Filtered Lie conformal algebras whose associated graded algebras are isomorphic to that of general conformal algebra gc_1 [1]

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Abstract. Let G be a filtered Lie conformal algebra whose associated graded conformal algebra is isomorphic to that of general conformal algebra gc_1. In this paper, we prove that $G \cong gc_1$ or $\mathrm{gr}\, gc_1$ (the associated graded conformal algebra of gc_1), by making use of some results on the second cohomology groups of the conformal algebra $\mathfrak{g}$ with coefficients in its module $M_{b,0}$ of rank 1, where $\mathfrak{g} = \mathrm{Vir} \ltimes M_{a,0}$ is the semi-direct sum of the Virasoro conformal algebra Vir with its module $M_{a,0}$. Furthermore, we prove that $\mathrm{gr}\, gc_1$ does not have a nontrivial representation on a finite $\mathbb{C}[\partial]$-module, this provides an example of a finitely freely generated simple Lie conformal algebra of linear growth that cannot be embedded into the general conformal algebra gc_N for any N.

Key words: Lie conformal algebras, filtered algebras, general conformal algebras, second cohomology groups

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§1. Introduction

The notion of conformal algebras, introduced in [14], encodes an axiomatic description of the operator product expansion (or rather its Fourier transform) of chiral fields in conformal field theory. Conformal algebras play important roles in quantum field theory and vertex operator algebras, which is also an adequate tool for the study of infinite-dimensional Lie algebras and associative algebras (and their representations), satisfying the locality property in [15]. There has been a great deal of work towards understanding of the algebraic structure underlying the notion of the operator product expansion of chiral fields of a conformal field theory (e.g., [5, 16]). The singular part of the operator product expansion encodes the commutation relations of fields, which leads to the notion of Lie conformal algebras.

The structure theory, representation theory and cohomology theory of finite Lie conformal algebras has been developed in the past few years (e.g., [3, 4, 6–11, 19–23]). Simple finite Lie conformal algebras were classified in [8], which shows that a simple finite conformal algebra is isomorphic either to the Virasoro conformal algebra or to the current Lie conformal algebra $\mathrm{Cur}\,\mathfrak{g}$ associated to a simple finite-dimensional Lie algebra $\mathfrak{g}$. The theory of conformal modules has been developed in [6] and of their extensions in [7]. In particular, all finite simple irreducible representations of simple finite Lie conformal algebras were constructed in [6]. The cohomology theory of conformal algebras with coefficients in an arbitrary module has

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been developed in [4]. However, the structure theory, representation theory and cohomology theory of simple infinite Lie conformal algebras is far from being well developed.

In order to better understand the theory of simple infinite Lie conformal algebras, it is very natural to first study some important examples. As is pointed out in [14], the general Lie conformal algebras and the associative conformal algebras are the most important examples of simple infinite conformal algebras. It is well-known that the general Lie conformal algebra gc_N plays the same important role in the theory of Lie conformal algebras as the general Lie algebra gl_N does in the theory of Lie algebras: any module $M = \mathbb{C}[\partial]^N$ over a Lie conformal algebra R is obtained via a homomorphism $R \to gc_N$. Thus the study of Lie conformal algebras gc_N has drawn some authors' attentions (e.g., [1, 2, 17]).

In order to study simple infinite Lie conformal algebras, it is also very important to find the filtered infinite Lie conformal algebras whose associated graded Lie conformal algebras are some known infinite Lie conformal algebras. The determination of filtered algebras with associated graded algebras isomorphic to some known algebras is in general a highly nontrivial problem, as can be seen from examples in [12, 13]. Thus one of our motivations in the present paper is to investigate some simple infinite Lie conformal algebras by determining filtered Lie conformal algebras whose associated graded conformal algebras are isomorphic to that of general conformal algebra gc_N. Due to the reason stated in Remark 2.3(2), we have to treat gc_1 separately. Thus in the present paper, we shall only consider the case gc_1. The consideration of the general case gc_N for $N > 1$ will be our next goal.

The main result in the present paper is the following theorem.

Theorem 1.1 (Main Theorem)

(1) *Let G be a filtered Lie conformal algebra whose associated graded conformal algebra is isomorphic to that of general conformal algebra gc_1. Then $G \cong gc_1$ or $\mathrm{gr}\, gc_1$ (the associated graded conformal algebra of gc_1).*

(2) *The graded conformal algebra $\mathrm{gr}\, gc_1$ does not have a nontrivial representation on any finite $\mathbb{C}[\partial]$-module. In particular, $\mathrm{gr}\, gc_1 \not\cong gc_1$, and $\mathrm{gr}\, gc_1$ is a finitely freely generated simple Lie conformal algebra of linear growth that is not embedded into gc_N for any N.*

Theorem 1.1(1) will follow from some computations (see Section 4) and some results (Theorem 3.2) on the second cohomology groups of the conformal algebra $\mathfrak{g}$ with coefficients in its modules $M_{b,0}$ of rank 1, where $\mathfrak{g} = \mathrm{Vir} \ltimes M_{a,0}$ is the semi-direct sum of the Virasoro conformal algebra Vir with its module $M_{a,0}$. Here in general, Vir and $M_{\Delta,\alpha}$ (which is a simple Vir-module if and only if $\Delta \neq 0$) are defined by

$$\mathrm{Vir} = \mathbb{C}[\partial]L: \quad L_\lambda L = (\partial + 2\lambda)L. \tag{1.1}$$

$$M_{\Delta,\alpha} = \mathbb{C}[\partial]v: \quad L_\lambda v = (\alpha + \partial + \Delta\lambda)v. \tag{1.2}$$

We will prove Theorem 1.1(2) in Section 5.

§2. The general Lie conformal algebra gc_1

Definition 2.1 A *Lie conformal algebra* is a $\mathbb{C}[\partial]$-module A with a λ-bracket $[x_\lambda y]$ which defines a linear map $A\times A\to A[\lambda]$, where $A[\lambda]=\mathbb{C}[\lambda]\otimes A$ is the space of polynomials of λ with coefficients in A, such that for $x,y,z\in A$,

$$[\partial x_\lambda y]=-\lambda[x_\lambda y],\quad [x_\lambda \partial y]=(\partial+\lambda)[x_\lambda y]\quad \text{(conformal sesquilinearity)}, \tag{2.1}$$

$$[x_\lambda y]=-[y_{-\lambda-\partial}x]\quad \text{(skew-symmetry)}, \tag{2.2}$$

$$[x_\lambda[y_\mu z]]=[[x_\lambda y]_{\lambda+\mu}z]+[y_\mu[x_\lambda z]]\quad \text{(Jacobi identity)}. \tag{2.3}$$

The *general Lie conformal algebra* gc_N can be defined as the infinite rank $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial,x]\otimes gl_N$, with the λ-bracket

$$[f(\partial,x)A\,_\lambda\, g(\partial,x)B]=f(-\lambda,x+\partial+\lambda)g(\partial+\lambda,x)AB-f(-\lambda,x)g(\partial+\lambda,x-\lambda)BA, \tag{2.4}$$

for $f(\partial,x),g(\partial,x)\in\mathbb{C}[\partial,x]$, $A,B\in gl_N$, where gl_N is the space of $N\times N$ matrices, and we have identified $f(\partial,x)\otimes A$ with $f(\partial,x)A$. If we set $J_A^n=x^nA$, then

$$[J_A^m\,_\lambda J_B^n]=\sum_{s=0}^m\binom{m}{s}(\lambda+\partial)^sJ_{AB}^{m+n-s}-\sum_{s=0}^n\binom{n}{s}(-\lambda)^sJ_{BA}^{m+n-s},$$

for $m,n\in\mathbb{Z}_+$, $A,B\in gl_N$, where $\binom{m}{s}=m(m-1)\cdots(m-s+1)/s!$ if $s\geq 0$ and $\binom{m}{s}=0$ otherwise, is the binomial coefficient. The formal distribution Lie algebra corresponding to gc_N is the well-known Lie algebra $\mathcal{D}^N$ of $N\times N$-matrix differential operators on the circle.

In particular, the *general Lie conformal algebra* gc_1 is the infinite rank free $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial,x]$ with a generating set $\{J_n=x^{n+1}\mid -1\leq n\in\mathbb{Z}\}$, such that

$$[J_{m_\lambda}J_n]=\sum_{s=0}^m\binom{m+1}{s+1}(\lambda+\partial)^{s+1}J_{m+n-s}-\sum_{s=0}^n\binom{n+1}{s+1}(-\lambda)^{s+1}J_{m+n-s}, \tag{2.5}$$

for $-1\leq m,n\in\mathbb{Z}$. Naturally, gc_1 is a filtered algebra with filtration

$$\{0\}=gc_1^{(-2)}\subset gc_1^{(-1)}\subset\cdots\subset gc_1 \text{ with } gc_1^{(n)}=\mathrm{span}\{J_i\mid -1\leq i\leq n\} \text{ for } n\geq -1, \tag{2.6}$$

such that

$$[J_{i_\lambda}J_j]\equiv\Big((i+j+2)\lambda+(i+1)\partial\Big)J_{i+j}\ \ (\bmod\ gc_1^{(i+j-1)})\ \text{ for }\ i,j\geq -1. \tag{2.7}$$

Definition 2.2 Let $G=\cup_{i\in\mathbb{Z}}G^{(i)}$ be a Lie conformal algebra with a filtration

$$\cdots\subseteq G^{(-1)}\subseteq G^{(0)}\subseteq G^{(1)}\subseteq\cdots,$$

such that $[G^{(i)}{}_\lambda G^{(j)}] \subseteq G^{(i+j)}$ for $i,j\in\mathbb{Z}$. Denote $\mathrm{gr}\, G = \oplus_{i\in\mathbb{Z}} \overline{G^{(i)}}$, where $\overline{G^{(i)}} = G^{(i)}/G^{(i-1)}$, then $\mathrm{gr}\, G$ has a natural Lie conformal algebra structure, called the *associated graded conformal algebra* of G.

Remark 2.3 (1) The $\mathrm{gr}\, gc_1$ is the Lie conformal algebra with free $\mathbb{C}[\partial]$-generating set $\{J_i \,|\, -1\le i\in\mathbb{Z}\}$, and the λ-bracket (cf. (2.7))

$$[J_{i_\lambda} J_j] = \big((i+j+2)\lambda + (i+1)\partial\big) J_{i+j} \ \text{ for } \ i,j\ge -1. \tag{2.8}$$

It is straightforward to verify that $\mathrm{gr}\, gc_1$ is a simple conformal algebra, whose corresponding formal distribution Lie algebra is a well-known Block type Lie algebra studied in [18], thus we refer this conformal algebra to as a *Block type conformal algebra.*

(2) Note from (2.4) that when $N>1$, the filtration of gc_N has to be taken as

$$\{0\}=gc_N^{(-1)}\subset gc_N^{(0)}\subset\cdots\subset gc_N \text{ with } gc_N^{(n)}=\mathrm{span}\{J_A^i \,|\, A\in gl_N,\ 0\le i\le n\} \text{ for } n\ge 0, \tag{2.9}$$

such that for $i,j,\in\mathbb{Z}_+$, $A,B\in gl_N$,

$$[J^i_{A_\lambda} J^j_B] \equiv J^{i+j}_{[A,B]} \ \ (\bmod\ gc_N^{(i+j-1)}), \ \text{ where } [A,B]=AB-BA. \tag{2.10}$$

Thus the filtration of gc_N for $N>1$ is different from that of gc_1. This is why we have to treat gc_1 separately (if we use the filtration (2.9) for gc_1, then we obtain from (2.10) that the associated graded conformal algebra of this filtration is simply nothing but trivial).

The main problem to be addressed in this paper is to determine filtered Lie conformal algebras whose associated graded conformal algebras are isomorphic to that of general conformal algebra gc_1.

§3. Some second cohomology groups

To prove Theorem 1.1(1), we need some results on the second cohomology groups. We refer to [4] for definition of conformal cohomology. Let $\mathrm{Vir} = \mathbb{C}[\partial]L$ and $M_{\Delta,\alpha} = \mathbb{C}[\partial]v$ be respectively the Virasoro conformal algebra and its module defined in (1.1) and (1.2). By [4, Theorem 7.2(3)], we have

$$\dim H^2(\mathrm{Vir}, M_{\Delta,\alpha}) = \begin{cases} 2 & \text{if } \ \Delta=-1,0 \text{ and } \alpha=0,\\ 1 & \text{if } \ \Delta=-6,-4,1 \text{ and } \alpha=0,\\ 0 & \text{otherwise.}\end{cases} \tag{3.1}$$

$$H^2(\mathrm{Vir}, M_{1,0}) = \mathbb{C}\phi, \ \text{ where } \phi_{\lambda_1,\lambda_2}(L,L) = \lambda_1-\lambda_2. \tag{3.2}$$

Let $\mathrm{Vir} \ltimes M_{a,0} = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]J$ be the semi-direct sum of the Virasoro conformal algebra $\mathrm{Vir} = \mathbb{C}[\partial]L$ and its module $M_{a,0} = \mathbb{C}[\partial]J$. Then a Vir-module $M_{b,0} = \mathbb{C}[\partial]v$ becomes a $\mathrm{Vir} \ltimes M_{a,0}$-module with the trivial λ-action of J. As in [4, §7.2], for any 2-cochain $\psi \in C^2(\mathrm{Vir} \ltimes M_{a,0}, M_{b,0})$, there are 3 unique polynomials $P_{JJ}(\lambda_1, \lambda_2)$, P_{λ_1,λ_2}, $P_{LL}(\lambda_1, \lambda_2)$ such that $P_{JJ}(\lambda_1, \lambda_2)$, $P_{LL}(\lambda_1, \lambda_2)$ are skew-symmetric, and

$$\psi_{\lambda_1,\lambda_2}(J,J) \equiv P_{JJ}(\lambda_1,\lambda_2)v, \quad \psi_{\lambda_1,\lambda_2}(L,J) \equiv P_{\lambda_1,\lambda_2}v, \quad \psi_{\lambda_1,\lambda_2}(L,L) \equiv P_{LL}(\lambda_1,\lambda_2)v, \tag{3.3}$$

where "$\equiv$" means "equality under modulo $\partial + \lambda_1 + \lambda_2$".

Lemma 3.1 *Suppose $\psi \in C^2(\mathrm{Vir} \ltimes M_{a,0}, M_{b,0})$ is a 2-cocycle. Then for some $c_0 \in \mathbb{C}$,*

$$P_{JJ}(\lambda_1, \lambda_2) = \begin{cases} c_0(\lambda_1 - \lambda_2) & \text{if } \ b = 2a - 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. From the definition of differential operator d, and using $[J_\lambda J] = 0$ and $J_\lambda v = 0$, we obtain

$$\begin{aligned} 0 &= (d\psi)_{\lambda_1,\lambda_2,\lambda_3}(L,J,J) = L_{\lambda_1}\psi_{\lambda_2,\lambda_3}(J,J) - \psi_{\lambda_1+\lambda_2,\lambda_3}([L_{\lambda_1}J],J) + \psi_{\lambda_1+\lambda_3,\lambda_2}([L_{\lambda_1}J],J) \\ &\equiv (-\lambda_1 - \lambda_2 - \lambda_3 + b\lambda_1)P_{JJ}(\lambda_2,\lambda_3)v - (-\lambda_1 - \lambda_2 + a\lambda_1)P_{JJ}(\lambda_1+\lambda_2,\lambda_3)v \\ &\quad + (-\lambda_1 - \lambda_3 + a\lambda_1)P_{JJ}(\lambda_1+\lambda_3,\lambda_2)v. \end{aligned} \tag{3.5}$$

First assume $a \neq 1$. Letting $\lambda_3 = 0$, we obtain

$$P_{JJ}(\lambda_1,\lambda_2) = \frac{1}{(a-1)\lambda_1}\Big(((1-b)\lambda_1+\lambda_2)P_{JJ}(\lambda_2,0) + ((a-1)\lambda_1-\lambda_2)P_{JJ}(\lambda_1+\lambda_2,0)\Big). \tag{3.6}$$

If $a = b = 2$, then $P_{JJ}(\lambda_1,\lambda_2) = \frac{\lambda_1-\lambda_2}{\lambda_1}(P_{JJ}(\lambda_1+\lambda_2,0) - P_{JJ}(\lambda_2,0))$, using this in (3.5), we easily obtain that $P_{JJ}(\lambda,0) = c_0\lambda$ for some $c_0 \in \mathbb{C}$, and the first case of (3.4) holds. If $a = 2 \neq b$, using $P_{JJ}(\lambda,\lambda) = 0$ in (3.6), we immediately obtain that $P_{JJ}(\lambda,0) = 0$, thus $P_{JJ}(\lambda_1,\lambda_2) = 0$ by (3.6). Hence, we suppose $a \neq 2$. Using $P_{JJ}(\lambda,\lambda) = 0$, we have $P_{JJ}(2\lambda,0) = \frac{b-2}{a-2}P_{JJ}(\lambda,0)$, which implies $P_{JJ}(\lambda,0) = P_{JJ}(1,0)\lambda^m$ is a homogenous polynomial of degree, say m, such that $\frac{b-2}{a-2} = 2^m$. Then by (3.5) and (3.6), we obtain that $m = 1$, $P_{JJ}(\lambda,0) = P_{JJ}(1,0)\lambda$, and $P_{JJ}(\lambda_1,\lambda_2) = \frac{(a-1)\lambda_1+(a-1-b)\lambda_2}{a-1}P_{JJ}(1,0)$. This gives (3.4).

Finally assume $a = 1$. Letting $\lambda_3 = 1$ in (3.5) gives

$$P_{JJ}(\lambda_1,\lambda_2) = ((b-1)\lambda_1 - \lambda_2 - b - 1)R(\lambda_2) + \lambda_2 R(\lambda_1+\lambda_2-1), \tag{3.7}$$

where $R(\lambda) = P_{JJ}(\lambda, 1)$. Using $P_{JJ}(\lambda, 0) + P_{JJ}(0, \lambda) = 0$, and using (3.7) in (3.5) with $\lambda_2 = 0$ and $\lambda_1 = \lambda_3 = \lambda$, we obtain respectively

$$(\lambda + b)R(\lambda) - \lambda R(\lambda - 1) + ((1 - b)\lambda + b)R(0) = 0, \tag{3.8}$$

$$((b - 2)\lambda - b)R(\lambda) + \lambda R(2\lambda - 1) = 0. \tag{3.9}$$

If $b \neq 0$, then (3.9) shows that $R(0) = 0$, and from (3.8) and (3.9), we can obtain $R(\lambda) = 0$ for infinite many λ's, thus $R(\lambda) = 0$. If $b = 0$, then (3.8) gives $R(1) = 0$ and $R(\lambda) = R(0)(1-\lambda)$, and we have (3.4) by (3.7). $\square$

Now we determine the polynomial P_{λ_1,λ_2} below. Similar to (3.5), we have

$$\begin{aligned}
0 &= (d\psi)_{\lambda_1,\lambda_2,\lambda_3}(L, L, J)\\
&= L_{\lambda_1}\psi_{\lambda_2,\lambda_3}(L, J) - L_{\lambda_2}\psi_{\lambda_1,\lambda_3}(L, J)\\
&\quad -\psi_{\lambda_1+\lambda_2,\lambda_3}([L_{\lambda_1}L], J) + \psi_{\lambda_1+\lambda_3,\lambda_2}([L_{\lambda_1}J], L) - \psi_{\lambda_2+\lambda_3,\lambda_1}([L_{\lambda_2}J], L)\\
&\equiv ((b - 1)\lambda_1 - \lambda_2 - \lambda_3)P_{\lambda_2,\lambda_3}v - (-\lambda_1 + (b - 1)\lambda_2 - \lambda_3)P_{\lambda_1,\lambda_3}v\\
&\quad -(\lambda_1-\lambda_2)P_{\lambda_1+\lambda_2,\lambda_3}v-((a-1)\lambda_1-\lambda_3)P_{\lambda_2,\lambda_1+\lambda_3}v+((a-1)\lambda_2-\lambda_3)P_{\lambda_1,\lambda_2+\lambda_3}v.
\end{aligned} \tag{3.10}$$

Denote by $P^{(m)}_{\lambda_1,\lambda_2}$ the homogenous part of P_{λ_1,λ_2} of degree m. Then (3.10) is satisfied by $P^{(m)}_{\lambda_1,\lambda_2}$. Note that for any polynomial $Q(\lambda)$, we can define a 1-cochain $f \in C^1(\mathrm{Vir} \ltimes M_{a,0}, M_{b,0})$ by $f_\lambda(L) = 0, f_\lambda(J) = Q(\lambda)v$. Let $\bar\psi = df$ be the corresponding 2-coboundary, and set $\psi' = \psi - df$. Then

$$\begin{aligned}
&\psi'_{\lambda_1,\lambda_2}(L, J) = P'_{\lambda_1,\lambda_2}v, \quad \text{where}\\
&P'_{\lambda_1,\lambda_2} = P_{\lambda_1,\lambda_2} - (-\lambda_1 - \lambda_2 + b\lambda_1)Q(\lambda_2) + (-\lambda_1 - \lambda_2 + a\lambda_1)Q(\lambda_1 + \lambda_2).
\end{aligned} \tag{3.11}$$

If we replace ψ by ψ', then P_{λ_1,λ_2} is replaced by P'_{λ_1,λ_2}. Thus, by some suitable choice of $Q(\lambda)$, we can always suppose

$$\begin{aligned}
&P^{(m)}_{\lambda,0} = 0 \ \text{ if } \ m \geq 1,\ a \neq 1, \quad \text{or} \quad (m, a, b) = (1, 1, 0),\\
&P^{(m)}_{\lambda,-\lambda} = 0 \ \text{ if } \ m \geq 1,\ a = 1,\ b \neq 0,\\
&P^{(m)}_{\lambda,\lambda} = 0 \ \text{ if } \ m \geq 3,\ a = 1,\ b = 0.
\end{aligned} \tag{3.12}$$

Assume $P^{(m)}_{\lambda_1,\lambda_2} \neq 0$. For $m = 0, 1, 2$, one can directly check that for some $c_i \in \mathbb{C}$,

$$P^{(m)}_{\lambda_1,\lambda_2} = \begin{cases} c_1 & \text{if } \ m = 0,\ a = b,\\ c_2\lambda_1 & \text{if } \ m = 1,\ a = b,\\ c_3\lambda_2 & \text{if } \ (m, a, b) = (1, 1, 0),\\ c_4\lambda_1\lambda_2 & \text{if } \ m = 2,\ a = 1,\ b \neq 0,\\ c_5\lambda_1^2 + c_6\lambda_1\lambda_2 & \text{if } \ (m, a, b) = (2, 1, 0). \end{cases} \tag{3.13}$$

From now on, we suppose $m \geq 3$. First assume $a \neq 1, b$. Putting $\lambda_2 = \lambda_3 = 0$ in (3.10), we obtain $(a-1)P_{0,\lambda_1} = (b-1)P_{0,0}$, thus $P_{0,\lambda_2} = 0$. This together with (3.12) proves $P^{(m)}_{\lambda_1,\lambda_2}$ is divided by $\lambda_1\lambda_2$, and we can suppose $P^{(m)}_{\lambda_1,\lambda_2} = \lambda_1\lambda_2 P'_{\lambda_1,\lambda_2}$ for some polynomial P'_{λ_1,λ_2}. Now putting $\lambda_3 = 0$ in (3.10) gives $\lambda_2 P'_{\lambda_1,\lambda_2} = \lambda_1 P'_{\lambda_2,\lambda_1}$. Thus P'_{λ_1,λ_2} is divided by λ_1. Therefore, we can suppose

$$P^{(m)}_{\lambda_1,\lambda_2} = \lambda_1^2\lambda_2 S_{\lambda_1,\lambda_2}, \tag{3.14}$$

where S_{λ_1,λ_2} is a homogenous symmetric polynomial of degree $m-3$. Write S_{λ_1,λ_2} as $S_{\lambda_1,\lambda_2} = \sum_{i=0}^{m-3} s_i\lambda_1^{m-3-i}\lambda_2^i$. Comparing the coefficients of $\lambda_1\lambda_2^{m-1-j}\lambda_3^{j+1}$ on the both sides of (3.10), we have $(m-1+b-a)s_j = 0$ for $0 \leq j \leq m-3$. Thus $S_{\lambda_1,\lambda_2} = 0$ if $m \neq a-b+1$. Now suppose $m = a-b+1$. For $m = 3,4,5,6,7$, one can easily solve

$$P^{(m)}_{\lambda_1,\lambda_2} = \begin{cases} c_7\lambda_1^2\lambda_2 & \text{if } m=a-b+1=3,\\ c_8\lambda_1^2\lambda_2(\lambda_1+\lambda_2) & \text{if } m=a-b+1=4,\\ \lambda_1^2\lambda_2(c_9(\lambda_1^2+\lambda_2^2)+c_{10}\lambda_1\lambda_2) & \text{if } m=a-b+1=5,\\ c_{11}\lambda_1^2\lambda_2(\lambda_1^3+\lambda_2^3+\frac{20}{7}\lambda_1\lambda_2(\lambda_1+\lambda_2)) & \text{if } (m,a,b)=(6,5,0),\\ \lambda_1^2\lambda_2\big(c_{12}(\lambda_1^5+\lambda_2^5)+c_{13}(\lambda_1^4\lambda_2+\lambda_1\lambda_2^4)+c_{14}(\lambda_1^3\lambda_2^2+\lambda_1^2\lambda_2^3)\big) & \text{if } m=a-b+1=7, \end{cases} \tag{3.15}$$

for some $c_i \in \mathbb{C}$, with the following conditions,

$$\begin{aligned} &(10+3b)c_9 = (5+2b)c_{10} \text{ if } m=5,\\ &(9+2b)c_{13}=(33+5b)c_{12},\ (9+2b)c_{14}=(51+16b)c_{12},\ b=\tfrac{1}{2}(-5\pm\sqrt{19}) \text{ if } m=7. \end{aligned} \tag{3.16}$$

Now suppose $m = a-b+1 \geq 8$. By comparing the coefficients of $\lambda_1^2\lambda_2^{a-b-j}\lambda_3^j$ on the both sides of (3.10), we obtain

$$(a-b-2-j)(a-b+1-j)s_{j-1} + (j+1)(j-2a+2)s_j = 2\big(\tbinom{a-b}{j} - a\tbinom{a-b-1}{j}\big)s_0, \tag{3.17}$$

$$(a-b-j)(-a-b+1-j)s_{j-1} + (j-1)(j+2)s_j = 2\big(\tbinom{a-b}{j+1} - a\tbinom{a-b-1}{j}\big)s_0, \tag{3.18}$$

for $1 \leq j \leq a-b-2$, where (3.18) is obtained from (3.17) by symmetry of S_{λ_1,λ_2}. Similarly, by comparing the coefficients of $\lambda_1^3\lambda_2^{a-b-j}\lambda_3^{j-1}$, we have

$$\begin{aligned} &(a-b-3-j)(a-b+1-j)(a-b+2-j)s_{j-2}+j(j+1)(j-3a+2)s_j\\ &\qquad = 6\big(\tbinom{a-b-1}{j-1} - a\tbinom{a-b-2}{j-1}\big)s_1, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &(j+b-a)(a-b+1-j)(2a+b+j-2)s_{j-2}+(j-3)(j+1)(j+2)s_j\\ &\qquad = 6\big(\tbinom{a-b-1}{j} - a\tbinom{a-b-2}{j-1}\big)s_1, \end{aligned} \tag{3.20}$$

for $3 \leq j \leq a-b-2$. Note that we can recursively use (3.18) to solve s_j in terms of s_0 and s_1 for $j \geq 2$. If $a = \frac{3}{2}$, then from (3.17), we obtain $s_0 = 0$, and one can then easily check from (3.17)–(3.20) that $s_j=0$ for all j. Thus suppose $a\neq\frac{3}{2}$. Use (3.17) to solve s_1 in term of s_0, we can then obtain s_j in term of s_0. Taking $j=2$ in (3.17) and $j=4$ in (3.20), one immediately obtain $s_0=0$. Thus $S_{\lambda_1,\lambda_2}=0$.

Now assume $a=b$. By (3.12), we can suppose $P^{(m)}_{\lambda_1,\lambda_2}=\lambda_2R_{\lambda_1,\lambda_2}$ if $a\neq1$ or $P^{(m)}_{\lambda_1,\lambda_2}=(\lambda_1+\lambda_2)R_{\lambda_1,\lambda_2}$ if $a=1$, for some homogenous polynomial R_{λ_1,λ_2} of degree $m-1$. Then using (3.10) and discussing as above, we can prove $P^{(m)}_{\lambda_1,\lambda_2}=0$.

Next assume $a=1$, $b=0$. As above, we can suppose $P^{(m)}_{\lambda_1,\lambda_2}=(\lambda_1-\lambda_2)R_{\lambda_1,\lambda_2}$ for some homogenous polynomial R_{λ_1,λ_2} of degree $m-1$, from which we can deduce that $P^{(m)}_{\lambda_1,\lambda_2}=0$.

Finally assume $a=1$, $b\neq0,1$. As before, we can suppose $P^{(m)}_{\lambda_1,\lambda_2}=(\lambda_1+\lambda_2)R_{\lambda,\lambda_2}$ for some homogenous polynomial R_{λ_1,λ_2} of degree $m-1$. For $m=3,4,5,6$, one can easily solve

$$P^{(m)}_{\lambda_1,\lambda_2}=\begin{cases}c_{15}\lambda_1^2(\lambda_1+\lambda_2) & \text{if } (m,a,b)=(3,1,-1),\\ c_{16}\lambda_1^2\lambda_2(\lambda_1+\lambda_2) & \text{if } (m,a,b)=(4,1,-2),\\ \lambda_1^2\lambda_2^2(\lambda_1+\lambda_2) & \text{if } (m,a,b)=(5,1,-3),\\ c_{17}\lambda_1^2\lambda_2(\lambda_1+\lambda_2)(\lambda_1^2+\lambda_1\lambda_2+7\lambda_2^2) & \text{if } (m,a,b)=(6,1,-4).\end{cases}\tag{3.21}$$

If $m\geq7$, then similar to the arguments after (3.16), we obtain $P^{(m)}_{\lambda_1,\lambda_2}=0$.

Note that every 2-cochain $\psi\in C^2(\mathrm{Vir}\ltimes M_{a,0},M_{b,0})$ can be restricted to a 2-cochain $\psi':=\psi|_{\mathrm{Vir}\times\mathrm{Vir}}\in C^2(\mathrm{Vir},M_{b,0})$, and conversely, every 2-cochain $\psi'\in C^2(\mathrm{Vir},M_{b,0})$ can be extended to a 2-cochain $\psi\in C^2(\mathrm{Vir}\ltimes M_{a,0},M_{b,0})$ by taking the corresponding polynomials $P_{JJ}(\lambda_1,\lambda_2)$, P_{λ_1,λ_2} to be zero. From this, we obtain an embedding $H^2(\mathrm{Vir},M_{b,0})\to H^2(\mathrm{Vir}\ltimes M_{a,0},M_{b,0})$. Thus, we can regard $H^2(\mathrm{Vir},M_{b,0})$ as a subspace of $H^2(\mathrm{Vir}\ltimes M_{a,0},M_{b,0})$. Now we can state the main result in this section.

Theorem 3.2 *We have*

$$\dim H^2(\mathrm{Vir}\ltimes M_{a,0},M_{b,0})=\dim H^2(\mathrm{Vir},M_{b,0})+\delta_{b,2a-2}+\tau_{a,b},\ \textit{where}\tag{3.22}$$

$$\tau_{a,b}=\begin{cases}3 & \textit{if}\ \ (a,b)=(1,0),\\ 2 & \textit{if}\ \ a=b,\ \ \textit{or}\ \ a=1,\ \ b=-3,-4,-5,-6,\\ 1 & \textit{if}\ \ a=1,\ \ b\neq1,0,-3,-4,-5,-6,\ \ \textit{or}\ \ a\neq1,\ \ a-b=2,3,4,\\ & \quad \textit{or}\ \ a=5,\ \ b=0,\ \ \textit{or}\ \ a=6+b,\ \ b=\frac{1}{2}(-5\pm\sqrt{19}),\\ 0 & \textit{otherwise}.\end{cases}$$

Furthermore, every 2-cocycle $\psi'\in C^2(\mathrm{Vir}\ltimes M_{a,0},M_{b,0})$ is equivalent to a 2-cocycle ψ such that the corresponding polynomial $P_{JJ}(\lambda_1,\lambda_2)$ defined in (3.3) *has the form in* (3.1)*, and the homogenous part $P^{(m)}_{\lambda_1,\lambda_2}$ of P_{λ_1,λ_2} has the form in* (3.13)*,* (3.15) *or* (3.21)*.*

§4. Proof of Theorem 1.1(1)

Now in order to prove the main theorem, we need some preparations. Since G is a filtered Lie conformal algebra and $\mathrm{gr}\, G \cong \mathrm{gr}\, gc_1$, by (2.7), we can suppose

$$[J_{-1}{}_\lambda J_{-1}] = 0 \ , \tag{4.1}$$

$$[J_{-1}{}_\lambda J_0] = \lambda J_{-1} \ , \tag{4.2}$$

$$[J_0{}_\lambda J_0] = (2\lambda + \partial)J_0 + g_1(\lambda, \partial)J_{-1} \ , \tag{4.3}$$

$$[J_{-1}{}_\lambda J_1] = 2\lambda J_0 + g_2(\lambda, \partial)J_{-1} \ , \tag{4.4}$$

$$[J_0{}_\lambda J_1] = (3\lambda + \partial)J_1 + g_3(\lambda, \partial)J_0 + g_4(\lambda, \partial)J_{-1} \ , \tag{4.5}$$

$$[J_{-1}{}_\lambda J_2] = 3\lambda J_1 + h_1(\lambda, \partial)J_0 + h_2(\lambda, \partial)J_{-1} \ , \tag{4.6}$$

$$[J_0{}_\lambda J_2] = (4\lambda + \partial)J_2 + h_3(\lambda, \partial)J_1 + h_4(\lambda, \partial)J_0 + h_5(\lambda, \partial)J_{-1} \ , \tag{4.7}$$

$$[J_1{}_\lambda J_1] = 2(2\lambda + \partial)J_2 + f_1(\lambda, \partial)J_1 + f_2(\lambda, \partial)J_0 + f_3(\lambda, \partial)J_{-1} \ , \tag{4.8}$$

where $g_i(\lambda, \partial)$ for $1 \le i \le 4$, $h_i(\lambda, \partial)$ for $1 \le i \le 5$ and $f_i(\lambda, \partial)$ for $1 \le i \le 3$ are all polynomials of λ and ∂. Then our aim is to determine all these polynomials of λ and ∂ by making use of Theorem 3.2 and by computations.

Lemma 4.1 *In* (4.3), *by re-choosing the generator J_0, we can suppose $g_1(\lambda, \partial) = 0$.*

Proof. Noting from (4.2), we see $P_{LL}(\lambda_1, \lambda_2) := g_1(\lambda_1, -\lambda_1 - \lambda_2)$ is a skew-symmetric polynomial, thus we can define a 2-cochain $\psi \in C^2(\mathrm{Vir}, M_{1,0})$ with $\psi_{\lambda_1,\lambda_2}(L, L) = g_1(\lambda_1, -\lambda_1 - \lambda_2)$, which is in fact a 2-cocycle by Jacobi identity. Note that for any polynomial $p(\partial)$, if we replace the generator J_0 by $J_0' = J_0 - p(\partial)J_{-1}$, it is equivalent to replacing the 2-cocycle ψ by $\psi' = \psi - d\phi$, where $\phi \in C^1(\mathrm{Vir}, M_{1,0})$ is the 1-cochain defined by $\phi_\lambda(L) = p(-\lambda)$. Thus by (3.2), we can suppose $g_1(\lambda_1, -\lambda_1 - \lambda_2) = a_0(\lambda_1 - \lambda_2)$ for some $a_0 \in \mathbb{C}$, i.e., $g_1(\lambda, \partial) = a_0(\partial + 2\lambda)$. Applying the operator $J_{0\mu}$ to (4.4) and (4.5), using the Jacobi identity and comparing the coefficients of J_0 and J_{-1} respectively, we obtain

$$\begin{aligned}&(3\mu + \lambda + \partial)g_3(\lambda, \partial) + (2\lambda + \partial)g_3(\mu, \lambda + \partial) - (\lambda - \mu)g_3(\lambda + \mu, \partial)\\ &= (3\lambda + \mu + \partial)g_3(\mu, \partial) + (2\mu + \partial)g_3(\lambda, \partial + \mu) + 2a_0(\lambda + \mu)(\lambda - \mu).\end{aligned}$$

$$\begin{aligned}&2\mu a_0(\partial + 2\lambda) + (\lambda + \partial)g_2(\mu, \lambda + \partial) + \mu g_2(\lambda + \mu, \partial)\\ &= (3\lambda + \mu + \partial)g_2(\mu, \partial) + \mu g_3(\lambda, \mu + \partial).\end{aligned}$$

To prove $a_0 = 0$, we only need to suppose $g_3(\lambda, \partial), g_2(\lambda, \partial)$ are homogenous polynomials of degree 1 (since coefficients of terms of other degrees do not contribute). Then one can immediately check from the above two equations that $a_0 = 0$. The lemma follows. $\square$

Similar to the proof of Lemma 4.1, from (4.5), we can use $g_3(\lambda, \partial)$ to define a 2-cocycle $\psi \in C^2(\mathrm{Vir} \ltimes M_{3,0}, M_{2,0})$ such that the corresponding polynomials defined in (3.3) have the forms,

$$P_{LL}(\lambda_1, \lambda_2) = P_{JJ}(\lambda_1, \lambda_2) = 0, \quad \text{and} \quad P_{\lambda_1,\lambda_2} = g_3(\lambda_1, -\lambda_1 - \lambda_2).$$

Thus, using Theorem 3.2, by replacing J_1 by $J_1' = J_1 + p(\partial)J_0$ for some polynomial $p(\partial)$, we can suppose $g_3(\lambda, \partial) = 0$. Similarly, $g_4(\lambda, \partial)$ defines a 2-cocycle in $C^2(\mathrm{Vir} \ltimes M_{3,0}, M_{1,0})$, using Theorem 3.2 and the first case of (3.15), by replacing J_1 by $J_1' = J_1 + p(\partial)J_{-1}$ for some polynomial $p(\partial)$ (this replacement does not affect $g_3(\lambda, \partial)$), we can suppose

$$g_4(\lambda, \partial) = a_1\lambda^2(\lambda + \partial) \ \text{ for some } \ a_1 \in \mathbb{C}.$$

Lemma 4.2 *In* (4.5), *there exists some* $a_2 \in \mathbb{C}$ *such that* $g_2(\lambda, \partial) = a_2\lambda(\lambda - \partial)$.

Proof. Comparing the coefficients of J_{-1} on the both sides of the Jacobi identity $[J_{0\lambda}[J_{-1\mu}J_1]] = [[J_{0\lambda}J_{-1}]_{\lambda+\mu}J_1] + [J_{-1\mu}[J_{0\lambda}J_1]]$, by (4.2)–(4.5), Lemma 4.1 and $g_3(\lambda, \partial) = 0$, we have

$$(\lambda + \partial)g_2(\mu, \lambda + \partial) - (3\lambda + \mu + \partial)g_2(\mu, \partial) + \mu g_2(\lambda + \mu, \partial) = 0. \tag{4.9}$$

Taking $\partial = 0$ gives

$$g_2(\mu, \lambda) = \frac{1}{\lambda}\big((3\lambda + \mu)g_2(\mu, 0) - \mu g_2(\lambda + \mu, 0)\big). \tag{4.10}$$

Using this in (4.9) and taking $\partial = \mu = -\lambda$, we obtain $g_2(2\lambda, 0) = 4g_2(\lambda, 0) + 3g_2(0, 0)$. Inductively, we obtain that $g_2(x\lambda, 0) = x^2 g_2(\lambda, 0) + (x^2 - 1)g_2(0, 0)$ holds for all $x = 2^k$, $k = 1, 2, ...,$ thus holds for all $x \in \mathbb{C}$ since $g_2(\lambda, \mu)$ is a polynomial. Taking $x = 0$, we obtain $g_2(0, 0) = 0$. Thus $g_2(x, 0) = a_2x^2$, where $a_2 = g_2(1, 0)$. Using this in (4.10), we obtain $g_2(\lambda, \partial) = a_2\lambda(\lambda - \partial)$. $\square$

For later convenience (in order to let other polynomials have some suitable forms), we respectively re-denote a_2 by $\frac{c}{2}$, a_1 by $-\frac{b}{2}$ and replace J_1 by $J_1' = J_1 + \frac{c}{2}\partial J_0 + \frac{b}{2}\partial^2 J_{-1}$, so that $g_2(\lambda, \partial)$, $g_3(\lambda, \partial)$ and $g_4(\lambda, \partial)$ have the following forms,

$$g_2(\lambda, \partial) = c\lambda^2, \quad g_3(\lambda, \partial) = c\lambda^2, \quad g_4(\lambda, \partial) = b\lambda^2\partial \ \text{ for some } \ b, c \in \mathbb{C}. \tag{4.11}$$

Similarly, by (4.7), $h_3(\lambda, \partial)$, $h_4(\lambda, \partial)$ and $h_5(\lambda, \partial)$ define 2-cocycles $\psi_3 \in C^2(\mathfrak{g}, M_{3,0})$, $\psi_4 \in C^2(\mathfrak{g}, M_{2,0})$ and $\psi_5 \in C^2(\mathfrak{g}, M_{1,0})$ respectively with $\mathfrak{g} = \mathrm{Vir} \ltimes M_{4,0}$. Thus using Theorem

3.2, by replacing J_2 by $J_2' = J_2 + p_0(\partial)J_1 + p_1(\partial)J_0 + p_2(\partial)J_{-1}$ for some polynomials $p_0(\partial)$, $p_1(\partial)$, $p_2(\partial)$, we can suppose for some $a_3, t \in \mathbb{C}$,

$$h_3(\lambda, \partial) = 0, \quad h_4(\lambda, \partial) = a_3\lambda^2(\lambda + \partial), \quad h_5(\lambda, \partial) = t\lambda^2(\lambda + \partial)\partial. \tag{4.12}$$

Lemma 4.3 *In* (4.6), *we have* $h_1(\lambda, \partial) = c\lambda(\lambda - 2\partial)$.

Proof. Comparing the coefficients of J_{-1} and J_0 on the both sides of the Jacobi identities $[J_{-1\,\lambda}[J_{-1\,\mu}J_2]] = [J_{-1\,\mu}[J_{-1\,\lambda}J_2]]$ and $[J_{-1\,\lambda}[J_{0\,\mu}J_2]] = [[J_{-1\,\lambda}J_0]_{\lambda+\mu}J_2]] + [J_{0\,\mu}[J_{-1\,\lambda}J_2]]$ respectively, we deduce

$$3\mu g_2(\lambda, \partial) + \lambda h_1(\mu, \lambda + \partial) = 3\lambda g_2(\mu, \partial) + \mu h_1(\lambda, \mu + \partial), \tag{4.13}$$

$$\begin{aligned} &(\lambda + 4\mu + \partial)h_1(\lambda, \partial) + 2\lambda h_3(\mu, \lambda + \partial) \\ &= \lambda h_1(\lambda + \mu, \partial) + 3\lambda g_3(\mu, \partial) + (2\mu + \partial)h_1(\lambda, \mu + \partial). \end{aligned} \tag{4.14}$$

Set $\mu = \partial = 0$ in (4.13), by (4.11), we get $\lambda h_1(0, \lambda) = 2\lambda g_1(0, 0) = 0$, i.e., $h_1(0, \lambda) = 0$. Taking $\partial = 0$ in (4.14), by (4.11) and (4.12), we immediately obtain

$$h_1(\lambda, \mu) = \frac{1}{2\mu}\big((\lambda + 4\mu)h_1(\lambda, 0) - \lambda h_1(\lambda + \mu, 0) - 3c\lambda\mu^2\big). \tag{4.15}$$

Using this and (4.11) in (4.13), we obtain

$$\begin{aligned} &\lambda(\mu + \partial)(4\lambda + \mu + 4\partial)h_1(\mu, 0) + \lambda\mu(\lambda - \mu)h_1(\lambda + \partial + \mu, 0) \\ &= \mu(\lambda + \partial)(\lambda + 4\mu + 4\partial)h_1(\lambda, 0) + 3c\lambda\mu(\mu - \lambda)(\lambda + \partial)(\mu + \partial). \end{aligned}$$

Setting $\partial = -\lambda - 1$, $\mu = 1$, and using $h_1(0, 0) = 0$, we obtain $h_1(\lambda, 0) = \lambda h_1(1, 0) + c\lambda(\lambda - 1)$. Thus (4.15) turns into $h_1(\lambda, \partial) = c\lambda(\lambda - 2\partial) + \frac{3}{2}\lambda(h_1(1, 0) - c)$. Taking $\lambda = 1$ and $\partial = 0$ gives $h_1(1, 0) = c$. Hence we have the lemma. □

For later convenience (in order to let other polynomials have some suitable forms), we re-denote a_3 by $\frac{5k}{3}$, and replace J_2 by $J_2' = J_2 + c\partial J_1 - \frac{k}{3}\partial^2 J_0$, so that by (4.12) and Lemma 4.3, $h_1(\lambda, \partial)$, $h_3(\lambda, \partial)$, $h_4(\lambda, \partial)$, $h_5(\lambda, \partial)$ have the following forms

$$h_1(\lambda, \partial) = 3c\lambda^2, \quad h_3(\lambda, \partial) = 3c\lambda^2, \quad h_4(\lambda, \partial) = k\lambda^3, \quad h_5(\lambda, \partial) = t\lambda^2(\lambda + \partial)\partial. \tag{4.16}$$

Lemma 4.4 *In* (4.6), *we have*

$$h_2(\lambda, \partial) = \frac{1}{2}(k - b - c^2)\lambda\partial^2 - \frac{3}{2}(k - c^2 + b)\lambda^2\partial + \frac{1}{2}(k + b + c^2)\lambda^3. \tag{4.17}$$

$$f_1(\lambda, \partial) = -c(2\lambda\partial + \partial^2). \tag{4.18}$$

Proof. Comparing the coefficients of J_{-1} on the both sides of $[J_{0\lambda}[J_{-1\mu}J_2]] = [[J_{0\lambda}J_{-1}]_{\lambda+\mu}J_2] + [J_{-1\mu}[J_{0\lambda}J_2]]$, we get

$$\begin{aligned}
&3\mu g_4(\lambda,\partial) - g_2(\mu,\partial)h_3(\lambda,\mu+\partial) - \mu h_4(\lambda,\mu+\partial) \\
&= -\mu h_2(\lambda+\mu,\partial) + (4\lambda+\mu+\partial)h_2(\mu,\partial) - (\lambda+\partial)h_2(\mu,\lambda+\partial).
\end{aligned}$$

Since we already get $g_4(\lambda,\partial) = b\lambda^2\partial$, $g_2(\lambda,\partial) = c\lambda^2$, $h_3(\lambda,\partial) = 3c\lambda^2$ and $h_4(\lambda,\partial) = k\lambda^3$, the above equation turns into

$$\mu h_2(\lambda+\mu,\partial) - (4\lambda+\mu+\partial)h_2(\mu,\partial) + (\lambda+\partial)h_2(\mu,\lambda+\partial) = -3b\lambda^2\mu\partial + 3c^2\lambda^2\mu^2 + k\lambda^3\mu.$$

Setting $\mu = 0$ gives $(4\lambda+\partial)h_2(0,\partial) = (\lambda+\partial)h_2(0,\lambda+\partial)$, which means $h_2(0,\partial)$ is divided by $\lambda+\partial$. Thus $h_2(0,\partial) = 0$. Then taking $\lambda = -\mu$ and $\partial = 0$ respectively in the above equation, we obtain

$$\begin{aligned}
&(3\mu-\partial)h_2(\mu,\partial) - (\mu-\partial)h_2(\mu,\partial-\mu) = -3b\mu^3\partial + (3c^2-k)\mu^4, \\
&h_2(\mu,\lambda) = \frac{1}{\lambda}(-\mu h_2(\lambda+\mu,0) + (4\lambda+\mu)h_2(\mu,0) + 3c^2\lambda^2\mu^2 + k\lambda^3\mu).
\end{aligned} \tag{4.19}$$

Using the second equation in the first equation with $\lambda = \partial = \mu$, we obtain

$$h_2(2\mu,0) = 5h_2(\mu,0) + \frac{3}{2}(k+b+c^2)\mu^3. \tag{4.20}$$

If $h_2(\mu,0)$ has degree, say m, greater than 3, then by comparing coefficients of μ^m in (4.20), we obtain $2^m = 5$, a contradiction. Thus $m \leq 3$, and we can suppose $h_2(\mu,0) = d_3\mu^3 + d_2\mu^2 + d_1\mu + d_0$. By (4.20), we immediately get $d_3 = \frac{1}{2}(k+b+c^2)$ and $d_2 = d_1 = d_0 = 0$. Now using this in (4.19), we obtain (4.17).

By comparing the coefficients of J_0 on the both sides of the Jacobi identity

$$[J_{-1\lambda}[J_{1\mu}J_1]] = [[J_{-1\lambda}J_1]_{\lambda+\mu}J_1] + [J_{1\mu}[J_{-1\lambda}J_1]], \tag{4.21}$$

we deduce

$$\begin{aligned}
&(2\mu+\lambda+\partial)h_1(\lambda,\partial) + \lambda g_3(-\mu-\partial,\partial) + \lambda f_1(\mu,\lambda+\partial) \\
&= \lambda g_3(\lambda+\mu,\partial) + (\lambda+\mu)g_2(\lambda,-\lambda-\mu) + (\mu+\partial)g_2(\lambda,\mu+\partial).
\end{aligned}$$

Taking $\partial = 0$ and sing $g_3(\lambda,\partial) = c\lambda^2$, $g_2(\lambda,\partial) = c\lambda^2$ and $h_1(\lambda,\partial) = 3c\lambda^2$, we immediately obtain (4.18). □

Lemma 4.5 *We have* $f_2(\lambda,\partial) = \frac{2b}{5}(2\lambda\partial^2 + \partial^3)$, *and* $k = c^2 - \frac{7}{5}b$.

Proof. Comparing the coefficients of J_{-1} on the both sides of (4.21), we deduce

$$\begin{aligned}
&2(2\mu+\lambda+\partial)h_2(\lambda,\partial)+f_1(\mu,\lambda+\partial)g_2(\lambda,\partial)+\lambda f_2(\mu,\lambda+\partial)\\
&=2\lambda g_4(\lambda+\mu,\partial)-2\lambda g_4(-\mu-\partial,\partial)-g_2(\lambda,\mu+\partial)g_2(-\mu-\partial,\partial)\\
&\quad+g_2(\lambda,-\lambda-\mu)g_2(\lambda+\mu,\partial).
\end{aligned}$$

Using (4.11), (4.16) and (4.18), we obtain

$$\lambda f_2(\mu,\lambda+\partial)=-\lambda(\lambda+2\mu+\partial)\big((k+b-c^2)(\lambda^2+\partial^2)+(-5b+3c^2-3k)\lambda\partial\big).$$

Taking $\partial=0$ and $\partial=\lambda$ respectively, we immediately obtain the result. □

Now we collect information we obtain in Lemmas 4.1–4.5 as follows

$$\begin{aligned}
&g_1(\lambda,\partial)=0,\qquad g_2(\lambda,\partial)=c\lambda^2,\qquad g_3(\lambda,\partial)=c\lambda^2,\qquad g_4(\lambda,\partial)=b\lambda^2\partial,\\
&h_1(\lambda,\partial)=3c\lambda^2,\quad h_2(\lambda,\partial)=-\frac{6}{5}b\lambda\partial^2+\frac{3}{5}b\lambda^2\partial+(c^2-\frac{1}{5}b)\lambda^3,\quad h_3(\lambda,\partial)=3c\lambda^2,\\
&h_4(\lambda,\partial)=(-\frac{7}{5}b+c^2)\lambda^3,\quad h_5(\lambda,\partial)=t(\lambda^2\partial^2+\lambda^3\partial),\\
&f_1(\lambda,\partial)=-c(2\lambda\partial+\partial^2),\quad f_2(\lambda,\partial)=\frac{2}{5}b(\partial^3+2\lambda\partial^2).
\end{aligned}\tag{4.22}$$

Next we should determine $f_3(\lambda,\partial)$. By comparing the coefficients of J_{-1} on the both sides of the Jacobi identity $[J_{0\lambda}[J_{1\mu}J_1]]=[[J_{0\lambda}J_1]_{\lambda+\mu}J_1]+[J_{1\mu}[J_{0\lambda}J_1]]$, we obtain that

$$\begin{aligned}
&2(2\mu+\lambda+\partial)h_5(\lambda,\partial)+f_1(\mu,\lambda+\partial)g_4(\lambda,\partial)+(\lambda+\partial)f_3(\mu,\lambda+\partial)\\
&=(2\lambda-\mu)f_3(\lambda+\mu,\partial)+g_3(\lambda,-\lambda-\mu)g_4(\lambda+\mu,\partial)+g_4(\lambda,-\lambda-\mu)g_2(\lambda+\mu,\partial)\\
&\quad+(3\lambda+\mu+\partial)f_3(\mu,\partial)-g_3(\lambda,\mu+\partial)g_4(-\mu-\partial,\partial)-g_4(\lambda,\mu+\partial)g_2(-\mu-\partial,\partial).
\end{aligned}\tag{4.23}$$

Lemma 4.6 *We have* $f_3(\lambda,\partial)=-bc(\partial^4+3\lambda\partial^3+3\lambda^2\partial^2+2\lambda^3\partial)$ *and* $t=\frac{3}{2}bc$.

Proof. Using (4.22) in (4.23) gives

$$\begin{aligned}
&(\lambda+\partial)f_3(\mu,\lambda+\partial)-(2\lambda-\mu)f_3(\lambda+\mu,\partial)-(3\lambda+\mu+\partial)f_3(\mu,\partial)\\
&=-\lambda^2(\lambda+2\mu+\partial)\big(bc(\lambda^2+\lambda\mu+\mu^2-3\lambda\partial+\mu\partial+\partial^2)+2t(\partial^2+\lambda\partial)\big).
\end{aligned}\tag{4.24}$$

Taking $\partial=0$ gives

$$\begin{aligned}
f_3(\mu,\lambda)=&\frac{1}{\lambda}\big((2\lambda-\mu)f_3(\lambda+\mu,0)-(3\lambda+\mu)f_3(\mu,0)\big)\\
&-\lambda(\lambda+2\mu)bc(\lambda^2+\lambda\mu+\mu^2).
\end{aligned}\tag{4.25}$$

Using this in (4.24) with $\partial = -\mu = -\lambda$, we obtain $f_3(2\lambda, 0) = -10f_3(\lambda, 0) - 9f_3(0, 0)$. If $f_3(\lambda, 0)$ has degree, say m, greater than 0, then by comparing coefficients of λ^m, we obtain $2^m = -10$, a contradiction. Thus $m = 0$, and so $f_3(\lambda, 0) = f_3(0, 0) = 0$. Now using this in (4.25), we obtain that $f_3(\lambda, \partial) = -bc(2\lambda + \partial)\partial(\partial^2 + \lambda\partial + \lambda^2)$. Using this in (4.24) gives $t = \frac{3}{2}bc$. Therefore the lemma holds. $\square$

In order to prove Theorem 1.1(1), we need the following lemma whose proof seems to be rather technical.

Lemma 4.7 *In* (4.22), *we have* $b = 0$.

Proof. First we assume $c = 0$. By (2.7), we can suppose

$$[J_{1\lambda}J_2] = (5\lambda + 2\partial)J_3 + f_4(\lambda, \partial)J_2 + f_5(\lambda, \partial)J_1 + f_6(\lambda, \partial)J_0 + f_7(\lambda, \partial)J_{-1},$$

where, $f_i(\lambda, \partial)$ are polynomials of λ and ∂ for $4 \leq i \leq 7$. From (2.3), we can get the Jacobi identity

$$[J_{1\lambda}[J_{1\mu}J_1]] = [[J_{1\lambda}J_1]_{\lambda+\mu}J_1] + [J_{1\mu}[J_{1\lambda}J_1]]. \tag{4.26}$$

Since our purpose is to prove $b = 0$, by comparing the coefficients of J_1 and J_{-1} on the both sides of (4.26) respectively, we see that the coefficients of terms of $f_5(\lambda, \partial)$ (resp., $f_7(\lambda, \partial)$) with degree not equal to 3 (resp., 5) do not have relations with b. Thus we can suppose that $f_5(\lambda, \partial) = \sum_{i=0}^{3} a_i\lambda^{3-i}\partial^i$ and $f_7(\lambda, \partial) = \sum_{i=0}^{5} b_i\lambda^{5-i}\partial^i$. Set $J_3' = J_3 + c_1\partial^2 J_1 + c_2\partial^4 J_{-1}$ for some c_1, $c_2 \in \mathbb{C}$, then we can choose suitable complex numbers c_1 and c_2 such that by replacing J_3 by J_3', we can suppose $a_3 = 0$, $b_5 = 0$. Thus

$$f_5(\lambda, \partial) = \sum_{i=0}^{2} a_i\lambda^{3-i}\partial^i, \quad f_7(\lambda, \partial) = \sum_{i=0}^{4} b_i\lambda^{5-i}\partial^i. \tag{4.27}$$

Since $c = 0$, by (4.22) and Lemma 4.6, comparing the coefficients of J_1 and J_{-1} on the both sides of (4.26) respectively, we obtain that

$$\begin{aligned}
[J_{1\lambda}J_2] = {} & (5\lambda + 2\partial)J_3 + f_4(\lambda, \partial)J_2 + (a_0\lambda^3 + (a_0 - b)\lambda^2\partial + a_2\lambda\partial^2)J_1 \\
& + f_6(\lambda, \partial)J_0 + (\frac{1}{5}b^2\lambda^4\partial + b_2\lambda^3\partial^2 + (b_2 + \frac{2}{5}b^2)\lambda^2\partial^3)J_{-1}.
\end{aligned} \tag{4.28}$$

By (2.3), we also have the Jacobi identity

$$[J_{0\lambda}[J_{1\mu}J_2]] = [[J_{0\lambda}J_1]_{\lambda+\mu}J_2] + [J_{1\mu}[J_{0\lambda}J_2]]. \tag{4.29}$$

By (4.22) (with $c = 0$) and (4.28), we obtain

$$(5\mu + 2\lambda + 2\partial)[J_{0\lambda} J_3] = (5\mu + 2\lambda + 2\partial)(5\lambda + \partial)J_3 + l_1(\lambda, \mu, \partial)J_2 + l_2(\lambda, \mu, \partial)J_1 \\ + l_3(\lambda, \mu, \partial)J_0 + l_4(\lambda, \mu, \partial)J_{-1}, \tag{4.30}$$

where $l_i(\lambda, \mu, \partial)$ for $1 \leq i \leq 4$ are polynomials of λ, μ, ∂. Using (4.29), by a little lengthy calculation, we also get

$$l_2(\lambda, \mu, \partial) = -\frac{1}{5}\lambda^2\big((15b - 10a_0)\lambda^2 + (51b - 25a_0 + 15a_2)\lambda\mu + (24b - 10a_0)\lambda\partial \\ + (15b - 15a_0 + 35a_2)\mu\partial - 10a_2\partial^2\big), \tag{4.31}$$

$$l_4(\lambda, \mu, \partial) = \frac{1}{5}\lambda^2\big(b^2\lambda^4 + 4b^2\lambda^3\mu + (4b^2 - 5b_2)\lambda^2\mu^2 + (4b^2 - 5b_2)\lambda\mu^3 - b^2\lambda^3\partial \\ - (2b^2 + 5ba_2)\lambda^2\mu\partial + (3b^2 - 5ba_0 - 20b_2)\lambda\mu^2\partial - (b^2 + 5ba_0 + 15b_2)\mu^3\partial \\ + (6b^2 + 10b_2)\lambda^2\partial^2 + (26b^2 - 10ba_2 + 25b_2)\lambda\mu\partial^2 - (b^2 + 5ba_0 + 15b_2)\mu^2\partial^2 \\ + (11b^2 + 10b_2)\lambda\partial^3 + (6b^2 - 5ba_2 + 15b_2)\mu\partial^3\big). \tag{4.32}$$

From (4.30), we know that $5\mu+2\lambda+2\partial$ must be a common factor of $l_2(\lambda, \mu, \partial)$ and $l_4(\lambda, \mu, \partial)$. Therefore by (4.31) and (4.32), we can deduce that b must be zero.

Now assume $c \neq 0$. By comparing the coefficients of J_{-1} on the both sides of (4.26), we obtain

$$b_0 = 0, \quad b_1 = \frac{1}{5}b^2 - \frac{5}{2}bc^2, \quad b_2 = -\frac{2}{5}b^2, \quad b_3 = -\frac{1}{2}bc^2.$$

Then by comparing the coefficients of J_{-1} on the both sides of (4.30) (this time, instead of (4.32), we need to assume that $l_4(\lambda, \mu, \partial) = (5\mu + 2\lambda + 2\partial)\sum_{i=0}^{5} d_i\lambda^{5-i}\partial^i)$, we obtain an equation, then by comparing coefficients of ∂^6, ∂^5, ∂^4 in this equation, we can obtain $d_5 = d_4 = b_4 = 0$ and $bc = 0$. Thus $b = 0$. □

Now we are ready to prove the main result of this paper.

Proof of Theorem 1.1. By (4.22), Lemma 4.6 and Lemma 4.7, we have

$$\begin{aligned} &[J_{-1\lambda} J_{-1}] = 0, && [J_{-1\lambda} J_0] = \lambda J_{-1}, \\ &[J_{0\lambda} J_0] = (2\lambda + \partial)J_0, && [J_{-1\lambda} J_1] = 2\lambda J_0 + c\lambda^2 J_{-1}, \\ &[J_{0\lambda} J_1] = (3\lambda + \partial)J_1 + c\lambda^2 J_0, && [J_{-1\lambda} J_2] = 3\lambda J_1 + 3c\lambda^2 J_0 + c^2\lambda^3 J_{-1}, \\ &[J_{0\lambda} J_2] = (4\lambda + \partial)J_2 + 3c\lambda^2 J_1 + c^2\lambda^3 J_0, && [J_{1\lambda} J_1] = 2(2\lambda + \partial)J_2 - c(2\lambda\partial + \partial^2)J_1, \end{aligned} \tag{4.33}$$

for some $c \in \mathbb{C}$. If $c \neq 0$, by replacing J_{-1} and J_1 by $J'_{-1} = -cJ_{-1}$ and $J'_1 = -c^{-1}J_1$ respectively, we can suppose $c = -1$. Thus, we can suppose c is either equal to 0 or -1. We want to prove

$$[J_{m_\lambda} J_n] = \begin{cases} \Big((m+n+2)\lambda + (m+1)\partial\Big) J_{m+n} & \text{if } \ c=0, \\ \sum\limits_{s=0}^{m} \binom{m+1}{s+1}(\lambda+\partial)^{s+1} J_{m+n-s} - \sum\limits_{s=0}^{n} \binom{n+1}{s+1}(-\lambda)^{s+1} J_{m+n-s} & \text{if } \ c=-1, \end{cases} \tag{4.34}$$

for $m,n \geq -1$. By (4.33), we see (4.34) holds for all m,n with $\max\{m,n,n+m\} \leq 2$. Now inductively assume that for $N \geq 2$, (4.34) holds for all m,n with $\max\{m,n,n+m\} \leq N$. Denote the right hand side of (4.34) by $\sum_{k=-1}^{m+n} F_{m,n,k}(\lambda,\partial) J_k$ for some polynomials $F_{m,n,k}(\lambda,\partial)$. Assume

$$[J_{1_\lambda} J_N] - \sum_{k=-1}^{N+1} F_{1,N,k}(\lambda,\partial) J_k = \sum_{k=-1}^{N} p_k(\lambda,\partial) J_k, \tag{4.35}$$

$$[J_{0_\lambda} J_{N+1}] - \sum_{k=-1}^{N+1} F_{0,N+1,k}(\lambda,\partial) J_k = \sum_{k=-1}^{N} q_k(\lambda,\partial) J_k, \tag{4.36}$$

$$[J_{-1_\lambda} J_{N+1}] - \sum_{k=-1}^{N} F_{-1,N+1,k}(\lambda,\partial) J_k = \sum_{k=-1}^{N-1} r_k(\lambda,\partial) J_k, \tag{4.37}$$

for some polynomials $p_k(\lambda,\partial), q_k(\lambda,\partial), r_k(\lambda,\partial)$. Applying the μ-brackets J_{0_μ}, J_{-1_μ} to (4.35) respectively, using inductive assumption, we obtain

$$\begin{aligned} -F_{1,N,N+1}(\lambda,\partial) \sum_{k=-1}^{N} q_k(\mu,\partial) J_k &= \sum_{k=-1}^{N} p_k(\lambda,\partial) \sum_{k'=-1}^{k} F_{0,k,k'}(\mu,\partial) J_{k'} \\ &= \sum_{k=-1}^{N} \Big(\sum_{k'=k}^{N} p_{k'}(\lambda,\partial) F_{0,k',k}(\mu,\partial)\Big) J_k, \end{aligned} \tag{4.38}$$

$$\begin{aligned} -F_{1,N,N+1}(\lambda,\partial) \sum_{k=-1}^{N-1} r_k(\mu,\partial) J_k &= \sum_{k=-1}^{N} p_k(\lambda,\partial) \sum_{k'=-1}^{k-1} F_{-1,k,k'}(\mu,\partial) J_{k'} \\ &= \sum_{k=-1}^{N-1} \Big(\sum_{k'=k+1}^{N} p_{k'}(\lambda,\partial) F_{-1,k',k}(\mu,\partial)\Big) J_k. \end{aligned} \tag{4.39}$$

Note from the right hand side of (4.34) that $F_{1,N,N+1}(\lambda,\partial) = (N+3)\lambda + 2\partial$, and $F_{-1,k',k}(\mu,\partial) = \mu F'_{k',k}(\mu)$ for some polynomial $F'_{k',k}(\mu)$ on μ. Comparing the coefficients of J_k for $k = N, N-1, ...,$ in (4.39) shows that μ and $(N+3)\lambda+2\partial$ are factors of $r_k(\mu,\partial)$ and $p_k(\lambda,\partial)$ respectively. Thus $r_k(\mu,\partial) = \mu r'_k(\mu,\partial)$ and $p_k(\lambda,\partial) = ((N+3)\lambda+2\partial)p'_k(\lambda,\partial)$ for some $r'_k(\mu,\partial)$ and $p'_k(\lambda,\partial)$. Furthermore, we see that $r'_k(\mu,\partial) = r'_k(\partial)$ and $p'_k(\lambda,\partial) = p'_k(\partial)$ do not depend on λ,μ. Thus in (4.38), if we replace J_{N+1} by $J_{N+1} - \sum_{k=-1}^{N} p'_k(\partial) J_k$, we see that the right hand side of (4.38) becomes zero, i.e., by re-choosing the generator J_{N+1}, we can suppose all

$p_k(\lambda,\partial)=0$. Then (4.38) and (4.39) show that all $q_k(\mu,\partial), r_k(\mu,\partial)$ are zero. Hence (4.34) holds for all m,n with $\max\{m,n,n+m\}=N+1$ and $m\le 1$. To prove (4.34) for $2\le m\le n$ with $m+n=N+1$, we use $J_m=\frac{1}{F_{1,m-1,m}(\mu,\partial)}\big([J_{1_\mu}J_{m-1}]-\sum_{k=-1}^{m-1}F_{1,m-1,k}(\mu,\partial)J_k\big)$, and Jacobi identity and induction on m. This proves (4.34) and Theorem 1.1(1). $\square$

§5. Proof of Theorem 1.1(2)

Assume V is a finitely freely $\mathbb{C}[\partial]$-generated nontrivial $\mathrm{gr}\, gc_1$-module. Regarding V as a module over Vir, by [6, Theorem 3.2(1)], we can choose a composition series,

$$V=V_N\supset V_{N-1}\supset\cdots\supset V_1\supset V_0=0,$$

such that for each $i=1,2,...,N$, the composition factor $\overline{V}_i=V_i/V_{i-1}$ is either a rank one free module M_{Δ_i,α_i} with $\Delta_i\neq 0$, or else a 1-dimensional trivial module $\mathbb{C}_{\alpha_i}$ with trivial λ-action and with ∂ acting as the scalar α_i. Denote by $\bar v_i$ a $\mathbb{C}[\partial]$-generator of $\overline{V}_i$ and $v_i\in V_i$ the preimage of $\bar v_i$. Then $\{v_i\,|\,1\le i\le N\}$ is a $\mathbb{C}[\partial]$-generating set of V, such that the λ-action of J_0 on v_i is a $\mathbb{C}[\lambda,\partial]$-combination of $v_1,...,v_i$.

Lemma 5.1 *For all $i\gg 0$, the λ-action of J_i on v_1 is trivial, namely, $J_{i\lambda}v_1=0$.*

Proof. Assume $i\gg 0$ is fixed and suppose $J_{i\lambda}v_1\neq 0$, and let $k_i\ge 1$ be the largest integer such that $J_{i\lambda}v_1\not\subset V_{k_i-1}$. We consider the following possibilities.

Case 1. $V_1=M_{\Delta_1,\alpha_1}$, $\overline{V}_{k_i}=M_{\Delta_{k_i},\alpha_{k_i}}$.

We can write

$$J_{i\lambda}v_1\equiv p_i(\lambda,\partial)v_{k_i}\ (\mathrm{mod}\, V_{k_i-1})\ \text{ for some }\ p_i(\lambda,\partial)\in\mathbb{C}[\lambda,\partial]. \tag{5.1}$$

Applying the operator J_{0_μ} to (5.1), we obtain

$$p_i(\lambda,\mu+\partial)(\alpha_{k_i}+\partial+\Delta_{k_i}\mu)=((1+i)\mu-\lambda)p_i(\lambda+\mu,\partial)+(\alpha_1+\lambda+\partial+\Delta_1\mu)p_i(\lambda,\partial). \tag{5.2}$$

Letting $\partial=0$, we obtain

$$p_i(\lambda,\mu)=-\frac{1}{\alpha_{k_i}+\Delta_{k_i}\mu}\Big(((1+i)\mu-\lambda)p_i(\lambda+\mu,0)+(\alpha_1+\lambda+\Delta_1\mu)p_i(\lambda,0)\Big). \tag{5.3}$$

Using this in (5.2) with $\lambda=(1+i)\mu$ and $\partial=-\alpha_{k_i}-\Delta_{k_i}\mu$, we obtain

$$\begin{aligned}&(i+1)\big((\Delta_{k_i}+1)\mu+\alpha_{k_i}\big)p_i\big((i+1-\Delta_{k_i})\mu-\alpha_{k_i},0\big)\\&=\big((i+1-\Delta_1\Delta_{k_i})\mu+\alpha_1-\alpha_{k_i}\Delta_1\big)p_i\big((i+1)\mu,0\big).\end{aligned}$$

Suppose $p_i(\lambda,0)$ has degree m_i. Comparing the coefficients of μ^{m_i+1} in the above equation, we obtain (note that the following equation does not depend on the coefficients of $p_i(\lambda,\mu)$)

$$(i+1)(\Delta_{k_i}+1)(i+1-\Delta_{k_i})^{m_i}=(i+1-\Delta_1\Delta_{k_i})(i+1)^{m_i}. \tag{5.4}$$

When i is sufficient large, one can easily see that (5.4) cannot hold if $m_i > 1$ (note that $\Delta_1, \Delta_{k_i} \neq 0$, and Δ_{k_i} has only a finite possible choices since $1 \leq k_i \leq N$). Thus $m_i \leq 1$ if $i \gg 0$. Then from (5.3), we obtain that $p_i(\lambda, \mu)$ is a polynomial of degree ≤ 1. Thus suppose $p_i(\lambda, \mu) = a_{i,0} + a_{i,1}\lambda + a_{i,2}\mu$. Then (5.2) immediately gives $p_i(\lambda, \mu) = 0$.

Case 2. $V_1 = \mathbb{C}_{\alpha_1}$, $\overline{V}_{k_i} = M_{\Delta_{k_i}, \alpha_{k_i}}$.

In this case, we can still assume (5.1). Applying the operator $J_{0\mu}$ to (5.1), we obtain

$$p_i(\lambda, \mu + \partial)(\alpha_{k_i} + \partial + \Delta_{k_i}\mu) = ((1+i)\mu - \lambda)p_i(\lambda + \mu, \partial). \tag{5.5}$$

Letting $\mu = \partial = 0$, we obtain $p_i(\lambda, 0) = 0$. Then letting $\partial = 0$, we obtain $p_i(\lambda, \mu) = 0$.

Case 3. $V_1 = M_{\Delta_1, \alpha_1}$, $\overline{V}_{k_i} = \mathbb{C}_{\alpha_{k_i}}$.

In this case, since ∂ acts on $\bar{v}_{k_i}$ as the scalar α_{k_i}, i.e., $\partial v_{k_i} \equiv \alpha_{k_i} v_{k_i} \,(\mathrm{mod}\, V_{k_i - 1})$, we can write

$$J_{1\lambda} v_1 \equiv p_i(\lambda) v_{k_i} \,(\mathrm{mod}\, V_{k_i-1}) \ \text{ for some } \ p_i(\lambda) \in \mathbb{C}[\lambda]. \tag{5.6}$$

Applying the operator $J_{0\mu}$ to (5.6), we obtain

$$0 = ((1+i)\mu - \lambda)p_i(\lambda + \mu) + (\alpha_1 + \lambda + \partial + \Delta_1\mu)p_i(\lambda). \tag{5.7}$$

By comparing the coefficients of ∂, we immediately obtain $p_i(\lambda) = 0$.

Case 4. $V_1 = \mathbb{C}_{\alpha_1}$, $\overline{V}_{k_i} = \mathbb{C}_{\alpha_{k_i}}$.

As above, we immediately obtain $p_i(\lambda) = 0$. $\square$

By induction on $j \leq N$, we obtain $J_{i\lambda} v_j = 0$, i.e., the λ-action of J_i is trivial. From this, we immediately obtain that the λ-action of $\mathrm{gr}\, gc_1$ on V is trivial since $\mathrm{gr}\, gc_1$ is a simple conformal algebra. This proves Theorem 1.1(2).

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