ON $p-$RING

MOHAMMED KABBOUR

ABSTRACT. In this paper, we introduced the concept of a p-ideal for a given ring. We provide necessary and sufficient condition for $\frac{R[x]}{(f(x))}$ to be a p-ring, where R is a finite p-ring. It is also shown that the amalgamation of rings, $A \bowtie^f J$ is a p-ring if and only if so is A and J is a p-ideal. Finally, we establish the transfer of this notion to trivial ring extensions.

1. INTRODUCTION

All rings considered below are commutative with identity element $\neq 0$; and all modules are unital. Following N.H. McCoy and D. Montgomery [8], a ring R is said to be a p-ring (p is a prime integer) if $x^p = x$ and $px = 0$, for each $x \in R$. Thus a Boolean ring, as a ring in which every element is idempotent, is simply 2-ring ($p = 2$). Recall that a ring is said to be reduced if its nilradical is zero.

The following conditions on a ring R are equivalent:
(1) For each a in R, there is some $b \in R$ such that $a = a^2 b$.
(2) R is a reduced ring and every prime ideal is maximal (i.e R is a reduced 0-dimensional ring).
(3) For any maximal ideal $\mathfrak{m}$ of R, the localization $R_{\mathfrak{m}}$ at $\mathfrak{m}$ is a field.

A ring satisfying the conditions as above is called a von Neumann regular ring. See for instance [4, 5].

Let A be a ring, E an A-module and let $R = A \propto E$ be the set of pair (a, e) with pairwise addition and multiplication is giving by $(a, e)(b, f) = (ab, af + be)$, R is called the trivial ring extension of A by E (also called the idealization of E over A). Considerable work, part of it summarized in Glaz's book [4] and Huckaba's book [5], has been concerned with trivial ring extensions.

Let A and B be a pair of rings, J an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. The following sub-ring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \ ; \ a \in A, j \in J\}$$

is said to be amalgamation of A with B along J with respect to f. Motivations and some applications of this construction, introduced by M. D'Anna, C.A. Finocchiaro and M. Fontana, are well discussed with more detail in the recent paper [2].

2000 *Mathematics Subject Classification.* 13D05, 13D02.
Key words and phrases. von Neumann regular ring, p-ring, trivial rings extension and amalgamation of rings.

The main purpose of this paper is to give new and original families of examples of p-rings. Also we investigate the transfer of this notion to trivial ring extensions and amalgamation of rings.

2. Main results

We state formally the definition of a p-ideal for a given ring.

Definition 2.1. *Let R be a ring and let p be a prime integer. An ideal I of A is called a p-ideal if for each $x \in I$:*

$$x^p = x \quad \text{and} \quad px = 0.$$

From this definition, we can deduce that a ring R is a p-ring if and only if every principal ideal of R is a p-ideal.
In the next theorem, we give a necessary and sufficient condition for $\mathbb{Z}/n\mathbb{Z}$ to have a nonzero p-ideal.

Theorem 2.2. *Let n be a nonnegative integer and let p be a prime integer.*

- *If $v_p(n) = 1$ ($v_p(n)$ is the p-valuation of n) then $\mathbb{Z}/n\mathbb{Z}$ has a unique nonzero p-ideal.*
- *Otherwise (0) is the unique p-ideal of $\mathbb{Z}/n\mathbb{Z}$.*

Proof. We say that every ideal of $\mathbb{Z}/n\mathbb{Z}$ has the form $k\mathbb{Z}/n\mathbb{Z}$, where $k \in \{0, ..., n-1\}$. Assume that $v_p(n) = 0$ and let k be an integer such that $1 \leq k \leq n-1$. Suppose that $k\mathbb{Z}/n\mathbb{Z}$ is p-ideal, then $pkx \in n\mathbb{Z}$ for each element x in $\mathbb{Z}$. Thus n divides pk and so n divides k, witch is absurd. We deduce that (0) is the unique p-ideal of $\mathbb{Z}/n\mathbb{Z}$.

We shall need to use the following property:
Let $R_1, ..., R_n$ be rings then every ideal of $R_1 \times \cdots \times R_n$ has the form $I_1 \times \cdots \times I_n$, where I_k is an ideal of R_k for each $k \in \{1, ..., n\}$.
On the other hand, it is easy to see that $I_1 \times \cdots \times I_n$ is a p-ideal if and only if so is I_k for all $k \in \{1, ..., n\}$. Now suppose that $v_p(n) = 1$ and let q be the integer such that $pq = n$. We denote $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, the Galois field of order p. From the assumption (since p is relatively prime to q), we can write $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{F}_p \times \mathbb{Z}/q\mathbb{Z}$. From the previous part of the proof $\mathbb{F}_p \times \mathbb{Z}/q\mathbb{Z}$ has a unique nonzero p-ideal, which is $\mathbb{F}_p \times (0)$.

Assume that $\alpha = v_p(n) \geq 2$. There is some positive integer q, relatively prime to p, such that $n = p^\alpha q$. Hence $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/p^\alpha\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$. Let I be a p-ideal of $\mathbb{Z}/p^\alpha\mathbb{Z}$. Since $px = 0$ for each $x \in I$, there exists some integer $k \in \{0, ..., p-1\}$ such that $I = kp^{\alpha-1}\mathbb{Z}/p^\alpha\mathbb{Z}$. From the assumption p divides $kx\left((kxp^{\alpha-1})^{p-1} - 1\right)$ for every integer x. But p is relatively prime to $(kxp^{\alpha-1})^{p-1} - 1$, thus p must divides k, therefore $k = 0$. We conclude that $I = (0)$, and so (0) is the unique p-ideal of $\mathbb{Z}/n\mathbb{Z}$. $\square$

For example let R be the ring $\mathbb{Z}/60\mathbb{Z}$. Then we have, as follow, the list of all p-ideals of R, where p ranges over the set of prime integers:

- $20\mathbb{Z}/60\mathbb{Z}$ is the unique nonzero 3-ideal of R.
- $12\mathbb{Z}/60\mathbb{Z}$ is the unique nonzero 5-ideal of R.
- R has not a nonzero 2-ideal (since $v_2(60)=2$).
- (0) is a p-ideal for each prime integer p.

Theorem 2.3. *Let p be a prime integer and let $f(x)\in\mathbb{F}_p[x]$ be a nonconstant polynomial over $\mathbb{F}_p$. Then $\dfrac{\mathbb{F}_p[x]}{(f(x))}$ contains a nonzero p-ideal if and only if $f(x)$ has at last one simple zero in $\mathbb{F}_p$.*

We need the following lemmas before proving Theorem2.3

Lemma 2.4. *Let R be a p-ring, $R[x]$ the polynomial ring over R in the indeterminate x and let $f(x)$ be an element of $R[x]$. Then $\dfrac{R[x]}{(f(x))}$ is a p-ring if and only if $f(x)$ divides x^p-x.*

Proof. We suppose that $\dfrac{R[x]}{(f(x))}$ is a p-ring. Then $(x+(f(x)))^p=x+(f(x))$, therefore $x^p-x\in(f(x))$. Conversely, assume that $f(x)$ divides x^p-x and let $0\neq g(x)\in R[x]$. By induction on $n=\deg g$, the degree of the polynomial $g(x)$, we claim that $(g(x))^p=g\left(x^p\right)$. Indeed, it is certainly true for $n=0$. Assume that the statement is true for each $k\leq n$ and that $\deg g=n+1$. We put $g(x)=a_{n+1}x^{n+1}+g_1(x)$, where $0\neq a_{n+1}\in R$ and $g_1(x)\in R[x]$ such that $\deg g_1\leq n$. By the binomial theorem,

$$(g(x))^p=\left(a_{n+1}x^{n+1}\right)^p+(g_1(x))^p=a_{n+1}^p x^{p(n+1)}+g_1\left(x^p\right).$$

Thus $(g(x))^p=g\left(x^p\right)$, as desired.
On the other hand, x^p-x divides $x^{kp}-x^k$ for each positive integer k. Hence x^p-x divides $(g(x))^p-g(x)$, and so $(g(x)+(f(x)))^p=g(x)+(f(x))$. Finally, it is easy to see that $p(g(x)+(f(x)))=0$, so we have the desired result. □

Lemma 2.5. *Let $f(x)$ be an irreducible polynomial over $\mathbb{F}_p$ and let k be a nonnegative integer. Then the following statements are equivalent:*

(1) *The ring $\dfrac{\mathbb{F}_p[x]}{(f^k(x))}$ contains a nonzero p-ideal.*
(2) *$\dfrac{\mathbb{F}_p[x]}{(f^k(x))}$ is a p-ring.*
(3) *$\dfrac{\mathbb{F}_p[x]}{(f^k(x))}$ is isomorphic (as a ring) to $\mathbb{F}_p$.*

In this case $k=1$ and $\deg f=1$.

Proof. (1) $\Longrightarrow$ (2): Let I be a nonzero p-ideal of $\dfrac{\mathbb{F}_p[x]}{(f^k(x))}$. There is some $j \in \{0, ..., k-1\}$ such that $I = \dfrac{f^j(x)\mathbb{F}_p[x]}{(f^k(x))}$. We get that $f^k(x)$ divides

$$f^j(x)g(x)\left((f^j(x)g(x))^{p-1} - 1\right),$$

for all $g(x)$ in $\mathbb{F}_p[x]$. Hence $f^{k-j}(x)$ divides $f^{j(p-1)}(x) - 1$ (since $\mathbb{F}_p[x]$ is an integral domain). It follows that $j = 0$, since $f(x)$ is relatively prime with $f^m(x) - 1$ for every nonnegative integer m. We conclude that $I = \dfrac{\mathbb{F}_p[x]}{(f^k(x))}$, as desired.

(2) $\Longrightarrow$ (3): By using the above lemma, we get that $f^k(x)$ divides $x^p - x$. We denote $\mathbb{F}_p = \{a_0, ..., a_{p-1}\}$. For each $i \in \{0, ..., p-1\}$, a_i is a root of the polynomial $x^p - x$. Therefore $x^p - x = (x - a_0) \cdots (x - a_{p-1})$. We conclude that $k = 1$ and $f(x) = x - a_i$ for some i in $\{0, ..., p-1\}$, and so

$$\frac{\mathbb{F}_p[x]}{(f^k(x))} = \frac{\mathbb{F}_p[x]}{(x - a_i)} \simeq \mathbb{F}_p$$

(3) $\Longrightarrow$ (1): Clear $\square$

Proof of Theorem 2.3. Suppose that $f(x) = (x - a)g(x)$, where a is an element of $\mathbb{F}_p$ and $g(x) \in \mathbb{F}_p[x]$ such that $g(a) \neq 0$. Then

$$\frac{\mathbb{F}_p[x]}{(f(x))} \simeq \frac{\mathbb{F}_p[x]}{(x - a)} \times \frac{\mathbb{F}_p[x]}{(g(x))} \simeq \mathbb{F}_p \times \frac{\mathbb{F}_p[x]}{(g(x))}.$$

But $\mathbb{F}_p \times \dfrac{\mathbb{F}_p[x]}{(g(x))}$ has a nonzero p-ideal which is $\mathbb{F}_p \times (0)$. The sufficient condition is now straightforward.

Conversely, suppose that $\dfrac{\mathbb{F}_p[x]}{(f(x))}$ contains a nonzero p-ideal. We may assume that $f(x)$ is monic polynomial. Let $f(x) = f_1^{k_1}(x)...f_n^{k_n}(x)$ be the irreducible factors decomposition of $f(x)$ ($f_i(x)$ is a monic irreducible polynomial and $k_i \in \mathbb{N}^*$, for each $i \in \{1, ..., n\}$). By applying Chinese remainder theorem, we deduce that

$$\frac{\mathbb{F}_p[x]}{(f(x))} \simeq \frac{\mathbb{F}_p[x]}{\left(f_1^{k_1}(x)\right)} \times ... \times \frac{\mathbb{F}_p[x]}{\left(f_n^{k_n}(x)\right)}.$$

On the other hand, the finite product $I_1 \times ... \times I_n$ of ideals is a p-ideal if and only if so is I_k for each $k \in \{1, ..., n\}$. We deduce that there exists $i \in \{1, ..., n\}$ such that $\dfrac{\mathbb{F}_p[x]}{\left(f_i^{k_i}(x)\right)}$ has a nonzero p-ideal. By Lemma 2.5, $k_i = 1$ and $\deg f_i = 1$. This completes the proof of Theorem 2.3. $\square$

Our next theorem is due to N.H. McCoy, for instance see [7, Theorem 1] in the case where $p = 2$, and [7, Theorem 8] in the general case. It is shown that any finite p-ring is isomorphic to a direct sum of copies of $\mathbb{F}_p$. For the convenience of reader, we include here a sketch of the proof.

Theorem 2.6. *Let R be a finite ring. Then R is a p-ring with n maximal ideals if and only if $R \simeq (\mathbb{F}_p)^n$.*

Proof. $\Longleftarrow$) Since every finite direct product $R_1 \times \cdots \times R_n$ of rings is p-ring if and only if so is R_k for each $k \in \{1, ..., n\}$, then R is a p-ring. On the other hand, every maximal ideal of $R_1 \times \cdots \times R_n$ has the form $R_1 \times \cdots \times R_{k-1} \times \mathfrak{m}_k \times R_{k+1} \times \cdots \times R_n$, where $\mathfrak{m}_k$ is a maximal ideal of R_k, and $k \in \{1, ..., n\}$. We denote

$$J_k = \mathbb{F}_p \times \cdots \times \mathbb{F}_p \times (0) \times \mathbb{F}_p \times \cdots \times \mathbb{F}_p,$$

(0) in its kth place and $\mathbb{F}_p$ elsewhere, for each $k \in \{1, ..., n\}$. Then $\{J_1, ..., J_n\}$ is the set of all maximal ideals of $(\mathbb{F}_p)^n$. We conclude that R is a p-ring with n maximal ideals.

$\Longrightarrow$) Since $a^p = a$ for each a in R, then R is a von Neumann regular ring. Therefore every prime ideal of R is maximal and R is a reduced ring. It follows that $\bigcap_{1 \leq i \leq n} \mathfrak{m}_i = (0)$, where $\{\mathfrak{m}_1, ..., \mathfrak{m}_n\}$ is the set of all maximal ideals of R. By using Chinese remainder theorem we deduce that:

$$R = \frac{R}{\mathfrak{m}_1 \cap ... \cap \mathfrak{m}_n} \simeq \frac{R}{\mathfrak{m}_1} \times \cdots \times \frac{R}{\mathfrak{m}_n}.$$

Now, we need only shows that every $\frac{R}{\mathfrak{m}_k}$ is isomorphic to $\mathbb{F}_p$. Let $k \in \{1, ..., n\}$, we denote $\frac{R}{\mathfrak{m}_k} = \{a_1, ..., a_q\}$. By the assumption every field $\frac{R}{\mathfrak{m}_k}$ is a p-ring, then each a_i is a root of the polynomial $x^p - x$ and so $q \leq p$. But $\frac{R}{\mathfrak{m}_k}$ is a finite field of characteristic p, then there exists a nonnegative integer α such that $q = p^\alpha$. Thus $q = p$ and $\frac{R}{\mathfrak{m}_k} \simeq \mathbb{F}_p$, completing the proof of the theorem. □

Remark 2.7. *Let R be a semi local p-ring with n maximal ideals. Then*

1. *R is a finite p-ring and has p^n elements.*
2. *R has 2^n ideals which are all p-ideals.*

Proof. Under the notations of the above proof, it suffices to show that $\frac{R}{\mathfrak{m}_k}$ is a finite field for each $k \in \{1, ..., n\}$. Since every element of $\frac{R}{\mathfrak{m}_k}$ is a root of the polynomial $x^p - x \in \frac{R}{\mathfrak{m}_k}[x]$, we have the required property. □

Now, we give a characterization that $\frac{R[x]}{(f(x))}$ is a p-ring, in the case when R is a finite p-ring.

Theorem 2.8. *Let R be a finite p-ring and let $\{\mathfrak{m}_1, ..., \mathfrak{m}_n\}$ be the set of all maximal ideals of R. For every polynomial $f(x)$ in $R[x]$ and $j \in \{1, ..., n\}$, we denote by $f_j(x)$ the reduction of $f(x)$ modulo $\mathfrak{m}_j$ i.e $f_j(x) = \sum_{i=0}^{k} (a_i + \mathfrak{m}_j)\, x^i \in \frac{R}{\mathfrak{m}_j}[x]$, where*

$f(x)=\sum_{i=0}^{k} a_i x^i$. *Then* $\dfrac{R[x]}{(f(x))}$ *is a* p*-ring if and only if for each* $j \in \{1,...,n\}$, $f_j(x)$ *splits with distinct roots in the field* $\dfrac{R}{\mathfrak{m}_j}$.

Proof. Under the above hypothesis, we get that $R[x] \simeq \dfrac{R}{\mathfrak{m}_1}[x] \times ... \times \dfrac{R}{\mathfrak{m}_n}[x]$, as a ring, via the map: $g(x) \mapsto (g_1(x),...,g_n(x))$, where $g_j(x)$ is the reduction of $g(x)$ modulo $\mathfrak{m}_j$. For each $j \in \{1,...,n\}$, we put $R_j = \dfrac{R}{\mathfrak{m}_j}$. Then R_j is a p-ring. On the other hand, the map

$$\varphi : R[x] \longrightarrow \frac{R_1[x]}{(f_1(x))} \times ... \times \frac{R_n[x]}{(f_n(x))}$$

defined by $\varphi(g(x)) = (g_1(x) + (f_1(x)),...,g_n(x) + (f_n(x)))$ is a surjective ring homomorphism. Also we have the following equality $\ker \varphi = (f(x))$. Thus $\dfrac{R[x]}{(f(x))}$ is isomorphic to $\dfrac{R_1[x]}{(f_1(x))} \times ... \times \dfrac{R_n[x]}{(f_n(x))}$. It follows that $\dfrac{R[x]}{(f(x))}$ is a p-ring if and only if so is $\dfrac{R_j[x]}{(f_j(x))}$, for each $j \in \{1,...,n\}$. Now we can apply Lemma 2.4 to prove that $\dfrac{R_j[x]}{(f_j(x))}$ is a p-ring if and only if $f_j(x)$ has $\deg f_j$ distinct roots in R_j. This completes the proof of Theorem 2.8. □

The next example illustrates the above results.

Example 2.9. Let p be a prime integer of the form $8n+1$, for some non negative integer n. Consider the polynomial over $\mathbb{F}_p^4$ defined by

$$f(x) = (1,-1,2,-2) + (0,0,1,1)x^2 + (1,1,0,0)x^n.$$

Then $\dfrac{\mathbb{F}_p^4[x]}{(f(x))}$ is a finite p-ring with $2n+4$ maximal ideals and p^{2n+4} elements.

Proof. Under the above notations, we have

$$f_1(x) = 1 + x^n,\ f_2(x) = -1 + x^n,\ f_3(x) = 2 + x^2 \text{ and } f_4(x) = -2 + x^2.$$

It is easy to see that $f_1(x)$ and $f_2(x)$ divide $x^{p-1}-1$, hence $f_1(x)$ and $f_2(x)$ split with distinct roots in $\mathbb{F}_p$. Also $x^{4n}+1$ divides $x^p - x$, then $x^{4n}+1$ splits. Let a be a root of the polynomial $x^{4n}+1 \in \mathbb{F}_p[x]$, hence $\left(\dfrac{a^{2n}+1}{a^n}\right)^2 = 2$ and $\left(\dfrac{a^{2n}-1}{a^n}\right)^2 = -2$. Therefore $f_3(x)$ and $f_4(x)$ have distinct zeros in $\mathbb{F}_p$, since $f_3'(a) = f_4'(a) \neq 0$, ($f_j'(x)$ is the derivative of $f_j(x)$). The result then follows from Theorem 2.8. □

In the next theorem we give our main result about the transfer of p-ring property to amalgamation of rings.

Theorem 2.10. *Let A and B be a pair of rings, J an ideal of B, $f : A \longrightarrow B$ a ring homomorphism and let $A \bowtie^f J$ be the amalgamation of A with B along J with respect to f. Then $A \bowtie^f J$ is a p-ring if and only if so is A and J is a p-ideal of B.*

Proof. $\Longrightarrow$) Let $a \in A$ and $j \in J$. It is easy to see that $(a, f(a))^p = (a^p, f(a^p))$ and $(0, j)^p = (0, j^p)$. But $(a, f(a))^p = (a, f(a))$ and $(0, j)^p = (0, j)$, then $a^p = a$ et $j^p = j$. Obviously $pa = 0$ and $pj = 0$, since $p(a, f(a) + j) = 0$. We have the desired implication.

$\Longleftarrow$) Assume that A is a p-ring and J is a p-ideal of B. Let $(a, f(a)+j) \in A \bowtie^f J$. By the binomial theorem (which is valid in any commutative ring),

$$(a, f(a) + j)^p = \left(a^p, f(a^p) + j^p + \sum_{k=1}^{p-1} \binom{p}{k} j^k f(a^{p-k})\right).$$

Since $j^k f(a^{p-k}) \in J$ and p divides $\binom{p}{k}$, for each $k \in \{1, ..., p-1\}$, then

$$(a, f(a) + j)^p = (a^p, f(a^p) + j^p) = (a, f(a) + j).$$

On the other hand, $p(a, f(a) + j) = (pa, f(pa) + pj) = 0$. It follows that $A \bowtie^f J$ is a p-ring. □

Example 2.11. Let A be the set of all sequences of elements of $\mathbb{F}_p$ and let $B = \mathbb{Z}/n\mathbb{Z}$, with $n = p(p + 1)$. By using Theorem2.2, the principal ideal $(p + 1)B$ is a p-ideal of B. Consider the mapping $f : A \longrightarrow B$ defined by $f(a) = (p+1)a_0$, where $a = (a_k + p\mathbb{Z})_{k\in\mathbb{N}}$. It is easy to see that f is a ring homomorphism. On the other hand, the set all functions of a non empty set X into a p-ring is also a p-ring. Hence A is a p-ring. From the above theorem $A \bowtie^f (p + 1)B$ is a p-ring.

The following corollary is an immediate consequence of the above theorem.

Corollary 2.12. *Let A be a ring and let $A \bowtie I$ be the amalgamated duplication of A along an ideal I of A. Then $A \bowtie I$ is a p-ring if and only if so is A.*

We end this paper by giving a necessary and sufficient condition for the trivial ring extension, $A \propto E$, to be a von Neumann regular ring (resp., a p-ring).

Theorem 2.13. *Let A be a ring, E an A-module and let $A \propto E$ be the trivial ring extension of A by E. Then*

1. *$A \propto E$ is a von Neumann regular ring if and only if so is A and $E = \{0\}$.*
2. *$A \propto E$ is a p-ring if and only if so is A and $E = \{0\}$.*

Proof. (1) We say that every maximal ideal of $A \propto E$ has the form $\mathfrak{m} \propto E$, where $\mathfrak{m}$ is a maximal ideal of A. Let M be a maximal ideal of $A \propto E$. By [1, Theorem 4.1], $(A \propto E)_M \simeq A_\mathfrak{m} \propto E_\mathfrak{m}$, where $M = \mathfrak{m} \propto E$. Thus $(A \propto E)_M$ is a field if and only if so is $A_\mathfrak{m}$ and $E_\mathfrak{m} = \{0\}$. We deduce that $A \propto E$ is a von Neumann regular ring if and only if so is A and $E_\mathfrak{m} = \{0\}$, for all maximal ideal $\mathfrak{m}$ of A. We have the desired result.

(2) Assume that $A \propto E$ is a p-ring. It is easy to see that every sub-ring of p-ring is also a p-ring. It follows that A is a p-ring. On the other hand, $E = \{0\}$ since $A \propto E$ is a von Neumann regular ring. We can also deduce this result from the following equalities:

$$(a, x) = (a, x)^p = (a^p, pa^{p-1}x) = (a^p, 0),$$

since $(a, x)^n = (a^n, na^{n-1}x)$ for every nonnegative integer n, and $p(b, y) = 0$ for every element (b, y) of $A \propto E$.

The sufficient condition is obvious. □

Acknowledgements. The author thank the referee for his/her careful reading of this work.

References

1. D. D. Anderson and M. Winders; *Idealization of a module*, J. Commutative Algebra, Volume 1, (2009), 3-56.
2. M. D'Anna, C.A. Finocchiaro, and M. Fontana; *Properties of chains of prime ideals in amalgamated algebra along an ideal*, arxiv 1001.0472v1(2010).
3. M. D'Anna and M. Fontana; *An amalgamated duplication of a ring along an ideal: the basic proerties*, J. Algebra Appl.6 (2007), 443-459.
4. S. Glaz; *Commutative coherent rings*, Springer-Verlag, Lecture Notes in Mathematics, 1371 (1989).
5. J.A. Huckaba; *Commutative rings with zero divisors*, Marcel Dekker, New York-Basel, 1988.
6. S. Lang; *Undergraduate algebra*, Springer, Ungraduate Texts in Mathematics, Third edition (2005).
7. N.H. McCoy; *Subdirect sums of rings*, Bull. Amer. Math. Soc. Volume 55, No. 9. (1947), 856-877.
8. N.H. McCoy and D. Montgomery; *A representation of generalised Boolean rings*, Duk. Math, No. 3. (1937), 455-459.

Mohammed Kabbour, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.

E − mail address : *mkabbour@gmail.com*