On n-strongly Gorenstein rings

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Abstract

This paper introduces and studies a particular subclass of the class of commutative rings with finite Gorenstein global dimension.

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1 Introduction

Throughout the paper, all rings are commutative with identity, and all modules are unitary.
Let R be a ring, and let M be an R-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of M.

For a two-sided Noetherian ring R, Auslander and Bridger [1] introduced the G-dimension, $\mathrm{Gdim}_R(M)$, for every finitely generated R-module M. They showed that $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for all finitely generated R-modules M, and equality holds if $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [8, 9] introduced the notion of Gorenstein projective dimension (G-projective dimension for short), as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda, and Torrecillas [11] introduced the Gorenstein flat dimension. Some references are [3, 6, 7, 8, 9, 11, 12].

Recall that an R-module M is called Gorenstein projective, if there exists an exact sequence of projective R-modules:

$$\mathbf{P} : ... \to P_1 \to P_0 \to P^0 \to P^1 \to ...$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that the functor $\mathrm{Hom}_R(-, Q)$ leaves $\mathbf{P}$ exact whenever Q is a projective R-module. The complex $\mathbf{P}$ is called a complete projective resolution. The Gorenstein injective R-modules are defined dually.
The Gorenstein projective and injective dimensions are defined in terms of resolutions and denoted by $\mathrm{Gpd}(-)$ and $\mathrm{Gid}(-)$, respectively ([6, 10, 12]).

In [3], the authors proved, for any associative ring R, the equality

$$\sup\{\mathrm{Gpd}_R(M) \mid \text{M is a (left) R-module}\} = \sup\{\mathrm{Gid}_R(M) \mid \text{M is a (left) R-module}\}.$$

They called the common value of the above quantities the left Gorenstein global dimension of R and denoted it by $l.\mathrm{Ggldim}(R)$. Since in this paper all rings are commutative, we drop the letter l.

Recently, in [15], particular modules of finite Gorenstein projective, injective, and flat dimensions are defined as follows:

Definitions 1.1. *Let n be a positive integer.*

1. *An R-module M is said to be strongly n-Gorenstein projective, if there exists a short exact sequence of R-modules $0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$ where $\mathrm{pd}_R(P) \leq n$ and $\mathrm{Ext}_R^{n+1}(M, Q) = 0$ whenever Q is projective.*

2. *An R-module M is said to be strongly n-Gorenstein injective, if there exists a short exact sequence of R-modules $0 \longrightarrow M \longrightarrow I \longrightarrow M \longrightarrow 0$ where $\mathrm{id}_R(I) \leq n$ and $\mathrm{Ext}_R^{n+1}(E, M) = 0$ whenever E is injective.*

Clearly, strongly 0-Gorenstein projective and injective are just the strongly Gorenstein projective, injective, and flat modules, respectively ([3, Propositions 2.9 and 3.6]).

In this paper, we investigate these modules to characterize a new class of rings with finite Gorenstein global dimension, which we call n-strongly Gorenstein rings.

2 n-Strongly Gorenstein rings

In [15], the authors proved the following proposition:

Proposition 2.1 (Proposition 2.16, [15]). *Let R be a ring. The following statements are equivalent:*

1. *Every module is strongly n-Gorenstein projective.*
2. *Every module is strongly n-Gorenstein injective.*

Thus, we give the following definition:

Definition 2.2. *Let n be a positive integer. A ring R is called n-strongly Gorenstein (n-SG ring for short), if R satisfies one of the equivalent conditions of Proposition 2.1.*

The 0-SG rings and 1-SG rings are already studied in [5, 14] and they are called strongly Gorenstein semi-simple rings and strongly Gorenstein hereditary rings, respectively. Clearly, by definition, every n-SG ring is m-SG whenever $n \leq m$.

Our first result gives a characterization of strongly n-Gorenstein rings.

Proposition 2.3. *For a ring R and a positive integer n, the following statements are equivalent:*

1. *R is an n-SG ring.*
2. $\mathrm{Ggldim}(R) \leq n$ *and for every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$$
where $\mathrm{pd}_R(P) < \infty$.
3. $\mathrm{Ggldim}(R) < \infty$ *and for every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$$
where $\mathrm{pd}_R(P) \leq n$.

Proof. ($1 \Rightarrow 2$) Clear since for every n-SG ring R we have $\mathrm{Ggldim}(R) \leq n$ (by [15, Proposition 2.2(1)]).

($2 \Rightarrow 3$) Follows directly from [3, Corollary 2.7].

($3 \Rightarrow 1$) Follows from [15, Proposition 2.10].

The next result studies the direct product of n-SG rings.

Theorem 2.4. *Let $\{R_i\}_{i=1}^{m}$ be a family of rings and set $R := \prod_{i=1}^{m} R_i$. Then, R is an n-SG ring if and only if R_i is an n-SG ring for each $i = 1, .., m$.*

Proof. By induction on m it suffices to prove the assertion for $m = 2$. First suppose that $R_1 \times R_2$ is an n-SG ring. We claim that R_1 is an n-SG ring. Let M be an arbitrary R_1 module. $M \times 0$ can be viewed as an $R_1 \times R_2$-module. For such module and since $R_1 \times R_2$ is an n-SG ring, there is an exact sequence $0 \longrightarrow M \times 0 \longrightarrow P \longrightarrow M \times 0 \longrightarrow 0$ where $\mathrm{pd}_{R_1 \times R_2}(P) \leq n$. Thus, since R_1 is a projective $R_1 \times R_2$ module, by applying $- \otimes_{R_1 \times R_2} R_1$ to the sequence above, we find the short exact sequence of R-modules: $0 \longrightarrow M \times 0 \otimes_{R_1 \times R_2} R_1 \longrightarrow P \otimes_{R_1 \times R_2} R_1 \longrightarrow M \times 0 \otimes_{R_1 \times R_2} R_1 \longrightarrow 0$. Clearly $\mathrm{pd}_{R_1}(P \otimes_{R_1 \times R_2} R_1) \leq \mathrm{pd}_{R_1 \times R_2}(P) \leq n$. Moreover, we have the isomorphism of R-modules:

$$M \times 0 \otimes_{R_1 \times R_2} R_1 \cong M \times 0 \otimes_{R_1 \times R_2} (R_1 \times R_2)/(0 \times R_2) \cong M.$$

Thus, we obtain an exact sequence of R-module with the form: $0 \longrightarrow M \longrightarrow P \otimes_{R_1 \times R_2} R_1 \longrightarrow M \longrightarrow 0$. On the other hand, by [4, Theorem 3.1], we have $\mathrm{Ggldim}(R_1) \leq \mathrm{Ggldim}(R_1 \times R_2) \leq n$. Thus, using Proposition 2.3, R_1 is an n-SG ring, as desired. By the same argument, R_2 is also an n-SG ring.
Now, suppose that R_1 and R_2 are an n-SG rings and we claim that $R_1 \times R_2$ is an n-SG ring. Let M be an $R_1 \times R_2$-module. We have

$$M \cong M \otimes_{R_1 \times R_2} (R_1 \times R_2) \cong M \otimes_{R_1 \times R_2} ((R_1 \times 0) \oplus (R_2 \times 0)) \cong M_1 \times M_2$$

where $M_i = M \otimes_{R_1 \times R_2} R_i$ for $i = 1, 2$. For each $i = 1, 2$, there is an exact sequence $0 \longrightarrow M_i \longrightarrow P_i \longrightarrow M_i \longrightarrow 0$ where $\mathrm{pd}_{R_i}(P_i) \leq n$ since R_i is an n-SG ring. Thus, we have the exact sequence of $R_1 \times R_2$-modules:

$$0 \longrightarrow M_1 \times M_2 \longrightarrow P_1 \times P_2 \longrightarrow M_1 \times M_2 \longrightarrow 0.$$

On the other hand, $\mathrm{pd}_{R_1 \times R_2}(P_1 \times P_2) = \sup\{\mathrm{pd}_{R_i}(P_i)\}_{1,2} \leq n$ (by [13, Lemma 2.5 (2)]). Moreover, by [4, Theorem 3.1], $\mathrm{Ggldim}(R_1 \times R_2) = \sup\{\mathrm{Ggldim}(R_i)\}_{1,2} \leq n$. Thus, from Proposition 2.3, $R_1 \times R_2$ is an n-SG ring, as desired.

Let $T := R[X_1, X_2, ..., X_n]$ be the polynomial ring in n indeterminates over R. If we suppose that T is an m-SG ring, it is easy, by [4, Theorem 2.1], to see that $n \leq m$.

Theorem 2.5. *If $R[X_1, X_2, ..., X_n]$ is an m-SG ring then R is an $(m - n)$-SG ring.*

Proof. By induction on n is suffices to prove the result for $n = 1$. So, suppose that $R[X]$ is an m-SG ring. Let M be an arbitrary R-module. For the $R[X]$-module $M[X] := M \otimes_R R[X]$ there is an exact sequence of $R[X]$-modules $0 \longrightarrow M[X] \longrightarrow P \longrightarrow M[X] \longrightarrow 0$ where $\mathrm{pd}_{R[X]}(P) \leq m$. Applying $- \otimes_{R[X]} R$ to the short exact sequence above and seeing that $M \cong_R$

$M[X] \otimes_{R[X]} R$, we obtain a short exact sequence of R-modules with the form $0 \longrightarrow M \longrightarrow P \otimes_{R[X]} R \longrightarrow M \longrightarrow 0$ (see that R is a projective $R[X]$-module). Moreover, $\mathrm{pd}_R(P \otimes_{R[X]} R) \leq \mathrm{pd}_{R[X]}(P) < \infty$. On the other hand, by [4, Theorem 2.1], $\mathrm{Ggldim}(R) = \mathrm{Ggldim}(R[X]) - 1 \leq m - 1$. Hence, by Proposition 2.3, R is an $(m-1)$-SG ring.

Trivial examples of n-SG-ring are the rings with global dimension $\leq n$. The following example gives a new family of n-SG rings with infinite weak global dimension.

Example 2.6. *Consider the non semi-simple quasi-Frobenius rings* $R_1 := K[X]/(X^2)$ *and* $R_2 := K[X]/(X^3)$ *where* K *is a field, and let* S *be a non Noetherian ring such that gldim*$(S) = n$. *Then,*

1. $\mathrm{Ggldim}(R_1) = \mathrm{Ggldim}(R_2) = 0$ *and* R_1 *is* 0*-SG ring but* R_2 *is not.*
2. $R_1 \times S$ *is a non Noetherian* n*-SG ring with infinite weak global dimension.*
3. $\mathrm{Ggldim}(R_2 \times S) = n$ *but* $R_2 \times S$ *is not an* n*-SG ring.*

Proof. From [5, Corollary 3.9] and [3, Proposition 2.6], $\mathrm{Ggldim}(R_1) = \mathrm{Ggldim}(R_2) = 0$ and R_1 is 0-SG ring but R_2 is not. So, (1) is clear. Moreover R_1 and R_2 have infinite weak global dimensions. By [4, Theorems 2.1] and Theorem 2.4, it is easy to see that, $\mathrm{Ggldim}(R_2 \times S) = n$ and that $R_2 \times S$ is not an n-SG ring.

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