on universal localizations is well known.

Lemma 2.3. (see [12], [29]) *Let R be a ring and Σ a set of homomorphisms between finitely generated projective R-modules. Then there is a ring R_Σ and a homomorphism $\lambda : R \to R_\Sigma$ of rings with the following properties:*

(1) *λ is Σ-inverting, that is, if $\alpha : P \to Q$ belongs to Σ, then $R_\Sigma \otimes_R \alpha : R_\Sigma \otimes_R P \to R_\Sigma \otimes_R Q$ is an isomorphism of R_Σ-modules, and*

(2) *λ is universal Σ-inverting, that is, if S is a ring such that there exists a Σ-inverting homomorphism $\varphi : R \to S$, then there exists a unique homomorphism $\psi : R_\Sigma \to S$ of rings such that $\varphi = \lambda\psi$.*

(3) *The homomorphism $\lambda : R \to R_\Sigma$ is a ring epimorphism with* $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$.

We call $\lambda : R \to R_\Sigma$ in Lemma 2.3 the universal localization of R at Σ. Recall that, by [2, Theorem 2.5], if λ is injective and the R-module R_Σ has projective dimension at most one, then $R_\Sigma \oplus R_\Sigma/R$ is a tilting R-module.

Of particular interest are the following two kinds of universal localizations.

The first one is associated with subsets of elements in rings. Let Φ be a non-empty subset of R. Then we consider the universal localization of R at all homomorphisms ρ_r with $r \in \Phi$, where ρ_r is the right multiplication map $R \to R$ defined by $x \mapsto xr$ for $x \in R$. For simplicity, we write R_Φ for this universal localization, and say that R_Φ is the universal localization of R at Φ. Note that, by the property of universal localizations, R_Φ is also isomorphic to the "right" universal localization of R at all left multiplication maps $\sigma_r : R_R \to R_R$ defined by $x \mapsto rx$ for $x \in \Phi$, which are regarded as homomorphisms of right R-modules. Clearly, if $0 \in \Phi$, then $R_\Phi = 0$. If $0 \notin \Phi$, then we consider the smallest multiplicative subset of R containing Φ, and get $R_\Phi = R_{\Phi_1}$. Recall that a subset Φ of R is said to be multiplicative if $0 \notin \Phi$, $1 \in \Phi$, and it is closed under multiplication.

From now on, we assume that Φ is a multiplicative subset of R.

Under some extra assumptions on Φ, the ring R_Φ can be characterized by Ore localizations which generalizes the notion of localizations in commutative rings. To explain this point in detail, we first recall some relevant definitions about Ore localizations. For more details, we refer to [23, Chapter 4].

Definition 2.4. A subset Φ of R is called a left denominator subset of R if Φ satisfies the following two conditions: (i) For any $a \in R$ and $s \in \Phi$, there holds $\Phi a \cap Rs \neq \emptyset$, and (ii) for any $r \in R$, if $rt = 0$ for some $t \in \Phi$, then there exists some $t' \in \Phi$ such that $t'r = 0$. If Φ satisfies only the condition (i), then Φ is called a left Ore subset of R.

Similarly, we can define the notions of right denominator sets and right Ore sets, respectively. Clearly, if R is commutative, then every multiplicative subset of R is a left and right denominator set. Furthermore, if R is a domain, that is, R is a (not necessarily commutative) ring which has neither left zero-divisors nor right zero-divisors, then $R\setminus\{0\}$ is a left denominator set if and only if it is a left Ore set if and only if $Rr_1\cap Rr_2\neq\{0\}$ for any non-zero elements $r_1,r_2\in R$. We say that R is a left Ore domain if $R\setminus\{0\}$ is a left denominator set.

The following lemma explains how left Ore localizations arise, and establishes a relationship between left Ore localizations and universal localizations.

Lemma 2.5. [23, Theorem 10.6, Corollary 10.11] *Let Φ be a left denominator subset of R and $\lambda: R\to R_\Phi$ the universal localization of R at Φ. Then there is a ring, denoted by $\Phi^{-1}R$, and a ring homomorphism $\mu: R\to\Phi^{-1}R$ such that*

(1) *μ is Φ-invertible, that is, $(s)\mu$ is a unit in $\Phi^{-1}R$ for each $s\in\Phi$,*
(2) *every element of $\Phi^{-1}R$ has the form $\big((t)\mu\big)^{-1}(r)\mu$ for some $t\in\Phi$ and some $r\in R$,*
(3) $\ker(\mu)=\{r\in R\mid sr=0$ *for some* $s\in\Phi\}$, *and*
(4) *there is a unique isomorphism $\nu:\Phi^{-1}R\to R_\Phi$ of rings such that $\lambda=\mu\nu$.*

The ring $\Phi^{-1}R$ in Lemma 2.5 is called a left ring of fractions of R (with respect to $\Phi\subseteq R$), or alternatively, a left Ore localization of R at Φ. Clearly, for commutative rings, Ore localizations and the usual localizations at multiplicative subsets coincide.

Similarly, when Φ is a right denominator subset of R, we can define a right ring $R\Phi^{-1}$ of fractions of R. If Φ is a left and right denominator subset of R, then $\Phi^{-1}R$ is called the ring of fractions of R, or the Ore localization of R at Φ. Actually, in this case, both $\Phi^{-1}R$ and $R\Phi^{-1}$ are isomorphic to R_Φ. Furthermore, if R is a left and right Ore domain R, then the ring of fractions of R with respect to $R\setminus\{0\}$ is usually denoted by $Q(R)$. Notice that, up to isomorphism, $Q(R)$ is the smallest division ring containing R as a subring. So we call $Q(R)$ the division ring of fractions of R.

The other kind of universal localizations is provided by universal localizations at injective homomorphisms between finitely generated projective modules, and therefore related to finitely presented modules of projective dimension at most one.

Suppose that $\mathcal{U}$ is a set of finitely presented R-modules of projective dimension at most one. For each $U\in\mathcal{U}$, there is a finitely generated projective presentation of U, that is, an exact sequence of R-modules

$$(*)\quad 0\longrightarrow P_1\xrightarrow{f_U}P_0\longrightarrow U\longrightarrow 0,$$

such that P_1 and P_0 are finitely generated and projective. Set $\Sigma:=\{f_U\mid U\in\mathcal{U}\}$, and let $R_{\mathcal{U}}$ be the universal localization of R at Σ. If $f'_U: Q_1\to Q_0$ is another finitely generated projective presentation of U, then the universal localization of R at $\Sigma':=\{f'_U\mid U\in\mathcal{U}\}$ is isomorphic to $R_{\mathcal{U}}$. Hence $R_{\mathcal{U}}$ does not depend on the choices of the injective homomorphisms f_U, and we may say that $R_{\mathcal{U}}$ is the universal localization of R at $\mathcal{U}$.

Clearly, we have $\mathrm{Tor}_i^R(R_{\mathcal{U}},U)=0$ for all $i\geq 0$ and $U\in\mathcal{U}$, and therefore $\mathrm{Tor}_i^R(R_{\mathcal{U}},X)=0$ for all $i\geq 0$ and $X\in\mathcal{F}(\mathcal{U})$.

Now, we state the promised example of recollements as a proposition which is a consequence of [8, Lemma 6.2, Corollary 6.6]. It is worthy to notice that the recollement in this proposition is, in general, different from the one obtained from the structure of triangular matrix rings.

Proposition 2.6. *Let $\mathcal{U}$ be a set of finitely presented R-modules of projective dimension one, and let $\lambda: R\to R_{\mathcal{U}}$ be the universal localization of R at $\mathcal{U}$. Suppose that λ is injective and that the R-module $R_{\mathcal{U}}$ has projective dimension at most one. Set $S:=\mathrm{End}_R(R_{\mathcal{U}}/R)$, $B:=\mathrm{End}_R(R_{\mathcal{U}}\oplus R_{\mathcal{U}}/R)$ and $\Sigma:=\{S\otimes_R f_U\mid U\in\mathcal{U}\}$. Then there is a recollement of derived module categories:*

$$\mathscr{D}(S_\Sigma)\longrightarrow\mathscr{D}(B)\longrightarrow\mathscr{D}(R)\ ,$$

where S_Σ is the universal localization of S at Σ.

In many cases we can use this proposition repeatedly because the following result states that iterated universal localizations are again universal localizations.

Lemma 2.7. [29, Theorem 4.6] *Let Σ and Γ be sets of homomorphisms between finitely generated projective R-modules. Set $\overline{\Gamma} := \{R_\Sigma \otimes_R f \mid f \in \Gamma\}$. Then the universal localization of R at $\Sigma \cup \Gamma$ is isomorphic to the universal localization of R_Σ at $\overline{\Gamma}$, that is, $R_{\Sigma\cup\Gamma} \simeq (R_\Sigma)_{\overline{\Gamma}}$ as rings.*

Next, we recall the definition of discrete valuation rings.

Definition 2.8. A ring R is called a discrete valuation ring (which may not be commutative) if the following conditions hold true:

(1) R is a local ring, that is, R has a unique maximal left ideal $\mathfrak{m}$;
(2) $\bigcap_{i\geq 1} \mathfrak{m}^i = 0$;
(3) $\mathfrak{m} = pR = Rp$, where p is some non-nilpotent element of R.

We remark that an equivalent definition of discrete valuation rings is the following: A non-division ring R is called a discrete valuation ring if it is a local domain with $\mathfrak{m}$ the unique maximal ideal of R such that the only left ideals and the only right ideals of R are of the form $\mathfrak{m}^i$ for $i \in \mathbb{N}$.

The element p in the above condition (3) is called a prime element of R. Clearly, for each invertible element v of R, both vp and pv are prime elements. A discrete valuation ring is said to be complete if the canonical map $R \to \varprojlim_i R/\mathfrak{m}^i$ is an isomorphism. Note that every discrete valuation ring can be embedded into a complete discrete valuation ring.

The following lemma collects some basic properties of discrete valuation rings, which will be frequently used in our proofs.

Lemma 2.9. ([22, Chapter 1], [23]) *Let R be a discrete valuation ring, $\mathfrak{m}$ the unique maximal ideal of R, and p a prime element of R. Then the following statements are true:*

(1) *The ideals $\mathfrak{m}^i$ $(i \in \mathbb{N})$ are the only left ideals and the only right ideals of R.*

(2) *For any non-zero element $x \in R$, there are unique elements $x_1, x_2 \in R \setminus \mathfrak{m}$ such that $x = x_1 p^n = p^n x_2$ for some $n \in \mathbb{N}$.*

(3) *R is a left and right Ore domain. In particular, the division ring $Q(R)$ of fractions of R exists.*

(4) *$Q(R)$ is isomorphic to the universal localization of R at the map $\rho_p : R \to R$ defined by $r \mapsto rp$ for $r \in R$.*

Finally, we prepare several homological results for our later proofs.

Lemma 2.10. *Let R be a ring and let $0 \longrightarrow X \xrightarrow{(f,g)} Y \oplus Z \xrightarrow{h} W \longrightarrow 0$ be an exact sequence of R-modules. Assume that $f : X \to Y$ is injective and that there is a homomorphism $\tilde{g} : Y \to Z$ with $g = f\tilde{g} : X \to Z$. Then there exists an automorphism φ of the module $Y \oplus Z$ and an isomorphism $\psi : W \to \mathrm{Coker}(f) \oplus Z$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{(f,g)} & Y \oplus Z & \xrightarrow{\quad h \quad} & W & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & X & \xrightarrow{(f,0)} & Y \oplus Z & \xrightarrow{\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)} & \mathrm{Coker}(f) \oplus Z & \longrightarrow & 0, \end{array}$$

where $\pi : Y \to \mathrm{Coker}(f)$ stands for the canonical surjection.

Proof. Set $\varphi := \begin{pmatrix} 1 & -\tilde{g} \\ 0 & 1 \end{pmatrix}$. Then φ is an automorphism of the module $Y \oplus Z$. Since $g = f\tilde{g}$, we have $(f,g)\varphi = (f,0)$. Thus, there exists a unique homomorphism $\psi : W \longrightarrow \mathrm{Coker}(f) \oplus Z$, such that the exact diagram in Lemma 2.10 is commutative. Clearly, ψ is an isomorphism. This completes the proof. $\square$

The following homological facts are well known in the literature (see, for example, the book [17]).

Lemma 2.11. *Let R be a ring.*

(1) *If $\{X_\alpha\}_{\alpha\in I}$ is a direct system of R-modules, then*

(i) $\mathrm{Hom}_R(\varinjlim_\alpha X_\alpha, M) \simeq \varprojlim_\alpha \mathrm{Hom}_R(X_\alpha, M)$ *for any R-module M.*

(ii) For any finitely presented R-module M, we have $\mathrm{Hom}_R(M, \varinjlim_\alpha X_\alpha) \simeq \varinjlim_\alpha \mathrm{Hom}_R(M, X_\alpha)$.

(iii) Let $n \geq 0$. If M is an R-module with a projective resolution $\cdots \to P_{n+1} \to \cdots \to P_1 \to P_0 \to M \to 0$ such that all P_j, with $0 \leq j \leq n+1$, are finitely generated, then

$$\mathrm{Ext}^i_R(M, \varinjlim_\alpha X_\alpha) \simeq \varinjlim_\alpha \mathrm{Ext}^i_R(M, X_\alpha)$$

for all $i \leq n$.

(iv) If M is a pure-injective R-module (for example, M is of finite length over its endomorphism ring), then

$$\mathrm{Ext}^i_R(\varinjlim_\alpha X_\alpha, M) \simeq \varprojlim_\alpha \mathrm{Ext}^i_R(X_\alpha, M)$$

for all $i \geq 0$. Conversely, if this isomorphism is true for $i = 1$ and for every directed system X_α, then M is pure-injective.

(2) *If $\{Y_\alpha\}_{\alpha\in I}$ is an inverse system of R-modules, then, for any R-module M,*

$$\mathrm{Hom}_R(M, \varprojlim_\alpha Y_\alpha) \simeq \varprojlim_\alpha \mathrm{Hom}_R(M, Y_\alpha).$$

Remarks. (1) The statement (iv) is due to Maurice Auslander.

(2) The class of all pure-injective R-modules is closed under products, direct summands and finite direct sums. In general, it is not closed under extensions.

Lemma 2.12. *Let A be a finite-dimensional k-algebra over a field k, M a finite-dimensional A-module and N an arbitrary A-module.*

(1) If $\mathrm{proj.dim}(M) \leq 1$, *then* $D\mathrm{Ext}^1_A(M,N) \simeq \mathrm{Hom}_A(N, \tau M)$, *where* $\mathrm{proj.dim}(M)$ *stands for the projective dimension of M.*

(2) If $\mathrm{inj.dim}(M) \leq 1$, *then* $\mathrm{Ext}^1_A(N,M) \simeq D\mathrm{Hom}_A(\tau^{-1}M, N))$, *where* $\mathrm{inj.dim}(M)$ *stands for the injective dimension of M.*

Proof. It is known that every A-module N is a direct limit of finitely presented A-modules $\{X_\alpha\}_{\alpha\in I}$, and that (1) and (2) hold true for finitely generated modules N. Then, it follows from Lemma 2.11 that

$$\begin{aligned} D\mathrm{Ext}^1_A(M,N) &\simeq D\mathrm{Ext}^1_A(M, \varinjlim_\alpha X_\alpha) \simeq D\, \varinjlim_\alpha \mathrm{Ext}^1_A(M, X_\alpha) \simeq \varprojlim_\alpha D\mathrm{Ext}^1_A(M, X_\alpha) \\ &\simeq \varprojlim_\alpha \mathrm{Hom}_A(X_\alpha, \tau M) \simeq \mathrm{Hom}_A(\varinjlim_\alpha X_\alpha, \tau M) = \mathrm{Hom}_A(N, \tau M). \end{aligned}$$

This proves (1). The statement (2) can be shown similarly. □

3 Proof of the main result

Unless stated otherwise, we assume from now on that R is an indecomposable finite-dimensional tame hereditary algebra over an arbitrary but fixed field k.

Let $\mathscr{S} := \mathscr{S}(R)$ be a fixed complete set of isomorphism classes of all simple regular R-modules. For each $U \in \mathscr{S}$ and $n > 0$, we denote by $U[n]$ the R-module of regular length n on the ray

$$(*) \quad U = U[1] \subset U[2] \subset \cdots \subset U[n] \subset U[n+1] \subset \cdots,$$

and let $U[\infty] = \varinjlim_n U[n]$ be the Prüfer module corresponding to U. Note that $U[\infty]$ has a unique regular submodule $U[n]$ of regular length n, and therefore admits a unique chain of regular submodules, and that each endomorphism of $U[\infty]$ restricts to an endomorphism of $U[n]$ for any $n > 0$. For further information on regular modules and Prüfer modules over tame hereditary algebras, we refer to [27, Section 4, 5] and [15].

Recall that we have defined an equivalence relation $\sim$ on $\mathscr{S}$ in Section 1. It is known that two simple regular modules lie in the same clique if and only if they lie in the same tube. Thus a clique is just the set of all simple regular modules belonging to a fixed tube.

Let $U \in \mathscr{S}$ and $\mathcal{U} \subseteq \mathscr{S}$. We denote by $\mathscr{C}(U)$ the clique containing U, and by $c(U)$ the cardinality of $\mathscr{C}(U)$. Similarly, we denote by $\mathscr{C}(\mathcal{U})$ the union of all cliques $\mathscr{C}(U)$ with $U \in \mathcal{U}$, and by $c(\mathcal{U})$ the cardinality of $\mathscr{C}(\mathcal{U})$. As mentioned before, $c(U)$ is always finite, and furthermore, $c(U) = 1$ for almost all $U \in \mathscr{S}$. In fact, there are at most 3 cliques consisting of more than one element. Also, we know that all cliques consist of one simple regular R-module if and only if R has only two isomorphism classes of simple modules. If k is an algebraically closed field, this is equivalent to that R is Morita equivalent to the Kronecker algebra.

3.1 Endomorphism rings of direct sums of Prüfer modules

In this subsection, we shall consider the endomorphism ring of the direct sum of all Prüfer modules obtained from a given tube. This ring was calculated already in [27]. For convenience of the reader and also for the later proof of our main result, we include here some details of this calculation.

Throughout this subsection, let $\mathcal{C}$ be a clique of R-mod, $U \in \mathcal{C}$, and $\mathbf{t}$ the tube of rank $m \geq 1$ containing $\mathcal{C}$. Set $U_i := \tau^{-(i-1)}U$ for $i \in \mathbb{Z}$. Then $\tau^{-m}U \simeq U$, and $\mathcal{C} = \{U_1, U_2, \cdots, U_{m-1}, U_m\}$ which is a complete set of non-isomorphic simple regular modules in $\mathbf{t}$. Since $U_j \simeq U_{j+m}$ for any $j \in \mathbb{Z}$, the subscript of U_j is always modulo m in our discussion below. It is well known that $\mathrm{End}_R(U_i)$ is a division algebra and $\mathrm{Hom}_R(U_i, U_j) = 0$ for $1 \leq i \neq j \leq m$, and that $D\mathrm{Ext}^1_R(U_i, U_j) \simeq \mathrm{End}_R(U_i)$ if $j = i-1$, and zero otherwise. Furthermore, $\mathbf{t}$ is an exact abelian subcategory of R-mod, and every indecomposable module in $\mathbf{t}$ is serial, that is, it has a unique regular composition series in $\mathbf{t}$. For example, for any $i \in \mathbb{Z}$ and $j > 0$, the module $U_i[j]$ admits successive regular composition factors $U_i, U_{i+1}, \cdots, U_{i+j-1}$ with U_i as its unique regular socle and with U_{i+j-1} as its unique regular top. For details, see [28, Section 3.1].

Now, we mention some properties of Prüfer modules.

Lemma 3.1. *The following statements hold true for the tube* $\mathbf{t}$.

(1) *For any $1 \leq i \leq m$ and for any regular module X in* $\mathbf{t}$, *we have* $\mathrm{Hom}_R(U_i[\infty], X) = 0 = \mathrm{Ext}^1_R(X, U_i[\infty])$. *Further, if $1 \leq i < j \leq m$, then* $\mathrm{Hom}_R(U_i[n], U_j[\infty]) = 0$ *for* $1 \leq n \leq j-i$, *and* $\mathrm{Hom}_R(U_j[n], U_i[\infty]) = 0$ *for* $1 \leq n \leq m-j+i$.

(2) *Let $i, j \in \mathbb{N}$ with $1 \leq i < j$. Then, for any $n > j-i$, there is a canonical exact sequence of R-modules:*

$$0 \longrightarrow U_i[j-i] \longrightarrow U_i[n] \xrightarrow{\varepsilon_{i,j}[n]} U_j[n-(j-i)] \longrightarrow 0.$$

In particular, we get a canonical exact sequence

$$0 \longrightarrow U_i[j-i] \longrightarrow U_i[\infty] \xrightarrow{\varepsilon_{i,j}} U_j[\infty] \longrightarrow 0,$$

where $\varepsilon_{i,j} := \varinjlim_n \varepsilon_{i,j}[n]$. *Moreover, we have* $\varepsilon_{i,j} = \varepsilon_{i+m,\, j+m}$ *and* $\varepsilon_{i,j}\,\varepsilon_{j,p} = \varepsilon_{i,p}$ *for any* $p > j$.

(3) *Let $i, j \in \mathbb{N}$ with $1 \leq j-i < m$. Then $\varepsilon_{i,j}$ induces an isomorphism of left* $\mathrm{End}_R(U_i[\infty])$*-modules:*

$$(\varepsilon_{i,j})^* : \mathrm{End}_R(U_i[\infty]) \xrightarrow{\sim} \mathrm{Hom}_R(U_i[\infty], U_j[\infty]),$$

and an isomorphism of right $\mathrm{End}_R(U_j[\infty])$*-modules:*

$$(\varepsilon_{i,j})_* : \mathrm{End}_R(U_j[\infty]) \xrightarrow{\sim} \mathrm{Hom}_R(U_i[\infty], U_j[\infty]).$$

In particular, we get a ring isomorphism $\varphi_{i,j} : \mathrm{End}_R(U_i[\infty]) \to \mathrm{End}_R(U_j[\infty])$, $f \mapsto f'$ *for* $f \in \mathrm{End}_R(U_i[\infty])$ *and* $f' \in \mathrm{End}_R(U_j[\infty])$,*with* $f\varepsilon_{i,j} = \varepsilon_{i,j} f'$.

(4) *Suppose* $1 \le r, s, t \le m$. *Set* $\Delta_{r,s} := \begin{cases} 0 & \text{if } r < s, \\ 1 & \text{if } r \ge s, \end{cases}$ *and define* $\pi_{r,s} := \varepsilon_{r,s+\Delta_{r,s}m} \in \mathrm{Hom}_R(U_r[\infty], U_{s+\Delta_{r,s}m}[\infty])$. *Then*

$$\pi_{r,s}\,\pi_{s,t} = \begin{cases} \pi_{r,t} & \text{if } \Delta_{r,s} + \Delta_{s,t} = \Delta_{r,t}, \\ \pi_{r,r}\,\pi_{r,t} & \text{otherwise}. \end{cases}$$

In particular, we have $(\pi_{i,i})\varphi_{i,j} = \pi_{j,j}$ *for any* $1 \le i < j \le m$.

(5) *The ring* $\mathrm{End}_R(U_i[\infty])$ *is a complete discrete valuation ring with* $\pi_{i,i}$ *as a prime element. If k is an algebraically closed field, then there is a ring isomorphism* $\varphi_i : \mathrm{End}_R(U_i[\infty]) \to k[[x]]$ *which sends* $\pi_{i,i}$ *to x.*

Proof. (1) Note that $D\mathrm{Ext}^1_R(X, U_i[\infty]) \simeq \mathrm{Hom}_R(U_i[\infty], \tau X)$ for any $X \in \mathbf{t}$ by Lemma 2.12(1), and that every indecomposable module in $\mathbf{t}$ is serial. This means that, to prove the first statement in (1), it suffices to show $\mathrm{Hom}_R(U_i[\infty], U_j) = 0$ for all $1 \le j \le m$. In fact, since the inclusion map $U_i[n] \to U_i[n+1]$ induces a zero map from $\mathrm{Hom}_R(U_i[n+1], U_j)$ to $\mathrm{Hom}_R(U_i[n], U_j)$ for all n. This implies that

$$\mathrm{Hom}_R(U_i[\infty], U_j) = \mathrm{Hom}_R(\varinjlim_n U_i[n], U_j) \simeq \varprojlim_n \mathrm{Hom}_R(U_i[n], U_j) = 0.$$

The last statement in (1) follows from the fact that the abelian category $\mathbf{t}$ is serial.

(2) For any $n > j - i$, we can easily see from the structure of the tube $\mathbf{t}$ that there is an exact commutative diagram of R-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_i[j-i] & \longrightarrow & U_i[n] & \xrightarrow{\varepsilon_{i,j}[n]} & U_j[n-(j-i)] & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_i[j-i] & \longrightarrow & U_i[n+1] & \xrightarrow{\varepsilon_{i,j}[n+1]} & U_j[n-(j-i)+1] & \longrightarrow & 0, \end{array}$$

where the map $\varepsilon_{i,j}[n]$ is induced by the canonical inclusion $U_i[j-i] \hookrightarrow U_i[n]$. Thus, by taking the direct limit of the above diagram, we obtain the following canonical exact sequence

$$(*) \quad 0 \longrightarrow U_i[j-i] \longrightarrow U_i[\infty] \xrightarrow{\varepsilon_{i,j}} U_j[\infty] \longrightarrow 0,$$

where $\varepsilon_{i,j} := \varinjlim_n \varepsilon_{i,j}[n]$. This finishes the proof of the first assertion in (2). In the following, we shall show that $\varepsilon_{i,j} = \varepsilon_{i+m,\,j+m}$ and $\varepsilon_{i,j}\,\varepsilon_{j,p} = \varepsilon_{i,p}$ for any $p > j$. In fact, the former clearly follows from $\varepsilon_{i,j}[n] = \varepsilon_{i+m,\,j+m}[n]$ for any $n > j - i$, since $U_i = U_{i+m}$ and $U_j = U_{j+m}$ by our convention. As for the latter, one can check that, for any $u > p - i$, the composition of

$$\varepsilon_{i,j}[u] : U_i[u] \longrightarrow U_j[u-(j-i)] \quad \text{and} \quad \varepsilon_{j,p}[u-(j-i)] : U_j[u-(j-i)] \longrightarrow U_p[u-(p-i)]$$

coincides with $\varepsilon_{i,p}[u] : U_i[u] \longrightarrow U_p[u-(p-i)]$. So, we have $\varepsilon_{i,j}[u]\,\varepsilon_{j,p}[u-(j-i)] = \varepsilon_{i,p}[u]$. Consequently, by taking the direct limit of the two-sides of the equality, we have $\varepsilon_{i,j}\,\varepsilon_{j,p} = \varepsilon_{i,p}$ for any $p > j$. This completes the proof of (2).

(3) If we apply $\mathrm{Hom}_R(U_i[\infty], -)$ to the sequence $(*)$ in the proof of (2), then we can get the following exact sequence:

$$0 \to \mathrm{Hom}_R(U_i[\infty], U_i[j-i]) \to \mathrm{Hom}_R(U_i[\infty], U_i[\infty]) \xrightarrow{(\varepsilon_{i,j})^*} \mathrm{Hom}_R(U_i[\infty], U_j[\infty]) \to \mathrm{Ext}^1_R(U_i[\infty], U_i[j-i]).$$

Note that $\mathrm{Hom}_R(U_i[\infty], U_i[j-i]) = 0$ by (1). Thus, to prove that $(\varepsilon_{i,j})^*$ is an isomorphism, it suffices to show $\mathrm{Ext}^1_R(U_i[\infty], U_i[j-i]) = 0$. In fact, this follows from $\mathrm{Ext}^1_R(U_i[\infty], U_i[j-i]) \simeq D\mathrm{Hom}_R\big(\tau^-(U_i[j-i]), U_i[\infty]\big) \simeq D\mathrm{Hom}_R(U_{i+1}[j-i], U_i[\infty] = 0$, where the last equality holds for $1 \le j-i < m$ by (1).

Next, if we apply $\mathrm{Hom}_R(-,U_j[\infty])$ to the sequence $(*)$, then we get the following exact sequence:

$$0 \to \mathrm{End}_R(U_j[\infty]) \xrightarrow{(\varepsilon_{i,j})_*} \mathrm{Hom}_R(U_i[\infty],U_i[\infty]) \longrightarrow \mathrm{Hom}_R(U_i[j-i],U_j[\infty]).$$

Since $1 \leq j-i < m$, we have $\mathrm{Hom}_R(U_i[j-i],U_j[\infty]) = 0$, and therefore $(\varepsilon_{i,j})_*$ is an isomorphism.

Now, it follows from the isomorphisms $(\varepsilon_{i,j})^*$ and $(\varepsilon_{i,j})_*$ that the map

$$\varphi_{i,j} : \mathrm{End}_R(U_i[\infty]) \to \mathrm{End}_R(U_j[\infty])$$

in (3) is well-defined and thus a ring isomorphism.

(4) By definition, for $1 \leq r, s, t \leq m$, one can check

$$\pi_{r,s}\,\pi_{s,t} = \varepsilon_{r,s+\Delta_{r,s}m}\,\varepsilon_{s,t+\Delta_{s,t}m} = \varepsilon_{r,s+\Delta_{r,s}m}\,\varepsilon_{s+\Delta_{r,s}m,t+(\Delta_{s,t}+\Delta_{r,s})m} = \varepsilon_{r,t+(\Delta_{r,s}+\Delta_{s,t})m}.$$

On the one hand, for any $p > r$ and $q > r$, we infer from (2) that $\varepsilon_{r,p} = \varepsilon_{r,q}$ if and only if $p = q$. On the other hand, we always have $\Delta_{r,s} + \Delta_{s,t} - \Delta_{r,t} \in \{0,1\}$. Consequently, the first statement in (4) follows. In particular, this implies that $\pi_{i,j}\pi_{j,j} = \pi_{i,i}\pi_{i,j}$ for $1 \leq i < j \leq m$. By the definition of $\varphi_{i,j}$ in (3), we can prove the second statement in (4).

(5) Set $D_i := \mathrm{End}_R(U_i[\infty])$. It follows from [27, Section 4.4] that D_i is a complete discrete valuation ring. Let $\mathfrak{m}$ be the unique maximal ideal of D_i. We shall prove that $\pi_{i.i}$ is a prime element of D_i, that is, $\mathfrak{m} = \pi_{i,i}D_i = D_i\pi_{i,i}$. Indeed, by applying $\mathrm{Hom}_R(-,U_i[\infty])$ to the following exact sequence:

$$0 \longrightarrow U_i[m] \longrightarrow U_i[\infty] \xrightarrow{\pi_{i,i}} U_i[\infty] \longrightarrow 0,$$

we obtain another exact sequence of right D_i-modules:

$$0 \longrightarrow D_i \xrightarrow{(\pi_{i,i})_*} D_i \longrightarrow \mathrm{Hom}_R(U_i[m],U_i[\infty]) \longrightarrow 0,$$

due to $\mathrm{Ext}^1_R(U_i[\infty],U_i[\infty]) = 0$, which follows from [27, Section 4.5]. To show $\mathfrak{m} = \pi_{i,i}D_i$, we first claim that $\mathrm{Hom}_R(U_i[m],U_i[\infty]) \simeq \mathrm{Hom}_R(U_i,U_i[\infty]) \simeq D_i/\mathfrak{m}$ as right D_i-modules.

Let

$$0 \longrightarrow U_i \longrightarrow U_i[m] \xrightarrow{\varepsilon_{i,i+1}[m]} U_{i+1}[m-1] \longrightarrow 0$$

be the exact sequence defined in (2). Then we get the following exact sequence of k-modules:

$$\mathrm{Hom}_R(U_{i+1}[m-1],U_i[\infty]) \longrightarrow \mathrm{Hom}_R(U_i[m],U_i[\infty]) \longrightarrow \mathrm{Hom}_R(U_i,U_i[\infty]) \longrightarrow \mathrm{Ext}^1_R(U_{i+1}[m-1],U_i[\infty]).$$

Since $\mathrm{Hom}_R(U_{i+1}[m-1],U_i[\infty]) = 0 = \mathrm{Ext}^1_R(U_{i+1}[m-1],U_i[\infty])$ by (1), we have $\mathrm{Hom}_R(U_i[m],U_i[\infty]) \simeq \mathrm{Hom}_R(U_i,U_i[\infty])$ as right D_i-modules.

It remains to show $\mathrm{Hom}_R(U_i,U_i[\infty]) \simeq D_i/\mathfrak{m}$ as right D_i-modules. Let

$$0 \longrightarrow U_i \xrightarrow{\zeta} U_i[\infty] \xrightarrow{\varepsilon_{i,i+1}} U_{i+1}[\infty] \longrightarrow 0$$

be the exact sequence defined in (2) with ζ the canonical inclusion. Since $\mathrm{Ext}^1_R((U_{i+1})[\infty],U_i[\infty]) = 0$ by [27, Section 4.5], we infer that, for any $f : U_i \to U_i[\infty]$, there is $g \in D_i$ such that $f = \zeta g$. This means $\mathrm{Hom}_R(U_i,U_i[\infty]) = \zeta D_i$. Clearly, $\zeta D_i \simeq D_i/N$ as right D_i-modules, where $N := \{h \in D_i \mid \zeta h = 0\}$. As the canonical ring homomorphism from D_i to $\mathrm{End}_R(U_i)$ via the map ζ induces a ring isomorphism from $D_i/\mathfrak{m}$ to $\mathrm{End}_R(U_i)$, we have $\zeta\mathfrak{m} = 0$, that is, $\mathfrak{m} \subseteq N$. Since D_i is a local ring and $N \subsetneq D_i$, we get $N = \mathfrak{m}$, and therefore $\mathrm{Hom}_R(U_i,U_i[\infty]) \simeq D_i/\mathfrak{m}$ as right D_i-modules. This finishes the claim.

From the above claim, we conclude that $\mathfrak{m}$ coincides with the image of $(\pi_{i,i})_*$, that is, $\mathfrak{m} = \pi_{i,i}D_i$. Similarly, we can prove $\mathfrak{m} = D_i\pi_{i,i}$. This means that $\pi_{i.i}$ is a prime element of D_i. As for the second statement in

(5), we note that, for any $p \in \mathbb{N}$ and $1 \le q < m$, the canonical inclusion map $U_i[pm+q] \to U_i[pm+q+1]$ induces an isomorphism:

$$\mathrm{Hom}_R(U_i[pm+q+1], U_i[\infty]) \xrightarrow{\simeq} \mathrm{Hom}_R(U_i[pm+q], U_i[\infty]).$$

Consequently, we have the following isomorphisms of abelian groups:

$$\begin{aligned} D_i &= \mathrm{Hom}_R\big(\varinjlim_n U_i[n], U_i[\infty]\big) \simeq \varprojlim_n \mathrm{Hom}_R\big(U_i[n], U_i[\infty]\big) \\ &\simeq \varprojlim_n \mathrm{Hom}_R\big(U_i[(n-1)m+1], U_i[\infty]\big) \simeq \varprojlim_n k[x]/(x^n) \simeq k[[x]]. \end{aligned}$$

Here we need the assumption that k is algebraically closed field. Now, one can check directly that the composition of the above isomorphisms yields a ring isomorphism $\varphi_i : D_i \to k[[x]]$, which sends $\pi_{i,i}$ to x. This finishes the proof. □

By Lemma 3.1(3), the rings $\mathrm{End}_R(U_i[\infty])$, with $1 \le i \le m$, are all isomorphic. From now on, we always identify these rings, and simply denote them by $D(\mathcal{C})$. Further, we write $\mathfrak{m}(\mathcal{C})$ and $Q(\mathcal{C})$ for the maximal ideal of $D(\mathcal{C})$ and the division ring of fractions of $D(\mathcal{C})$, respectively. In particular, $\mathfrak{m}(\mathcal{C}) = \pi_{i,i}D(\mathcal{C}) = D(\mathcal{C})\pi_{i,i}$.

Suppose that C is a $\mathbb{Z}$-module and $c \in C$. For $1 \le i, j \le m$, we denote by $E_{i,j}(c)$ the $m \times m$ matrix which has the (i, j)-entry c, and the other entries 0. For simplicity, we write $E_{i,j}$ for $E_{i,j}(1)$ if C is a ring with the identity 1.

Lemma 3.2. *For $1 \le i, j \le m$, let $\pi_{i,j}$ be the homomorphisms defined in Lemma 3.1(4). Then there is a ring isomorphism*

$$\rho: \ \mathrm{End}_R\Big(\bigoplus_{i=1}^m U_i[\infty]\Big) \longrightarrow \Gamma(\mathcal{C}) := \begin{pmatrix} D(\mathcal{C}) & D(\mathcal{C}) & \cdots & D(\mathcal{C}) \\ \mathfrak{m}(\mathcal{C}) & D(\mathcal{C}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & D(\mathcal{C}) \\ \mathfrak{m}(\mathcal{C}) & \cdots & \mathfrak{m}(\mathcal{C}) & D(\mathcal{C}) \end{pmatrix}_{m\times m}$$

which sends $E_{m,1}(\pi_{m,1})$ to $E_{m,1}(\pi_{m,m})$ and $E_{r,r+1}(\pi_{r,r+1})$ to $E_{r,r+1}$ for $1 \le r < m$, where the maximal ideal $\mathfrak{m}(\mathcal{C})$ of the ring $D(\mathcal{C})$ is generated by the element $\pi_{m,m}$.

Proof. For any $1 \le i < m$, by Lemma 3.1(2) and (4), we have the following exact sequence of R-modules:

$$0 \longrightarrow U_i[m-i] \longrightarrow U_i[\infty] \xrightarrow{\pi_{i,m}} U_m[\infty] \longrightarrow 0.$$

Summing up these sequences, we can get the following exact sequence:

$$0 \longrightarrow \bigoplus_{i=1}^{m-1} U_i[m-i] \longrightarrow \bigoplus_{j=1}^{m} U_j[\infty] \xrightarrow{\xi} U_m[\infty]^{(m)} \longrightarrow 0,$$

where $\xi := \mathrm{diag}\big(\pi_{1,m}, \pi_{2,m}, \cdots, \pi_{m-1,m}, 1\big)$, the $m \times m$ diagonal matrix with $\pi_{i,m}$ in the (i,i)-position for $1 \le i < m$, and with 1 in the (m,m)-position.

Let $D := \mathrm{End}_R(U_m[\infty])$, and let $\mathfrak{m}$ be the unique maximal ideal of D. Set $\Lambda := \mathrm{End}_R(\bigoplus_{j=1}^m U_j[\infty])$. Since $\mathrm{Hom}_R\big(U_i[m-i], U_m[\infty]\big) = 0$ for $1 \le i < m$, we see that, for any $g \in \Lambda$, there exists a unique homomorphism

f and a unique homomorphism h such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^{m-1} U_i[m-i] & \longrightarrow & \bigoplus_{j=1}^{m} U_j[\infty] & \xrightarrow{\xi} & U_m[\infty]^{(m)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \bigoplus_{i=1}^{m-1} U_i[m-i] & \longrightarrow & \bigoplus_{j=1}^{m} U_j[\infty] & \xrightarrow{\xi} & U_m[\infty]^{(m)} & \longrightarrow & 0. \end{array}$$

This yields a ring homomorphism $\rho : \Lambda \to M_m(D)$ defined by $g \mapsto h$. More precisely, if $g = \big(g_{u,v}\big)_{1\leq u,v\leq m} \in \Lambda$ with $g_{u,v} \in \mathrm{Hom}_R(U_u[\infty], U_v[\infty])$, then $h = \big(h_{u,v}\big)_{1\leq u,v\leq m} \in M_m(D)$ with $h_{u,v} \in D$ satisfying

(a) $g_{u,v}\pi_{v,m} = \pi_{u,m}h_{u,v}$ if $u < m$ and $v < m$,

(b) $h_{m,v} = g_{m,v}\pi_{v,m}$ if $u = m$ and $v < m$,

(c) $g_{u,m} = \pi_{u,m}h_{u,m}$ if $u < m$ and $v = m$, and

(d) $h_{m,m} = g_{m,m}$.

In particular, the map ρ sends $E_{u,u}$ in Λ to $E_{u,u}$ in $M_m(D)$. In this sense, we may write $\rho = \big(\rho_{u,v}\big)_{1\leq u,v\leq m}$, where $\rho_{u,v} : \mathrm{Hom}_R(U_u[\infty], U_v[\infty]) \to D$ is defined by $g_{u,v} \mapsto h_{u,v}$.

Clearly, ρ is injective since $\mathrm{Hom}_R(U_j[\infty], U_i[m-i]) = 0$ for $1 \leq j \leq m$ and $1 \leq i < m$ by Lemma 3.1(1). In the following, we shall determine the image of ρ, which is clearly a subring of $M_m(D)$.

On the one hand, for any $a \in \mathrm{End}_R(U_u[\infty])$, $b \in \mathrm{Hom}_R(U_u[\infty], U_v[\infty])$ and $c \in \mathrm{End}_R(U_v[\infty])$, we have $(abc)\rho_{u,v} = (a)\rho_{u,u}(b)\rho_{u,v}(c)\rho_{v,v}$. On the other hand, it follows from Lemma 3.1(3) that $\rho_{u,u}$ is always a ring isomorphism, and the left $\mathrm{End}_R(U_u[\infty])$-module $\mathrm{Hom}_R(U_u[\infty], U_v[\infty])$ is freely generated by $\pi_{u,v}$ for $1 \leq u \neq v \leq m$. This implies that the image of ρ coincides with the $m\times m$ matrix ring having $D(\pi_{u,v})\rho_{u,v}$ in the (u,v)-position if $1 \leq u \neq v \leq m$, and D otherwise. Moreover, by Lemmata 3.1(3) and (4), if $1 \leq s < t < m$ and $1 \leq w < m$, we can form the following commutative diagrams:

$$\begin{array}{ccc} U_s[\infty] & \xrightarrow{\pi_{s,m}} & U_m[\infty] \\ \downarrow{\scriptstyle \pi_{s,t}} & & \| \\ U_t[\infty] & \xrightarrow{\pi_{t,m}} & U_m[\infty], \end{array} \quad \begin{array}{ccc} U_t[\infty] & \xrightarrow{\pi_{t,m}} & U_m[\infty] \\ \downarrow{\scriptstyle \pi_{t,s}} & & \downarrow{\scriptstyle \pi_{m,m}} \\ U_s[\infty] & \xrightarrow{\pi_{s,m}} & U_m[\infty], \end{array} \quad \begin{array}{ccc} U_m[\infty] & = & U_m[\infty] \\ \downarrow{\scriptstyle \pi_{m,w}} & & \downarrow{\scriptstyle \pi_{m,m}} \\ U_w[\infty] & \xrightarrow{\pi_{w,m}} & U_m[\infty], \end{array} \quad \begin{array}{ccc} U_w[\infty] & \xrightarrow{\pi_{w,m}} & U_m[\infty] \\ \downarrow{\scriptstyle \pi_{w,m}} & & \| \\ U_m[\infty] & = & U_m[\infty]. \end{array}$$

In other words, we have $(\pi_{s,t})\rho_{s,t} = 1 = (\pi_{w,m})\rho_{w,m}$ and $(\pi_{t,s})\rho_{t,s} = \pi_{m,m} = (\pi_{m,w})\rho_{m,w}$. Thus, the image of ρ is equal to the $m\times m$ matrix ring having $D\pi_{m,m}$ as the (p,q)-entry for $1 \leq q < p \leq m$, and D as the other entries. By Lemma 3.1(5), we know $\mathfrak{m} = D\pi_{m.m}$. Now, by identifying D with $D(\mathcal{C})$ and $\mathfrak{m}$ with $\mathfrak{m}(\mathcal{C})$, we infer that the image of σ coincides with the ring $\Gamma\big(\mathcal{C}\big)$ defined in Lemma 3.2. Therefore, we conclude that $\rho : \Lambda \to \Gamma\big(\mathcal{C}\big)$ is a ring isomorphism which sends $E_{m,1}(\pi_{m,1})$ to $E_{m,1}(\pi_{m,m})$ and $E_{r,r+1}(\pi_{r,r+1})$ to $E_{r,r+1}$ for $1 \leq r < m$. This completes the proof. $\square$

Combining Lemma 3.1(5) with Lemma 3.2, we then obtain the following result which will be used for the calculation of stratifications of derived module categories in the next section.

Corollary 3.3. *For $1 \leq i, j \leq m$, let $\pi_{i,j}$ be the homomorphisms defined in Lemma 3.1(4). Assume that k is an algebraically closed field. Then there exists a ring isomorphism*

$$\sigma : \ \mathrm{End}_R\Big(\bigoplus_{i=1}^{m} U_i[\infty]\Big) \longrightarrow \Gamma(m) := \begin{pmatrix} k[[x]] & k[[x]] & \cdots & k[[x]] \\ (x) & k[[x]] & \ddots & \vdots \\ \vdots & \ddots & \ddots & k[[x]] \\ (x) & \cdots & (x) & k[[x]] \end{pmatrix}_{m\times m}$$

which sends $E_{m,1}(\pi_{m,1})$ to $E_{m,1}(x)$ and $E_{r,r+1}(\pi_{r,r+1})$ to $E_{r,r+1}$ for $1 \leq r < m$.

3.2 Universal localizations at simple regular modules

From now on, let us fix a non-empty subset $\mathcal{U}$ of $\mathscr{S}$, where $\mathscr{S}$ is a complete set of isomorphism classes of all simple regular R-modules. Denote by $\lambda : R \to R_{\mathcal{U}}$ the universal localization of R at $\mathcal{U}$. It follows from [29, Theorems 4.9, 5.1, and 5.3] that λ is injective and $R_{\mathcal{U}}$ is hereditary. Moreover, it is shown in [2, Corollary 4.6(2), 4.7] and [3] that the R-module

$$T_{\mathcal{U}} := R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$$

is a tilting module with $\mathrm{Hom}_R(R_{\mathcal{U}}/R, R_{\mathcal{U}}) = 0$.

Suppose

$$(*) \quad 0 \longrightarrow R \xrightarrow{\lambda} R_{\mathcal{U}} \xrightarrow{\pi} R_{\mathcal{U}}/R \longrightarrow 0,$$

is the canonical exact sequence of R-modules with π the canonical surjection. Set $B := \mathrm{End}_R(T_{\mathcal{U}})$, $S := \mathrm{End}_R(R_{\mathcal{U}}/R)$ and $\Sigma := \{S \otimes_R f_U \mid U \in \mathcal{U}\}$. Recall that the right multiplication map $\mu : R \to S$ defined by $r \mapsto (y \mapsto yr)$ for $r \in R$ and $y \in S/R$, is a ring homomorphism, which endows S with a natural R-R-bimodule structure.

Let $\mathcal{U}^+$ be the full subcategory of R-Mod, defined by

$$\mathcal{U}^+ := \{X \in R\text{-Mod} \mid \mathrm{Ext}^i_R(U,X) = 0 \text{ for all } U \in \mathcal{U} \text{ and all } i \in \mathbb{N}\}.$$

For example, the Prüfer module $V[\infty]$ for $V \in \mathscr{S} \setminus \mathcal{U}$ lies in $\mathcal{U}^+$ by Lemma 3.1(1).

This subcategory has the following characterization, due to [2, Proposition 3.8].

Lemma 3.4. *$\mathcal{U}^+$ coincides with the image of the restriction functor $\lambda_* : R_{\mathcal{U}}\text{-Mod} \to R\text{-Mod}$. In particular, for any $Y \in \mathcal{U}^+$, the unit adjunction $\eta_Y : Y \to R_{\mathcal{U}} \otimes_R Y$, defined by $y \mapsto 1 \otimes y$ for $y \in Y$, is an isomorphism of R-modules.*

Thus, for an R-module $Y \in \mathcal{U}^+$, we may endow it with an $R_{\mathcal{U}}$-module structure via the isomorphism η_Y, and in this way, we consider the R-module Y as an $R_{\mathcal{U}}$-module. Note that this $R_{\mathcal{U}}$-module structure on Y extended from the R-module structure is unique.

Concerning the universal localization $R_{\mathcal{U}}$ of R at $\mathcal{U}$, we have the following facts (see [3, Proposition 1.10], [29] and [13]).

Lemma 3.5. (1) *Suppose that $\mathcal{U}$ contains no cliques. Then $R_{\mathcal{U}}$ is a finite-dimensional tame hereditary k-algebra. In particular, the tilting R-module $T_{\mathcal{U}}$ is classical. Moreover, $\{R_{\mathcal{U}} \otimes_R V \mid V \in \mathscr{S}\setminus\mathcal{U}\}$ is a complete set of non-isomorphic simple regular $R_{\mathcal{U}}$-modules, and $(R_{\mathcal{U}} \otimes_R V)[\infty] \simeq V[\infty]$ as $R_{\mathcal{U}}$-modules for each $V \in \mathscr{S}\setminus\mathcal{U}$.*

(2) *Suppose that $\mathcal{U}$ contains cliques. Then $R_{\mathcal{U}}$ is a hereditary order. Moreover, $\{R_{\mathcal{U}} \otimes_R V \mid V \in \mathscr{S}\setminus\mathcal{U}\}$ is a complete set of non-isomorphic simple $R_{\mathcal{U}}$-modules, and the injective envelope of the $R_{\mathcal{U}}$-module $R_{\mathcal{U}} \otimes_R V$ is isomorphic to $V[\infty]$ for each $V \in \mathscr{S}\setminus\mathcal{U}$.*

(3) *Suppose $\mathcal{V} \subseteq \mathscr{S}\setminus\mathcal{U}$. Then $R_{\mathcal{U}\cup\mathcal{V}} = (R_{\mathcal{U}})_{\overline{\mathcal{V}}}$, where $\overline{\mathcal{V}} := \{R_{\mathcal{U}} \otimes_R V \mid V \in \mathcal{V}\}$. In particular, there are injective ring epimorphisms $R_{\mathcal{U}} \longrightarrow R_{\mathcal{U}\cup\mathcal{V}}$ and $R_{\mathcal{U}\cup\mathcal{V}} \longrightarrow R_{\mathscr{S}}$.*

As remarked in [13], in the case of Lemma 3.5(1), the set of simple regular $R_{\mathcal{U}}$-modules in a clique is of the form

$$\{R_{\mathcal{U}} \otimes_R V \mid V \in \mathcal{C}, V \notin \mathcal{U}\},$$

where $\mathcal{C}$ is a clique of R. Further, by Lemma 3.5(1), for each $V \in \mathcal{C}\setminus\mathcal{U}$, the Prüfer modules corresponding to $R_{\mathcal{U}} \otimes_R V$ and to V are isomorphic. In particular, they have the isomorphic endomorphism ring.

Thus, if $\mathcal{C}_1, \mathcal{C}_2, \cdots,$ and $\mathcal{C}_s$ are all cliques from non-homogeneous tubes and if $\mathcal{U}$ is a union of $c(\mathcal{C}_i) - 1$ simple regular R-modules from each $\mathcal{C}_i$, then each clique of $R_{\mathcal{U}}$ consists of only one single element. This implies that $R_{\mathcal{U}}$ has only two isomorphism classes of simple modules. If, in addition, the field k is algebraically closed, then $R_{\mathcal{U}}$ is Morita equivalent to the Kronecker algebra. In this case, since the set of cliques of the Kronecker algebra are parameterized by $\mathbb{P}^1(k)$, we see that the set of cliques of an arbitrary tame hereditary k-algebra can be indexed by $\mathbb{P}^1(k)$.

A description of the structure of the module $R_{\mathcal{U}}/R$ was first given in [30], and a further substantial discussion is carried out recently in [3]. Especially, the following lemma is proved in [3], where the field k is required to be algebraically closed. In fact, one can check that, if k is an arbitrary field, all of the arguments in the proof of the lemma in [3] are still valid except some mild changes. For instance, the field k should be replaced by certain division rings in most of the proofs.

Lemma 3.6. (1) *The R-module $R_{\mathcal{U}}/R$ is a direct union of finite extensions of modules in $\mathcal{U}$.*

(2) *Let $\mathbf{t} \subset R$-mod be a tube of rank $m > 1$, and let $\mathcal{U} = \{U_1, U_2, \cdots, U_{m-1}\}$ be a set of $m-1$ simple regular modules in $\mathbf{t}$ such that $U_{i+1} = \tau^- U_i$ for all $1 \le i \le m-1$. Then*

$$R_{\mathcal{U}}/R \simeq U_1[m-1]^{(\delta_{U_1})} \oplus U_2[m-2]^{(\delta_{U_2})} \oplus \cdots \oplus U_{m-1}[1]^{(\delta_{U_{m-1}})},$$

with $\delta_{U_j} := \dim_{\mathrm{End}_R(U_j)} \mathrm{Ext}^1_R(U_j, R)$ for $1 \le j \le m-1$. Moreover, $R_{\mathcal{U}} \otimes_R U_m \simeq U_m[m]$ as $R_{\mathcal{U}}$-modules.

(3) *If $\mathcal{U}$ is a union of cliques, then, for any finitely generated projective R-module P,*

$${}_R(R_{\mathcal{U}}/R) \otimes_R P \simeq \bigoplus_{U \in \mathcal{U}} U[\infty]^{(\delta_{U,P})},$$

where $\delta_{U,P} := \dim_{\mathrm{End}_R(U)} \mathrm{Ext}^1_R(U, P)$.

Next, we shall show that $R_{\mathcal{U}}$ and $\mathrm{End}_R(R_{\mathcal{U}}/R)$ can be interpreted as the tensor product and direct sum of some rings, respectively.

Lemma 3.7. *Let $\mathcal{U} = \mathcal{U}_0 \dot\cup \mathcal{U}_1 \subseteq \mathscr{S}$ such that $\mathcal{U}_0$ contains no cliques and $\mathcal{U}_1$ is a union of cliques. Then the following statements are true:*

(1) *$\mathcal{U}_0 \subseteq \mathcal{U}_1^+$, $\mathcal{U}_1 \subseteq \mathcal{U}_0^+$, $R_{\mathcal{U}} \simeq R_{\mathcal{U}_1} \otimes_R R_{\mathcal{U}_0}$ as $R_{\mathcal{U}_1}$-$R_{\mathcal{U}_0}$-bimodules, and $R_{\mathcal{U}}/R_{\mathcal{U}_1} \simeq R_{\mathcal{U}_1} \otimes_R (R_{\mathcal{U}_0}/R)$ as $R_{\mathcal{U}_1}$-R-bimodules.*

(2) *There is a ring isomorphism*

$$\varphi : \mathrm{End}_R(R_{\mathcal{U}}/R) \longrightarrow \mathrm{End}_R(R_{\mathcal{U}_0}/R) \times \mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0}).$$

Proof. (1) By the assumption on $\mathcal{U}$, if $U \in \mathcal{U}_0$ and $V \in \mathcal{U}_1$ then they belong to different tubes, and therefore $\mathcal{U}_0 \subseteq \mathcal{U}_1^+$ and $\mathcal{U}_1 \subseteq \mathcal{U}_0^+$.

By Lemma 3.4, the unit adjunction $\eta_U : U \to R_{\mathcal{U}_0} \otimes_R U$ is an isomorphism of R-modules for any $U \in \mathcal{U}_1$. This implies that every module in $\mathcal{U}_1$ can be endowed with a unique $R_{\mathcal{U}_0}$-module structure that preserves the given R-module structure via the universal localization $\lambda_0 : R \to R_{\mathcal{U}_0}$. Consequently, it follows from Lemma 3.5(3) that $R_{\mathcal{U}} = (R_{\mathcal{U}_0})_{\mathcal{U}_1}$. Moreover, we can construct the following exact commutative diagram of R-modules:

$$(*) \qquad \begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{\lambda_0} & R_{\mathcal{U}_0} & \xrightarrow{\pi_0} & R_{\mathcal{U}_0}/R & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_2} & & \\
0 & \longrightarrow & R & \xrightarrow{\lambda} & R_{\mathcal{U}} & \xrightarrow{\pi} & R_{\mathcal{U}}/R & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \\
 & & & & R_{\mathcal{U}}/R_{\mathcal{U}_0} & = & R_{\mathcal{U}}/R_{\mathcal{U}_0} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

where λ_1 is the universal localization of $R_{\mathcal{U}_0}$ at $\mathcal{U}_1$, and λ_2 is the canonical injection induced by λ_1, and where π_0, π_1 and π_2 are canonical surjections.

Clearly, $R_{\mathcal{U}_0}$ is a finite-dimensional tame hereditary algebra by Lemma 3.5(1). From $R_{\mathcal{U}} = (R_{\mathcal{U}_0})_{\mathcal{U}_1}$ we see that $R_{\mathcal{U}}/R_{\mathcal{U}_0}$ is a direct union of finite extensions of modules in $\mathcal{U}_1$ by Lemma 3.6(1). Since $R_{\mathcal{U}_1}$ is the universal localization of R at $\mathcal{U}_1$, we have $\mathrm{Tor}_i^R(R_{\mathcal{U}_1}, V) = 0$ for any $i \geq 0$ and $V \in \mathcal{U}_1$. Note that the i-th left derived functor $\mathrm{Tor}_i^R(R_{\mathcal{U}_1}, -) : R\text{-Mod} \to \mathbb{Z}\text{-Mod}$ commutes with direct limits. Thus $\mathrm{Tor}_i^R(R_{\mathcal{U}_1}, R_{\mathcal{U}}/R_{\mathcal{U}_0}) = 0$ for any $i \geq 0$, which implies that the homomorphisms $R_{\mathcal{U}_1} \otimes_R \lambda_1$ and $R_{\mathcal{U}_1} \otimes_R \lambda_2$ are isomorphisms. Moreover, by Lemma 2.7, we have $R_{\mathcal{U}} = (R_{\mathcal{U}_1})_{\overline{\mathcal{U}}_0}$ with $\overline{\mathcal{U}}_0 := \{R_{\mathcal{U}_1} \otimes_R U \mid U \in \mathcal{U}_0\}$, and therefore $R_{\mathcal{U}}$ can be regarded as an $R_{\mathcal{U}_1}$-module. Consequently, the canonical multiplication map $\nu_2 : R_{\mathcal{U}_1} \otimes_R R_{\mathcal{U}} \to R_{\mathcal{U}}$ is an isomorphism.

Now we apply the tensor functor $R_{\mathcal{U}_1} \otimes_R -$ to the diagram $(*)$, and get the following exact commutative diagram of $R_{\mathcal{U}_1}$-R-bimodules:

$$\begin{array}{ccccccccc}
 & & R_{\mathcal{U}_1}\otimes_R R & \xrightarrow{R_{\mathcal{U}_1}\otimes_R\lambda_0} & R_{\mathcal{U}_1}\otimes_R R_{\mathcal{U}_0} & \xrightarrow{R_{\mathcal{U}_1}\otimes_R\pi_0} & R_{\mathcal{U}_1}\otimes_R (R_{\mathcal{U}_0}/R) & \longrightarrow & 0\\
 & & \| & & \simeq\downarrow{\scriptstyle R_{\mathcal{U}_1}\otimes_R\lambda_1} & & \simeq\downarrow{\scriptstyle R_{\mathcal{U}_1}\otimes_R\lambda_2} & & \\
 & & R_{\mathcal{U}_1}\otimes_R R & \xrightarrow{R_{\mathcal{U}_1}\otimes_R\lambda} & R_{\mathcal{U}_1}\otimes_R R_{\mathcal{U}} & \xrightarrow{R_{\mathcal{U}_1}\otimes_R\pi} & R_{\mathcal{U}_1}\otimes_R (R_{\mathcal{U}}/R) & \longrightarrow & 0\\
 & & \simeq\downarrow{\scriptstyle \nu_1} & & \simeq\downarrow{\scriptstyle \nu_2} & & \downarrow & & \\
0 & \longrightarrow & R_{\mathcal{U}_1} & \longrightarrow & R_{\mathcal{U}} & \longrightarrow & R_{\mathcal{U}}/R_{\mathcal{U}_1} & \longrightarrow & 0,
\end{array}$$

where ν_1 is the multiplication map. Thus $R_{\mathcal{U}} \simeq R_{\mathcal{U}_1} \otimes_R R_{\mathcal{U}_0}$ as $R_{\mathcal{U}_1}$-$R_{\mathcal{U}_0}$-bimodules, and $R_{\mathcal{U}}/R_{\mathcal{U}_1} \simeq R_{\mathcal{U}_1} \otimes_R (R_{\mathcal{U}_0}/R)$ as $R_{\mathcal{U}_1}$-R-bimodules.

(2) Note that $\mathcal{U}_1 \subseteq \mathcal{U}_0^+$ and $\mathcal{U}_0 \subseteq \mathcal{U}_1^+$, and that $\mathcal{U}_0$ and $\mathcal{U}_1$ consist of finitely presented modules of projective dimension one. By Lemmata 3.5(1) and 3.6(1), we can write $R_{\mathcal{U}}/R_{\mathcal{U}_0} = \varinjlim_{\alpha} X_\alpha$ with $X_\alpha \in \mathcal{F}(\mathcal{U}_1)$. Then, by Lemma 2.11, we have the following isomorphisms:

$$(**)\quad \mathrm{Ext}_R^j(R_{\mathcal{U}_0}/R, R_{\mathcal{U}}/R_{\mathcal{U}_0}) \simeq \varinjlim_{\alpha} \mathrm{Ext}_R^j(R_{\mathcal{U}_0}/R, X_\alpha) = 0 = \varprojlim_{\alpha} \mathrm{Ext}_R^j(X_\alpha, R_{\mathcal{U}_0}/R) \simeq \mathrm{Ext}_R^j(R_{\mathcal{U}}/R_{\mathcal{U}_0}, R_{\mathcal{U}_0}/R)$$

for any $j \geq 0$. Particularly, the canonical exact sequence

$$0 \longrightarrow R_{\mathcal{U}_0}/R \xrightarrow{\lambda_2} R_{\mathcal{U}}/R \xrightarrow{\pi_2} R_{\mathcal{U}}/R_{\mathcal{U}_0} \longrightarrow 0$$

splits in R-Mod, that is, $R_{\mathcal{U}}/R \simeq R_{\mathcal{U}_0}/R \oplus R_{\mathcal{U}}/R_{\mathcal{U}_0}$ as R-modules. Since $R \to R_{\mathcal{U}_0}$ is a ring epimorphism, we have $\mathrm{End}_R(R_{\mathcal{U}}/R_{\mathcal{U}_0}) = \mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0})$. Thus it follows from $(**)$ for $j = 0$ that

$$\mathrm{End}_R(R_{\mathcal{U}}/R) \simeq \mathrm{End}_R(R_{\mathcal{U}_0}/R) \times \mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0}).$$

This completes the proof of (2). $\square$

3.3 Proof of Theorem 1.1

Before we start with the proof of the main result, Theorem 1.1, we have to make the following preparations.

Lemma 3.8. *Let $\mathcal{U} = \mathcal{U}_0 \dot{\cup} \mathcal{U}_1 \subseteq \mathscr{S}$ such that $\mathcal{U}_1$ is a union of cliques and $\mathcal{U}_0$ does not contain any cliques. Set $\Lambda := \mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0})$ and $\Theta := \{\Lambda \otimes_{R_{\mathcal{U}_0}} (R_{\mathcal{U}_0} \otimes_R f_V) \mid V \in \mathcal{U}_1\}$, $S := \mathrm{End}_R(R_{\mathcal{U}}/R)$ and $\Sigma := \{S \otimes_R f_U \mid U \in \mathcal{U}\}$. Then S_Σ is isomorphic to the universal localization Λ_Θ of Λ at Θ.*

Proof. By Lemma 3.4, we have $R_{\mathcal{U}_0}\otimes_R V\simeq V$ as R-modules for each $V\in\mathcal{U}_1$. Combining this with Lemma 3.5(1), we see that $\mathcal{U}_1$ can be seen as a set of simple regular $R_{\mathcal{U}_0}$-modules, and therefore $R_{\mathcal{U}}=(R_{\mathcal{U}_0})_{\mathcal{U}_1}$ by Lemma 3.5(3). More precisely, for any $V\in\mathcal{U}_1$, we fix a minimal projective presentation

$$0\longrightarrow P_1\xrightarrow{f_V}P_0\longrightarrow V\longrightarrow 0$$

of V in R-mod, and get a projective presentation of V in $R_{\mathcal{U}_0}$-mod :

$$0\longrightarrow R_{\mathcal{U}_0}\otimes_R P_1\xrightarrow{R_{\mathcal{U}_0}\otimes_R f_V}R_{\mathcal{U}_0}\otimes_R P_0\longrightarrow V\longrightarrow 0.$$

This is due to the fact that $\mathrm{Tor}_1^R(R_{\mathcal{U}_0},V)\simeq\mathrm{Tor}_1^R(R_{\mathcal{U}_0},R_{\mathcal{U}_0}\otimes_R V)\simeq\mathrm{Tor}_1^{R_{\mathcal{U}_0}}(R_{\mathcal{U}_0},R_{\mathcal{U}_0}\otimes_R V)=0$. Thus, $R_{\mathcal{U}}$ is the universal localization of $R_{\mathcal{U}_0}$ at the set $\{R_{\mathcal{U}_0}\otimes_R f_V\mid V\in\mathcal{U}_1\}$. Note that $R_{\mathcal{U}_0}$ is a tame hereditary k-algebra by Lemma 3.5(1).

Let $\Lambda:=\mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0})$ and $\Theta:=\{\Lambda\otimes_{R_{\mathcal{U}_0}}(R_{\mathcal{U}_0}\otimes_R f_V)\mid V\in\mathcal{U}_1\}$. In the following, we shall show that S_Σ is isomorphic to Λ_Θ.

Let $\Gamma:=\mathrm{End}_R(R_{\mathcal{U}_0}/R)$ and $\varphi=(\varphi_0,\varphi_1):S\to\Gamma\times\Lambda$, where $\varphi_0:S\to\Gamma$ and $\varphi_1:S\to\Lambda$ are the ring homomorphisms given in Lemma 3.7(2). Recall that $\mu:R\to S$ is the right multiplication map. Set $\mu_0=\mu\varphi_0:R\to\Gamma$ and $\mu_1=\mu\varphi_1:R\to\Lambda$. Clearly, both μ_0 and μ_1 are ring homomorphisms, through which Λ and Γ have a right R-module structure, respectively. Now, we write $\Sigma:=\{S\otimes_R f_U\mid U\in\mathcal{U}\}$ as $\Phi\times\Psi$ with $\Phi:=\{\Gamma\otimes_R f_U\mid U\in\mathcal{U}\}$ and $\Psi:=\{\Lambda\otimes_R f_U\mid U\in\mathcal{U}\}$. Consequently, the ring isomorphism φ implies that $S_\Sigma\simeq\Gamma_\Phi\times\Lambda_\Psi$. To finish the proof, it suffices to prove that $\Gamma_\Phi=0$ and $\Lambda_\Psi\simeq\Lambda_\Theta$.

Indeed, we write $\Phi=\Phi_0\cup\Phi_1$ with $\Phi_0:=\{\Gamma\otimes_R f_U\mid U\in\mathcal{U}_0\}$ and $\Phi_1:=\{\Gamma\otimes_R f_U\mid U\in\mathcal{U}_1\}$. Then, by Lemma 2.7, we have $\Gamma_\Phi\simeq(\Gamma_{\Phi_0})_{\overline{\Phi}_1}$, where $\overline{\Phi}_1:=\{\Gamma_{\Phi_0}\otimes_R f_U\mid U\in\mathcal{U}_1\}$. To prove $\Gamma_\Phi=0$, it suffices to prove $\Gamma_{\Phi_0}=0$. Consider the canonical exact sequence of R-modules:

$$0\longrightarrow R\xrightarrow{\lambda_0}R_{\mathcal{U}_0}\xrightarrow{\pi_0}R_{\mathcal{U}_0}/R\longrightarrow 0.$$

By Lemma 3.5(1), the module $T_{\mathcal{U}_0}:=R_{\mathcal{U}_0}\oplus R_{\mathcal{U}_0}/R$ is a classical tilting R-module, and therefore $\mathscr{D}(R)$ is triangle equivalent to $\mathscr{D}(\mathrm{End}_R(T_{\mathcal{U}_0}))$ in the recollement of $\mathscr{D}(R)$, $\mathscr{D}(\mathrm{End}_R(T_{\mathcal{U}_0}))$ and $\mathscr{D}(\Gamma_{\Phi_0})$ by Proposition 2.6. Thus $\Gamma_{\Phi_0}=0$ and $\Gamma_\Phi=0$.

It remains to show $\Lambda_\Psi\simeq\Lambda_\Theta$. Let $\mu_2:R_{\mathcal{U}_0}\to\Lambda$ be the right multiplication map defined by $r\mapsto(x\mapsto xr)$ for $r\in R_{\mathcal{U}_0}$ and $x\in R_{\mathcal{U}}/R_{\mathcal{U}_0}$. Then, along the diagram $(*)$ in the proof of Lemma 3.7, one can check that the following diagram of ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\lambda_0} & R_{\mathcal{U}_0} \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu_2} \\ S & \xrightarrow{\varphi_1} & \Lambda \end{array}$$

commutes. Now, we write $\Psi=\Psi_0\cup\Psi_1$ with

$$\Psi_0:=\{\Lambda\otimes_R f_U\mid U\in\mathcal{U}_0\}\quad\text{and}\quad\Psi_1:=\{\Lambda\otimes_R f_V\mid V\in\mathcal{U}_1\},$$

and claim $\Lambda_{\Psi_0}=\Lambda$. It suffices to show that $\Lambda\otimes_R f_U$ is an isomorphism for any $U\in\mathcal{U}_0$. However, this follows from $\Lambda\otimes_R f_U\simeq\Lambda\otimes_{R_{\mathcal{U}_0}}(R_{\mathcal{U}_0}\otimes_R f_U)$ and $R_{\mathcal{U}_0}\otimes_R f_U$ being an isomorphism by the definition of universal localizations. Hence $\Lambda_{\Psi_0}=\Lambda$.

Now, we have $\overline{\Psi}_1:=\{\Lambda_{\Psi_0}\otimes_\Lambda h\mid h\in\Psi_1\}=\Psi_1$. It follows from Lemma 2.7 that $\Lambda_\Psi\simeq(\Lambda_{\Psi_0})_{\overline{\Psi}_1}\simeq\Lambda_{\Psi_1}$. Further, we have $\Lambda\otimes_R f_V\simeq\Lambda\otimes_{R_{\mathcal{U}_0}}(R_{\mathcal{U}_0}\otimes_R f_V)$ for any $V\in\mathcal{U}_1$. By comparing the elements in Θ with the ones in Ψ_1, one knows immediately that $\Lambda_\Psi\simeq\Lambda_\Theta$, and therefore $S_\Sigma\simeq\Lambda_\Theta$, finishing the proof. $\square$

Next, we shall show that the universal localizations of interest for us take actually the form of adèle rings in the algebraic number theory [24]. Before stating the following lemma, we first recall some notations.

Let $\mathcal{C}$ be a clique of R-mod. Recall that $D(\mathcal{C})$ stands for the endomorphism ring of a Prüfer module $V[\infty]$ with $V \in \mathcal{C}$. Note that $D(\mathcal{C})$ is a discrete valuation ring with the division ring $Q(\mathcal{C})$ of fractions of $D(\mathcal{C})$. Clearly, $Q(\mathcal{C})$ contains $D(\mathcal{C})$ as a subring.

Lemma 3.9. *Suppose that $\mathcal{U} \subseteq \mathscr{S}$ is a union of cliques, say $\mathcal{U} = \cup_{i\in I}\mathcal{C}_i$ with I an index set. Let $S := \mathrm{End}_R(R_{\mathcal{U}}/R)$ and $\Sigma := \{S\otimes_R f_U \mid U \in \mathcal{U}\}$. Then S_Σ is Morita equivalent to the adèle ring*

$$\mathbb{A}_{\mathcal{U}} := \Big\{ \big(f_i\big)_{i\in I} \in \prod_{i\in I} Q(\mathcal{C}_i) \;\Big|\; f_i \in D(\mathcal{C}_i) \textit{ for almost all } i \in I \Big\}.$$

Proof. For any finitely generated projective R-module P, we always have $S\otimes_R P \simeq \mathrm{Hom}_R(R_{\mathcal{U}}/R, (R_{\mathcal{U}}/R)\otimes_R P)$ as S-modules. Thus, we can rewrite $\Sigma = \{\mathrm{Hom}_R\big(R_{\mathcal{U}}/R, (R_{\mathcal{U}}/R)\otimes_R f_V\big) \mid V \in \mathcal{U}\}$. The whole proof of Lemma 3.9 will be proceeded in three steps.

Step (1). We provide an alternative form of the homomorphism $(R_{\mathcal{U}}/R)\otimes_R f_V$ for any $V \in \mathcal{U}$.

In fact, this procedure can be done for each clique $\mathcal{C}$ in $\mathcal{U}$. Let us give the details: Fix a clique $\mathcal{C} \subseteq \mathcal{U}$ and an element $U \in \mathcal{C}$, and choose a projective resolution $0 \longrightarrow P_1 \xrightarrow{f_U} P_0 \longrightarrow U \longrightarrow 0$ of U in R-mod, where P_1 and P_0 are finitely generated projective R-modules. As $\lambda: R \to R_{\mathcal{U}}$ is the universal localization of R at $\mathcal{U}$, we know that $R_{\mathcal{U}}\otimes_R f_U : R_{\mathcal{U}}\otimes_R P_1 \to R_{\mathcal{U}}\otimes_R P_0$ is an isomorphism. This yields the following exact and commutative diagram of R-modules:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & U & & \\
 & & \downarrow & & & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & P_1 & \xrightarrow{\lambda\otimes_R P_1} & R_{\mathcal{U}}\otimes_R P_1 & \xrightarrow{\pi\otimes_R P_1} & (R_{\mathcal{U}}/R)\otimes_R P_1 & \longrightarrow & 0 \\
 & & {\scriptstyle f_U}\downarrow & & \simeq\downarrow{\scriptstyle R_{\mathcal{U}}\otimes_R f_U} & & \downarrow{\scriptstyle (R_{\mathcal{U}}/R)\otimes_R f_U} & & \\
0 & \longrightarrow & P_0 & \xrightarrow{\lambda\otimes_R P_0} & R_{\mathcal{U}}\otimes_R P_0 & \xrightarrow{\pi\otimes_R P_0} & (R_{\mathcal{U}}/R)\otimes_R P_0 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & U & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

Consider the following short exact sequence of R-modules:

$$(a) \quad 0 \longrightarrow U \xrightarrow{\psi} (R_{\mathcal{U}}/R)\otimes_R P_1 \xrightarrow{(R_{\mathcal{U}}/R)\otimes_R f_U} (R_{\mathcal{U}}/R)\otimes_R P_0 \longrightarrow 0.$$

On the one hand, by Lemma 3.6(3), we have

$$(R_{\mathcal{U}}/R)\otimes_R P_1 \simeq \bigoplus_{i\in I}\bigoplus_{V\in\mathcal{C}_i} V[\infty]^{(n_V)}$$

for some $n_V \in \mathbb{N}$, where n_U is non-zero since U can be embedded into $(R_{\mathcal{U}}/R)\otimes_R P_1$. On the other hand, for $W \in \mathcal{U}$, we have $\mathrm{Hom}_R(U, W[\infty]) = 0$ if $W \not\simeq U$, and $\mathrm{Hom}_R(U, U[\infty]) \simeq \mathrm{End}_R(U)$. Now, let

$$0 \longrightarrow U \xrightarrow{\zeta_U} U[\infty] \xrightarrow{\pi_U} (\tau^- U)[\infty] \longrightarrow 0$$

be the canonical exact sequence defined in Lemma 3.1(2), where ζ_U is the canonical inclusion. Set $D := \mathrm{End}_R(U[\infty])$. Then D is a discrete valuation ring. Particularly, it is a local ring with a unique maximal ideal $\mathfrak{m}$. By the proof of Lemma 3.1(5), we know that $\mathrm{Hom}_R(U, U[\infty]) = \zeta_U D \simeq D/\mathfrak{m}$ as right D-modules. This means that, for any $\alpha : U \to U[\infty]$, there is a homomorphism $\beta \in D$ such that $\alpha = \zeta_U \beta$. Moreover, if the above homomorphism α is non-zero, then β must be an isomorphism.

Keeping these details in mind, we can form the following commutative diagram:

$$(b) \qquad \begin{array}{ccc} U & \xrightarrow{\psi} & (R_{\mathcal{U}}/R)\otimes_R P_1 \\ \| & & \downarrow \simeq \\ U & \xrightarrow{(\zeta_U, g)} & U[\infty]\oplus E, \end{array}$$

where E is an R-module and $g : U \to E$ is an R-homomorphism which factorizes through ζ_U. Then, by applying Lemma 2.10 and combining (a) with (b), we can construct the following exact and commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & U & \xrightarrow{\psi} & (R_{\mathcal{U}}/R)\otimes_R P_1 & \xrightarrow{(R_{\mathcal{U}}/R)\otimes_R f_U} & (R_{\mathcal{U}}/R)\otimes_R P_0 & \to & 0 \\ & & \| & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \to & U & \xrightarrow{(\zeta_U, 0)} & U[\infty]\oplus E & \xrightarrow{\left(\begin{smallmatrix} \pi_U & 0 \\ 0 & 1 \end{smallmatrix}\right)} & (\tau^- U)[\infty]\oplus E & \to & 0. \end{array}$$

Suppose $\mathcal{C} = \{U_1, U_2, \cdots, U_{m-1}, U_m\}$ with $m \geq 1$ such that $\tau^- U_i = U_{i+1}$ for any $1 \leq i \leq m$, where the subscript of U_i is always modulo m. Suppose $U = U_j$ for some $1 \leq j \leq m$. This means that π_U coincides with $\pi_{j,j+1} : U_j[\infty] \to U_{j+1}[\infty]$ defined in Lemma 3.1(4), where $\pi_{m,m+1} = \pi_{m,1}$ by our convention.

Set

$$M := \bigoplus_{i=1}^{m} U_i[\infty], \quad \Lambda := \mathrm{End}_R(M) \quad \text{and} \quad \Pi := \{\mathrm{Hom}_R(M, \pi_{s,s+1}) \mid 1 \leq s \leq m\}.$$

Step (2). We prove $\Lambda_\Pi \simeq M_m\big(Q(\mathcal{C})\big)$, the $m \times m$ matrix ring over the division ring $Q(\mathcal{C})$.

For convenience, if $1 \leq u, v \leq m$, we denote by $E_{u,v}$ the $m \times m$ matrix unit which has 1 in the (u,v) position, and 0 elsewhere.

By Lemma 3.2, there is a ring isomorphism $\rho : \Lambda \to \Gamma(\mathcal{C})$, which sends $E_{m,1}(\pi_{m,1})$ to $E_{m,1}(\pi_{m,m})$ and $E_{s,s+1}(\pi_{s,s+1})$ to $E_{s,s+1}$ for $1 \leq s \leq m-1$ (see Lemma 3.2 for notations). Let $\varphi_m : \Gamma(\mathcal{C})E_{m,m} \to \Gamma(\mathcal{C})E_{1,1}$ and $\varphi_s : \Gamma(\mathcal{C})E_{s,s} \to \Gamma(\mathcal{C})E_{s+1,s+1}$ be the canonical homomorphisms induced by multiplying on the right by $E_{m,1}(\pi_{m,m})$ and $E_{s,s+1}$ for $1 \leq s \leq m-1$, respectively, and define $\Theta := \{\varphi_m\} \cup \{\varphi_s \mid 1 \leq s \leq m-1\}$. As a result, we get $\Lambda_\Pi \simeq \Gamma(\mathcal{C})_\Theta$. It remains to prove $\Gamma(\mathcal{C})_\Theta \simeq M_m\big(Q(\mathcal{C})\big)$. In fact, by Lemma 2.3, one can check that the canonical inclusion from $\Gamma(\mathcal{C})$ to $M_m\big(D(\mathcal{C})\big)$ is the universal localization of $\Gamma(\mathcal{C})$ at $\{\varphi_s \mid 1 \leq s \leq m-1\}$. Observe that the universal localization $D(\mathcal{C})_{\pi_{m,m}}$ of $D(\mathcal{C})$ at $\pi_{m,m}$ is equal to $Q(\mathcal{C})$ by Lemma 2.9. Now, combining Lemma 2.7 with Corollary [8, Corollary 3.4], we have

$$\Gamma(\mathcal{C})_\Theta \simeq M_m\big(D(\mathcal{C})\big)_{\varphi'_m} \simeq M_m\big(D(\mathcal{C})_{\pi_{m,m}}\big) \simeq M_m\big(Q(\mathcal{C})\big),$$

where $\varphi'_m : M_m\big(D(\mathcal{C})\big)E_{m,m} \to M_m\big(D(\mathcal{C})\big)E_{1,1}$ is the canonical homomorphism induced by $E_{m,1}(\pi_{m,m})$. Thus $\Lambda_\Pi \simeq M_m\big(Q(\mathcal{C})\big)$.

Step (3). We show that S_Σ is Morita equivalent to the adèle ring $\mathbb{A}_{\mathcal{U}}$ defined in Lemma 3.9.

Indeed, by Lemma 3.6(3), we have $R_{\mathcal{U}}/R \simeq \bigoplus_{i\in I} \bigoplus_{V\in \mathcal{C}_i} V[\infty]^{(\delta_V)}$, where $\delta_V := \dim_{\mathrm{End}_R(V)} \mathrm{Ext}^1_R(V,R) = \dim_{\mathrm{End}_R(V)^{op}}(\tau V) \neq 0$. We claim that there exists $d \in \mathbb{N}$ such that $\delta_V \leq d$ for all $V \in \mathcal{U}$.

In fact, let $\{S_j \mid 1 \leq j \leq r\}$ be a complete set of isomorphism classes of simple R-modules for some $r \in \mathbb{N}$. For each $X \in R$-mod, denote by $\underline{\dim}\, X \in \mathbb{N}^r$ the dimension vector of X. Now, let $< -, - >: \mathbb{N}^r \times \mathbb{N}^r \to \mathbb{Z}$ be the Euler form of the tame hereditary k-algebra R, that is, $< \underline{\dim}\, Y, \underline{\dim}\, Z > := \dim_k \mathrm{Hom}_R(Y,Z) - \dim_k \mathrm{Ext}^1_R(Y,Z)$

with $Y,Z \in R$-mod, and further, let $q : \mathbb{N}^r \to \mathbb{Z}$ be the quadratic form of R, that is, $q(\underline{\dim} Y) := <\underline{\dim} Y, \underline{\dim} Y>$, and let $h = (h_i)_{1\leq i\leq r}$ be the minimal positive radical vector of q. It is known that h is equal to the sum of the dimension vectors of all simple regular R-modules in $\mathbf{t}'$ for an arbitrary tube $\mathbf{t}'$ of R. Therefore, we have $\delta_U \leq \dim_k(\tau U) \leq (\sum_i h_i)(\sum_j \dim_k S_j) < \infty$ for $U \in \mathscr{S}$. In particular, if we take $d = (\sum_i h_i)(\sum_j \dim_k S_j)$, then $\delta_V \leq d$ for all $V \in \mathcal{U}$, as claimed.

Set

$$N := \bigoplus_{i\in I} \bigoplus_{V\in \mathcal{C}_i} V[\infty] \quad \text{and} \quad \Gamma := \mathrm{End}_R(N).$$

By the above claim, one can check that $\mathrm{Hom}_R(R_{\mathcal{U}}/R, N)$ is a finitely generated, projective generator for S-Mod, and therefore S is Morita equivalent to Γ. Note that Morita equivalences preserve universal localizations by [8, Corollary 3.4]. Thus, we conclude from Step (1) and the definition of Σ that S_Σ is Morita equivalent to Γ_Φ with

$$\Phi := \{\mathrm{Hom}_R(N, \pi_V) \mid V \in \mathcal{U}\}.$$

Now, let $\mathcal{U} = \mathcal{L}\dot{\cup}\mathcal{W}$ be an arbitrary decomposition such that $\mathcal{L}$ is a union of cliques $\mathcal{C}_i$ with i in an index set I_0 and that $\mathcal{W}$ is a union of cliques $\mathcal{C}_j$ with j in an index set I_1. Note that $I = I_0\dot{\cup}I_1$. Moreover, if $i, j \in I$ with $i \neq j$, then $\mathrm{Hom}_R(U[\infty], V[\infty]) = 0$ for all $U \in \mathcal{C}_i$ and $V \in \mathcal{C}_j$. Thus, by Lemma 3.2, we get the following isomorphisms:

$$(*) \quad \Gamma \simeq \prod_{i\in I} \mathrm{End}_R\Big(\bigoplus_{V\in\mathcal{C}_i} V[\infty]\Big) \simeq \prod_{i\in I}\Gamma(\mathcal{C}_i) \simeq \prod_{i\in I_0}\Gamma(\mathcal{C}_i) \times \prod_{i\in I_1}\Gamma(\mathcal{C}_i).$$

We write $\Gamma_0 := \prod_{i\in I_0}\Gamma(\mathcal{C}_i)$ and $\Gamma_1 := \prod_{i\in I_1}\Gamma(\mathcal{C}_i)$ and decompose $\Phi = \Phi_0 \cup \Phi_1$ where

$$\Phi_0 := \{\mathrm{Hom}_R(N, \pi_V) \mid V \in \mathcal{L}\} \quad \text{and} \quad \Phi_1 := \{\mathrm{Hom}_R(N, \pi_W) \mid W \in \mathcal{W}\}.$$

Under these isomorphisms $(*)$, we can regard Φ_0 (respectively, Φ_1) as the set of homomorphisms between finitely generated projective Γ_0-modules (respectively, Γ_1-modules). With these identifications, one can prove $\Gamma_\Phi \simeq (\Gamma_0)_{\Phi_0} \times (\Gamma_1)_{\Phi_1}$.

Next, we assume that each clique in $\mathcal{W}$ is of rank one, and each clique $L \in \mathcal{L}$ is of rank greater than one. Clearly, $\mathcal{L}$ is a finite set.

On the one hand, by the foregoing discussion and Step (2), we obtain

$$(\Gamma_0)_{\Phi_0} \simeq \prod_{i\in I_0} M_{c(\mathcal{C}_i)}\big(Q(\mathcal{C}_i)\big).$$

On the other hand, we have $\Gamma_1 = \prod_{i\in I_1} D(\mathcal{C}_i)$. Now, we claim $(\Gamma_1)_{\Phi_1} \simeq \mathbb{A}_{\mathcal{W}}$, where

$$\mathbb{A}_{\mathcal{W}} := \Big\{(f_i)_{i\in I_1} \in \prod_{i\in I_1} Q(\mathcal{C}_i) \mid f_i \in D(\mathcal{C}_i) \text{ for almost all } i \in I_1\Big\}.$$

This ring is similar to the so called adèle ring appearing in the algebraic number theory (see [24, Chapter 5, Section 1]).

Actually, for each $i \in I_1$, the clique $\mathcal{C}_i$ consists of only one simple regular module. Hence we write $D(\mathcal{C}_i) = \mathrm{End}_R(\mathcal{C}_i)$, which is a discrete valuation ring with a unique maximal ideal generated by π_i.

We define $e_i := \big(\beta_j\big)_{j\in I_1} \in \Gamma_1$ by $\beta_i = 1$ and $\beta_j = 0$ if $j \neq i$, and define $\varepsilon_i := \big(\theta_j\big)_{j\in I_1} \in \Gamma_1$ by $\theta_i = \pi_i$ and $\theta_j = 1$ if $j \neq i$. Let $\varphi_i : \Gamma_1 e_i \to \Gamma_1 e_i$ be the right multiplication map defined by $g \mapsto g\pi_i$ for any $g \in D(\mathcal{C}_i)$.

Under the isomorphisms $(*)$, we can identify Φ_1 with $\{\varphi_j \mid j \in I_1\}$. Note that the right multiplication map $\overline{\varepsilon}_i$ defined by ε_i has the following form:

$$\overline{\varepsilon}_i = \begin{pmatrix} \varphi_i & 0 \\ 0 & 1 \end{pmatrix} : \Gamma_1 e_i \oplus \Gamma_1(1-e_i) \longrightarrow \Gamma_1 e_i \oplus \Gamma_1(1-e_i).$$

Set $\Psi := \{\bar{\varepsilon}_j \mid j \in I_1\}$. It is easy to see that $(\Gamma_1)_{\Phi_1}$ is isomorphic to the universal localization $(\Gamma_1)_\Psi$ of Γ_1 at Ψ. We consider the minimal multiplicative subset Υ of Γ_1 containing all ε_j for $j \in I_1$. Clearly, $(\Gamma_1)_\Psi$ is isomorphic to the universal localization of Γ_1 at Υ, that is, the universal localization of Γ_1 at all right multiplication maps induced by the elements of Υ. One can check

$$\Upsilon = \left\{ (f_i)_{i\in I_1} \in \prod_{i\in I_1} \{(\pi_i)^n \mid n \in \mathbb{N}\} \;\middle|\; f_i = 1 \text{ for almost all } i \in I_1 \right\} \subseteq \Gamma_1.$$

We claim that Υ is a left and right denominator subset of Γ_1 (see Definition 2.4).

Indeed, let $a = (a_i)_{i\in I_1} \in \Gamma_1$ and $s = (\pi_i^{n_i})_{i\in I_1} \in \Upsilon$ with $n_i \in \mathbb{N}$. Since $D(\mathcal{C}_i)$ is a discrete valuation ring for each $i \in I_1$, we have $D(\mathcal{C}_i)\pi_i^{n_i} = \pi_i^{n_i} D(\mathcal{C}_i)$, and therefore $\Gamma_1 s = \prod_{i\in I_1} D(\mathcal{C}_i)\pi_i^{n_i} = \prod_{i\in I_1} \pi_i^{n_i} D(\mathcal{C}_i)$. This means $sa \in \Upsilon a \cap \Gamma_1 s \neq \emptyset$, which verifies the condition (i) in Definition 2.4. On the other hand, if $as = 0$, then $a_i\pi_i^{n_i} = 0$. Since $\pi_i^{n_i} \neq 0$ and $D(\mathcal{C}_i)$ is a domain for $i \in I_1$, we have $a_i = 0$, and so $a = 0$, which verifies the condition (ii) in Definition 2.4. Thus, Υ is a left denominator subset of Γ_1. Similarly, we can prove that Υ is also a right denominator subset of Γ_1.

It remains to prove $\Upsilon^{-1}\Gamma_1 \simeq \mathbb{A}_{\mathcal{W}}$. In fact, it follows from Lemma 2.5 that the universal localization of Γ_1 at Υ is the same as the Ore localization $\Upsilon^{-1}\Gamma_1$ of Γ_1 at Υ. Moreover, by Lemma 2.9, we see that, for each $j \in I_1$, the Ore localization of $D(\mathcal{C}_j)$ at $\{(\pi_j)^n \mid n \in \mathbb{N}\}$ is the division ring $Q(\mathcal{C}_j)$ of fractions of $D(\mathcal{C}_j)$. Thus, by the definition of Ore localizations (see Lemma 2.5), one can easily prove $\Upsilon^{-1}\Gamma_1 \simeq \mathbb{A}_{\mathcal{W}}$.

Summing up what we have proved, we get

$$\Gamma_\Phi \simeq (\Gamma_0)_{\Phi_0} \times (\Gamma_1)_{\Phi_1} \simeq \prod_{i\in I_0} M_{c(\mathcal{C}_i)}\big(Q(\mathcal{C}_i)\big) \times \mathbb{A}_{\mathcal{W}},$$

the latter is Morita equivalent to $\mathbb{A}_{\mathcal{U}}$. As S_Σ is Morita equivalent to Γ_Φ, we see that S_Σ is Morita equivalent to $\mathbb{A}_{\mathcal{U}}$. This completes the whole proof. $\square$

Proof of Theorem 1.1. Recall that $B = \mathrm{End}_R(R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R)$ and $S := \mathrm{End}_R(R_{\mathcal{U}}/R)$. By Corollary 2.6, there is a recollement of derived module categories:

$$(*) \qquad \mathscr{D}(S_\Sigma) \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(R),$$

where S_Σ is the universal localization of S at $\Sigma := \{S \otimes_R f_U \mid U \in \mathcal{U}\}$.

Now we write $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1 \subseteq \mathscr{S}$ such that $\mathcal{U}_0$ contains no cliques and $\mathcal{U}_1$ is a union of cliques $\mathcal{C}_i$ with $i \in I$, an index set. We conclude from Lemma 3.8 that S_Σ is isomorphic to the universal localization Λ_Θ of Λ at Θ with $\Lambda := \mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0})$ and $\Theta := \{\Lambda \otimes_{R_{\mathcal{U}_0}} (R_{\mathcal{U}_0} \otimes_R f_V) \mid V \in \mathcal{U}_1\}$. Note that $R_{\mathcal{U}_0}$ is a finite-dimensional tame hereditary k-algebra, and that $\mathcal{U}_1$ is a union of cliques when regarded as a set of simple regular $R_{\mathcal{U}_0}$-modules. Now, by applying Lemma 3.9 to $R_{\mathcal{U}_0}$ and $\mathcal{U}_1$, we can deduce that Λ_Θ is Morita equivalent to the adèle ring $\mathbb{A}_{\mathcal{U}}$ in Theorem 1.1.

Thus, we have proved that S_Σ is Morita equivalent to $\mathbb{A}_{\mathcal{U}}$. If we substitute $\mathscr{D}(S_\Sigma)$ by $\mathscr{D}(\mathbb{A}_{\mathcal{U}})$ in $(*)$, then we obtain the desired recollement of derived module categories in Theorem 1.1:

$$\mathscr{D}(\mathbb{A}_{\mathcal{U}}) \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(R).$$

This completes the proof of the first part of Theorem 1.1.

As for the second part, we note that, if k is algebraically closed, then, for each clique $\mathcal{C}$ of R, the rings $D(\mathcal{C})$ and $Q(\mathcal{C})$ are isomorphic to $k[[x]]$ and $k((x))$ by Lemma 3.1(5), respectively. Now, combining this with the first part of Theorem 1.1, we know that $\mathbb{A}_{\mathcal{U}}$ is isomorphic to $\mathbb{A}_I$. This finishes the proof. $\square$

If we take $\mathcal{U} = \mathscr{S}$, then the tilting R-module $R_{\mathscr{S}} \oplus R_{\mathscr{S}}/R$ is a Reiten-Ringel tilting module (see [27]). This tilting module is actually of the form $G^{(n)} \oplus \bigoplus_{U\in\mathscr{S}} U[\infty]^{(\delta_U)}$, where G is the unique generic R-module with $n = \dim_{\mathrm{End}_R(G)} G$, and $\delta_U = \dim_{\mathrm{End}_R(U)} \mathrm{Ext}^1_R(U,R)$ for $U \in \mathscr{S}$ (see [3, Proposition 1.8]). Recall that $\mathscr{S}$ is parameterized by the projective line $\mathbb{P}^1(k)$ if k is algebraically closed. As a consequence of Theorem 1.1, we have the following corollary.

Corollary 3.10. *If k is an algebraically closed field and T is the Reiten-Ringel tilting R-module $T_{\mathscr{S}}$, then there is a recollement*

$$\mathscr{D}(\mathbb{A}_{\mathbb{P}^1(k)}) \longrightarrow \mathscr{D}(\mathrm{End}_R(T)) \longrightarrow \mathscr{D}(R)\ .$$

4 Stratifications of derived module categories

In this section, we shall use Theorem 1.1 to get stratifications of the derived categories of the endomorphism rings of tilting modules of the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$. It turns out that our consideration for general tame hereditary algebras is converted into understanding the case of special tame hereditary algebras consisting of two isomorphism classes of simple modules. In particular, if k is an algebraically closed field, we are led to the Kronecker algebra. In this way, we shall prove Corollary 1.2 in this section.

4.1 Universal localizations of general tame hereditary algebras

In this subsection, we shall discuss the endomorphism algebras of tilting modules associated with universal localizations of tame hereditary algebras at simple regular modules. The consideration here will be served as a part of preparations for stratifications of derived categories in Subsection 4.3.

Throughout this subsection, R is an indecomposable finite-dimensional tame hereditary algebra over an arbitrary field k, and $\mathscr{S} := \mathscr{S}(R)$ is the complete set of isomorphism classes of all simple regular R-modules.

Let $\mathcal{U}$ be an arbitrary subset of $\mathscr{S}$. The following result gives a characterization of the universal localization $R_{\mathcal{U}}$ of R at $\mathcal{U}$ from the view of derived equivalences.

Lemma 4.1. *Let $\mathcal{U} \subseteq \mathscr{S}$. Then there exists $\mathcal{V} \subseteq \mathscr{S}$ with $\mathcal{U} \cap \mathcal{V} = \emptyset$ such that, for $\mathcal{W} := \mathcal{U} \cup \mathcal{V}$, the following statements are true.*

(1) *There is a finite-dimensional tame hereditary k-algebra Λ with only two non-isomorphic simple modules, and a set $\mathcal{S}$ of simple regular Λ-modules such that $R_{\mathcal{W}}$ coincides with the universal localization $\Lambda_{\mathcal{S}}$ of Λ at $\mathcal{S}$.*

(2) *The $R_{\mathcal{U}}$-module $T := R_{\mathcal{W}} \oplus R_{\mathcal{W}}/R_{\mathcal{U}}$ is a classical tilting module. In particular, $R_{\mathcal{U}}$ and $\mathrm{End}_{R_{\mathcal{U}}}(T)$ are derived-equivalent.*

Proof. Suppose $\mathcal{U} = \mathcal{U}_0 \dot\cup \mathcal{U}_1 \subseteq \mathscr{S}$ such that $\mathcal{U}_0$ contains no cliques and $\mathcal{U}_1$ is a union of cliques. Observe that we may assume $\mathcal{U}_0 = \emptyset$. In fact, if $\mathcal{U}_0$ is not empty, we can replace R by $R_{\mathcal{U}_0}$ and $\mathcal{U}$ by $\mathcal{U}_1$ since $R_{\mathcal{U}_0}$ is a tame hereditary algebra and $\mathcal{U}_1$ can be seen as a set of simple regular $R_{\mathcal{U}_0}$-modules.

From now on, we suppose $\mathcal{U}_0 = \emptyset$, that is, $\mathcal{U}$ is a union of cliques. Let $\mathcal{V}$ be a maximal subset of $\mathscr{S}$ with respect to the following property: $\mathcal{V} \cap \mathcal{U} = \emptyset$ and $\mathcal{V}$ contains no cliques. In other words, from each clique $\mathcal{C}$ not contained in $\mathcal{U}$, we choose $c(\mathcal{C}) - 1$ elements, and let $\mathcal{V}$ be the union of all these elements. Clearly, the choice of $\mathcal{V}$ is not unique in general.

Let $\mathcal{W} := \mathcal{U} \dot\cup \mathcal{V}$, and let $\mathcal{U}_{>1}$ be the union of all cliques $\mathcal{C}_{i\in I}$ in $\mathcal{U}$ of rank greater than one, where I is a finite set. We choose $c(\mathcal{C}_i) - 1$ elements from each $\mathcal{C}_i$ for $i \in I$, and let $\mathcal{V}'$ be the set consisting of all of these elements. Now, we define $\mathcal{L} = \mathcal{V} \cup \mathcal{V}'$ and write $\mathcal{W} = \mathcal{L} \dot\cup \mathcal{M}$.

We claim that the statement (1) holds true. Indeed, it follows from Lemma 3.5(1) that $R_{\mathcal{L}}$ is a tame hereditary algebra such that all cliques of $R_{\mathcal{L}}$ consist of only one simple regular module. This means that

$R_{\mathcal{L}}$ has exactly two isomorphism classes of simple modules. By Lemma 3.5(3), we have $R_{\mathcal{W}} = (R_{\mathcal{L}})_{\overline{\mathcal{M}}}$ with $\overline{\mathcal{M}} := \{R_{\mathcal{L}} \otimes_R L \mid L \in \mathcal{M}\}$. Thus, setting $\Lambda := R_{\mathcal{L}}$ and $\mathcal{S} := \overline{\mathcal{M}}$, we get the statement (1).

In the following, we shall show the statement (2). Note that $\mathcal{V}$ contains no cliques. Thus, it follows from Lemma 3.5(1) that $R_{\mathcal{V}}$ is a tame hereditary algebra and $R_{\mathcal{V}}/R$ is a finitely presented R-module. By Lemma 3.7(1), $R_{\mathcal{W}}/R_{\mathcal{U}} \simeq R_{\mathcal{U}} \otimes_R (R_{\mathcal{V}}/R)$ as $R_{\mathcal{U}}$-R-bimodules. This implies that $R_{\mathcal{W}}/R_{\mathcal{U}}$ is a finitely presented $R_{\mathcal{U}}$-module, and so is the $R_{\mathcal{U}}$-module T. Hence, T is a classical $R_{\mathcal{U}}$-module. □

As a consequence of Lemma 4.1, we obtain the following result, which describes $R_{\mathcal{U}}$ up to derived equivalence by a triangular matrix ring such that the rings in the diagonal are relatively simple.

Corollary 4.2. *Suppose that $\mathcal{U} \subseteq \mathscr{S}$ is a union of cliques $\mathcal{C}_{i\in I}$ with I an index set. Let $\mathcal{V}$ be a maximal subset of $\mathscr{S}$ such that $\mathcal{V} \cap \mathcal{U} = \emptyset$ and $\mathcal{V}$ contains no cliques, and let $\mathscr{C}(\mathcal{V}) = \dot{\cup}_{j\in J}\mathcal{C}_j$ with J an index set. Define $\mathcal{W} := \mathcal{U} \cup \mathcal{V}$ and $T_{\mathcal{U}} := R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$. Then the following statements hold true:*

(1) *There is a canonical ring isomorphism:*

$$\mathrm{End}_R(T_{\mathcal{U}}) \simeq \begin{pmatrix} R_{\mathcal{U}} & \mathrm{Hom}_R(R_{\mathcal{U}}, R_{\mathcal{U}}/R) \\ 0 & \mathrm{End}_R(R_{\mathcal{U}}/R) \end{pmatrix}.$$

(2) $\mathrm{End}_R(R_{\mathcal{U}}/R)$ *is Morita equivalent to $\prod_{i\in I}\Gamma(\mathcal{C}_i)$, where $\Gamma(\mathcal{C})$ is defined in Lemma 3.2 for each clique $\mathcal{C}$ of R.*

(3) $R_{\mathcal{U}}$ *is derived-equivalent to the following triangular matrix ring*

$$\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}} \oplus R_{\mathcal{W}}/R_{\mathcal{U}}) = \begin{pmatrix} R_{\mathcal{W}} & \mathrm{Hom}_{R_{\mathcal{U}}}(R_{\mathcal{W}}, R_{\mathcal{W}}/R_{\mathcal{U}}) \\ 0 & \mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}}) \end{pmatrix}$$

such that

(a) *$R_{\mathcal{W}}$ is the universal localization $\Lambda_{\mathcal{S}}$ of a finite-dimensional tame hereditary k-algebra Λ, which has two isomorphism classes of simple modules, at a set $\mathcal{S}$ of simple regular Λ-modules, and*

(b) $\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}})$ *is Morita equivalent to $\prod_{j\in J} T_{c(\mathcal{C}_j)-1}\big(\mathrm{End}_R(V_j)\big)$, where $V_j \in \mathcal{C}_j$ is a fixed element for each $j \in J$, and $T_n(A)$ stands for the $n\times n$ upper triangular matrix ring over a ring A.*

Proof. Clearly, (1) follows from $\lambda : R \to R_{\mathcal{U}}$ being a ring epimorphism and $\mathrm{Hom}_R(R_{\mathcal{U}}/R, R_{\mathcal{U}}) = 0$. (2) follows from $(*)$ in Step (3) of the proof of Lemma 3.9. As to (3), we first show the statement (b). In fact, by the proof of Lemma 4.1, we know $R_{\mathcal{W}}/R_{\mathcal{U}} \simeq R_{\mathcal{U}} \otimes_R (R_{\mathcal{V}}/R)$ as $R_{\mathcal{U}}$-R-bimodules. Since $\mathcal{V} \subseteq \mathcal{U}^+$, we have $R_{\mathcal{U}} \otimes_R (R_{\mathcal{V}}/R) \simeq R_{\mathcal{V}}/R$ as R-modules by Lemma 3.4, and therefore $R_{\mathcal{W}}/R_{\mathcal{U}} \simeq R_{\mathcal{V}}/R$ as R-modules. This implies that $\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}}) \simeq \mathrm{End}_R(R_{\mathcal{W}}/R_{\mathcal{U}}) \simeq \mathrm{End}_R(R_{\mathcal{V}}/R)$. Now, by Lemma 3.6(2), one can prove

$$R_{\mathcal{V}}/R \simeq \bigoplus_{j\in J} \bigoplus_{i=1}^{c(\mathcal{C}_j)-1} U_{i,j}\,[c(\mathcal{C}_j)-i]^{(\delta_{i,j})},$$

where $\delta_{i,j} > 0$ and $\mathcal{V} \cap \mathcal{C}_j = \{U_{i,j} \mid 1 \le i < c(\mathcal{C}_j)\}$ such that $U_{i+1,j} = \tau^- U_{i,j}$ for all $1 \le i < c(\mathcal{C}_j)-1$. Further, for any $j \in J$, one can check

$$\mathrm{End}_R\Big(\bigoplus_{i=1}^{c(\mathcal{C}_j)-1} U_{i,j}\,[c(\mathcal{C}_j)-i]\Big) \simeq T_{c(\mathcal{C}_j)-1}\big(\mathrm{End}_R(V_j)\big),$$

where V_j is a fixed element of $\mathcal{C}_j$ with $j \in J$. Note that $\mathrm{End}_R(V_j)$ is independent of the choice of elements of $\mathcal{C}_j$ up to isomorphism. Thus $\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}})$ is Morita equivalent to $\prod_{j\in J} T_{c(\mathcal{C}_j)-1}\big(\mathrm{End}_R(V_j)\big)$, since there is no non-trivial homomorphism between two different tubes.

Note that the other conclusions in (3) are consequences of Lemma 4.1 and of properties of injective ring epimorphisms (see also [8, Lemma 6.4(2)]). This completes the proof. □

Thus, by Corollary 4.2(3), the consideration of the derived category $\mathscr{D}(R_{\mathcal{U}})$ needs first to understand universal localizations of tame hereditary algebras with two isomorphism classes of simple modules, at simple regular modules. If k is an algebraically closed field, then each tame hereditary algebra with two isomorphism classes of simple modules is Morita equivalent to the Kronecker algebra. So, in the next subsection, we shall focus our attention on the universal localizations of the Kronecker algebra.

4.2 Universal localizations of the Kronecker algebra at simple regular modules

In this subsection, we shall consider the particular tame hereditary algebra, the Kronecker algebra. The results obtained here will be served again as a preparation for the discussion of stratifications of derived module categories in the next subsection.

Throughout this subsection, k is a field, and R is the Kronecker algebra $\begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$, where the k-k-bimodule structure of k^2 is given by $a(b,c)d = (abd, acd)$ with $a,b,c,d \in k$. It is known that R can be interpreted as the path algebra of the quiver

$$Q:\ 2 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 1\ ,$$

and that R-Mod (respectively, R-mod) is equivalent to the category of (respectively, finite-dimensional) representations of Q over k.

In this subsection, we denote by V the representation $k \underset{1}{\overset{0}{\rightrightarrows}} k$. By Lemma 2.3, one can check that $R_V = M_2(k[x])$, and the universal localization $\lambda : R \to R_V$ is given by $\begin{pmatrix} a & (c,d) \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} a & c+dx \\ 0 & b \end{pmatrix}$ for $a,b,c,d \in k$. In particular, the restriction functor $\lambda_* : R_V\text{-Mod} \to R\text{-Mod}$ induced by λ is fully faithful. Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R_V$. Clearly, the tensor functor $R_V e \otimes_{k[x]} - : k[x]\text{-Mod} \to R_V\text{-Mod}$ is an equivalence. Now, we define $F : k[x]\text{-Mod} \to R\text{-Mod}$ to be the composition of the functors $R_V e \otimes_{k[x]} -$ and λ_*. Then F is a fully faithful exact functor, and sends each $k[x]$-module M to the representation $M \underset{x}{\overset{1}{\rightrightarrows}} M$. Moreover, we have the following result.

Lemma 4.3. [26, Theorem 4] *The functor F induces an equivalence between the category of finite-dimensional $k[x]$-modules and the category of finite-dimensional regular R-modules with regular composition factors not isomorphic to V.*

Let $\mathcal{P}$ be the set of all monic irreducible polynomials in $k[x]$. For each $p(x) \in \mathcal{P}$, we denote by $k_{p(x)}$ the extension field $k[x]/(p(x))$ of k, and by $V_{p(x)}$ the representation $k_{p(x)} \underset{x}{\overset{1}{\rightrightarrows}} k_{p(x)}$, which is the image of $k_{p(x)}$ under F. Since simple $k[x]$-modules are parameterized by monic irreducible polynomials, it follows from Lemma 4.3 that $\mathscr{S} := \{V\} \cup \{V_{p(x)} \mid p(x) \in \mathcal{P}\}$ is a complete set of isomorphism classes of simple regular R-modules. If k is algebraically closed, then $\mathcal{P} = \{x-a \mid a \in k\}$, and therefore $\mathscr{S}$ can be identified with the projective line $\mathbb{P}^1(k)$.

The following corollary gives a characterization of the endomorphisms rings of Prüfer modules.

Corollary 4.4. *Let t be a variable and $p(x) \in \mathcal{P}$. Then there are isomorphisms of rings:*

$$\mathrm{End}_R\big(V[\infty]\big) \simeq k[[t]] \quad \textit{and} \quad \mathrm{End}_R\big(V_{p(x)}[\infty]\big) \simeq k_{p(x)}[[t]].$$

Proof. Recall that, for any simple regular R-module U, we have $\mathrm{End}_R\big(U[\infty]\big) \simeq \varprojlim_n \mathrm{End}_R(U[n])$ as rings. If $U = V$, then $\mathrm{End}_R(U[n]) \simeq k[t]/(t^n)$ for any $n > 0$, and therefore $\mathrm{End}_R\big(U[\infty]\big) \simeq \varprojlim_n k[t]/(t^n) \simeq k[[t]]$. Suppose $U = V_{p(x)}$. It follows from Lemma 4.3 that $U[n] \simeq F\big(k[x]/(p(x)^n)\big)$ as R-modules, and that $\mathrm{End}_R(U[n]) \simeq \mathrm{End}_{k[x]}\big(k[x]/(p(x)^n)\big) \simeq k[x]/(p(x)^n)$ for any $n > 0$. Thus $\mathrm{End}_R\big(U[\infty]\big) \simeq \varprojlim_n k[x]/(p(x)^n)$. This implies that $\mathrm{End}_R\big(U[\infty]\big)$ is a complete commutative discrete valuation ring (see Lemma 3.1(5)), and therefore it is a regular ring of Krull dimension 1. Recall that a regular ring is by definition a commutative noetherian ring of finite global dimension. For regular rings, the global dimension agrees with the Krull dimension.

It remains to prove $\varprojlim_n k[x]/(p(x)^n) \simeq k_{p(x)}[[t]]$. Actually, this follows straightforward from the following classical result (see [11, Theorem 15] for details):

Let S be a complete regular local ring of Krull dimension m with the residue class field K. If S contains a field, then S is isomorphic to the formal power series ring $K[[t_1, \cdots, t_m]]$ over K in variables $t_1, \cdots, t_m$.

Hence $\mathrm{End}_R\big(U[\infty]\big) \simeq \varprojlim_n k[x]/(p(x)^n) \simeq k_{p(x)}[[t]]$, which finishes the proof. $\square$

In the remainder of this subsection, let Δ be a subset of $\mathcal{P}$, and let $\mathcal{U} := \{V\} \cup \{V_{p(x)} \mid p(x) \in \Delta\}$. We define the Δ-adèle ring of $k[x]$ as follows:

$$\mathbb{A}(\Delta) := k((t)) \times \Big\{ \big(\theta_{p(x)}\big)_{p(x)\in\Delta} \in \prod_{p(x)\in\Delta} k_{p(x)}((t)) \;\Big|\; \theta_{p(x)} \in k_{p(x)}[[t]] \text{ for almost all } p(x) \in \Delta \Big\}.$$

Combining Theorem 1.1 with Corollary 4.4, we get the following result.

Corollary 4.5. *Let B be the endomorphism ring of the tilting R-module $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$. Then there is a recollement of derived categories:*

$$\xymatrix{ \mathscr{D}(\mathbb{A}(\Delta)) \ar[r] & \mathscr{D}(B) \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathscr{D}(R) \ar@/_1pc/[l] \ar@/^1pc/[l] }\,.$$

In Corollary 4.5, if $\Delta = \mathcal{P}$, then the $\mathcal{P}$-adèle ring $\mathbb{A}(\mathcal{P})$ of $k[x]$ coincides with the adèle ring $\mathbb{A}_{k(x)}$ of the fraction field $k(x)$, which appears in global class field theory (see [24, Chapter VI] and [16, Theorem 2.1.4]).

Finally, we prove the following lemma as the last preparation for the proof of Corollary 1.2.

Lemma 4.6. *Let D be the smallest subring of the fraction field $k(x)$ of $k[x]$ containing both $k[x]$ and $\frac{1}{p(x)}$ with all $p(x) \in \Delta$. Then $R_{\mathcal{U}} \simeq M_2(D)$, the 2×2 matrix ring over D. In particular, $R_{\mathcal{U}}$ is Morita equivalent to the Dedekind integral domain D.*

Proof. Define $\mathcal{W} := \{R_V \otimes_R V_{p(x)} \mid p(x) \in \Delta\}$. Then $R_{\mathcal{U}} = (R_V)_{\mathcal{W}}$ by Lemma 3.5(3). Recall that $R_V = M_2(k[x])$ and $\lambda : R \to R_V$ is the universal localization of R at V. On the one hand, for each $p(x) \in \Delta$, it follows from $V_{p(x)} = F\big(k_{p(x)}\big) = \lambda_*\big(R_V\, e \otimes_{k[x]} k_{p(x)}\big)$ that

$$R_V \otimes_R V_{p(x)} \simeq V_{p(x)} = R_V\, e \otimes_{k[x]} k_{p(x)} = \begin{pmatrix} k_{p(x)} \\ k_{p(x)} \end{pmatrix}$$

as R_V-modules. On the other hand, by [8, Corollary 3.4], Morita equivalences preserve universal localizations. Consequently, we have $R_{\mathcal{U}} = \big(M_2(k[x])\big)_{\mathcal{W}} \simeq M_2\big(k[x]_\Theta\big)$ with $\Theta := \{k_{p(x)} \mid p(x) \in \Delta\} \subseteq k[x]$-Mod. Now, one may readily see that $k[x]_\Theta$ coincides with the localization of $k[x]$ at the smallest multiplicative subset of $k[x]$ containing $\{p(x) \mid p(x) \in \Delta\}$, which is exactly the ring D defined in Lemma 4.6. Since $k[x]$ is a

Dedekind integral domain and since localizations of Dedekind integral domains are again Dedekind integral domains, we see that D is a Dedekind integral domain. As a result, we have $R_{\mathcal{U}} \simeq M_2(D)$. This completes the proof. $\square$

Remarks. (1) If k is an algebraically closed field, then, for any simple regular R-module U, we can choose an automorphism $\sigma : R \to R$, such that the induced functor $\sigma_* : R\text{-Mod} \to R\text{-Mod}$ by σ is an equivalence with $\sigma_*(U) \simeq V$. This implies that, up to isomorphism, Lemma 4.6 provides a complete description of $R_{\mathcal{V}}$ for any subset $\mathcal{V}$ of $\mathscr{S}$. In particular, $R_{\mathcal{V}}$ is Morita equivalent to a Dedekind integral domain.

(2) If we localize R at all non-isomorphic simple regular modules $\mathscr{S}$ which is indexed by all monic irreducible polynomials, then, by Lemma 4.6, we have $R_{\mathscr{S}} \simeq M_2(k(x))$ since the smallest subring containing the inverses of all irreducible polynomials $p(x)$ is just $k(x)$.

4.3 Stratifications of derived module categories

The main purpose of this subsection is to prove Corollary 1.2. We first recall the definition of stratifications of derived categories of rings.

As in [1], the derived module category $\mathscr{D}(A)$ of a ring A is called derived simple if it is not a non-trivial recollement of any derived categories of rings. A stratification of $\mathscr{D}(A)$ of a ring A by derived categories of rings is defined to be a sequence of iterated recollements of the following form: a recollement of A, if it is not derived simple,

$$\xymatrix{\mathscr{D}(A_1) \ar[r] & \mathscr{D}(A) \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathscr{D}(A_2) \ar@/_1pc/[l] \ar@/^1pc/[l]}\ ,$$

a recollement of the ring A_1, if it is not derived simple,

$$\xymatrix{\mathscr{D}(A_{11}) \ar[r] & \mathscr{D}(A_1) \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathscr{D}(A_{12}) \ar@/_1pc/[l] \ar@/^1pc/[l]}\ ,$$

and a recollement of the ring A_2, if it is not derived simple,

$$\xymatrix{\mathscr{D}(A_{21}) \ar[r] & \mathscr{D}(A_2) \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathscr{D}(A_{22}) \ar@/_1pc/[l] \ar@/^1pc/[l]}$$

and recollements of the rings A_{ij} with $1 \leq i, j \leq 2$, if they are not derived simple, and so on, until one arrives at derived simple rings at all positions, or continue to infinitum. All the derived simple rings appearing in this procedure are called composition factors of the stratification. The cardinality of the set of all composition factors (counting the multiplicity) is called the length of the stratification. If the length of a stratification is finite, we say that this stratification is finite or of finite length.

Proof of Corollary 1.2. Under the assumption that k is an algebraically closed field, the following two facts are known: (a) For any simple regular R-module U, the algebras $\text{End}_R(U)$ and $\text{End}_R(U[\infty])$ are isomorphic to k and $k[[x]]$ (see Lemma 3.1(5)), respectively, and (b) each tame hereditary algebra having two isomorphism classes of simple modules is Morita equivalent to the Kronecker algebra.

One the one hand, it follows from Theorem 1.1 that $\mathscr{D}(B)$ is stratified by $\mathscr{D}(R)$ and $\mathscr{D}(\mathbb{A}_I)$, where $I = \{1, 2, \cdots, s\}$ is an index set of the cliques contained in $\mathcal{U}$, and the ring $\mathbb{A}_I$ is defined in Introduction. Since $\mathcal{U}$ is a union of finitely many cliques of $\mathscr{S}$, we know that $\mathbb{A}_I$ is equal to $k((x))^s$, the direct product of s copies of $k((x))$. Thus $\mathscr{D}(\mathbb{A}_I)$ has a stratification by derived module categories with s composition factors $k((x))$. Note that $\mathscr{D}(R)$ has a stratification by derived module categories with r copies of the composition factor k, where r is the number of non-isomorphic simple R-modules. Thus $\mathscr{D}(B)$ has a stratification of length $r+s$ with the composition factor k of multiplicity r, and the composition factor $k((x))$ of multiplicity s.

On the other hand, by Corollary 4.2, we know that $\mathscr{D}(B)$ can be stratified by $\mathscr{D}(R_{\mathcal{W}})$, $\mathscr{D}(\text{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}}))$ and $\mathscr{D}(\text{End}_R(R_{\mathcal{U}}/R))$, where $\mathcal{W}$ is defined in Corollary 4.2. Here, we have used the known fact that every

2×2 triangular matrix ring yields a recollement of derived module categories of the rings in the diagonal. In the following, we shall calculate composition factors of $\mathscr{D}(B)$.

First, it follows from Corollary 4.2(3) and Lemma 4.6 that $R_{\mathcal{W}}$ is Morita equivalent to a Dedekind integral domain and that $\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}})$ is Morita equivalent to $\prod_{j\in J} T_{c(\mathcal{C}_j)-1}(k)$. It is known from [1] that every Dedekind domain is derived simple. Thus $R_{\mathcal{W}}$ contributes one composition factor to $\mathscr{D}(B)$. It is easy to see that $\mathscr{D}(T_{c(\mathcal{C}_j)-1}(k))$ has a stratification with $c(\mathcal{C}_j)-1$ copies of the composition factor k. Thus $\mathscr{D}(\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}}))$ admits a stratification with $\sum_{j\in J}\big(c(\mathcal{C}_j)-1\big)$ copies of the composition factor k.

Second, combining Corollary 4.2(2) with Corollary 3.3, we see that $\mathrm{End}_R(R_{\mathcal{U}}/R)$ is Morita equivalent to $\prod_{i=1}^{s}\Gamma\big(c(\mathcal{C}_i)\big)$, where $\mathcal{U}$ is assumed to be a union of s cliques $\mathcal{C}_i$ with $1\leq i\leq s$, and where $\Gamma(m)$ is defined in Corollary 3.3 for each positive integer m. Note that the canonical inclusion f of $\Gamma(m)$ into $M_m(k[[x]])$ is a ring epimorphism and that $M_m(k[[x]])$ is projective as a left $\Gamma(m)$-module. Thus the sequence

$$0\to\Gamma(m)\xrightarrow{f} M_m(k[[x]])\to \mathrm{coker}(f)\to 0$$

is an $\mathrm{add}\big(\Gamma(m)E_{m,m}\big)$-split sequence in the category of all left $\Gamma(m)$-modules, and therefore $\mathrm{End}_{\Gamma(m)}\big(\Gamma(m)\oplus M_m(k[[x]])\big)$ and $\mathrm{End}_{\Gamma(m)}\big(M_m(k[[x]])\oplus\mathrm{coker}(f)\big)$ are derived-equivalent by [20, Theorem 1.1]. Clearly, the former ring is Morita equivalent to $\Gamma(m)$ and the latter is Morita equivalent to $\mathrm{End}_{\Gamma(m)}\big(M_m(k[[x]])E_{m,m}\oplus\mathrm{coker}(f)\big)$. Hence $\Gamma(m)$ is derived-equivalent to $\mathrm{End}_{\Gamma(m)}\big(M_m(k[[x]])\oplus\mathrm{coker}(f)\big)$ which is just the following matrix ring:

$$\begin{pmatrix} k[[x]] & 0 & \cdots & 0\\ k & k & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ k & \cdots & k & k\end{pmatrix}_{m\times m}.$$

For a general consideration of derived equivalences between subrings of matrix rings, we refer to [9]. Thus, we see that $\mathscr{D}(\Gamma(m))$ has a stratification with the composition factor $k[[x]]$ of multiplicity one, and the composition factor k of multiplicity $m-1$. Therefore, $\mathscr{D}(\mathrm{End}_R(R_{\mathcal{U}}/R))$ admits a stratification with the composition factors: s copies of $k[[x]]$ and $\sum_{i=1}^{s}\big(c(\mathcal{C}_i)-1\big)$ copies of k.

Finally, by summarizing up the above discussions, we conclude that $\mathscr{D}(B)$ has a stratification of length $r+s-1$ with the following composition factors: $r-2$ copies of k, s copies of $k[[x]]$ and one copy of a fixed Dedekind domain. Here, we use the well known fact: $\sum_{\mathcal{C}}\big(c(\mathcal{C})-1\big)=r-2$, where $\mathcal{C}$ runs over all of the cliques of R. Thus the proof is completed. $\square$

Let us end this section by mentioning the following questions suggested by our results.

(1) For tilting modules of the form $R_{\mathcal{U}}\oplus R_{\mathcal{U}}/R$, we have provided a recollement of the derived categories of their endomorphism rings. It would be interesting to have a similar result for tilting modules of other types described in [3].

(2) In Corollary 1.2, it would be nice to know that $\mathscr{D}(B)$ has no other composition factors (up to derived equivalence) except the ones displayed there.

(3) It would be interesting to generalize the results in this paper to hereditary orders.

(4) Suppose the derived category $\mathscr{D}(A)$ of a ring A admits a stratification of finite length by derived categories of rings. Does $\mathscr{D}(A)$ then have only finitely many derived composition factors? (up to derived equivalence).

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