RANDOM ITERATION WITH PLACE DEPENDENT PROBABILITIES

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Abstract. Markov chains arising from random iteration of functions $S_\theta : X \to X$, $\theta \in \Theta$, where X is a Polish space and Θ is arbitrary set of indices are considerd. At $x \in X$, θ is sampled from distribution ϑ_x on Θ and ϑ_x are different for different x. Exponential convergence to a unique invariant measure is proved. This result is applied to case of random affine transformations on $\mathbb{R}^d$ giving existence of exponentially attractive perpetuities with place dependent probabilities.

1. Introduction

We consider Markov chain of the form $X_0 = x_0$, $X_1 = S_{\theta_0}(x_0)$, $X_2 = S_{\theta_1} \circ S_{\theta_0}(x_0)$ and inductively

$$X_{n+1} = S_{\theta_n}(X_n), \tag{1}$$

where S_{θ_0}, S_{θ_1},...,S_{θ_n} are randomly chosen from a family $\{S_\theta : \theta \in \Theta\}$ of functions that map a state space X into itself. If chain is at $x \in X$ then $\theta \in \Theta$ is sampled from distribution ϑ_x on Θ, where ϑ_x are, in general, different for different x. We are interested in the rate of convergence to stationary distribution μ_* on X, i.e.

$$P\{X_n \in A\} \to \mu_*(A) \qquad \text{as} \qquad n \to \infty. \tag{2}$$

In case of constant probabilities, i.e. $\vartheta_x = \vartheta_y$ for $x, y \in X$, the basic tool when studying asymptotics of (1) are backward iterations

$$Y_{n+1} = S_{\theta_0} \circ S_{\theta_1} \circ ... \circ S_{\theta_n}(x).$$

Since X_n and Y_n are identically distributed and, under suitable conditions, Y_n converge almost surely at exponential rate to some random element Y, one obtains exponential convergence in (2) (see [6] for bibliography and excellent survey of the field). For place dependent ϑ_x we need different approach because distributions of X_n and Y_n are not equal.

The simplest case when $\Theta = \{1, ..., n\}$ is treated in [2] and [20], where existence of a unique attractive invariant measure is established. Similar result

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holds true when $\Theta = [0,T]$ and ϑ_x are absolutely continuous (see [13]). Recently it was shown that the rate of convergence in case of $\Theta = \{1, ..., n\}$ is exponential (see [21]).

In this paper we treat general case of place dependent ϑ_x for arbitrary Θ and prove the existence of a unique exponentially attractive invariant measure for (1). Our approach is based on coupling method which can be briefly described as follows. For arbitrary starting points $x, \bar{x} \in X$ we consider chains $(X_n)_{n\in\mathbb{N}_0}$, $(\bar{X}_n)_{n\in\mathbb{N}_0}$ with $X_0 = x_0$, $\bar{X}_0 = \bar{x}_0$ and try to build correlations between $(X_n)_{n\in\mathbb{N}_0}$ and $(\bar{X}_n)_{n\in\mathbb{N}_0}$ in order to make their trajectories as close as possible. This can be done because transition probability function $\mathbf{B}_{x,y}(A) = P\{(X_{n+1}, \bar{X}_{n+1}) \in A \,|\, (X_n, \bar{X}_n) = (x,y)\}$ of the coupled chain $(X_n, \bar{X}_n)_{n\in\mathbb{N}_0}$ taking values in X^2 can be decomposed (see [11]) in the following way

$$\mathbf{B}_{x,y} = \mathbf{Q}_{x,y} + \mathbf{R}_{x,y},$$

where sub-probabilistic measures $\mathbf{Q}_{x,y}$ are contractive in metric d on X:

$$\int_{X^2} d(u,v)\, \mathbf{Q}_{x,y}(du, dv) \leq \alpha d(x,y)$$

for some constant $\alpha \in (0,1)$.

Since transition probabilities for (1) can be mutually singular for even very close points, one cannot expect that chains $(X_n)_{n\in\mathbb{N}_0}$ and $(\bar{X}_n)_{n\in\mathbb{N}_0}$ couple in finite time ($X_n = \bar{X}_n$ for some $n \in \mathbb{N}_0$) as in classical coupling constructions ([16]) leading to convergence in total variation norm. On the contrary, they only couple at infinity ($d(X_n, \bar{X}_n) \to 0$ as $n \to \infty$) so this method is sometimes called asymptotic coupling ([12]) and gives convergence in *-weak topology.

The paper is organized as follows. In Section 2 we formulate and prove theorem which assures exponential convergence to invariant measure for a class of Markov chains. This theorem is applied in Section 3 to chains generated by random iteration of functions. In Section 4 we discuss special class of such functions, random affine transformations on $\mathbb{R}^d$, thus generalizing the notion of *perpetuity* to place dependent case.

2. An exponential convergence result

2.1. **Notation and basic definitions.** Let (X,d) be a *Polish* space, i.e. a complete and separable metric space and denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X. By $B_b(X)$ we denote the space of bounded Borel-measurable functions equipped with the supremum norm, $C_b(X)$ stands for subspace of bounded continuous functions. By $\mathcal{M}_{fin}(X)$ and $\mathcal{M}_1(X)$

we denote the sets of Borel measures on X such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}_{fin}(X)$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1(X)$. Elements of $\mathcal{M}_1(X)$ are called *probability* measures. Elements of $\mathcal{M}_{fin}(X)$ for which $\mu(X) \leq 1$ are called *sub-probabilistic*. By $supp\,\mu$ we denote the support of the measure μ. We also define

$$\mathcal{M}_1^L(X) = \{\mu \in \mathcal{M}_1(X) : \int_X L(x)\mu(dx) < \infty\}$$

where $L : X \to [0, \infty)$ is arbitrary Borel measurable function and

$$\mathcal{M}_1^1(X) = \{\mu \in \mathcal{M}_1(X) : \int_X d(\bar{x}, x)\mu(dx) < \infty\},$$

where $\bar{x} \in X$ is fixed. Definition of $\mathcal{M}_1^1(X)$ is independent of the choice of $\bar{x}$.

The space $\mathcal{M}_1(X)$ is equipped with the *Fourtet-Mourier metric*:

$$\|\mu_1 - \mu_2\|_{FM} = \sup\{|\int_X f(x)(\mu_1 - \mu_2)(dx)| : f \in \mathcal{F}\},$$

where

$$\mathcal{F} = \{f \in C_b(X) : |f(x) - f(y)| \leq 1 \quad \text{and} \quad |f(x)| \leq 1 \quad \text{for} \quad x, y \in X\}.$$

The space $\mathcal{M}_1^1(X)$ is equipped with the *Wasserstein metric*:

$$\|\mu_1 - \mu_2\|_W = \sup\{|\int_X f(x)(\mu_1 - \mu_2)(dx)| : f \in \mathcal{W}\},$$

where

$$\mathcal{W} = \{f \in C_b(X) : |f(x) - f(y)| \leq 1 \quad \text{for} \quad x, y \in X\}.$$

By $\|\cdot\|$ we denote the total variation norm. If the measure μ is nonnegative then $\|\mu\|$ is simply the total mass of μ.

Let $P : B_b(X) \to B_b(X)$ be the *Markov operator*, i.e. linear operator satisfying $P\mathbf{1}_X = \mathbf{1}_X$ and $Pf(x) \geq 0$ if $f \geq 0$. Denote by P^* the dual operator, i.e operator $P^* : \mathcal{M}_{fin}(X) \to \mathcal{M}_{fin}(X)$ defined as follows

$$P^*\mu(A) := \int_X P\mathbf{1}_A(x)\mu(dx) \qquad \text{for} \qquad A \in \mathcal{B}_X.$$

We say that $\mu_* \in \mathcal{M}_1(X)$ is *invariant* for P if

$$\int_X Pf(x)\mu_*(dx) = \int_X f(x)\mu_*(dx) \qquad \text{for every} \qquad f \in B_b(X)$$

or, alternatively, we have $P^*\mu_* = \mu_*$.

By $\{\mathbf{P}_x : x \in X\}$ we denote the *transition probability function* for P, i.e. the family of measures $\mathbf{P}_x \in \mathcal{M}_1(X)$ for $x \in X$ such that maps $x \mapsto \mathbf{P}_x(A)$ are measurable for every $A \in \mathcal{B}_X$ and

$$Pf(x) = \int_X f(y)\mathbf{P}_x(dy) \qquad \text{for} \qquad x \in X \quad \text{and} \quad f \in B_b(X)$$

or equivalently $P^*\mu(A) = \int_X \mathbf{P}_x(A)\mu(dx)$ for $A \in \mathcal{B}_X$ and $\mu \in \mathcal{M}_{fin}(X)$.

2.2. **Formulation of the theorem.**

Definition 2.1. *A coupling for* $\{\mathbf{P}_x : x \in X\}$ *is a family* $\{\mathbf{B}_{x,y} : x, y \in X\}$ *of probabilistic measures on* $X \times X$ *such that for every* $B \in \mathcal{B}_{X^2}$ *the map* $X^2 \ni (x, y) \mapsto \mathbf{B}_{x,y}(B)$ *is measurable and*

$$\mathbf{B}_{x,y}(A \times X) = \mathbf{P}_x(A), \qquad \mathbf{B}_{x,y}(X \times A) = \mathbf{P}_y(A)$$

for every $x, y \in X$ *and* $A \in \mathcal{B}_X$.

In the following we assume that there exists the family $\{\mathbf{Q}_{x,y} : x, y \in X\}$ of sub-probabilistic measures on X^2 such that maps $(x, y) \mapsto \mathbf{Q}_{x,y}(B)$ are measurable for every Borel $B \subset X^2$ and

$$\mathbf{Q}_{x,y}(A \times X) \leq \mathbf{P}_x(A) \qquad \text{and} \qquad \mathbf{Q}_{x,y}(X \times A) \leq \mathbf{P}_y(A)$$

for every $x, y \in X$ and Borel $A \subset X$.
Measures $\{\mathbf{Q}_{x,y} : x, y \in X\}$ allow us to construct coupling for $\{\mathbf{P}_x : x \in X\}$. Define on X^2 the family of measures $\{\mathbf{R}_{x,y} : x, y \in X\}$, which on rectangles $A \times B$ are given by

$$\mathbf{R}_{x,y}(A\times B) = \frac{1}{1 - \mathbf{Q}_{x,y}(X^2)}(\mathbf{P}_x(A) - \mathbf{Q}_{x,y}(A\times X))(\mathbf{P}_y(B) - \mathbf{Q}_{x,y}(X \times B)),$$

when $\mathbf{Q}_{x,y}(X^2) < 1$ and $\mathbf{R}_{x,y}(A \times B) = 0$ otherwise. A simple computation shows that family $\{\mathbf{B}_{x,y} : x, y \in X\}$ of measures on X^2 defined by

$$\mathbf{B}_{x,y} = \mathbf{Q}_{x,y} + \mathbf{R}_{x,y} \quad \text{for} \quad x, y \in X$$

is coupling for $\{\mathbf{P}_x : x \in X\}$.
For every $r > 0$ define $D_r = \{(x, y) \in X^2 : d(x, y) < r \}$.
Now we list assumptions on Markov operator P and transition probabilities $\{\mathbf{Q}_{x,y} : x, y \in X\}$.
A0 *P is a Feller operator, i.e.* $P(C_b(X)) \subset C_b(X)$.
A1 *There exists a Lapunov function for P, i.e. continuous function* $L : X \to [0, \infty)$ *such that L is bounded on bounded sets,* $\lim_{x\to\infty} L(x) = +\infty$ *and for some* $\lambda \in (0, 1)$, $c > 0$

$$PL(x) \leq \lambda L(x) + c \qquad for \qquad x \in X.$$

A2 *There exist* $F \subset X^2$ *and* $\alpha \in (0, 1)$ *such that* supp $\mathbf{Q}_{x,y} \subset F$ *and*

$$\int_{X^2} d(u, v)\mathbf{Q}_{x,y}(du, dv) \leq \alpha d(x, y) \qquad for \qquad (x, y) \in F. \tag{3}$$

A3 *There exist* $R > 0$, $\delta > 0$, $l > 0$ *and* $\nu \in (0, 1]$ *such that*

$$1 - \|\mathbf{Q}_{x,y}\| \leq l d(x, y)^\nu \qquad \text{and} \qquad \mathbf{Q}_{x,y}(D_{\alpha d(x,y)}) \geq \delta \tag{4}$$

for $(x,y) \in D_R \cap F$.

A4 *There exist* $\beta \in (0,1)$ *and* $\tilde{C} > 0$ *such that for*

$$\kappa(\,(x_n, y_n)_{n\in\mathbb{N}_0}\,) = \inf\{n \in \mathbb{N}_0 :\ (x_n, y_n) \in D_R \cap F\}$$

we have

$$\mathbb{E}_{x_0,y_0}\beta^{-\kappa} \le \tilde{C} \qquad \textit{whenever} \qquad L(x_0) + L(y_0) < \frac{4c}{1-\lambda},$$

where $\mathbb{E}_{x_0,y_0}$ *denotes here expectation with respect to chain starting from* (x_0, y_0) *and with trasition function* $\{\mathbf{B}_{x,y} : x, y \in X\}$.

Remark. Condition **A4** means that dynamics quickly enters the domain of contractivity F. In this paper we discuss Markov chains generated by random iteration of functions for which always $F = X^2$ and $L(x) = d(x, \bar{x})$ with some fixed $\bar{x} \in X$, so **A4** is trivially fulfilled when $R = \frac{4c}{1-\lambda}$. There are, however, examples of random dynamical systems for which F is a proper subset of X^2. Indeed, in *contractive Markov systems* introduced by I. Werner in [22] we have $X = \sum_{i=1}^n X_i$ but $F = \sum_{i=1}^n X_i \times X_i$. They will be studied in a subsequent paper.

Now we formulate the main result of this section. Its proof is given in Section 2.4.

Theorem 2.1. *Assume* **A0** – **A4**. *Then operator* P *possesses a unique invariant measure* $\mu_* \in \mathcal{M}_1^L(X)$, *which is attractive, i.e.*

$$\lim_{n\to\infty} \int_X P^n f(x)\,\mu(dx) = \int_X f(x)\,\mu(dx) \quad \textit{for} \quad f \in C_b(X),\ \mu \in \mathcal{M}_1(X).$$

Moreover, there exist $q \in (0,1)$ *and* $C > 0$ *such that*

$$\|P^{*n}\mu - \mu_*\|_{FM} \le q^n C(1 + \int_X L(x)\mu(dx)) \tag{5}$$

for $\mu \in \mathcal{M}_1^L(X)$ *and* $n \in \mathbb{N}$.

2.3. **Measures on the pathspace.** For fixed $(x_0, y_0) \in X^2$ the next step of the chain with transition probability function $\mathbf{B}_{x,y} = \mathbf{Q}_{x,y} + \mathbf{R}_{x,y}$ can be drawn according to $\mathbf{Q}_{x_0,y_0}$ or according to $\mathbf{R}_{x_0,y_0}$. To distinguish these two cases we introduce augmented space $\widehat{X} = X^2 \times \{0,1\}$ and transition function $\{\widehat{\mathbf{B}}_{x,y,\theta} : (x,y,\theta) \in \widehat{X}\}$ on $\widehat{X}$ given by

$$\widehat{\mathbf{B}}_{x,y,\theta} = \mathbf{Q}_{x,y} \times \delta_1 + \mathbf{R}_{x,y} \times \delta_0.$$

Parameter $\theta \in \{0,1\}$ is responsible for choosing measures $\mathbf{Q}_{x,y}$ and $\mathbf{R}_{x,y}$. If Markov chain with transition function $\widehat{\mathbf{B}}_{x,y,\theta}$ at time n stays in the set $X^2 \times \{1\}$ it means that the last step was drawn according to $\mathbf{Q}_{x,y}$, for some

$(x, y) \in X^2$.
For every $x \in X$ finite-dimensional distributions $\mathbf{P}_x^{0,\dots,n} \in \mathcal{M}_1(X^{n+1})$ are defined by

$$\mathbf{P}_x^{0,\dots,n}(B) = \int_X \mu(dx_0) \int_X \mathbf{P}_{x_1}(dx_2)\dots \int_X \mathbf{P}_{x_{n-1}}(dx_n)\mathbf{1}_B(x_0, \dots, x_n)$$

for $n \in \mathbb{N}_0$, $B \in \mathcal{B}_{X^{n+1}}$. By Kolmogorov extension theorem we obtain measure $\mathbf{P}_x^\infty$ on pathspace X^∞. Similarly we define measures $\mathbf{B}_{x,y}^\infty$, $\widehat{\mathbf{B}}_{x,y,\theta}^\infty$ on $(X \times X)^\infty$ and $\widehat{X}^\infty$. These measures have the following interpretation. Consider Markov chain $(X_n, Y_n)_{n\in\mathbb{N}_0}$ on $X \times X$, starting from (x_0, y_0), with transition function $\{\mathbf{B}_{x,y} : x, y \in X\}$, obtained by canonical Kolmogorov construction, i.e. $\Omega = (X \times X)^\infty$ is sample space equipped with probability measure $\mathbb{P} = \mathbf{B}_{x_0,y_0}^\infty$, $X_n(\omega) = x_n$, $Y_n(\omega) = y_n$, where $\omega = (x_k, y_k)_{k\in\mathbb{N}_0} \in \Omega$, and $n \in \mathbb{N}_0$. Then $(X_n)_{n\in\mathbb{N}_0}$, $(Y_n)_{n\in\mathbb{N}_0}$ are Markov chains in X, starting from x_0 and y_0, with transition function $\{\mathbf{P}_x : x \in X\}$, and $\mathbf{P}_x^\infty$, $\mathbf{P}_y^\infty$ are their measures on pathspace X^∞.
In this paper we often consider marginals of measures on the pathspace. If μ is a measure on a measurable space X and $f : X \to Y$ is a measurable map, then $f^\#\mu$ is the measure on Y defined by $f^\#\mu(A) = \mu(f^{-1}(A))$. So, if we denote by *pr* the projection map from a product space to its component, then $pr^\#\mu$ is simply the marginal of μ on this component.
In the following we consider Markov chains on $\widehat{X}$ with transition function $\{\widehat{\mathbf{B}}_{x,y,\theta} : x, y \in X, \theta \in \{0, 1\}\}$. We adopt as a convention that $\theta_0 = 1$, that is Φ always starts from $X^2 \times \{1\}$, and define

$$\widehat{\mathbf{B}}_{x,y}^\infty := \widehat{\mathbf{B}}_{x,y,1}^\infty.$$

For $b \in \mathcal{M}_{fin}(X^2)$ we write

$$\widehat{\mathbf{B}}_b^\infty(B) = \int_X \widehat{\mathbf{B}}_{x,y}^\infty(B)\, b(dx, dy), \qquad B \in \mathcal{B}_{\widehat{X}^\infty}$$

and

$$\mathbf{Q}_b(A) = \int_{X^2} \mathbf{Q}_{x,y}(A)\, b(dx, dy), \qquad A \in \mathcal{B}_{X^2}.$$

When studying asymptotics of chain $(X_n)_{n\in\mathbb{N}_0}$ with transition function $\{\mathbf{P}_x : x \in X\}$ it is particularly interesting whether coupled chain $(X_n, Y_n)_{n\in\mathbb{N}_0}$ is moving only accordingly to contractive part $\mathbf{Q}_{x,y}$ of transition function

$\mathbf{B}_{x,y}$. For every sub-probabilistic measure $b \in \mathcal{M}_{fin}(X^2)$ we define sub-probabilistic finite-dimensional distributions $\mathbf{Q}_b^{0,\dots,n} \in \mathcal{M}_{fin}((X \times X)^{n+1})$

$$\mathbf{Q}_b^{0,\dots,n}(B) = \int_{X^2} b(dx_0, dy_0) \int_{X^2} \mathbf{Q}_{x_0,y_0}(dx_1, dy_1)... \\ ... \int_{X^2} \mathbf{Q}_{x_{n-1},y_{n-1}} \mathbf{1}_B((x_0, y_0), ..., (x_n, y_n)),$$

where $B \in \mathcal{B}_{(X\times X)^{n+1}}$, $n \in \mathbb{N}_0$. Since family $\{\mathbf{Q}_b^{0,\dots,n} : n \in \mathbb{N}_0\}$ need not be consistent, we cannot use Kolmogorov extension theorem to obtain measure on the whole pathspace $\widehat{X}^\infty$. However, defining for every $b \in \mathcal{M}_{fin}(X^2)$ measure $\mathbf{Q}_b^\infty \in \mathcal{M}_{fin}(\widehat{X}^\infty)$ by

$$\mathbf{Q}_b^\infty(B) = \widehat{\mathbf{B}}_b^\infty(B \cap (X^2 \times \{1\})^\infty),$$

where $B \in \mathcal{B}_{\widehat{X}^\infty}$, one can easily check that for every cylindrical set $B = A \times \widehat{X}^\infty$, $A \in \mathcal{B}_{\widehat{X}^n}$, we have

$$\mathbf{Q}_b^\infty(B) = \lim_{n\to\infty} \mathbf{Q}_b^{0,\dots,n}(pr_{(X^2)^{n+1}}(A)).$$

2.4. **Proof of Theorem 2.1.** Before proceeding to the proof of Theorem 2.1 we formulate two lemmas. The proof of the first one is due to C. Odasso and can be found in [19] as a part of larger reasoning. Since it is very useful in coupling constructions we formulate it here explicitly and reproduce its proof.

Lemma 2.1. *Let Y be a metric space and $V : Y \to [0, \infty)$ a measurable function. Let $(Y_n^{y_0})_{n\in\mathbb{N}_0}$ be a family of Markov chains with common transition function, indexed by starting point $y_0 \in Y$. Suppose that there exist constants $V_0 > 0$, $\lambda \in (0, 1)$, $\tilde{C} > 0$ such that for*

$$\rho((y_k)_{k\in\mathbb{N}_0}) = \inf\{k \in \mathbb{N}_0 : V(y_k) < V_0\}$$

we have

$$\mathbb{E}_{y_0}\lambda^{-\rho} \le \tilde{C}(V(y_0) + 1)\} \qquad \textit{for} \qquad y_0 \in Y,$$

where $\mathbb{E}_{y_0}$ is expectation induced by $(Y_n^{y_0})_{n\in\mathbb{N}_0}$.
Let $B \subset Y^\infty$ be measurable and such that for some $p > 0$ we have $\mathbb{P}_{y_0}(B) > p$ for every y_0 satisfying $V(y_0) < V_0$. Then there exist constants $\gamma \in (0, 1)$ and $C > 0$ such that for

$$\tau_B((y_k)_{k\in\mathbb{N}_0}) = \inf\{n \in \mathbb{N}_0 : (y_{n+k})_{k\in\mathbb{N}_0} \in B\}$$

we have

$$\mathbb{E}_{y_0}\gamma^{-\tau_B} \le C(V(y_0) + 1) \qquad \textit{for} \qquad y_0 \in Y.$$

Proof of Lemma 2.1.
Fix $y_0 \in Y$. Define the time of n-th visit in $\{y \in Y : V(y) < V_0\}$:

$$\rho_1 = \rho$$
$$\rho_{n+1} = \rho_n + \rho \circ T_{\rho_n} \quad \text{for} \quad n > 1,$$

where $T_n((y_k)_{k\in\mathbb{N}_0}) = (y_{k+n})_{k\in\mathbb{N}_0}$. Strong Markov property implies that

$$\mathbb{E}_{y_0}(\lambda^{-\rho} \circ T_{\rho_n} | \mathcal{F}_{\rho_n}) = \mathbb{E}_{Y_{\rho_n}}(\lambda^{-\rho}) \quad \text{for} \quad n \in \mathbb{N},$$

where $\mathcal{F}_{\rho_n}$ is σ -algebra in Y^∞ generated by ρ_n. Since $V(Y_{\rho_n}) < V_0$ we have

$$\mathbb{E}_{y_0}(\lambda^{-\rho_{n+1}}) = \mathbb{E}_{y_0}(\lambda^{-\rho_n}\mathbb{E}_{y_0}(\lambda^{-\rho} \circ T_{\rho_n}|\mathcal{F}_{\rho_n})) = \mathbb{E}_{y_0}(\lambda^{-\rho_n}\mathbb{E}_{Y_{\rho_n}}(\lambda^{-\rho})) \leq$$
$$\leq \mathbb{E}_{y_0}(\lambda^{-\rho_n})[\tilde{C}(V_0 + 1)].$$

Taking $a = \tilde{C}(V_0 + 1)$ we obtain

$$\mathbb{E}_{y_0}(\lambda^{-\rho_{n+1}}) \leq a^n\tilde{C}(V(y_0) + 1).$$

Define

$$\widehat{\tau}_B((y_k)_{k\in\mathbb{N}_0}) = \inf\{n \in \mathbb{N}_0 : V(y_n) < V_0 \quad \text{and} \quad (y_{k+n})_{k\in\mathbb{N}_0} \in B\}$$

and

$$\sigma = \inf\{n \geq 1 : \widehat{\tau}_B = \rho_n\}.$$

By assumption we have $\mathbb{P}_{y_0}(\sigma = k) \leq (1 - p)^{k-1}$ for $k \geq 1$.
Let $r > 1$. Hölder inequality implies that

$$\mathbb{E}_{y_0}(\lambda^{-\frac{\widehat{\tau}_B}{r}}) \leq \sum_{k=1}^{\infty} \mathbb{E}_{y_0}(\lambda^{\frac{\rho_k}{r}}\mathbf{1}_{\sigma=k}) \leq$$
$$\leq \sum_{k=1}^{\infty}[\mathbb{E}_{y_0}(\lambda^{\rho_k})]^{\frac{1}{r}}\mathbb{P}_{y_0}(\sigma = k)^{1-\frac{1}{r}} \leq$$
$$\leq \sum_{k=1}^{\infty}[a^{k-1}\tilde{C}(V(y_0) + 1)]^{\frac{1}{r}}(1 - p)^{(1-k)(1-\frac{1}{r})} \leq$$
$$\leq \tilde{C}(1 + V(y_0))\sum_{k=1}^{\infty}[(\frac{a}{1-p})^{\frac{1}{r}}(1 - p)]^k.$$

Choosing sufficiently large r and setting $\gamma = \lambda^{\frac{1}{r}}$ we obtain

$$\mathbb{E}_{y_0}(\gamma^{-\widehat{\tau}_B}) \leq C(V(y_0) + 1)$$

for some $C > 0$. Since $\tau_B \leq \widehat{\tau}_B$, the proof is complete. □

Lemma 2.2. *Let $(Y_n^{y_0})_{n\in\mathbb{N}_0}$ with $y_0 \in Y$ be a family of Markov chains in metric space Y. Suppose that $V : Y \to [0, \infty)$ is Lapunov function for their* *transition function* $\{\pi_y : y \in Y\}$, *i.e. there exist* $a \in (0,1)$ *and* $b > 0$ *such that*

$$\int_Y V(x)\pi_y(dx) \le aV(y) + b \quad \textit{for} \quad y \in Y.$$

Then there exist $\lambda \in (0,1)$ *and* $\tilde{C} > 0$ *such that for*

$$\rho((y_k)_{k\in\mathbb{N}_0}) = \inf\{k \in \mathbb{N}_0 : V(y_k) < \frac{2b}{1-a}\}$$

we have

$$\mathbb{E}_{y_0}\lambda^{-\rho} \le \tilde{C}(V(y_0) + 1) \quad \textit{for} \quad y_0 \in Y.$$

Proof of Lemma 2.2.
Chains $(Y_n^{y_0})_{n\in\mathbb{N}_0}$, $y_0 \in Y$ are defined on common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Fix $max\{a, \frac{1+a}{2}\} < \alpha < 1$ and set $V_0 = \frac{b}{\alpha - a}$. Define

$$\tilde{\rho}((y_k)_{k\in\mathbb{N}_0}) = \inf\{k \in \mathbb{N}_0 : V(y_k) \le V_0\}$$

For every $y_0 \in Y$ let $\mathcal{F}_n \subset \mathcal{F}$, $n \in \mathbb{N}_0$ be filtration induced by $(Y_n^{y_0})_{n\in\mathbb{N}_0}$. Define

$$A_n = \{\omega \in \Omega : V(Y_n^{y_0}(\omega)) > V_0 \quad \text{for} \quad i = 0, 1, ..., n\}, \quad n \in \mathbb{N}_0.$$

Observe that $A_{n+1} \subset A_n$ and $A_n \in \mathcal{F}$. By the definition of V_0 we have $\mathbf{1}_{A_n}\mathbb{E}(V(Y_{n+1}^{y_0})|\mathcal{F}_n) \le \mathbf{1}_{A_n}(aV(Y_n^{y_0}) + b) < \alpha\mathbf{1}_{A_n}V(Y_n^{y_0})$ $\mathbb{P}$-a.e. in Ω. This gives

$$\int_{A_n} V(Y_n^{y_0})d\mathbb{P} \le \int_{A_{n-1}} V(Y_n^{y_0})d\mathbb{P} = \int_{A_{n-1}} \mathbb{E}(V(Y_n^{y_0})|\mathcal{F}_{n-1})d\mathbb{P}$$
$$\le \int_{A_{n-1}} (aV(Y_{n-1}^{y_0}) + b)d\mathbb{P} \le \alpha \int_{A_{n-1}} V(Y_{n-1}^{y_0})d\mathbb{P}.$$

By Chebyshev inequality

$$\mathbb{P}(V(Y_0^{y_0}) > V_0, ..., V(Y_n^{y_0}) > V_0) = \int_{A_{n-1}} \mathbb{P}(V(Y_n^{y_0}) > V_0|\mathcal{F}_{n-1})d\mathbb{P}$$
$$\le V_0^{-1} \int_{A_{n-1}} \mathbb{E}(V(Y_n^{y_0})|\mathcal{F}_{n-1})d\mathbb{P} \le \alpha^n V_0^{-1}(aV(y_0) + b),$$

and

$$\mathbb{P}_{y_0}(\tilde{\rho} > n) \le \alpha^n C(V(y_0) + 1) \quad \text{for} \quad n \in \mathbb{N}_0.$$

Fix $\gamma \in (0,1)$ and observe that for $\lambda = \alpha^\gamma$ we have

$$\mathbb{E}_{y_0}\lambda^{-\tilde{\rho}} \le 2 + \sum_{n=1}^{\infty} \mathbb{P}_{y_0}(\lambda^{-\tilde{\rho}} > n) \le 2 + \frac{C(V(y_0) + 1)}{\alpha} \sum_{n=1}^{\infty} n^{-\frac{1}{\gamma}} = \tilde{C}(V(y_0) + 1)$$

for properly chosen $\tilde{C}$. Since $\rho \le \tilde{\rho}$ the proof is finished. □

Proof of Theorem 2.1.

Step I: Define new metric $\bar{d}(x,y) = d(x,y)^\nu$ and observe that for $\bar{D}_r =$

$\{(x,y) \in X^2 : \bar{d}(x,y) < r\}$ we have $D_R = \bar{D}_{\bar{R}}$ with $\bar{R} = R^\nu$. By Jensen inequality (3) takes form

$$\int_{X^2} \bar{d}(u,v)\mathbf{Q}_{x,y}(du,dv) \le \bar{\alpha}\bar{d}(x,y) \qquad for \qquad (x,y) \in F, \tag{6}$$

with $\bar{\alpha} = \alpha^\nu$. Assumption **A3** implies that

$$1 - \|\mathbf{Q}_{x,y}\| \le l\bar{d}(x,y) \qquad \text{and} \qquad \mathbf{Q}_{x,y}(D_{\bar{\alpha}\bar{d}(x,y)}) \ge \delta \tag{7}$$

for $(x,y) \in \bar{D}_{\bar{R}} \cap F$.

Step II: Observe, that if $b \in \mathcal{M}_{fin}(X^2)$ satisfies $supp\, b \subset \bar{D}_{\bar{R}} \cap F$ then (7) implies

$$\|\mathbf{Q}_b\| \ge \|b\| - l\int_{X^2} \bar{d}(u,v)b(du,dv).$$

Iterating the above inequality we obtain

$$\|\mathbf{Q}_b^\infty\| \ge \|b\| - \frac{l}{1-\bar{\alpha}}\int_{X^2} \bar{d}(u,v)b(du,dv) \tag{8}$$

if $supp\, b \subset \bar{D}_{\bar{R}} \cap F$. Set $r_0 = min\{\bar{R}, \frac{1-\bar{\alpha}}{2l}\}$ and $n_0 = \min\{n \in \mathbb{N}_0 : \bar{\alpha}^n \bar{R} < r_0\}$. Now (7) and (8) imply, that for $(x,y) \in D_R \cap F$ we have

$$\|\mathbf{Q}_{x,y}^\infty\| \ge \frac{1}{2}\delta^{n_0}. \tag{9}$$

Step III: Define $\tilde{\rho} : (X^2)^\infty \to \mathbb{N}_0$

$$\tilde{\rho}((x_n,y_n)_{n\in\mathbb{N}_0}) = \inf\{n \in \mathbb{N}_0 : L(x_n) + L(y_n) < \frac{4c}{1-\lambda}\}.$$

Since $L(x)+L(y)$ is Lapunov function for Markov chain in X^2 with transition probabilities $\{\mathbf{B}_{x,y} : x,y \in X\}$, Lemma 2.2 shows that there exist constants $\lambda_0 \in (0,1)$ and C_0 such that

$$\mathbb{E}_{x,y}\, \lambda_0^{-\tilde{\rho}} \le C_0(L(x) + L(y) + 1) \qquad \text{for} \qquad (x,y) \in X^2. \tag{10}$$

Define

$$\rho((x_n,y_n,\theta_n)_{n\in\mathbb{N}_0}) = \inf\{n \in \mathbb{N}_0 : (x_n,y_n) \in D_R \cap F\}$$

and

$$\tau((x_n,y_n,\theta_n)_{n\in\mathbb{N}_0}) = \inf\{n \in \mathbb{N}_0 : (x_n,y_n) \in D_R \cap F \quad \text{and} \quad \forall_{k\ge n}\theta_k = 1\}.$$

Set $\lambda = max\{\beta, \lambda_0\}$. Since $\rho \le \tilde{\rho} + \kappa \circ T_{\tilde{\rho}}$, where $T_{\tilde{\rho}}((x_n,y_n,\theta_n)_{n\in\mathbb{N}_0}) = (x_{n+\tilde{\rho}}, y_{n+\tilde{\rho}}, \theta_{n+\tilde{\rho}})_{n\in\mathbb{N}_0}$, then strong Markov property, **A4** and (10) give

$$\mathbb{E}_{x,y,\theta}\, \lambda^{-\rho} \le \tilde{C}C_0(L(x) + L(y) + 1) \qquad \text{for} \qquad x,y \in X, \theta \in \{0,1\}.$$

Define $B = \{(x_n,y_n,\theta_n)_{n\in\mathbb{N}_0} : \theta_n = 1 \quad \text{for} \quad n \in \mathbb{N}_0\}$. From Step II we obtain that $\mathbb{P}_{x,y,\theta}(B) \ge \frac{1}{2}\delta^{n_0}$ for $(x,y,\theta) \in (D_R \cap F) \times \{0,1\}$. Finally Lemma 2.1 guarantees existence of constants $\gamma \in (0,1)$, $C_1 > 0$ such that

$$\mathbb{E}_{x,y,\theta}\, \gamma^{-\tau} \le C_1(L(x) + L(y) + 1) \qquad \text{for} \qquad x,y \in X, \theta \in \{0,1\}.$$

STEP IV: Define sets

$$G_{\frac{n}{2}} = \{t \in (X^2 \times \{0,1\})^\infty : \tau(t) \le \frac{n}{2}\}$$

and

$$H_{\frac{n}{2}} = \{t \in (X^2 \times \{0,1\})^\infty : \tau(t) > \frac{n}{2}\}.$$

For every $n \in \mathbb{N}$ we have

$$\widehat{\mathbf{B}}^\infty_{x,y,\theta} = \widehat{\mathbf{B}}^\infty_{x,y,\theta} \,|_{G_{\frac{n}{2}}} + \widehat{\mathbf{B}}^\infty_{x,y,\theta} \,|_{H_{\frac{n}{2}}} \qquad \text{for} \qquad x, y \in X, \theta \in \{0,1\}.$$

Fix $\theta = 1$ and $(x,y) \in X^2$. From the fact that $\|\cdot\|_{FM} \le \|\cdot\|_W$ it follows that

$$\begin{aligned}
&\|P^{*n}\delta_x - P^{*n}\delta_y\|_{FM} = \|\mathbf{P}^n_x - \mathbf{P}^n_y\|_{FM} \\
&= \sup_{f \in \mathcal{F}} \Big| \int_{X^2} (f(z_1) - f(z_2))(pr^{\#}_n \mathbf{B}^\infty_{x,y})(dz_1, dz_2)\Big| \\
&= \sup_{f \in \mathcal{F}} \Big| \int_{X^2} (f(z_1) - f(z_2))(pr^{\#}_{X^2} pr^{\#}_n \widehat{\mathbf{B}}^\infty_{x,y,\theta})(dz_1, dz_2)\Big| \\
&\le \sup_{f \in \mathcal{W}} \Big| \int_{X^2} (f(z_1) - f(z_2))(pr^{\#}_{X^2} pr^{\#}_n (\widehat{\mathbf{B}}^\infty_{x,y,\theta} \,|_{G_{\frac{n}{2}}}))(dz_1, dz_2)\Big| + 2\widehat{\mathbf{B}}^\infty_{x,y,\theta}(H_{\frac{n}{2}}).
\end{aligned}$$

From **A2** we obtain

$$\begin{aligned}
&\sup_{\mathcal{W}} \Big| \int_{X^2} (f(z_1) - f(z_2))(pr^{\#}_{X^2} pr^{\#}_n (\widehat{\mathbf{B}}^\infty_{x,y,\theta} \,|_{G_{\frac{n}{2}}}))(dz_1, dz_2)\Big| \\
&\le \int_{X^2} d(z_1, z_2)(pr^{\#}_{X^2} pr^{\#}_n (\widehat{\mathbf{B}}^\infty_{x,y,\theta} \,|_{G_{\frac{n}{2}}}))(dz_1, dz_2) \\
&\le \alpha^{\frac{n}{2}} \int_{X^2} d(z_1, z_2)(pr^{\#}_{X^2} pr^{\#}_{\frac{n}{2}} (\widehat{\mathbf{B}}^\infty_{x,y,\theta} \,|_{G_{\frac{n}{2}}}))(dz_1, dz_2) \le \alpha^{\frac{n}{2}} R.
\end{aligned}$$

Now Step III and Chebyshev inequality imply that

$$\widehat{\mathbf{B}}^\infty_{x,y,\theta}(H_{\frac{n}{2}}) \le \gamma^{\frac{n}{2}} C_1(L(x) + L(y) + 1) \qquad \text{for} \qquad n \in \mathbb{N}.$$

Taking $C_2 = 2C_1 + R$ and $q = \max\{\gamma^{\frac{n}{2}}, \alpha^{\frac{n}{2}}\}$ we obtain

$$\|P^{*n}\delta_x - P^{*n}\delta_y\|_{FM} \le \gamma^n C_1(L(x) + L(y) + 1) \qquad \text{for} \qquad x, y \in X, n \in \mathbb{N},$$

and so

$$\|P^{*n}\mu - P^{*n}\nu\|_{FM} \le \gamma^n C_1\Big(\int_X L(x)\mu(dx) + \int_X L(y)\nu(dy) + 1\Big) \tag{11}$$

for $\mu, \nu \in \mathcal{M}^L_1(X)$ and $n \in \mathbb{N}$.

Step V: Observe that Step IV and **A1** give

$$\begin{aligned}
&\|P^{*n}\delta_x - P^{*(n+k)}\delta_x\|_{FM} \le \int_X \|P^{*n}\delta_x - P^{*n}\delta_y\|_{FM} P^{*k}\delta_x(dy) \\
&\le q^n C_2 \int_X (L(x) + L(y)) P^{*k}\delta_x(dy) \le q^n C_3(1 + L(x)),
\end{aligned}$$

so $(P^{*n}\delta_x)_{n\in\mathbb{N}}$ is Cauchy sequence for every $x \in X$. Since $\mathcal{M}_1(X)$ equipped with norm $\|\cdot\|_{FM}$ is complete (see [8]), assumption **A0** implies the existence of invariant measure μ_*. Assumption **A1** gives $\mu_* \in \mathcal{M}_1^L(X)$. Applying inequality (11) we obtain (5). Observation that the space $\mathcal{M}_1^L(X)$ is dense in $\mathcal{M}_1(X)$ in the total variation norm finishes the proof.

□

Remark. In steps IV and V of the above proof we follow M. Hairer (see [11]).

3. Random iteration of functions

Let (X, d) be a Polish space and (Θ, Ξ) a measurable space with a family $\vartheta_x \in \mathcal{M}_1(\Theta)$ of distributions on Θ indexed by $x \in X$. Space Θ serves as a set of indices for a family $\{S_\theta : \theta \in \Theta\}$ of continuous functions acting on X into itself. We assume that $(\theta, x) \mapsto S_\theta(x)$ is product measurable. In this section we study some stochastically perturbed dynamical system $(X_n)_{n\in\mathbb{N}_0}$. Its intuitive description is following: if X_0 starts at x_0, then by choosing θ_0 at random from ϑ_{x_0} we define $X_1 = S_{\theta_0}(x_0)$. Having X_1 we select θ_1 according to the distribution ϑ_{X_1} and we put $X_2 = S_{\theta_1}(X_1)$ and so on. More precisely, the process $(X_n)_{n\in\mathbb{N}_0}$ can be written as

$$X_{n+1} = S_{Y_n}(X_n), \qquad n = 0, 1, \dots,$$

where $(Y_n)_{n\in\mathbb{N}_0}$ is a sequence of random elements defined on the probability space $(\Omega, \Sigma, prob)$ with values in Θ such that

$$prob\,(Y_n \in B | X_n = x) = \vartheta_x(B) \qquad \text{for} \qquad x \in X, B \in \Xi, n = 0, 1, \dots, \tag{12}$$

and $X_0 : \Omega \to X$ is a given random variable. Denoting by μ_n the probability law of X_n, we will give a recurrence relation between μ_{n+1} and μ_n. To this end fix $f \in B_b(X)$ and note that

$$\mathbb{E}f(X_{n+1}) = \int_X f d\mu_{n+1}.$$

But, by (12) we have

$$\int_A \vartheta_x(B)\mu_n(dx) = prob(\{Y_n \in B\} \cap \{X_n \in A\}) \qquad \text{for} \quad B \in \Xi, A \in \mathcal{B}_X,$$

hence

$$\mathbb{E}f(X_{n+1}) = \int_\Omega f(S_{Y_n(\omega)}(X_n(\omega))prob(d\omega) = \int_X \int_\Theta f(S_\theta(x))\vartheta_x(d\theta)\mu_n(dx).$$

Putting $f = \mathbf{1}_A$, $A \in \mathcal{B}_X$, we obtain $\mu_{n+1}(A) = P^*\mu_n(A)$, where

$$P^*\mu(A) = \int_X \int_\Theta \mathbf{1}_A(S_\theta(x))\vartheta_x(d\theta)\mu(dx) \qquad \text{for} \quad \mu \in \mathcal{M}_{fin}(X), A \in \mathcal{B}_X.$$

In other words this formula defines the transition operator for μ_n. Operator P^* is adjoint of the Markov operator $P : B_b(X) \to B_b(X)$ of the form

$$Pf(x) = \int_\Theta f(S_\theta(x))\vartheta_x(d\theta). \tag{13}$$

We take this formula as the precise formal definition of considered process. We will show that operator (13) has a unique invariant measure, provided the following conditions hold:

B1 *There exists* $\alpha \in (0,1)$ *such that*

$$\int_\Theta d(S_\theta(x), S_\theta(y))\vartheta_x(d\theta) \le \alpha d(x,y) \qquad \text{for} \quad x, y \in X.$$

B2 *There exists* $\bar{x} \in X$ *such that*

$$c := \sup_{x\in X} \int_\Theta d(S_\theta(\bar{x}), \bar{x})\vartheta_x(d\theta) < \infty.$$

B3 *A map* $x \mapsto \vartheta_x$, $x \in X$, *is Hölder continuous in the total variation norm, i.e. there exists* $l > 0$ and $\nu \in (0,1]$ *such that*

$$\|\vartheta_x - \vartheta_y\| \le l\, d(x,y)^\nu \qquad \text{for} \quad x, y \in X.$$

B4 *There exists* $\delta > 0$ *such that*

$$\vartheta_x \wedge \vartheta_y(\{\theta \in \Theta : d(S_\theta(x), S_\theta(y)) \le \alpha d(x,y)\}) > \delta \qquad \text{if} \quad d(x,\bar{x}) + d(y,\bar{x}) < \frac{4c}{1-\alpha},$$

where $\wedge$ *denotes the greatest lower bound in the lattice of finite measures.*

Remark. It is well known (see [15]) that replacing Hölder continuity in **B3** by slightly weaker condition of Dini continuity can lead to the lack of exponential convergence.

Proposition 3.1. *Assume* **B1** – **B4**. *Then operator* (13) *possesses a unique invariant measure* $\mu_* \in \mathcal{M}_1^1(X)$, *which is attractive in* $\mathcal{M}_1(X)$. *Moreover there exist* $q \in (0,1)$ *and* $C > 0$ *such that*

$$\|P^{*n}\mu - \mu_*\|_{FM} \le q^n C(1 + \int_X d(\bar{x}, x)\mu(dx))$$

for $\mu \in \mathcal{M}_1^1(X)$ *and* $n \in \mathbb{N}$.

Proof. Define an operator Q on $B_b(X^2)$ by

$$Q(f)(x,y) = \int_\Theta f(S_\theta(x), S_\theta(y))\vartheta_x \wedge \vartheta_y(d\theta).$$

Since

$$||\vartheta_{x'} \wedge \vartheta_{y'} - \vartheta_x \wedge \vartheta_y|| \le 2(||\vartheta_{x'} - \vartheta_x|| + ||\vartheta_{y'} - \vartheta_y||)$$

it follows that

$$\begin{aligned}|Q(f)(x',y')-Q(f)(x,y)| &\le \int_\Theta |f(S_\theta(x'),S_\theta(y'))|\,||\vartheta_{x'}\wedge\vartheta_{y'}-\vartheta_x\wedge\vartheta_y||(d\theta)\\ &+\int_\Theta |f(S_\theta(x'),S_\theta(y'))-f(S_\theta(x),S_\theta(y))|\vartheta_x\wedge\vartheta_y(d\theta)\\ &\le 2l\sup_{z\in X^2}|f(z)|\,(d(x,x')^\nu+d(y,y')^\nu)\\ &+\int_\Theta |f(S_\theta(x'),S_\theta(y'))-f(S_\theta(x),S_\theta(y))|\vartheta_x\wedge\vartheta_y(d\theta),\end{aligned}$$

for $f\in B_b(X^2)$, $x,y\in X$. Consequently, we see that $Q(C_b(X^2))\subset C_b(X^2)$, by Lebesgue's dominated convergence theorem. Put

$$\mathcal{F}=\{f\in B_b(X^2):\sup_{z\in X^2}|f(z)|\le M, Q(f)\in B_b(X^2)\},$$

where $M>0$ is fixed, and observe that the family $\mathcal{F}$ is closed in pointwise convergence. Therefore $\mathcal{F}$ consists the class of Baire functions bounded by M. By virtue of [17, Theorem 4.5.2] we obtain $Q(B_b(X^2))\subset B_b(X^2)$. In particular, for the family $\{Q_{x,y}:x,y\in X\}$ of (sub-probabilistic) measures given by

$$Q_{x,y}(C)=\int_\Theta \mathbf{1}_C(S_\theta(x),S_\theta(y))\vartheta_x\wedge\vartheta_y(d\theta),$$

we have that maps $(x,y)\mapsto Q_{x,y}(C)$ are measurable for every $C\in\mathcal{B}_{X^2}$.

Arguing similarly as above we show that (13) is well defined Feller operator. It has Lapunov function $L(x)=d(x,\bar{x})$, since

$$\int_\Theta d(S_\theta(x),\bar{x})\vartheta_x(d\theta)\le\alpha d(x,\bar{x})+c.$$

Now, observe that

$$\|Q_{x,y}\|=\vartheta_x\wedge\vartheta_y(\Theta)=1-\sup_{A\in\Theta}\{\vartheta_y(A)-\vartheta_x(A)\}\ge 1-l\,d(x,y)^\nu$$

for $x,y\in X$. Moreover, we have

$$\int_{X^2} d(u,v)Q_{x,y}(du,dv)=\int_\Theta d(S_\theta(x),S_\theta(y))\vartheta_x\wedge\vartheta_y(d\theta)\le\alpha d(x,y),$$

and

$$Q_{x,y}(D_{\alpha d(x,y)})=\vartheta_x\wedge\vartheta_y(\{\theta\in\Theta:d(S_\theta(x),S_\theta(y))\le\alpha d(x,y)\})>\delta$$

whenever $d(x,\bar{x})+d(y,\bar{x})<\frac{4c}{1-\alpha}$. In consequence **A0** – **A3** are fulfilled. The use of Theorem 2.1 (see also Remark concerning assumption **A4**) ends the proof. □

4. Perpetuities with place dependent probabilities

Let $X = \mathbb{R}^d$ and $G = \mathbb{R}^{d\times d} \times \mathbb{R}^d$, and consider a function $S_\theta : X \to X$ defined by $S_\theta(x) = M(\theta)x + Q(\theta)$, where (M, Q) is a random variable on (Θ, Ξ) with values in G. Then (13) may be written as

$$Pf(x) = \int_G f(mx + q)d\vartheta_x \circ (M, Q)^{-1}(m, q) \tag{14}$$

This operator is connected with random difference equation of the form

$$\Phi_n = M_n\Phi_{n-1} + Q_n, \qquad n = 1, 2, \ldots, \tag{15}$$

where $(M_n, Q_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables distributed as (M, Q). Namely, the process $(\Phi_n)_{n\in\mathbb{N}_0}$ is a homogeneous Markov chain with transition kernel P given by

$$Pf(x) = \int_G f(mx + q)d\mu(m, q), \tag{16}$$

where μ stands for a distribution of (M, Q). Equation (15) arises in various disciplines as economics, physics, nuclear technology, biology, sociology (see e.g. [23]). It is closely related to a sequence of backward iterations $(\Psi_n)_{n\in\mathbb{N}}$, given by $\sum_{k=1}^{n} M_1 \ldots M_{k-1}Q_k, n \in \mathbb{N}$ (see e.g. [9]). Under conditions ensuring the almost sure convergence of the sequence $(\Psi_n)_{n\in\mathbb{N}}$ the limiting random variable

$$\sum_{n=1}^{\infty} M_1 \ldots M_{n-1}Q_n \tag{17}$$

is often called *perpetuity*. It turns out that the probability law of (17) is a unique invariant measure for (16). The name perpetuity comes from perpetual payment streams and recently gained some popularity in the literature on stochastic recurrence equations (see [7]). In the insurance context a perpetuity represents the present value of a permanent commitment to make a payment at regular intervals, say annually, into the future forever. The Q_n represent annual payments, the M_n cumulative discount factors. Many interesting examples of perpetuities can be found in [1]. Due to significant papers [14], [10], [23] and [9] we have complete (in the dimension one) characterization of convergence of perpetuities. The rate of this convergence has recently been extensively studied by many authors (see for instance [3]-[5], [18]). The main result of this section concerns the rate of the convergence of the process $(X_n)_{n\in\mathbb{N}_0}$ associated with an operator $P : B_b(\mathbb{R}^d) \to B_b(\mathbb{R}^d)$ given by

$$Pf(x) = \int_G f(mx + q)d\mu_x(m, q), \tag{18}$$

where $\{\mu_x : x \in \mathbb{R}^d\}$ is a family of Borel probability measures on G. In contrast to $(\Phi_n)_{n\in\mathbb{N}_0}$, the process $(X_n)_{n\in\mathbb{N}_0}$ moves by choosing at random θ from a measure depending on x. Taking into considerations the concept of perpetuities we may say that $(X_n)_{n\in\mathbb{N}_0}$ forms *a perpetuity with place dependent probabilities.*

Corollary 4.1. *Assume that $\{\mu_x : x \in \mathbb{R}^d\}$ is a family of Borel probability measures on G such that*[1]

$$\alpha := \sup_{x\in\mathbb{R}^d} \int_G ||m|| d\mu_x(m,q) < 1, \qquad c := \sup_{x\in\mathbb{R}^d} \int_G |q| d\mu_x(m,q) < \infty. \quad (19)$$

Assume moreover that a map $x \mapsto \mu_x$, $x \in X$, is Hölder continuous in the total variation norm and there exists $\delta > 0$ such that

$$\mu_x \wedge \mu_y(\{(m,q) \in G : ||m|| \le \alpha\}) > \delta \qquad \text{if} \quad |x| + |y| < \frac{4c}{1-\alpha}.$$

Then operator (18) *possesses a unique invariant measure $\mu_* \in \mathcal{M}_1^1(\mathbb{R}^d)$, which is attractive in $\mathcal{M}_1(\mathbb{R}^d)$. Moreover there exist $q \in (0,1)$ and $C > 0$ such that*

$$\|P^{*n}\mu - \mu_*\|_{FM} \le q^n C(1 + \int_{\mathbb{R}^d} |x| \mu(dx))$$

for $\mu \in \mathcal{M}_1^1(\mathbb{R}^d)$ and $n \in \mathbb{N}$.

The proof of corollary is straightforward application of Proposition 3.1. We leave the details to the reader. We finish the paper by giving an example to illustrate Corollary 4.1.

Example. Let ν_0, ν_1 be distributions on $\mathbb{R}^2$. Assume that $p, q : \mathbb{R} \to [0,1]$ are Lipschitz functions (with Lipschitz constant L) summing up to 1, and $p(x) = 1$, for $x \le 0$, $p(x) = 0$, for $x \ge 1$. Define μ_x by

$$\mu_x = p(x)\nu_0 + q(x)\nu_1, \qquad x \in \mathbb{R}.$$

Then:

(1) $\|\mu_x - \mu_y\| \le 2L|x-y| \qquad \text{for} \quad x, y \in \mathbb{R}$.

(2) If $\int_{\mathbb{R}^2} |m| d\nu_i(m,q) < 1$ and $\int_{\mathbb{R}^2} |q| d\nu_i(m,q) < \infty$ for $i = 0, 1$, then (19) holds.

(3) For every $A \in \mathcal{B}_{\mathbb{R}^2}$, $x, y \in \mathbb{R}$ we have: $\mu_x \wedge \mu_y(A) \ge \nu_0 \wedge \nu_1(A) = (\nu_0 - \lambda^+)(A) = (\nu_1 - \lambda^-)(A) \ge \max\{\nu_0(A), \nu_1(A)\} - \|\nu_0 - \nu_1\|(A)$, where (λ^+, λ^-) is a Jordan decomposition of $\nu_1 - \nu_0$.

[1] $||m|| = \sup\{|mx| : x \in \mathbb{R}^d, |x| = 1\}$, and $|\cdot|$ is Euclidean norm in $\mathbb{R}^d$

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