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# TWISTED FROBENIUS–SCHUR INDICATORS FOR HOPF ALGEBRAS

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Abstract. The classical Frobenius–Schur indicators for finite groups are character sums defined for any representation and any integer $m \geq 2$. In the familiar case $m = 2$, the Frobenius–Schur indicator partitions the irreducible representations over the complex numbers into real, complex, and quaternionic representations. In recent years, several generalizations of these invariants have been introduced. Bump and Ginzburg, building on earlier work of Mackey, have defined versions of these indicators which are twisted by an automorphism of the group. In another direction, Linchenko and Montgomery have defined Frobenius–Schur indicators for semisimple Hopf algebras. In this paper, the authors construct twisted Frobenius–Schur indicators for semisimple Hopf algebras; these include all of the above indicators as special cases and have similar properties.

## 1. Introduction

Classically, the Frobenius–Schur indicator of a character of a finite group is the character evaluated at the sum of squares of the group elements divided by the order of the group. This indicator was introduced by Frobenius and Schur in their investigation of real representations. Indeed, they showed that the only possible values for an irreducible representation are 1, 0, and $-1$, corresponding to the partition of the irreducible representations into real, complex, and quaternionic representations [FS06]. Higher order versions can be obtained by replacing squares with other powers of group elements.

In recent years, there has been increasing interest in various generalizations of these invariants. In one direction, Bump and Ginzburg [BG04], building on earlier work of Mackey [Mac58] and Kawanaka-Matsuyama [KM90], have defined versions of Frobenius–Schur indicators which are twisted by an automorphism of the group. These indicators have applications to the study of multiplicity-free permutation representations, models for finite groups (in the sense of [BGG76]), and Shintani lifting of characters of finite reductive groups.

Another direction involves extending the theory from finite groups to suitable Hopf algebras. In 2000, Linchenko and Montgomery constructed Frobenius–Schur indicators for semisimple Hopf algebras over an algebraically closed field of characteristic zero and proved that the second indicator again only takes the values 0 or $\pm 1$ on irreducible representations [LM00]. The higher indicators were further

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studied by Kashina, Sommerhäuser, and Zhu [KSZ06]. These Frobenius–Schur indicators have been useful in classifying Hopf algebras [Kas03, NS08] and studying their representations [KSZ02]. More exotically, they arise in conformal field theory [Ban97, Ban00].

The goal of this paper is to construct twisted Frobenius–Schur indicators for semisimple Hopf algebras over an algebraically closed field of characteristic zero that include all of the above indicators as special cases and have similar properties. Given an automorphism of order $n$ of such a Hopf algebra, we define the $m^{th}$ *twisted Frobenius–Schur* indicator for $m$ any positive multiple of $n$. This definition is given in Section 2. In the next section, we consider the case of automorphisms of order at most two. We show that the second twisted Frobenius–Schur indicator gives rise to a partition of the simple modules into three classes; this partition involves the relationship between the module and its "twisted dual" (Theorem 3.3). In Section 4, we show that the $m^{th}$ twisted Frobenius–Schur indicator can be realized as the trace of an endomorphism of order $m$ (Theorem 4.7), so that the indicator is a cyclotomic integer. Finally, we compute a closed formula for the twisted indicator of the regular representation (Theorem 4.9).

## 2. Definition

Let $k$ be an algebraically closed field of characterisic 0, and let $H$ be a semisimple Hopf algebra over $k$ with comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$. The Hopf algebra $H$ contains a unique two-sided integral $\Lambda$ normalized so that $\varepsilon(\Lambda) = 1$. We will use the usual Sweedler notation for iterated comultiplication: $\Delta^{m-1}(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_m$. All $H$-modules considered will be finite-dimensional.

We are now ready to define the twisted indicators. Let $\tau$ be an automorphism of $H$ such that $\tau^m = \mathrm{Id}$ for some $m \in \mathbf{N}$. Let $(V, \rho)$ be an $H$-module with corresponding character $\chi$.

**Definition 2.1.** The *m-th twisted Frobenius–Schur indicator of* $(V, \rho)$ (or $\chi$) is defined to be the character sum

$$\nu_m(\chi, \tau) = \sum_{(\Lambda)} \chi\left(\Lambda_1 \tau\left(\Lambda_2\right) \cdots \tau^{m-1}\left(\Lambda_m\right)\right). \tag{2.1}$$

We note that this is only defined for $m$ divisible by the order of $\tau$. We will write $\widetilde{\nu}_m(\chi)$ instead of $\nu_m(\chi, \tau)$ when this does not cause confusion.

If $\tau = 1$, this formula coincides with the definition of Linchenko and Montgomery [LM00]. Moreover, suppose $H = k[G]$ for a finite group $G$. In this case, $\Lambda = \dfrac{1}{|G|} \sum_{g \in G} g$, and we recover Bump and Ginzburg's twisted Frobenius–Schur indicators for groups [BG04].

## 3. Twisted second Frobenius–Schur indicators

In this section, we will show that the second twisted Frobenius–Schur indicator gives rise to a partition of the irreducible $H$-modules into three classes, depending on the relationship between the module and its *twisted dual*. We also compute the indicators for all automorphisms of $H_8$–the smallest semisimple Hopf algebra that is neither commutative nor cocommutative.

3.1. **Twisted duals and the partition of the simple modules.** Let $\tau$ be an automorphism such that $\tau^2 = \mathrm{Id}$. We will let $T = \tau S$ denote the corresponding anti-involution. Let $(V, \rho)$ be a finite dimensional left $H$-module with character $\chi$. Using (2.1) for $m = 2$, we have

$$\widetilde{\nu}_2(\chi) = \sum_{(\Lambda)} \chi\left(\Lambda_1 TS(\Lambda_2)\right).$$

**Definition 3.1.** The *twisted dual $H$-module of $V$* is the dual space $V^*$ equipped with an $H$-module structure given by

$$(h \cdot f)(v) = f(T(h) \cdot v),$$

for all $h \in H, f \in V^*$ and $v \in V$. We denote it by $({}^*V, \widetilde{\rho})$.

**Proposition 3.2.** *The twisted dual $H$-module $({}^*V, \widetilde{\rho})$ satisfies $\widetilde{\rho}(h) = \rho(T(h))^t$.*

*Proof.* For $h \in H, f \in {}^*V$ and $v \in V$ we have,

$$\begin{aligned} \rho(T(h))^t(f)(v) &= f(\rho(T(h))(v)) \\ &= \widetilde{\rho}(h)f(v) \end{aligned}$$

Thus, $\rho(T(h))^t = \widetilde{\rho}(h)$, as required. □

We can now state the main theorem of this section.

**Theorem 3.3.** *Let $V$ be an irreducible representation with character $\chi$. Then the following properties hold:*

*(1)* $\widetilde{\nu}_2(\chi) = 0, 1,$ *or* $-1, \forall \chi \in \mathrm{Irr}(H)$.
*(2)* $\widetilde{\nu}_2(\chi) \neq 0$ *if and only if* $V \cong {}^*V$. *Moreover,* $\widetilde{\nu}_2(\chi) = 1$ *(resp.* $-1$*) if and only if there is a symmetric (resp. skew-symmetric) nonzero intertwining map* $V \to {}^*V$.

*Remark* 3.4. This result is well-known in two special cases. If we let $T = S$ (i.e., $\tau = \mathrm{Id}$), then we recover Theorem 3.1 in [LM00]. On the other hand, when $H$ is a group algebra, this is a theorem of Sharp [Sha60] and Kawanaka-Matsuyama [KM90]. See also [KS08].

We will some preliminary results before proving the theorem.

For future reference, we recall the orthogonality relations for irreducible characters. If the irreducible characters of $H$ are given by $\chi_1, \ldots, \chi_n$, then

$$\sum_{(\Lambda)} \chi_i\left(\Lambda_1\right) \chi_j\left(S\left(\Lambda_2\right)\right) = \delta_{ij}. \tag{3.1}$$

Equation (3.1) is the dual statement of Theorem 7.5.6 in [DNR00].

**Proposition 3.5.** *There is a canonical isomorphism of $H$-modules* $V \xrightarrow{\Psi} {}^{**}V$.

*Proof.* Let $\Psi : V \to {}^{**}V$ be the usual evaluation map, $\Psi(v)(f) = f(v)$. Then, $\Psi$ is a linear isomorphism. It remains to show that $\Psi$ is an $H$-map. For all $h \in H, f \in {}^*V$ and $v \in V$ we have,

$$\begin{aligned} (h \cdot \Psi(v))(f) &= \Psi(v)(T(h) \cdot f) \\ &= (T(h) \cdot f)(v) \\ &= f(T^2(h) \cdot v) \\ &= f(h \cdot v) \\ &= \Psi(h \cdot v)(f). \end{aligned}$$

Therefore, $h \cdot \Psi(v) = \Psi(h \cdot v)$. □

**Definition 3.6.** If $f : V \to W$ is a morphism of finite dimensional left $H$-modules, define ${}^*f : {}^*W \to {}^*V$ by $({}^*f(\beta))(v) = \beta(f(v))$, for $\beta \in {}^*W, v \in V$.

**Proposition 3.7.** *Twisted duality if an involutory auto-equivalence of the category of $H$-modules.*

*Proof.* We first check that $*f$ is a morphism of $H$-modules. Given $\alpha \in {}^*V$, $h \in H$, and $v \in V$, we have

$$\begin{aligned} {}^*f(h \cdot \alpha)(v) &= (h \cdot \alpha)(f(v)) \\ &= \alpha(T(h) \cdot f(v)) \\ &= \alpha(f(T(h) \cdot v)) \\ &= {}^*f(\alpha)(T(h) \cdot v) \\ &= (h \cdot {}^*f(\alpha))(v). \end{aligned}$$

It is now obvious that twisted duality is a functor. The fact that is an involutory auto-equivalence follows from Proposition 3.5. □

**Proposition 3.8.** *Let $V$ be a simple left $H$-module. Then ${}^*V$ is also simple.*

*Proof.* Let $X$ be a submodule of ${}^*V$, and $f : X \to {}^*V$ be the inclusion map. Then, ${}^*f : {}^{**}V \to {}^*X$ is a surjective morphism of left $H$-modules. We know that $V \simeq {}^{**}V$, so since $V$ is simple, it follows that ${}^{**}V$ is also simple. This implies that ${}^*f = 0$ or ${}^*f$ is an isomorphism. If ${}^*f = 0$, then ${}^*X = 0$, so $X = 0$. In the latter case, it follows that $f = \mathrm{Id}$ and $X = {}^*V$. Thus, $X = 0$ or $X = {}^*V$, so ${}^*V$ is simple. □

**Lemma 3.9.** *If $f \in \mathrm{Hom}_H({}^*V, V)$, then $f^t = \Psi^{-1} \circ {}^*f : {}^*V \to V$ is an $H$-map.*

*Proof.* We know that ${}^*f$ and $\Psi$ are both $H$-maps, so it follows that so is $f^t$. □

The standard decomposition of $\mathrm{Hom}({}^*V, V)$ into symmetric and anti-symmetric linear maps is no longer an $H$-decomposition. In fact, this is not true even when $\tau = \mathrm{Id}$ unless $H$ is cocommutative. However, the corresponding decomposition does hold for $H$-invariant maps.

It is well known (for example, see [Sch95, p. 42]) that

$$\mathrm{Hom}({}^*V, V)^H = \mathrm{Hom}_H({}^*V, V).$$

This means that

$$\mathrm{Sym}_H({}^*V, V) = \left\{ f \in \mathrm{Hom}_H({}^*V, V) | f^t = f \right\}$$

and

$$\mathrm{Alt}_H({}^*V, V) = \left\{ f \in \mathrm{Hom}_H({}^*V, V) | f^t = -f \right\}$$

are $H$-submodules.

**Proposition 3.10.** *Let $V$ be a finite dimensional simple left $H$-module. Then,*

$$\mathrm{Hom}_H({}^*V, V) = \mathrm{Sym}_H({}^*V, V) \oplus \mathrm{Alt}_H({}^*V, V).$$

*Proof.* Take $f \in \mathrm{Hom}_H({}^*V, V)$. The previous lemma shows that $f^t$ is an $H$-map, so

$$f = \frac{f + f^t}{2} + \frac{f - f^t}{2},$$

with $\frac{f+f^t}{2} \in \mathrm{Sym}_H({}^*V, V)$ and $\frac{f-f^t}{2} \in \mathrm{Alt}_H({}^*V, V)$. Since

$$\mathrm{Sym}_H({}^*V, V) \cap \mathrm{Alt}_H({}^*V, V) = \{0\}\,,$$

the result follows. □

Combining the proposition with Schur's lemma immediately gives:

**Corollary 3.11.** *If $V$ is a simple $H$-module, then*

$$\dim \mathrm{Sym}_H({}^*V, V) - \dim \mathrm{Alt}_H({}^*V, V) \in \{1, 0, -1\}\,.$$

*Moreover, it takes each value according to the conditions given in Theorem 3.3.*

Theorem 3.3 is now a consequence of the following proposition.

**Proposition 3.12.** *Let $V$ be a simple $H$-module with character $\chi$. Then*

$$\widetilde{\nu}_2\,(\chi) = \dim \mathrm{Sym}_H({}^*V, V) - \dim \mathrm{Alt}_H({}^*V, V). \tag{3.2}$$

*Proof.* We will compute matrix elements in terms of a fixed basis for $V$ and the dual basis for ${}^*V$. This means that elements of $\mathrm{Sym}({}^*V, V)$ (resp. $\mathrm{Alt}({}^*V, V)$) will be symmetric (resp. skew-symmetric). We temporarily denote the expression on the right side of (3.2) by $q(\chi)$.

Writing out $\widetilde{\nu}_2(\chi)$ gives

$$\begin{aligned}
\widetilde{\nu}_2\,(\chi) &= \sum_{(\Lambda)} \chi\left(\Lambda_1 T(S(\Lambda_2))\right) \\
&= \sum_{(\Lambda)} \mathrm{tr}\left(\rho(\Lambda_1)\rho(T(S(\Lambda_2)))\right) \\
&= \sum_{m,m'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'}\rho(T(S(\Lambda_2)))_{m'm} \\
&= \sum_{m,m'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'}\rho(T(S(\Lambda_2)))^t_{mm'} \\
&= \sum_{m,m'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'}\widetilde{\rho}(S(\Lambda_2))_{mm'}.
\end{aligned}$$

If $V \not\cong {}^*V$, then this expression is 0 by the orthogonality relations (3.1), hence coincides with $q(\chi)$. Otherwise, there exists a nonzero intertwiner $\varphi \in \mathrm{Hom}_H({}^*V, V)$, so that $\widetilde{\rho}(h) = \varphi^{-1}\rho(h)\varphi$. By Proposition 3.10, $\varphi$ is symmetric or skew-symmetric, and in fact, $\varphi_{mn} = q(\chi)\varphi_{nm}$ for all $n, m$. Applying these two facts to the previous equation, we have

$$\begin{aligned}
\widetilde{\nu}_2(\chi) &= \sum_{m,m'}\sum_{(\Lambda)}\rho(\Lambda_1)_{mm'}\widetilde{\rho}(S(\Lambda_2))_{mm'} \\
&= \sum_{m,m'}\sum_{(\Lambda)}\rho(\Lambda_1)_{mm'}\left(\varphi^{-1}\rho(S(\Lambda_2))\varphi\right)_{mm'} \\
&= \sum_{m,m',n,n'}\left(\varphi^{-1}\right)_{mn}\varphi_{n'm'}\sum_{(\Lambda)}\rho(\Lambda_1)_{mm'}\rho(S(\Lambda_2))_{nn'} \\
&= \sum_{m,m',n,n'}\left(\varphi^{-1}\right)_{mn}\varphi_{n'm'}\frac{\delta_{m',n}\delta_{m,n'}}{\dim V} \\
&= \sum_{m,n}\left(\varphi^{-1}\right)_{mn}\varphi_{mn}\frac{1}{\dim V} \\
&= \frac{q(\chi)}{\dim V}\sum_{m,n}\left(\varphi^{-1}\right)_{mn}\varphi_{nm} = q(\chi),
\end{aligned}$$

as desired. The fourth equality follows from the orthogonality relations for matrix elements given in [Lar71].

□

3.2. **The second twisted Frobenius–Schur indicators for $H_8$.** The smallest semisimple Hopf algebra which is neither commutative nor cocommutative has dimension 8. We denote it by $H_8$. As an algebra, $H_8$ is generated by elements $x$, $y$ and $z$, with relations:

$$x^2 = y^2 = 1,\ z^2 = \frac{1}{2}\left(1 + x + y - xy\right),\ xy = yx,\ xz = zy,\ \text{and } yz = zx.$$

The coalgebra structure of $H_8$ is given by the following:

$$\Delta(x) = x \otimes x,\ \varepsilon(x) = 1,\ \text{and } S(x) = x,$$
$$\Delta(y) = y \otimes y,\ \varepsilon(y) = 1,\ \text{and } S(y) = y,$$
$$\Delta(z) = \frac{1}{2}\left(1\otimes 1 + 1\otimes x + y\otimes 1 - y\otimes x\right)\left(z\otimes z\right),$$
$$\varepsilon(z) = 1,\ \text{and } S(z) = z.$$

The normalized integral is given by

$$\Lambda = \frac{1}{8}\left(1 + x + y + xy + z + xz + yz + xyz\right).$$

This Hopf algebra was first introduced by Kac and Paljutkin [KP66] and revisited later by Masuoka [Mas95].

The Hopf algebra $H_8$ has 4 one-dimensional representations and a single two-dimensional simple module. The characters for the irreducible representations of $H_8$ are listed in Table 1.

The automorphism group of $H_8$ is the Klein four-group. These automorphisms are given in Table 2.

All four automorphisms satisfy $\tau^2 = \mathrm{Id}$, so the second twisted Frobenius–Schur indicator is defined for all of them. These indicators are given in Table 3.

| | $1$ | $x$ | $y$ | $xy$ | $z$ | $xz$ | $yz$ | $xyz$ |
|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\chi_2$ | $1$ | $1$ | $1$ | $1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $\chi_3$ | $1$ | $-1$ | $-1$ | $1$ | $i$ | $-i$ | $-i$ | $i$ |
| $\chi_4$ | $1$ | $-1$ | $-1$ | $1$ | $-i$ | $i$ | $i$ | $-i$ |
| $\chi_5$ | $2$ | $0$ | $0$ | $-2$ | $0$ | $0$ | $0$ | $0$ |

TABLE 1. Characters for the Irreducible Representations of $H_8$

| | $1$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $\tau_1 = \mathrm{Id}$ | $1$ | $x$ | $y$ | $z$ |
| $\tau_2$ | $1$ | $x$ | $y$ | $xyz$ |
| $\tau_3$ | $1$ | $y$ | $x$ | $\frac{1}{2}(z + xz + yz - xyz)$ |
| $\tau_4$ | $1$ | $y$ | $x$ | $\frac{1}{2}(-z + xz + yz + xyz)$ |

TABLE 2. Automorphisms of $H_8$

| | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ | $\chi_5$ |
|---|---|---|---|---|---|
| $\nu_2(\chi, \tau_1) = \nu_2(\chi)$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\nu_2(\chi, \tau_2)$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\nu_2(\chi, \tau_3)$ | $1$ | $1$ | $0$ | $0$ | $1$ |
| $\nu_2(\chi, \tau_4)$ | $1$ | $1$ | $0$ | $0$ | $-1$ |

TABLE 3. Twisted Frobenius–Schur indicators for $H_8$

## 4. Higher order twisted Frobenius–Schur indicators

We now return to the general case. When $m > 2$, it is no longer true that the higher order twisted Frobenius–Schur indicators are integers. However, it is true that they are cyclotomic integers. We will show this by realizing $\widetilde{\nu}_m(\chi)$ as the trace of an endomorphism of order $m$. We will also compute a closed formula for the regular representation.

4.1. **A trace formula.** Let $(\widetilde{V^{\otimes m}}, \widetilde{\rho^m})$ be the $H$-module with underlying vector space $V^{\otimes m}$ and action given by

$$\widetilde{\rho^m}(h)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = \sum_h \rho(h_1) v_1 \otimes \rho(\tau(h_2)) v_2 \otimes \cdots \otimes \rho\left(\tau^{m-1}(h_m)\right) v_m.$$

Furthermore, let $\alpha : V^{\otimes m} \to V^{\otimes m}$ be defined by

$$\alpha(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_2 \otimes \cdots \otimes v_m \otimes v_1.$$

**Lemma 4.1.**

$$\widetilde{\nu}_m(\chi) = \mathrm{tr}_{V^{\otimes m}}\left(\alpha \circ \widetilde{\rho^m}(\Lambda)\right).$$

*Proof.*

$$\begin{aligned}\widetilde{\nu}_m(\chi) &= \sum_{(\Lambda)} \chi\left(\Lambda_1 \tau\left(\Lambda_2\right) \cdots \tau^{m-1}\left(\Lambda_m\right)\right) \\ &= \sum_{(\Lambda)} \operatorname{tr}_V\left(\rho\left(\Lambda_1\right) \rho\left(\tau\left(\Lambda_2\right)\right) \cdots \rho\left(\tau^{m-1}\left(\Lambda_m\right)\right)\right) \\ &= \operatorname{tr}_{V^{\otimes m}}\left(\alpha \circ\left(\rho \otimes \rho \tau \otimes \cdots \otimes \rho\left(\tau^{m-1}\right)\right) \Lambda\right) \\ &= \operatorname{tr}_{V^{\otimes m}}\left(\alpha \circ \widetilde{\rho^m}(\Lambda)\right).\end{aligned}$$

The third equality uses [KSZ02, Lemma2.3]. □

It is well known that the integral $\Lambda$ in $H$ is cocommutative, i.e.,

$$\Delta(\Lambda)=\sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2=\sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_1. \tag{4.1}$$

More generally, we have

**Proposition 4.2.** *For any $m \in \mathbb{N}$, $\Delta^m(\Lambda)$ is invariant under cyclic permutions:*

$$\Delta^m(\Lambda)=\sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_{m+1}=\sum_{(\Lambda)} \Lambda_2 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_{m+1} \otimes \Lambda_1.$$

*Proof.* The case $m=1$ is equation (4.1). Suppose the result is true for $m-1$, that is,

$$\Delta^{m-1}(\Lambda)=\sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_m=\sum_{(\Lambda)} \Lambda_2 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_1.$$

Then,

$$\begin{aligned}\Delta^m(\Lambda) &= \left(\Delta \otimes I^{\otimes(m-1)}\right)\left(\Delta^{m-1}(\Lambda)\right) \\ &= \left(\Delta \otimes I^{\otimes(m-1)}\right)\left(\sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_3 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_1\right) \\ &= \sum_{(\Lambda)} \Delta(\Lambda_2) \otimes \Lambda_3 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_1 \\ &= \sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_3 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_{m+1} \otimes \Lambda_1.\end{aligned}$$

□

**Proposition 4.3.** *Let $\sigma$ be an automorphism of $H$. If $h \in H$ is a left integral, then so is $\sigma(h)$.*

*Proof.* If $h \in H$ is a left integral, then

$$xh=\varepsilon(x)h$$

for all $x \in H$. Applying $\sigma$, we get

$$\sigma(xh)=\sigma(\varepsilon(x)h)=\varepsilon(x)\sigma(h), \tag{4.2}$$

for all $x \in H$. Let $y=\sigma(x)$, so that $x=\sigma^{-1}(y)$. Applying this change of variables, we can rewrite equation (4.2) as

$$y\sigma(h)=\varepsilon(\sigma^{-1}(y))\sigma(h)=\varepsilon(y)\sigma(h)$$

for all $y \in H$. Thus, $\sigma(h)$ is a left integral.

□

An analogous proof can be used to show that if $h$ is a right integral, then so is $\sigma(h)$.

**Corollary 4.4.** *If $\sigma$ is an automorphism of $H$, then $\sigma(\Lambda) = \Lambda$.*

*Proof.* We know that $\Lambda$ is the unique integral such that $\varepsilon(\Lambda) = 1$. However, $\sigma(\Lambda)$ is another integral satisfying $\varepsilon(\sigma(\Lambda)) = \varepsilon(\Lambda) = 1$.

□

**Lemma 4.5.**

$$\sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) = \sum_{(\Lambda)} \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) \otimes \Lambda_1.$$

*Proof.* By the previous corollary, $\Delta^{m-1}(\Lambda) = \Delta^{m-1}\left(\tau^{m-1}(\Lambda)\right)$. Since $\tau^{m-1}$ is a coalgebra morphism, we get

$$\begin{aligned}\sum_{(\Lambda)} \Lambda_1 \otimes \cdots \otimes \Lambda_m &= \sum_{(\Lambda)} \tau^{-1}(\Lambda)_1 \otimes \cdots \otimes \tau^{-1}(\Lambda)_m \\ &= \sum_{(\Lambda)} \tau^{-1}(\Lambda_1) \otimes \cdots \otimes \tau^{-1}(\Lambda_m).\end{aligned}$$

Combining this equation with Proposition 4.2, we get

$$\sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_3 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_1 = \sum_{(\Lambda)} \tau^{-1}(\Lambda_1) \otimes \tau^{-1}(\Lambda_2) \otimes \cdots \otimes \tau^{-1}(\Lambda_m).$$

Applying $\left(\tau \otimes \tau^2 \otimes \cdots \otimes \tau^m\right)$, we obtain

$$\sum_{(\Lambda)} \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) \otimes \Lambda_1 = \sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2) \cdots \otimes \tau^{m-1}(\Lambda_m),$$

as desired.

□

It is well-known that the action of $\Lambda$ on an $H$-module $W$ gives a projection onto its invariants. Let $\pi : \widetilde{V^{\otimes m}} \to \left(\widetilde{V^{\otimes m}}\right)^H$ defined by $\pi(w) = \Lambda \cdot w$ be this projection for $W = \widetilde{V^{\otimes m}}$.

**Proposition 4.6.** *The endomorphism $\alpha$ restricts to an endomorphism of $\left(\widetilde{V^{\otimes m}}\right)^H$.*

*Proof.* It is enough to show that $(\pi \circ \alpha)(w) = (\alpha \circ \pi)(w)$ for $w = v_1 \otimes \cdots \otimes v_m$.

Computing gives

$$\begin{aligned}(\pi \circ \alpha)(w) &= (\pi \circ \alpha)(v_1 \otimes \cdots \otimes v_m) \\ &= \pi(v_2 \otimes \cdots \otimes v_m \otimes v_1) \\ &= \sum_{(\Lambda)} \rho(\Lambda_1) v_2 \otimes \rho(\tau(\Lambda_2)) v_3 \otimes \cdots \otimes \rho\left(\tau^{m-1}(\Lambda_m)\right) v_1,\end{aligned}$$

and

$$\begin{aligned} (\alpha \circ \pi)(v) &= \alpha(\Lambda \cdot (v_1 \otimes \cdots \otimes v_m)) \\ &= \alpha\left(\sum_{(\Lambda)} \rho(\Lambda_1))v_1 \otimes \rho(\tau(\Lambda_2))v_2 \otimes \cdots \otimes \rho(\tau^{m-1}(\Lambda_m))v_m\right) \\ &= \sum_{(\Lambda)} \rho(\tau(\Lambda_2))v_2 \otimes \cdots \otimes \rho(\tau^{m-1}(\Lambda_m))v_m \otimes \rho(\Lambda_1)v_1). \end{aligned}$$

By Lemma 4.5, these two expressions are equal. □

**Theorem 4.7.**

$$\widetilde{\nu}_m(\chi) = \operatorname{tr}\left(\alpha|_{\left(\widetilde{V^{\otimes m}}\right)^H}\right).$$

*Proof.* By Proposition 4.6, the image of $\alpha \circ \widetilde{\nu}(h)$ is contained in $\left(\widetilde{V^{\otimes m}}\right)^H$. Moreover, its restriction to $\left(\widetilde{V^{\otimes m}}\right)^H$ coincides with the restriction of $\alpha$. The result now follows by Lemma 4.1.

□

**Corollary 4.8.** *Let $\zeta_m$ be a primitive $m$-th root of 1, then*

$$\widetilde{\nu}_m(\chi) \in \mathbb{Z}[\zeta_m].$$

*Proof.* The operator $\alpha$ is of order $m$, so its eigenvalues are $m^{th}$ roots of unity. It is now immediate from the theorem that the twisted indicators are cyclotomic integers. □

4.2. **The regular representation.** We now realize the twisted Frobenius–Schur indicators of the regular representation as the trace of an explicit linear endomorphism of $H$. Let $\chi_R$ denote the character of the left regular representation.

Let $\Omega^\tau_m : H \to H$ be the linear map defined by

$$\Omega^\tau_m(h) = \sum_{(h)} S\left(\tau^{m-1}(h_1)\tau^{m-2}(h_2)\cdots\tau^2(h_{m-2})\tau(h_{m-1})\right).$$

**Theorem 4.9.**

$$\widetilde{\nu}_m(\chi_R) = \operatorname{tr}(\Omega^\tau_m).$$

We will need two lemmas.

**Lemma 4.10.**

$$\sum_{(\Lambda)} \Lambda_1 h^1 \otimes \tau(\Lambda_2) h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1}) h^{m-1} \otimes \tau^{m-1}(\Lambda_m)$$

$$= \sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2 S(h^1_{m-1}))h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1}S(h^1_2))h^{m-1} \otimes \tau^{m-1}(\Lambda_m S(h^1_1)).$$

*Proof.* By [LR88, Lemma 1.2(b)], we have

$$\sum_{(\Lambda)} \Lambda_1 h^1 \otimes \Lambda_2 = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 S(h^1).$$

Applying $\mathrm{Id} \otimes \Delta^{m-1}$ to both sides, we get

$$\begin{aligned}&\sum_{(\Lambda)} \Lambda_1 h^1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_{m-1} \otimes \Lambda_m \\ &= \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 S(h^1_{m-1}) \otimes \Lambda_2 S(h^1_{m-2}) \otimes \cdots \otimes \Lambda_{m-1} S(h^1_2) \otimes \Lambda_m S(h^1_1).\end{aligned}$$

We then apply $\mathrm{Id} \otimes \tau \otimes \tau^2 \otimes \cdots \otimes \tau^{m-1}$ to get

$$\begin{aligned}&\sum_{(\Lambda)} \Lambda_1 h^1 \otimes \tau(\Lambda_2) \otimes \cdots \tau^{m-2}(\Lambda_{m-1}) \otimes \tau^{m-1}(\Lambda_m) \\ &= \sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2 S(h^1_{m-1})) \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1} S(h^1_2)) \otimes \tau^{m-1}(\Lambda_m S(h^1_1)).\end{aligned}$$

The lemma follows by right multiplying this equation by $h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1} \otimes 1$. $\square$

Next, define a linear map $\psi : \widetilde{H}^{\otimes(m-1)} \to \widetilde{H}^{\otimes(m-1)}$ by

$$\begin{aligned}&\psi\left(h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1}\right) \\ &= \sum_{(h^1)} \tau(S(h^1_{m-1}))h^2 \otimes \tau^2(S(h^1_{m-2}))h^3 \otimes \cdots \otimes \tau^{m-2}(S(h^1_2))h^{m-1} \otimes \tau^{m-1}(S(h^1_1)).\end{aligned}$$

**Lemma 4.11.**

$$\mathrm{tr}(\psi) = \mathrm{tr}\left(\alpha|_{\left(\widetilde{V^{\otimes m}}\right)^H}\right).$$

*Proof.* To prove the lemma, it suffices to find a linear isomorphism $\varphi : \widetilde{H}^{\otimes(m-1)} \to \left(H \otimes \widetilde{H}^{\otimes(m-1)}\right)^H$ making the diagram

$$\begin{array}{ccc} \widetilde{H}^{\otimes(m-1)} & \xrightarrow{\ \psi\ } & \widetilde{H}^{\otimes(m-1)} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ \left(H \otimes \widetilde{H}^{\otimes(m-1)}\right)^H & \xrightarrow[\ \alpha\ ]{} & \left(H \otimes \widetilde{H}^{\otimes(m-1)}\right)^H \end{array}$$

commute. Recall that for any $H$-module $W$, there is a linear isomorphism $W \to (H \otimes W)^H$ given by $w \mapsto \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 w$. Let $\varphi$ be this isomorphism for $W = \widetilde{H}^{\otimes(m-1)}$.

Calculating gives

$$
\begin{aligned}
&(\alpha\circ\varphi)\left(h^1\otimes h^2\otimes\cdots\otimes h^{m-1}\right)\\
&=\sum_{(\Lambda)}\tau(\Lambda_2)h^1\otimes\tau^2(\Lambda_3)h^2\cdots\otimes\tau^{m-1}(\Lambda_m)h^{m-1}\otimes\Lambda_1\\
&=\sum_{(\Lambda)}\Lambda_1h^1\otimes\tau(\Lambda_2)h^2\cdots\otimes\tau^{m-2}(\Lambda_{m-1})h^{m-1}\otimes\tau^{m-1}(\Lambda_m)\\
&=\sum_{(\Lambda)}\Lambda_1\otimes\tau(\Lambda_2S(h^1_{m-1}))h^2\otimes\cdots\otimes\tau^{m-2}(\Lambda_{m-1}S(h^1_2))h^{m-1}\otimes\tau^{m-1}(\Lambda_mS(h^1_1))\\
&=\sum_{(\Lambda)}\Lambda_1\otimes\tau(\Lambda_2)\tau(S(h^1_{m-1}))h^2\otimes\cdots\otimes\\
&\qquad\qquad\tau^{m-2}(\Lambda_{m-1})\tau^{m-2}(S(h^1_2))h^{m-1}\otimes\tau^{m-1}(\Lambda_m)\tau^{m-1}(S(h^1_1))\\
&=(\varphi\circ\psi)\left(h^1\otimes h^2\otimes\cdots\otimes h^{m-1}\right).
\end{aligned}
$$

Here, the second and third equalities use Lemmas 4.5 and 4.10 respectively. □

*Proof of Theorem 4.9.* By the previous lemma, we need only show that $\mathrm{tr}(\psi)=\mathrm{tr}(\Omega^\tau_m)$. Choose a basis $b^1,\cdots b^n\in H$ with dual basis $b^*_1,\cdots b^*_n\in H^*$. Writing out $\mathrm{tr}(\psi)$ in terms of the induced basis on $H^{\otimes m}$, we obtain

$$
\begin{aligned}
\mathrm{tr}\,(\psi)&=\sum_{i_1,\cdots,i_{m-1}=1}^n\left\langle b^*_{i_1}\otimes\cdots\otimes b^*_{i_{m-1}},\psi\left(b^{i_1}\otimes\cdots\otimes b^{i_{m-1}}\right)\right\rangle\\
&=\sum_{i_1,\cdots,i_{m-1}=1}^n b^*_{i_1}(\tau(S(b^{i_1}_{m-1}))b^{i_2})b^*_{i_2}(\tau^2(S(b^{i_1}_{m-2}))b^{i_3})\cdots\\
&\qquad b^*_{i_{m-2}}(\tau^{m-2}(S(b^{i_1}_2))b^{i_{m-1}})b^*_{i_m-1}(\tau^{m-1}(S(b^{i_1}_1)))\\
&=\sum_{i_1,\cdots,i_{m-2}=1}^n b^*_{i_1}(\tau(S(b^{i_1}_{m-1}))b^{i_2})b^*_{i_2}(\tau^2(S(b^{i_1}_{m-2}))b^{i_3})\cdots\\
&\qquad\qquad b^*_{i_{m-2}}\left(\tau^{m-2}(S(b^{i_1}_2))\tau^{m-1}(S(b^{i_1}_1))\right)\\
&=\cdots=\sum_{i_1=1}^n b^*_{i_1}\left(\tau(S(b^{i_1}_{m-1}))\tau^2(S(b^{i_1}_{m-2}))\cdots\tau^{m-2}(S(b^{i_1}_2))\tau^{m-1}(S(b^{i_1}_1))\right)\\
&=\sum_{i=1}^n b^*_i\left(\tau(S(b^i_{m-1}))\tau^2(S(b^i_{m-2}))\cdots\tau^{m-2}(S(b^i_2))\tau^{m-1}(S(b^i_1))\right)\\
&=\sum_{i=1}^n b^*_i\left(S(\tau(b^i_{m-1}))S(\tau^2(b^i_{m-2}))\cdots S(\tau^{m-2}(b^i_2))S(\tau^{m-1}(b^i_1))\right)\\
&=\sum_{i=1}^n b^*_i\left(S(\tau^{m-1}(b^i_1)\tau^{m-2}(b^i_2))\cdots\tau^2(b^i_{m-2})\tau(b^i_{m-1})\right)\\
&=\mathrm{tr}\,(\Omega^\tau_m),
\end{aligned}
$$

as desired.

□

| | $\Omega_2^{\tau_1}$ | $\Omega_2^{\tau_2}$ | $\Omega_2^{\tau_3}$ | $\Omega_2^{\tau_4}$ |
|---|---|---|---|---|
| $1$ | $1$ | $1$ | $1$ | $1$ |
| $x$ | $x$ | $x$ | $y$ | $y$ |
| $y$ | $y$ | $y$ | $x$ | $x$ |
| $xy$ | $xy$ | $xy$ | $xy$ | $xy$ |
| $z$ | $z$ | $xyz$ | $\frac{1}{2}(z+xz+yz-xyz)$ | $\frac{1}{2}(-z+xz+yz+xyz)$ |
| $xz$ | $yz$ | $xz$ | $\frac{1}{2}(z+xz-yz+xyz)$ | $\frac{1}{2}(z-xz+yz+xyz)$ |
| $yz$ | $xz$ | $yz$ | $\frac{1}{2}(z-xz+yz+xyz)$ | $\frac{1}{2}(z+xz-yz+xyz)$ |
| $xyz$ | $xyz$ | $z$ | $\frac{1}{2}(-z+xz+yz+xyz)$ | $\frac{1}{2}(z+xz+yz-xyz)$ |

TABLE 4. The linear maps $\Omega_2^\tau$ for $H_8$

**Example 4.12.** We revisit the Hopf algebra $H_8$ described in Section 3.2. The linear maps $\Omega_2^\tau$ from Theorem 4.9 are given in Table 4. Computing the traces, one obtains the twisted Frobenius–Schur indicators for the regular representation: $\nu_2(\chi_R,\tau_1)=6$, $\nu_2(\chi_R,\tau_2)=6$, $\nu_2(\chi_R,\tau_3)=4$, and $\nu_2(\chi_R,\tau_4)=0$. These can, of course, also be calculated from the information in Table 3.

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