LEX COLIMITS

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ABSTRACT. Many kinds of categorical structure require the existence of finite limits, of colimits of some specified type, and of "exactnesses" between the finite limits and the specified colimits. Some examples are the notions of regular, or Barr-exact, or lextensive, or coherent, or adhesive category. We introduce a general notion of exactness, of which each of the structures listed above, and others besides, are particular instances. The notion can be understood as a form of cocompleteness "in the lex world"—more precisely, in the 2-category of finitely complete categories and finite-limit preserving functors.

1. INTRODUCTION

Amongst the range of structures which it has been found mathematically useful to impose upon a category, we find a number which share the following common form. One requires the provision of finite limits; the provision of colimits of some specified type; and the validation of certain compatibilities between the finite limits and the specified colimits. For example, in asking that a category be *lextensive*, or *regular*, or *Barr-exact*, or *coherent*, or *adhesive*, we are asking for structure of this form, where the colimits in question comprise the finite coproducts, or the coequalisers of kernel-pairs, or the coequalisers of equivalence relations, or the coequalisers of kernel-pairs and the finite unions of subobjects, or the pushouts along monomorphisms. Though the precise nature of the limit-colimit compatibilities required varies from case to case, it is understood that these too share a common form—roughly speaking, they are just those compatibilities between the finite limits and the specified colimits which hold in the category of sets; more generally, in any Grothendieck topos; more generally still, in any ∞-pretopos.

The purpose of this paper is to describe a body of results which explains these similarities of form, by exhibiting each of the structures listed above as particular instances of a common notion: this notion being one of "cocompleteness in the lex world". Let us say a few words about what we mean by this. The term "lex" is here used with the meaning of "preserving finite limits"; the etymology of this usage is slightly complicated. Originally, an additive functor was called "left exact" if it preserved exact sequences on the left; equivalently, if it preserved kernels; equivalently, if it preserved all finite limits. Then "left exact" was abbreviated to "lex" and finally this came to be used to refer to the preservation of finite limits even in the non-additive context. There is a 2-category **LEX** comprising the finitely complete categories, the left exact functors and the natural transformations between them, and in working in this 2-category, we may consider that we are working "in the lex world". Thus in speaking of "cocompleteness in the lex world", we intend to refer to a notion of cocompleteness internal to this 2-category **LEX**.

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Let us now expand on how such a notion permits a uniform description of each of the structures listed above. There are two aspects to this. On the one hand, we must be able to express the kinds of cocompleteness appearing in our examples. On the other, we must be able to capture the corresponding limit-colimit compatibilities.

Regarding the first of these, we introduce the following concepts. By a *class of weights for lex colimits*, we mean a collection Φ of functors $\varphi\colon \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$, with each $\mathscr{K}$ small and finitely complete; and by saying that a finitely complete category $\mathscr{C}$ is Φ-*lex-cocomplete*, we mean that, for every $\varphi\colon \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$ in Φ and every finite-limit preserving $D\colon \mathscr{K} \to \mathscr{C}$, the weighted colimit $\varphi \star D$ exists in $\mathscr{C}$. For instance, if Φ_{ex} consists of the single functor $\varphi\colon \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathscr{K}$ is the free category with finite limits generated by an equivalence relation $(s,t)\colon R \rightarrowtail A\times A$, and where φ is the coequaliser in $[\mathscr{K}^{\mathrm{op}}, \mathbf{Set}]$ of the maps $\mathscr{K}(–, s)$ and $\mathscr{K}(–, t)$, then for a finitely complete category $\mathscr{C}$ to be Φ_{ex}-lex-cocomplete is for it to admit coequalisers of equivalence relations. Similarly, if Φ_{reg} consists of the single functor $\varphi\colon \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathscr{K}$ is the free category with finite limits on an arrow $f\colon X \to Y$, and where φ is the coequaliser in $[\mathscr{K}^{\mathrm{op}}, \mathbf{Set}]$ of the kernel pair of $\mathscr{K}(–, f)$, then for a finitely complete category $\mathscr{C}$ to be Φ_{reg}-lex-cocomplete is for it to admit coequalisers of kernel-pairs. In a similar way, we may express the having of finite coproducts, or of unions of subobjects, or of pushouts along monomorphisms, as notions of Φ-lex-cocompleteness for suitable classes Φ.

Our second problem is that of determining, for a given class Φ, the appropriate compatibilities to be imposed between finite limits and Φ-lex-colimits. In anticipation of a successful resolution to this, we reserve the term Φ-*exact* for a finitely complete and Φ-lex-cocomplete category satisfying these—as yet undetermined—compatibilities. In due course, we will imbue this term with meaning in such a way as to capture perfectly our examples: so that, for instance, a category is Φ_{reg}-exact just when it is regular, or Φ_{ex}-exact just when it is Barr-exact. It will turn out that there are several ways of characterising the notion of Φ-exactness. One of these is that a *small* $\mathscr{C}$ is Φ-exact just when it admits a full embedding into a Grothendieck topos via a functor preserving finite limits and Φ-lex-colimits: this captures the idea, stated above, that the limit-colimit compatibilities we impose should be just those that obtain in any Grothendieck topos. A second characterisation of Φ-exactness, valid for categories of any size, is given in terms of the Φ-*exact completion* of a finitely complete category $\mathscr{C}$: this being the category $\Phi_l\mathscr{C}$ obtained by closing the representables in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ under finite limits and Φ-colimits. Now the finitely complete $\mathscr{C}$ is Φ-exact just when the restricted Yoneda embedding $\mathscr{C} \to \Phi_l\mathscr{C}$ admits a finite-limit preserving left adjoint. This means that $\mathscr{C}$ reflective in $\Phi_l\mathscr{C}$ via a finite-limit-preserving reflector, and so as lex-cocomplete as $\Phi_l\mathscr{C}$ is; in particular, Φ-lex-cocomplete. Moreover, the same compatibilities between finite limits and Φ-lex-colimits as are affirmed in $\mathbf{Set}$ must be also affirmed in $\Phi_l\mathscr{C}$—since these limits and colimits are pointwise—and so also in $\mathscr{C}$. Thus this characterisation of Φ-exactness also accords with our motivating description.

Yet neither of the two *characterisations* of Φ-exactness given above are satisfactory as a *definition* of it: for whilst justifiable by their describing correctly the examples we have in mind, they fail to capture the essence of what Φ-exactness is. As anticipated above, this essence resides in the claim that Φ-exactness is a transposition "into the lex world" of the notion of cocompleteness with respect to a class of colimits. The force of this claim is most easily appreciated if we adopt the perspective of monad theory; in preparation for which, we first recast the standard notions of cocompleteness in these terms.

When we say that a category is cocomplete, we are asserting a property, but this property can be made into a structure: that of being equipped with a choice of colimits. This structure is algebraic, in the sense that there is a pseudomonad $\mathscr{P}$ on **CAT**, the 2-category of locally small categories, whose pseudoalgebras are categories equipped with such colimit structure; the value of $\mathscr{P}$ at a category $\mathscr{C}$ being given by the closure of the representables in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ under small colimits. This same perspective applies also to notions of partial cocompleteness. For any class of weights Φ—now meaning simply a collection of presheaves with small domain—we may again regard the property of being Φ-cocomplete, that is, of admitting φ-weighted colimits for all $\varphi \in \Phi$, as algebraic structure: we have a pseudomonad Φ on **CAT** whose pseudoalgebras are categories equipped with Φ-colimits. We could again describe this pseudomonad directly—its value at a category $\mathscr{C}$ being the closure of the representables in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ under Φ-colimits—but more pertinently, could also derive it from the pseudomonad $\mathscr{P}$: it is the smallest full submonad $\mathscr{Q}$ of $\mathscr{P}$ such that $\varphi \in \mathscr{Q}\mathscr{K}$ for all $\varphi \colon \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$ in the class Φ. In fact, we may recast even the definition of a class of weights solely in terms of $\mathscr{P}$; it is given by the specification, for each small category $\mathscr{K}$, of a full subcategory of $\mathscr{P}\mathscr{K}$. In other words, once we have the pseudomonad $\mathscr{P}$, representing a notion of cocompleteness, the corresponding notion of partial cocompleteness may be derived by a purely formal 2-categorical process.

This last observation allows us to give form to our claim that Φ-exactness constitutes a transposition "into the lex world" of the standard notion of cocompleteness with respect to a class of weights. We will exhibit a pseudomonad $\mathscr{P}_l$ on the 2-category **LEX**—as defined above—which represents a notion of small-exactness; its pseudoalgebras are the ∞-pretoposes. This $\mathscr{P}_l$ is in fact nothing other than the restriction and corestriction of $\mathscr{P}$ from **CAT** to **LEX**, and so represents an entirely canonical notion of "cocompleteness in the lex world". Now applying the formal 2-categorical process described above, we obtain the corresponding notion of "partial cocompleteness in the lex world": and this will constitute our definition of Φ-exactness.

(Let us remark here that our approach is related to, but different from, the work on *Yoneda structures* described in [30]. There, too, the authors consider an operation $\mathscr{P}$—this being one part of the definition of a Yoneda structure—which, in the case of their Example 7.3, resides on **LEX**. As for any Yoneda structure, one may define notions of cocompleteness or partial cocompleteness with respect to this $\mathscr{P}$; however, the notions so arising are not the same as our small-exactness or Φ-exactness. For instance, as was stated above, to be small-exact in our sense is to be an ∞-pretopos. The corresponding notion in the framework of [30] is the stronger one of being *lex-total* [28].)

As we have already said, the notion of Φ-exactness captures perfectly our motivating examples: however, it also allows us to move beyond those examples. For instance, we shall see that when Φ comprises the weights for finite unions, a category is Φ-exact just when those unions are stable under pullback and *effective*—calculated as the pushout over the intersection; and similarly, that if Φ comprises the weight for reflexive coequalisers, then a category is Φ-exact just when it is Barr-exact and the free equivalence relation on a reflexive one is calculated in the same way as it is in **Set**. Let us be clear that the properties just listed are neither new nor unexpected: what *is* new is the understanding that the structures they define stand on an equal footing with our motivating ones.

One further pleasant aspect of our theory is that it works just as well for enriched as for ordinary categories. In the $\mathscr{V}$-categorical setting, the notion of Φ-exactness involves inheriting limit-colimit compatibilities from $\mathscr{V}$, rather than from **Set**, which on a concrete level may cause the theory to look quite different, for different choices of $\mathscr{V}$. Our formal development will be given in the enriched context from the outset; when it comes to examples, however, we shall limit the scope of this paper to the motivating case $\mathscr{V} = \mathbf{Set}$, leaving applications over other bases for future investigation. Let us at least remark that amongst these applications are the case $\mathscr{V} = \mathbf{Ab}$—which should allow us to capture the various exactness notions of [12]—and the case $\mathscr{V} = \mathbf{Cat}$—which should allow us to provide a clear conceptual basis for various forms of 2-categorical exactness [4, 5, 7, 29]

Let us now give a brief overview of the content of this paper. We begin in Section 2 by recalling the construction of the pseudomonad $\mathscr{P}$ on $\mathscr{V}$-**CAT**, and describing its lifting to a pseudomonad $\mathscr{P}_l$ on $\mathscr{V}$-**LEX**. In Section 3, we go on to consider full submonads of $\mathscr{P}_l$, so arriving at our definition of Φ-exactness. Then in Section 4, we give the characterisation theorem described above—which, in the enriched context, states that a small, Φ-lex-cocomplete $\mathscr{C}$ is Φ-exact just when it admits an embedding in a "$\mathscr{V}$-topos"; that is, a $\mathscr{V}$-category reflective in a presheaf category by a finite-limit-preserving reflector. The more involved parts of the proof are deferred to Section 7 and an Appendix. In Section 5, we break off from the general theory in order to give a body of examples; as remarked above, these examples will be concerned solely with the case where $\mathscr{V} = \mathbf{Set}$. In Section 6, we show that, again in the case $\mathscr{V} = \mathbf{Set}$, it is possible to give a concrete characterisation of Φ-exactness for an *arbitrary* Φ, in terms of Anders Kock's notion of *postulated colimit* [21]. Then in Section 7, we resume our development of the general theory, describing the construction of *relative completions*: that is, of the free Ψ-exact category on a Φ-exact one, for suitable classes of weights Φ and Ψ. Finally, an Appendix proves some necessary technical results concerning localisations of locally presentable categories.

Acknowledgements. We should say some words about the prehistory of this project. Max Kelly observed that the regular completion and the exact completion of a category with finite limits [13] can be computed as full subcategories of the presheaf category. He proposed that this should be explained by the fact that these were "free cocompletions in the lex world", and he observed that the existence of coequalisers of equivalence relations could be seen as Φ-lex-cocompleteness, in essentially the same sense considered here, for a suitably chosen class of weights Φ. He planned to study this with the second-named author, with a view to explaining the construction of [13], but other things intervened and the project never progressed very far. Some years later, the first-named author encountered a remark asserting the existence of the project in the second author's [23]; observing that this was, in fact, the only trace of its existence, and perceiving how some basic aspects of the theory should go, he made contact with the second author and work on the project was begun anew, resulting in the present article. Let us observe that Kelly's original goal is fulfilled by our Corollary 3.7.

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2. Cocomplete and small-exact categories

In this section, we recall the construction of the free cocompletion pseudomonad $\mathscr{P}$ on **CAT**, and give various characterisations of its pseudoalgebras; they are, of course, the cocomplete categories. We then describe the lifting of $\mathscr{P}$ to a pseudomonad $\mathscr{P}_l$ on the

2-category of finitely complete categories, and give various analogous characterisations of the $\mathscr{P}_l$-pseudoalgebras; we call a category bearing such algebra structure *small-exact*.

As we have already said, we shall work from the outset in the context of the enriched category theory of [17]. Thus we fix a symmetric monoidal closed category $\mathscr{V}$, and henceforth write category to mean $\mathscr{V}$-category, functor to mean $\mathscr{V}$-functor, and so on; in particular, when we speak of limits and colimits, we mean the *weighted* (there called *indexed*) limits and colimits of [17, Chapter 3]. We shall also assume that $\mathscr{V}$ is *locally finitely presentable as a closed category* in the sense of [18]; which is to say that its underlying category $\mathscr{V}_0$ is locally finitely presentable—so in particular, complete and cocomplete—and that the finitely presentable objects are closed under the monoidal structure. This will be necessary later to ensure that we have a good notion of finite limit in our enriched setting.

It will also do us well to be clear on some foundational matters. We assume the existence of an inaccessible cardinal ∞, and call a set *small* if of cardinality $< \infty$, and a $\mathscr{V}$-category *small* if having only a small set of isomorphism-classes of objects. As usual, a set or category which is not small is called *large*; we may sometimes refer to a large set as a *class*. Now when we say that a category is complete or cocomplete, we really mean to say that it is small-complete or small-cocomplete, in the sense of having limits or colimits indexed by weights with small domain. **Set** is the category of *small* sets; which, means in particular that the case $\mathscr{V} = \mathbf{Set}$ of our general notions will be concerned with *locally small* ordinary categories.

Let **CAT** denote the 2-category of (possibly large) categories, functors and natural transformations; by which we mean, of course, $\mathscr{V}$-categories, $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations, so that our **CAT** is what might otherwise be denoted $\mathscr{V}$-**CAT**. Let **COCTS** denote the locally full sub-2-category of **CAT** comprising the cocomplete categories and cocontinuous functors. There is an inclusion 2-functor $\mathbf{COCTS} \to \mathbf{CAT}$, and by a *free cocompletion* of a category $\mathscr{C}$, we mean a bireflection of it along this 2-functor. This amounts to the provision of a category $\mathscr{P}\mathscr{C}$ and functor $Y\colon \mathscr{C} \to \mathscr{P}\mathscr{C}$ with the property that, for each cocomplete $\mathscr{D}$, the functor

$$\mathbf{COCTS}(\mathscr{P}\mathscr{C}, \mathscr{D}) \to \mathbf{CAT}(\mathscr{C}, \mathscr{D}) \tag{2.1}$$

induced by composition with Y is an equivalence of categories. As is explained in [17, §5.7], every category admits a free cocompletion; it may be described as follows. We declare a presheaf $\varphi\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$ to be *small* when it is the left Kan extension of its restriction to some small full subcategory $\mathscr{L}$ of $\mathscr{C}$, and define the category $\mathscr{P}\mathscr{C}$ to have as objects, the small presheaves on $\mathscr{C}$, and hom-objects $\mathscr{P}\mathscr{C}(\varphi, \psi)$ given by the usual end formula $\int_{X \in \mathscr{C}} [\varphi X, \psi X]$. Note that this large end, which *a priori* need not exist in $\mathscr{V}$, may be calculated as the small end $\int_{X \in \mathscr{L}} [\varphi J X, \psi J X]$ for any $J\colon \mathscr{L} \hookrightarrow \mathscr{C}$ witnessing the smallness of φ. The functor $Y\colon \mathscr{C} \to \mathscr{P}\mathscr{C}$ takes $X \in \mathscr{C}$ to the representable presheaf $\mathscr{C}(-, X)$. Observe that when $\mathscr{C}$ is small, every presheaf on $\mathscr{C}$ is small, so that $\mathscr{P}\mathscr{C} = [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ and Y is the Yoneda embedding. The following is now (a special case of) Theorem 5.35 of [17].

2.1. **Proposition.** *For every category $\mathscr{C}$, the category $\mathscr{P}\mathscr{C}$ of small presheaves on $\mathscr{C}$, together with its restricted Yoneda embedding $Y\colon \mathscr{C} \to \mathscr{P}\mathscr{C}$, is a free cocompletion of $\mathscr{C}$.*

The equivalence inverse of (2.1) takes a functor $F\colon \mathscr{C} \to \mathscr{D}$ with cocomplete domain to the functor $\bar{F}\colon \mathscr{P}\mathscr{C} \to \mathscr{D}$ which sends $\varphi \in \mathscr{P}\mathscr{C}$ to the weighted colimit $\varphi \star F$; as before, this large colimit, which *a priori* need not exist in $\mathscr{D}$, may be computed as the small colimit $\varphi J \star FJ$ for any $J\colon \mathscr{L} \hookrightarrow \mathscr{C}$ witnessing the smallness of φ. Observe that $\bar{F}$ is equally well the left Kan extension of F along $Y\colon \mathscr{C} \to \mathscr{P}\mathscr{C}$, by which we mean the pointwise left Kan extension; in this paper we shall consider no other kind.

The universal property of free cocompletion induces a pseudomonad structure on $\mathscr{P}$. The action of $\mathscr{P}$ on morphisms sends a functor $F\colon \mathscr{C} \to \mathscr{D}$ to the functor $\mathscr{P}\mathscr{C} \to \mathscr{P}\mathscr{D}$ obtained as the cocontinuous extension of $YF\colon \mathscr{C} \to \mathscr{P}\mathscr{D}$; this is equally well the functor sending $\varphi \in \mathscr{P}\mathscr{C}$ to $\mathrm{Lan}_{F^{\mathrm{op}}}(\varphi) \in \mathscr{P}\mathscr{D}$. The unit of the pseudomonad at $\mathscr{C}$ is $Y\colon \mathscr{C} \to \mathscr{P}\mathscr{C}$ whilst the multiplication $M\colon \mathscr{P}\mathscr{P}\mathscr{C} \to \mathscr{P}\mathscr{C}$ is the cocontinuous extension of the identity functor $\mathscr{P}\mathscr{C} \to \mathscr{P}\mathscr{C}$; thus $M(\varphi) \cong \varphi \star 1_{\mathscr{P}\mathscr{C}}$.

This pseudomonad is of the kind which has been called *Kock-Zöberlein*—see [22] and the references therein—but for which we adopt, following [19], the more descriptive name *lax-idempotent*. The characteristic property of such pseudomonads is that "structure is left adjoint to unit"; more precisely, this means that pseudoalgebra structures on an object correspond with left adjoint reflections for the unit map at that object. In the case of $\mathscr{P}$, the admission of such a left adjoint is easily seen to coincide with the property of being cocomplete; in more detail, we have the following result.

2.2. **Proposition.** *For a category $\mathscr{C}$, the following are equivalent:*

(1) $\mathscr{C}$ is (small-)cocomplete;
(2) For each $D\colon \mathscr{K} \to \mathscr{C}$ and $\varphi \in \mathscr{P}\mathscr{K}$, the colimit $\varphi \star D$ exists in $\mathscr{C}$;
(3) For every $\varphi \in \mathscr{P}\mathscr{C}$, the colimit $\varphi \star 1_{\mathscr{C}}$ exists in $\mathscr{C}$;
(4) For each $D\colon \mathscr{K} \to \mathscr{C}$ with $\mathscr{K}$ small, the Kan extension $\mathrm{Lan}_Y D\colon \mathscr{P}\mathscr{K} \to \mathscr{C}$ exists;
(5) For each $D\colon \mathscr{K} \to \mathscr{C}$, the Kan extension $\mathrm{Lan}_Y D\colon \mathscr{P}\mathscr{K} \to \mathscr{C}$ exists;
(6) The Kan extension $\mathrm{Lan}_Y(1_{\mathscr{C}})\colon \mathscr{P}\mathscr{C} \to \mathscr{C}$ exists;
(7) The functor $Y\colon \mathscr{C} \to \mathscr{P}\mathscr{C}$ admits a left adjoint;
(8) $\mathscr{C}$ admits a structure of $\mathscr{P}$-pseudoalgebra;
(9) $\mathscr{C}$ is reflective in some $\mathscr{P}\mathscr{D}$.

Proof. (1) ⇒ (2) by the preceding observations; (2) ⇒ (3) trivially. (1) ⇔ (4), (2) ⇔ (5) and (3) ⇔ (6) since $\mathrm{Lan}_Y D(\varphi) \cong \varphi \star D$. (6) ⇒ (7) since $\mathrm{Lan}_Y(1_{\mathscr{C}})\colon \mathscr{P}\mathscr{C} \to \mathscr{C}$ is, by basic properties of Kan extensions, left adjoint to Y. (7) ⇔ (8) because $\mathscr{P}$ is lax-idempotent. (7) ⇒ (9) since Y is fully faithful, so that if it admits a left adjoint, then $\mathscr{C}$ is reflective in $\mathscr{P}\mathscr{C}$. (9) ⇒ (1) since any category reflective in a cocomplete category is cocomplete. □

We now consider the interaction between the pseudomonad $\mathscr{P}$ and finite limit structure on a category. We begin by recalling from [18] some necessary definitions. A *finite weight* is a functor $\varphi\colon \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ such that $\mathscr{K}$ has a finite set of isomorphism-classes of objects, with each hom-object $\mathscr{K}(X, Y)$ and each $\varphi(X)$ being finitely presentable in $\mathscr{V}$. A weighted limit is called *finite* if its weight is finite, a category is *finitely complete* if it admits all finite limits, and a functor between finitely complete categories is *left exact* if it preserves finite limits; we sometimes write *lex* for *left exact*. The proof of the following result is now contained in Proposition 4.3 and Remark 6.6 of [9]; though in the case $\mathscr{V} = \mathbf{Set}$, the result is much older. Since we shall not need the details of the proof in what follows, we do not recount them here.

2.3. **Proposition.**

(1) If $\mathscr{C}$ is finitely complete, then so is $\mathscr{P}\mathscr{C}$;
(2) If $F\colon \mathscr{C} \to \mathscr{D}$ is left exact, then so is $\mathscr{P}F\colon \mathscr{P}\mathscr{C} \to \mathscr{P}\mathscr{D}$;

(3) For any $\mathscr{C}$*, both* $Y: \mathscr{C} \to \mathscr{P}\mathscr{C}$ *and* $M: \mathscr{P}\mathscr{P}\mathscr{C} \to \mathscr{P}\mathscr{C}$ *are left exact.*

As in the Introduction, let us write **LEX** for the locally full sub-2-category of **CAT** comprising the finitely complete categories and the left exact functors. It follows from Proposition 2.3 that $\mathscr{P}$ restricts and corestricts to a lax-idempotent pseudomonad on **LEX**, which, as in the Introduction, we shall denote by $\mathscr{P}_l$. We now wish to characterise the $\mathscr{P}_l$-pseudoalgebras. Since $\mathscr{P}_l$ is lax-idempotent, and all of its unit maps are fully faithful, pseudoalgebra structures on the finitely complete $\mathscr{C}$ may be identified with left adjoints in **LEX** for the unit $Y: \mathscr{C} \to \mathscr{P}_l\mathscr{C} = \mathscr{P}\mathscr{C}$. Such a left adjoint in **LEX** is of course also one in **CAT**, and so any $\mathscr{P}_l$-pseudoalgebra is cocomplete. The extra requirement that the left adjoint should be left exact may be rephrased in a number of ways, by analogy with Proposition 2.2.

2.4. **Proposition.** *For a finitely complete and cocomplete category* $\mathscr{C}$*, the following are equivalent:*

(1) For each lex $D: \mathscr{K} \to \mathscr{C}$ *with* $\mathscr{K}$ *small, the functor* $(-) \star D: \mathscr{P}\mathscr{K} \to \mathscr{C}$ *is also lex;*
(2) For each lex $D: \mathscr{K} \to \mathscr{C}$*, the functor* $(-) \star D: \mathscr{P}\mathscr{K} \to \mathscr{C}$ *is also lex;*
(3) The functor $(-) \star 1_{\mathscr{C}}: \mathscr{P}\mathscr{C} \to \mathscr{C}$ *is lex;*
(4) For each lex $D: \mathscr{K} \to \mathscr{C}$ *with* $\mathscr{K}$ *small, the functor* $\mathrm{Lan}_Y D: \mathscr{P}\mathscr{K} \to \mathscr{C}$ *is also lex;*
(5) For each lex $D: \mathscr{K} \to \mathscr{C}$*, the functor* $\mathrm{Lan}_Y D: \mathscr{P}\mathscr{K} \to \mathscr{C}$ *is also lex;*
(6) The functor $\mathrm{Lan}_Y(1_{\mathscr{C}}): \mathscr{P}\mathscr{C} \to \mathscr{C}$ *is lex;*
(7) The functor $Y: \mathscr{C} \to \mathscr{P}\mathscr{C}$ *admits a left exact left adjoint;*
(8) $\mathscr{C}$ *admits a structure of* $\mathscr{P}_l$*-pseudoalgebra;*
(9) $\mathscr{C}$ *is lex-reflective in some* $\mathscr{P}\mathscr{D}$ *with* $\mathscr{D}$ *finitely complete.*

In part (9), a category $\mathscr{A}$ is said to be *lex-reflective* in a category $\mathscr{B}$ if there is a fully faithful functor $\mathscr{A} \to \mathscr{B}$ which admits a left exact left adjoint; we may sometimes also say that $\mathscr{A}$ is a *localisation* of $\mathscr{B}$.

Proof. The only implications not exactly as before are (1) ⇒ (2) and (9) ⇒ (1). For the former, let $D: \mathscr{K} \to \mathscr{C}$ be lex; assuming (1), we must show that $(-) \star D: \mathscr{P}\mathscr{K} \to \mathscr{C}$ preserves finite limits. So given $\psi: \mathscr{M} \to \mathscr{V}$ a finite weight and $H: \mathscr{M} \to \mathscr{P}\mathscr{K}$, we are to show that $\{\psi, H\} \star D \cong \{\psi?, H? \star D\}$. Choose some $J: \mathscr{L} \hookrightarrow \mathscr{K}$ which witnesses the smallness of $HX \in \mathscr{P}\mathscr{K}$ for every $X \in \mathscr{M}$; without loss of generality, we may assume that $\mathscr{L}$ is closed under finite limits in $\mathscr{K}$, so that $\mathscr{L}$ is finitely complete and J lex. Write $\bar{H}: \mathscr{M} \to \mathscr{P}\mathscr{L}$ for the functor sending x to $(HX).J^{\mathrm{op}}: \mathscr{L}^{\mathrm{op}} \to \mathscr{V}$. Then $H \cong \mathrm{Lan}_{J^{\mathrm{op}}}.\bar{H}$, and $\mathrm{Lan}_{J^{\mathrm{op}}} \cong \mathscr{P}J$ preserves finite limits by Proposition 2.3, whence

$$\begin{aligned}\{\psi, H\} \star D &\cong \{\psi, \mathrm{Lan}_{J^{\mathrm{op}}}.\bar{H}\} \star D \cong (\mathrm{Lan}_{J^{\mathrm{op}}}\{\psi, \bar{H}\}) \star D \cong \{\psi, \bar{H}\} \star DJ \\ &\cong \{\psi?, \bar{H}? \star DJ\} \cong \{\psi?, \mathrm{Lan}_{J^{\mathrm{op}}}(\bar{H}?) \star D\} \cong \{\psi?, H? \star D\}\end{aligned}$$

where in passing from the first to the second line we use (1) applied to the lex DJ with small domain. This proves that (1) ⇒ (2); it remains only to show that (9) ⇒ (1). Let $\mathscr{D}$ be finitely complete, and let $L \dashv J: \mathscr{C} \to \mathscr{P}\mathscr{D}$ exhibit $\mathscr{C}$ as a localisation of $\mathscr{P}\mathscr{D}$. Now given $\mathscr{K}$ small and $D: \mathscr{K} \to \mathscr{C}$ lex, the functor $(-) \star D$ may be calculated to within isomorphism as the composite

$$\mathscr{P}\mathscr{K} \xrightarrow{\mathscr{P}D} \mathscr{P}\mathscr{C} \xrightarrow{\mathscr{P}J} \mathscr{P}\mathscr{P}\mathscr{D} \xrightarrow{M} \mathscr{P}\mathscr{D} \xrightarrow{L} \mathscr{C}$$

each of whose constituent parts is lex either by Proposition 2.3 or by assumption; whence the composite is lex as required. □

We shall call a category satisfying any of the equivalent conditions of this proposition *small-exact*. In the case where $\mathscr{V} = \mathbf{Set}$ we can give a concrete characterisation of the small-exact categories. Recall that an ∞*-pretopos* is a finitely complete and small-cocomplete **Set**-category in which colimits are stable under pullback, coproduct injections are disjoint, and every equivalence relation is effective.

2.5. **Proposition.** *A finitely complete and small-cocomplete* **Set***-category is small-exact if and only if it is an* ∞*-pretopos.*

Proof. **Set** is certainly an ∞-pretopos; whence also any **Set**-category $\mathscr{P}\mathscr{D}$ where $\mathscr{D}$ is lex, since finite limits and small colimits in $\mathscr{P}\mathscr{D}$ are computed pointwise. It is moreover easy to show that the exactness properties of an ∞-pretopos will be inherited by any localisation of it; and so every small-exact **Set**-category is an ∞-pretopos by clause (9) of Proposition 2.4. For the converse, we observe that any ∞-pretopos satisfies clause (4) of Proposition 2.4—see, for instance, [21, Corollary 3.3]—and so is small-exact. □

Returning to the case of a general $\mathscr{V}$, let us define a $\mathscr{V}$*-topos* to be any localisation of a presheaf category $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ on a small $\mathscr{C}$. The following result can be seen as a "Giraud theorem"; when $\mathscr{V} = \mathbf{Set}$, it recaptures [11]'s characterisation of the Grothendieck toposes as the ∞-pretoposes with a small generating family (bearing in mind that in an ∞-pretopos, the full subcategory spanned by any generating family is dense).

2.6. **Proposition.** *The finitely complete* $\mathscr{E}$ *is a* $\mathscr{V}$*-topos if and only if it is small-exact and has a small, dense subcategory.*

Proof. Suppose first that $\mathscr{E}$ is a localisation of $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ for some small $\mathscr{C}$. Certainly $\mathscr{E}$ has a small dense subcategory, given by the full image of the representables under the reflector $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}] \to \mathscr{E}$; we must show that $\mathscr{E}$ is also small-exact. So let $\mathscr{D}$ denote the closure of the representables in $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ under finite limits, and let $J\colon \mathscr{C} \to \mathscr{D}$ be the restricted Yoneda embedding. It is easy to see that $\mathscr{D}$ is again small; and now the composite of the fully faithful $\mathrm{Ran}_J\colon [\mathscr{C}^{\mathrm{op}}, \mathscr{V}] \to [\mathscr{D}^{\mathrm{op}}, \mathscr{V}]$ with $\mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ manifests $\mathscr{E}$ as lex-reflective in $[\mathscr{D}^{\mathrm{op}}, \mathscr{V}]$. Since $\mathscr{D}$ is finitely complete, $\mathscr{E}$ is small-exact by Proposition 2.4(9).

Conversely, suppose $\mathscr{E}$ is small-exact, with a small dense subcategory $J\colon \mathscr{C} \to \mathscr{E}$. Upon replacing $\mathscr{C}$ by its finite-limit closure in $\mathscr{E}$—which will again be small and dense—we may assume that $\mathscr{C}$ is finitely complete, and J left exact, so that by Proposition 2.4(4), $\mathrm{Lan}_Y J\colon [\mathscr{C}^{\mathrm{op}}, \mathscr{V}] \to \mathscr{E}$ is also left exact. But this functor has as right adjoint the singular functor $\tilde{J}\colon \mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$, which is fully faithful as $\mathscr{C}$ is dense; whence $\mathscr{E}$ is lex-reflective in $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$, and so a $\mathscr{V}$-topos as required. □

3. Φ-exactness

In the previous section we considered pseudoalgebras for the pseudomonad $\mathscr{P}_l$ on **LEX**. In this section, we consider pseudoalgebras for suitable full submonads of $\mathscr{P}_l$; these will be the Φ-exact categories which are the primary concern of this paper. Let us begin by defining what we mean by a full submonad of $\mathscr{P}_l$. Suppose that we are given, for each finitely complete $\mathscr{C}$, a subcategory $\mathscr{Q}\mathscr{C} \subseteq \mathscr{P}\mathscr{C}$ which is full, replete and closed under finite limits, with these choices being such that:

- For all finitely complete $\mathscr{C}$, the map $Y\colon \mathscr{C} \to \mathscr{P}\mathscr{C}$ factors through $\mathscr{Q}\mathscr{C}$ (i.e., $\mathscr{Q}\mathscr{C}$ contains the representables);
- For all lex $F\colon \mathscr{C} \to \mathscr{D}$, the map $\mathscr{Q}\mathscr{C} \hookrightarrow \mathscr{P}\mathscr{C} \xrightarrow{\mathscr{P}F} \mathscr{P}\mathscr{D}$ factors through $\mathscr{Q}\mathscr{D}$;

- For all finitely complete $\mathscr{C}$, the map $\mathscr{Q}\mathscr{Q}\mathscr{C} \hookrightarrow \mathscr{P}\mathscr{P}\mathscr{C} \xrightarrow{M} \mathscr{P}\mathscr{C}$ factors through $\mathscr{Q}\mathscr{C}$.

Under these circumstances, it is easy to see that $\mathscr{Q}\mathscr{C}$ is the value at $\mathscr{C}$ of a lax-idempotent pseudomonad $\mathscr{Q}$ on **LEX**, and that the inclusions $\mathscr{Q}\mathscr{C} \subseteq \mathscr{P}\mathscr{C}$ constitute a pseudomonad morphism $\mathscr{Q} \to \mathscr{P}_l$. We shall then call $\mathscr{Q}$ a *full submonad* of $\mathscr{P}_l$.

In order to generate full submonads of $\mathscr{P}_l$, we define, as in the Introduction, a *class of weights for lex colimits*—more briefly, a *class of lex-weights*—to be given by a collection Φ of functors $\varphi \colon \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$, with each $\mathscr{K}$ small and finitely complete. Before continuing, let us stress that by a lex-weight, we do not mean a weight that is lex; there is no requirement that the φ's should preserve finite limits, only that they should be presheaves on finitely complete categories. Let us also introduce some notation: given a class of lex-weights Φ, we write $\Phi[\mathscr{K}]$ for the full subcategory of $[\mathscr{K}^{\mathrm{op}}, \mathscr{V}]$ spanned by the functors in Φ with domain $\mathscr{K}^{\mathrm{op}}$.

From each class of lex-weights Φ we may generate a full submonad of $\mathscr{P}_l$: namely, the smallest full submonad $\mathscr{Q}$ such that $\Phi[\mathscr{K}] \subseteq \mathscr{Q}\mathscr{K}$ for each $\mathscr{K}$. We denote this submonad by Φ_l, and for each finitely complete $\mathscr{C}$, write

$$\mathscr{C} \xrightarrow{W} \Phi_l\mathscr{C} \xrightarrow{J} \mathscr{P}\mathscr{C}$$

for the corresponding factorisation of $Y \colon \mathscr{C} \to \mathscr{P}\mathscr{C}$. From this submonad Φ_l we obtain in turn a new class of lex-weights Φ^*, comprising all those $\varphi \colon \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ which lie in some $\Phi_l\mathscr{K}$. We call this class Φ^* the *saturation* of Φ, and say that Φ is *saturated* if $\Phi = \Phi^*$. Our notation is justified by the (easy) observation that saturation is a closure operator on classes of lex-weights. Let us warn the reader that the saturation of a class of lex-weights as we have just defined it is *not* the same as its saturation in the sense of [1, 20]; ours is a saturation "in the lex world", which, as the following result shows, involves closure under finite limits as well as under colimits from the specified class.

3.1. **Proposition.** *For each finitely complete $\mathscr{C}$, the category $\Phi_l\mathscr{C}$ is the closure of the representables in $\mathscr{P}\mathscr{C}$ under finite limits and Φ-lex-colimits.*

By this, we mean that $\Phi_l\mathscr{C}$ is the smallest full, replete subcategory of $\mathscr{P}\mathscr{C}$ which contains the representables and is closed under finite limits and Φ-lex-colimits.

Proof. Certainly, this closure exists for each finitely complete $\mathscr{C}$, obtained as the intersection of all full, replete subcategories of $\mathscr{P}\mathscr{C}$ with the stated properties. Let us write it as $\Phi'_l\mathscr{C}$; we are to show that for all finitely complete $\mathscr{C}$ we have $\Phi_l\mathscr{C} = \Phi'_l\mathscr{C}$. To show that $\Phi_l\mathscr{C} \subseteq \Phi'_l\mathscr{C}$, it is enough to show that the inclusions $J' \colon \Phi'_l\mathscr{C} \hookrightarrow \mathscr{P}\mathscr{C}$ constitute a full submonad of $\mathscr{P}_l$ with $\Phi[\mathscr{K}] \subseteq \Phi'_l\mathscr{K}$ for each small finitely complete $\mathscr{K}$. Clearly each $\Phi'_l\mathscr{C}$ is a full, replete, finite-limit-closed subcategory of $\mathscr{P}\mathscr{C}$, through which $Y \colon \mathscr{C} \to \mathscr{P}\mathscr{C}$ factors; we must in addition verify that:

(1) For each lex $F \colon \mathscr{C} \to \mathscr{D}$, the map $\Phi'_l\mathscr{C} \xrightarrow{J'} \mathscr{P}\mathscr{C} \xrightarrow{\mathscr{P}F} \mathscr{P}\mathscr{D}$ factors through $\Phi'_l\mathscr{D}$. For this, it suffices to show that the collection of objects $\varphi \in \mathscr{P}\mathscr{C}$ for which $\mathscr{P}F(\varphi)$ lies in $\Phi'_l\mathscr{D}$ contains the representables and is closed under finite limits and Φ-lex-colimits. But this follows by observing that $\mathscr{P}F$ is lex, cocontinuous, and sends representables to representables, and that $\Phi'_l\mathscr{D}$ contains the representables, and is closed under finite limits and Φ-lex-colimits in $\mathscr{P}\mathscr{D}$.

(2) For each finitely complete $\mathscr{C}$, the map $\Phi'_l\Phi'_l\mathscr{C} \xrightarrow{J'} \mathscr{P}\Phi'_l\mathscr{C} \xrightarrow{\mathscr{P}J'} \mathscr{P}\mathscr{P}\mathscr{C} \xrightarrow{M} \mathscr{P}\mathscr{C}$ factors through $\Phi'_l\mathscr{C}$. This follows similarly to (1), on observing that $M.\mathscr{P}J'$ is lex, cocontinuous, and maps the representables into $\Phi'_l\mathscr{C}$. to representables.

(3) For each small, finitely complete, $\mathscr{K}$, we have $\Phi[\mathscr{K}] \subseteq \Phi'_l\mathscr{K}$. For this we observe that each $\varphi \in \Phi[\mathscr{K}]$ can be expressed as $\varphi \star Y$—with $Y\colon \mathscr{K} \to [\mathscr{K}^{\mathrm{op}}, \mathscr{V}]$ the (lex) Yoneda embedding—and so as a Φ-lex-colimit of representables; hence $\varphi \in \Phi'_l\mathscr{K}$ as desired.

Thus we have shown that $\Phi_l\mathscr{C} \subseteq \Phi'_l\mathscr{C}$ for each finitely complete $\mathscr{C}$. For the converse inclusion, we must show that each $\Phi_l\mathscr{C}$ contains the representables and is closed in $\mathscr{P}\mathscr{C}$ under finite limits and Φ-lex-colimits. Only the closure under Φ-lex-colimits is non-trivial; to prove this, we must show that for each $\varphi \in \Phi[\mathscr{K}]$ and lex $D\colon \mathscr{K} \to \Phi_l\mathscr{C}$, the colimit $\varphi \star D$, computed in $\mathscr{P}\mathscr{C}$, lies again in $\Phi_l\mathscr{C}$. But this colimit is obtained by applying the composite functor

$$\Phi_l\mathscr{K} \xrightarrow{J} \mathscr{P}\mathscr{K} \xrightarrow{\mathscr{P}D} \mathscr{P}\Phi_l\mathscr{C} \xrightarrow{\mathscr{P}J} \mathscr{P}\mathscr{P}\mathscr{C} \xrightarrow{M} \mathscr{P}\mathscr{C}$$

to $\varphi \in \Phi_l\mathscr{K}$; and since $\mathscr{P}D.J = J.\Phi_l D$, this composite is equally well the functor

$$\Phi_l\mathscr{K} \xrightarrow{\Phi_l D} \Phi_l\Phi_l\mathscr{C} \xrightarrow{J} \mathscr{P}\Phi_l\mathscr{C} \xrightarrow{\mathscr{P}J} \mathscr{P}\mathscr{P}\mathscr{C} \xrightarrow{M} \mathscr{P}\mathscr{C}$$

whose latter part factors, by assumption, through $\Phi_l\mathscr{C}$. Hence $\Phi_l\mathscr{C}$ is closed under Φ-lex-colimits in $\mathscr{P}\mathscr{C}$, and so $\Phi'_l\mathscr{C} \subseteq \Phi_l\mathscr{C}$ as required. □

We shall also make use of the following result, which says that the pseudomonad generated by a class of lex-weights is "small-accessible":

3.2. **Proposition.** *If Φ is a class of lex-weights, and $\mathscr{C}$ a finitely complete category, then every $\varphi \in \Phi_l\mathscr{C}$ is of the form $\mathrm{Lan}_{J^{\mathrm{op}}}(\psi)$ for some small, finitely complete $\mathscr{L}$, some lex $J\colon \mathscr{L} \to \mathscr{C}$, and some $\psi \in \Phi_l\mathscr{L}$. In fact, we may always take J to be the inclusion of a full, finite-limit-closed subcategory, and ψ to be the composite φJ.*

Proof. Let $\mathscr{E}$ denote the subcategory of $\Phi_l\mathscr{C}$ spanned by those φ satisfying the stronger form of the stated condition. Clearly the representables lie in $\mathscr{E}$. Now suppose that $\psi\colon \mathscr{K} \to \mathscr{V}$ is a finite weight and $D\colon \mathscr{K} \to \Phi_l\mathscr{C}$ is such that every DX lies in $\mathscr{E}$; we shall show that $\{\psi, D\}$ does too. If the subcategory $J_X\colon \mathscr{L}_X \hookrightarrow \mathscr{C}$ witnesses the condition for DX, then taking $J\colon \mathscr{L} \hookrightarrow \mathscr{C}$ to be the lex closure of the union of these subcategories, we have $\{\psi, D\} \cong \{\psi, \mathrm{Lan}_{J^{\mathrm{op}}}(D(-)J)\} \cong \mathrm{Lan}_{J^{\mathrm{op}}}\{\psi, D(-)J\}$. By assumption, each $DX.J_X \in \Phi_l\mathscr{L}_X$, whence easily $DX.J \in \Phi_l\mathscr{L}$ and so $\{\psi, D(-)J\} \in \Phi_l\mathscr{L}$, as $\Phi_l\mathscr{L}$ is closed under finite limits in $\mathscr{P}\mathscr{L}$. Thus $\{\psi, D\}J \cong \{\psi, D(-)J\}$ is in $\Phi_l\mathscr{L}$ and $\mathrm{Lan}_{J^{\mathrm{op}}}(\{\psi, D\}J) \cong \{\psi, D\}$ as claimed. An entirely similar argument shows $\mathscr{E}$ is closed under Φ-lex-colimits in $\Phi_l\mathscr{C}$; and so by the preceding proposition, we have $\mathscr{E} = \Phi_l\mathscr{C}$. □

Given a class of lex-weights Φ, we now give a characterisation of the Φ_l-pseudoalgebras. Since Φ_l is lax-idempotent and all of its unit maps are fully faithful, the finitely complete $\mathscr{C}$ will admit Φ_l-pseudoalgebra structure just when the unit map $W\colon \mathscr{C} \to \Phi_l\mathscr{C}$ admits a left adjoint in **LEX**. The following two propositions characterise, firstly, those $\mathscr{C}$ for which a left adjoint to W exists in **CAT**, and secondly, those for which such a left adjoint is left exact. Given a class of lex-weights Φ, we say as in the Introduction that $\mathscr{C}$ is Φ-*lex-cocomplete* if it is finitely complete, and for every $\varphi \in \Phi[\mathscr{K}]$ and every lex $D\colon \mathscr{K} \to \mathscr{C}$, the colimit $\varphi \star D$ exists in $\mathscr{C}$.

3.3. **Proposition.** *Let Φ be a class of lex-weights, and let $\mathscr{C}$ be finitely complete. Then the following are equivalent:*

1. *$\mathscr{C}$ is Φ^*-lex-cocomplete;*
2. *For each lex $D\colon \mathscr{K} \to \mathscr{C}$ and $\varphi \in \Phi_l\mathscr{K}$, the colimit $\varphi \star D$ exists in $\mathscr{C}$;*
3. *For every $\varphi \in \Phi_l\mathscr{C}$, the colimit $\varphi \star 1_{\mathscr{C}}$ exists in $\mathscr{C}$;*
4. *For each lex $D\colon \mathscr{K} \to \mathscr{C}$ with $\mathscr{K}$ small, the Kan extension $\mathrm{Lan}_W D\colon \Phi_l\mathscr{K} \to \mathscr{C}$ exists;*
5. *For each lex $D\colon \mathscr{K} \to \mathscr{C}$, the Kan extension $\mathrm{Lan}_W D\colon \Phi_l\mathscr{K} \to \mathscr{C}$ exists;*
6. *The Kan extension $\mathrm{Lan}_W(1_{\mathscr{C}})\colon \Phi_l\mathscr{C} \to \mathscr{C}$ exists;*
7. *The functor $W\colon \mathscr{C} \to \Phi_l\mathscr{C}$ admits a left adjoint;*
8. *$\mathscr{C}$ is reflective in some $\Phi_l\mathscr{D}$.*

The proof of this result is entirely analogous to that of Proposition 2.2, though making essential use of Proposition 3.2 for the implication (1) $\Rightarrow$ (2). Let us remark on a further condition the finitely complete $\mathscr{C}$ may fulfil which is *not* on the preceding list, by virtue of its being strictly weaker: namely, the condition of being Φ-lex-cocomplete, as opposed to Φ^*-lex-cocomplete. Whilst the two are equivalent for many important classes of weights, it need not always be so. For example, in Section 5.7 below, we shall meet a class of lex-weights Φ_{rc} such that a finitely complete **Set**-category $\mathscr{C}$ is Φ_{rc}-lex-cocomplete just when it admits coequalisers of reflexive pairs. However, for such a $\mathscr{C}$ to be Φ^*_{rc}-lex-cocomplete, it must admit certain additional colimits, related to the construction of the free equivalence relation on a reflexive relation; see Proposition 5.3. The reason for the discrepancy is that, on closing the representables in $\mathscr{P}\mathscr{C}$ under coequalisers of reflexive pairs, the resultant subcategory need no longer closed under finite limits; and taking this closure—as we must do in forming $\Phi_{\mathrm{rc}}\mathscr{C}$—introduces new weights, not constructible from coequalisers of reflexive pairs alone, which any Φ^*-lex-cocomplete category must admit. Let us note, however, that if Φ is a class of lex-weights with the property that the closure of the representables in $\mathscr{P}\mathscr{C}$ under Φ-lex-colimits is already closed under finite limits—as happens in the case $\mathscr{V} = \mathbf{Set}$ when Φ is the class of weights for finite coproducts, or for coequalisers of kernel-pairs, or for coequalisers of equivalence relations—then Φ-lex-cocompleteness *does* coincide with Φ^*-lex-cocompleteness.

We now characterise, as promised, those finitely complete $\mathscr{C}$ for which $W\colon \mathscr{C} \to \Phi_l\mathscr{C}$ admits not just a left adjoint, but a left exact one. The proof is exactly analogous to that of Proposition 2.4.

3.4. **Proposition.** *Let Φ be a class of lex-weights, and let $\mathscr{C}$ be Φ^*-lex-cocomplete. Then the following are equivalent:*

1. *For each lex $D\colon \mathscr{K} \to \mathscr{C}$ with $\mathscr{K}$ small, the functor $(-) \star D\colon \Phi_l\mathscr{K} \to \mathscr{C}$ is also lex;*
2. *For each lex $D\colon \mathscr{K} \to \mathscr{C}$, the functor $(-) \star D\colon \Phi_l\mathscr{K} \to \mathscr{C}$ is also lex;*
3. *The functor $(-) \star 1_{\mathscr{C}}\colon \Phi_l\mathscr{C} \to \mathscr{C}$ is lex;*
4. *For each lex $D\colon \mathscr{K} \to \mathscr{C}$ with $\mathscr{K}$ small, the functor $\mathrm{Lan}_W D\colon \Phi_l\mathscr{K} \to \mathscr{C}$ is also lex;*
5. *For each lex $D\colon \mathscr{K} \to \mathscr{C}$, the functor $\mathrm{Lan}_W D\colon \Phi_l\mathscr{K} \to \mathscr{C}$ is also lex;*
6. *The functor $\mathrm{Lan}_W(1_{\mathscr{C}})\colon \Phi_l\mathscr{C} \to \mathscr{C}$ is lex;*
7. *The functor $W\colon \mathscr{C} \to \Phi_l\mathscr{C}$ admits a left exact left adjoint;*
8. *$\mathscr{C}$ admits a structure of Φ_l-pseudoalgebra;*
9. *$\mathscr{C}$ is lex-reflective in some $\Phi_l\mathscr{D}$.*

We shall call a category $\mathscr{C}$ satisfying any of the equivalent hypotheses of this proposition Φ-*exact*. We now consider what the appropriate notion of morphism between Φ-exact categories should be. We shall say that a functor $F\colon \mathscr{C} \to \mathscr{D}$ between Φ-lex-cocomplete categories is Φ-*lex-cocontinuous* if it is left exact, and for every $\varphi \in \Phi[\mathscr{K}]$ and lex $D\colon \mathscr{K} \to \mathscr{C}$, the canonical comparison map $\varphi \star FD \to F(\varphi \star D)$ is invertible.

3.5. **Proposition.** *Let $\mathscr{C}$ and $\mathscr{D}$ be Φ-exact categories, and $F\colon \mathscr{C} \to \mathscr{D}$ a left exact functor between them. Then the following are equivalent:*

(1) F admits a structure of algebra pseudomorphism with respect to some (equivalently, any) choice of Φ_l-pseudoalgebra structure on $\mathscr{C}$ and $\mathscr{D}$;

(2) The natural transformation

$$\begin{array}{ccc} \Phi_l\mathscr{C} & \xrightarrow{\Phi_l F} & \Phi_l\mathscr{D} \\ {\scriptstyle (-)\star 1_{\mathscr{C}}}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle (-)\star 1_{\mathscr{D}}} \\ \mathscr{C} & \xrightarrow[F]{} & \mathscr{D} \end{array}$$

obtained as the mate under the adjunctions $(-)\star 1_{\mathscr{C}} \dashv W_{\mathscr{C}}$ and $(-)\star 1_{\mathscr{D}} \dashv W_{\mathscr{D}}$ of the equality $\Phi_l F.W_{\mathscr{C}} = W_{\mathscr{D}}.F$, is invertible;

(3) F is Φ^*-lex-cocontinuous;

(4) F is Φ-lex-cocontinuous.

Proof. (1) $\Leftrightarrow$ (2) since the pseudomonad Φ_l is lax-idempotent. To see that (2) $\Rightarrow$ (3), suppose that the displayed 2-cell α is invertible; then for any $\varphi \in \Phi^*[\mathscr{K}] = \Phi_l\mathscr{K}$ and left exact $D\colon \mathscr{K} \to \mathscr{C}$, the canonical map $\varphi\star FD \to F(\varphi\star D)$ in $\mathscr{D}$ is, to within isomorphism, the component of α at $\Phi_l D(\varphi)$, and hence invertible: which gives (3). It is clear that (3) $\Rightarrow$ (4), and so it remains to show that (4) $\Rightarrow$ (2). It suffices by Proposition 3.1 to show that if F preserves Φ-lex-colimits, then the collection of $\varphi \in \Phi_l\mathscr{C}$ for which α_φ is invertible contains the representables and is closed under finite limits and Φ-lex-colimits. The first clause is immediate; the others follow on observing that the composites around both sides of the square in (2) preserve finite limits and Φ-lex-colimits: the only non-obvious fact being that $\Phi_l F$ preserves Φ-lex-colimits, which follows on observing that $\mathscr{P}F$ does so, being cocontinuous, and that $\Phi_l\mathscr{C}$ and $\Phi_l\mathscr{D}$ are closed in $\mathscr{P}\mathscr{C}$ and $\mathscr{P}\mathscr{D}$ under Φ-lex-colimits. □

We define a functor $F\colon \mathscr{C} \to \mathscr{D}$ between Φ-exact categories to be Φ-*exact* when it satisfies any of the equivalent conditions of this proposition. We may now verify the claim made in the Introduction that $\Phi_l\mathscr{C}$ constitutes a free Φ-exact completion of $\mathscr{C}$. In what follows, we write Φ-**EX** for the 2-category of Φ-exact categories, Φ-exact functors and arbitrary natural transformations.

3.6. **Proposition.** *Φ-**EX** is biequivalent to the 2-category **Ps**-Φ_l-**Alg** of Φ_l-pseudoalgebras, pseudoalgebra morphisms and algebra 2-cells.*

Proof. By Propositions 3.4 and 3.5, the forgetful 2-functor $U\colon$ **Ps**-Φ_l-**Alg** $\to$ **LEX** factors through the inclusion Φ-**EX** $\to$ **LEX**, as V, say. Since Φ_l is lax-idempotent, U is faithful and locally fully faithful, whence also V. By Proposition 3.5, V is also full, hence 2-fully faithful; since it is moreover clearly surjective on objects, it is a biequivalence. □

3.7. **Corollary.** *The forgetful 2-functor Φ-**EX** $\to$ **LEX** has a left biadjoint; the unit of this biadjunction at the left exact $\mathscr{C}$ may be taken to be $W\colon \mathscr{C} \to \Phi_l\mathscr{C}$.*

Combining this result with Proposition 3.1, we conclude, as was claimed in the Introduction, that the free Φ-exact completion of the finitely complete $\mathscr{C}$ may be obtained as the closure of the representables in $\mathscr{P}\mathscr{C}$ under finite limits and Φ-lex-colimits. In fact, we can be more precise about the nature of the left biadjoint we have just described.

3.8. **Proposition.** *For every finitely complete $\mathscr{C}$ and Φ-exact category $\mathscr{E}$, the functor*

$$W^* \colon \Phi\text{-}\mathbf{EX}(\Phi_l\mathscr{C}, \mathscr{E}) \to \mathbf{LEX}(\mathscr{C}, \mathscr{E})$$

induced by precomposition with W—which Corollary 3.7 assures us is an equivalence of categories—has equivalence pseudoinverse given by left Kan extension along W.

Proof. Suppose given $F \colon \mathscr{C} \to \mathscr{E}$ lex. By Propositions 3.3 and 3.4, $\mathrm{Lan}_W F$ exists and is left exact. Moreover, we have $\mathrm{Lan}_W F \cong J{-} \star F$; and so $\mathrm{Lan}_W F$ preserves Φ-lex-colimits because J does and taking colimits is cocontinuous in the weight. Thus $\mathrm{Lan}_W F$ is Φ-exact, and since $(\mathrm{Lan}_W F)W \cong F$, as W is fully faithful, Lan_W is an equivalence pseudoinverse for W^* as claimed. □

4. The embedding theorem

In the next section, we shall begin to describe, in elementary terms, what the notion of Φ-exactness amounts to for some particular choices of Φ. In doing so, we will make repeated use of one further result, which characterises the Φ-exact categories in terms of the embeddings they admit.

4.1. **Theorem.** *Let Φ be a class of lex-weights, and $\mathscr{C}$ a small Φ-lex-cocomplete category. Then the following conditions are equivalent:*

(1) $\mathscr{C}$ admits a full Φ-lex-cocontinuous embedding into a $\mathscr{V}$-topos;
(2) $\mathscr{C}$ admits a full Φ-lex-cocontinuous embedding into a small-exact category;
(3) $\mathscr{C}$ admits a full Φ-lex-cocontinuous embedding into a Φ-exact category;
(4) $\mathscr{C}$ is Φ-exact.

Moreover, even when $\mathscr{C}$ is not small, we still have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

In the statement of this theorem, recall that we defined a *$\mathscr{V}$-topos* to be any category lex-reflective in a presheaf category.

Proof. We begin by showing that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), regardless of $\mathscr{C}$'s size. The first two implications are straightforward, since every $\mathscr{V}$-topos is small-exact by Proposition 2.6, whilst every small-exact category is clearly Φ-exact. For (3) $\Rightarrow$ (4), let there be given a Φ-lex-cocontinuous embedding $J \colon \mathscr{C} \to \mathscr{E}$ into a Φ-exact category. By replacing $\mathscr{C}$ by its replete image in $\mathscr{E}$, we may assume that J exhibits $\mathscr{C}$ as a full, replete, finite-limit- and Φ-lex-colimit-closed subcategory of $\mathscr{E}$. Now by Proposition 3.8, since $\mathscr{E}$ is Φ-exact, the left Kan extension $\mathrm{Lan}_W J \colon \Phi_l\mathscr{C} \to \mathscr{E}$ exists and is Φ-exact. We claim that $\mathrm{Lan}_W J$ factors through the subcategory $\mathscr{C}$; given this, the factorisation $\Phi_l\mathscr{C} \to \mathscr{C}$ will be $\mathrm{Lan}_W(1_{\mathscr{C}})$, and left exact, since $\mathrm{Lan}_W J$ is, whence $\mathscr{C}$ will be Φ-exact by Proposition 3.4(6). To prove the claim, observe that the collection of $\varphi \in \Phi_l\mathscr{C}$ at which $\mathrm{Lan}_W J$ lands in $\mathscr{C}$ contains the representables, since $W^*.\mathrm{Lan}_W \cong 1$, and is closed under finite limits and Φ-lex-colimits, since $\mathrm{Lan}_W J$ preserves them, and $\mathscr{C}$ is closed in $\mathscr{E}$ under them. This proves (3) $\Rightarrow$ (4); it remains to show that, when $\mathscr{C}$ is small, we have (4) $\Rightarrow$ (1). We shall in fact defer this task until Section 7 below. There, we will see that any small Φ-exact $\mathscr{C}$ admits a *small-exact completion* $V \colon \mathscr{C} \to \mathscr{P}_\Phi\mathscr{C}$, and this V will provide the required full Φ-exact embedding of $\mathscr{C}$ into a $\mathscr{V}$-topos; see Corollary 7.4. □

An obvious limitation of this theorem is that its full strength is only available for a small $\mathscr{C}$. The following result allows us to work around this; though it does so at the cost of introducing a further size constraint, this time on Φ. We call a class of lex-weights Φ *small* if $\Phi_l\mathscr{C}$ is small whenever $\mathscr{C}$ is (this condition was called *locally small* in [20]). This will certainly be the case if the cardinality of Φ is small, as is easily seen upon giving a transfinite construction of $\Phi_l\mathscr{C}$ from $\mathscr{C}$ in the manner of [17, §3.5].

4.2. **Proposition.** *Let Φ be a small class of lex-weights, and $\mathscr{C}$ a Φ-lex-cocomplete category. Now $\mathscr{C}$ is Φ-exact if and only if every small, full, finite-limit- and Φ-lex-colimit-closed subcategory of $\mathscr{C}$ is Φ-exact.*

Proof. For brevity's sake, let us temporarily agree to call any $\mathscr{D} \subseteq \mathscr{C}$ as in the statement of this proposition a Φ-*subcategory* of $\mathscr{C}$. By Theorem 4.1, if $\mathscr{C}$ is Φ-exact then so are all of its Φ-subcategories. Conversely, suppose that every Φ-subcategory of $\mathscr{C}$ is Φ-exact: to show that $\mathscr{C}$ is too, it is enough by Proposition 3.4(4) to show that, for every small finitely complete $\mathscr{K}$ and lex $F\colon \mathscr{K} \to \mathscr{C}$, the Kan extension $\mathrm{Lan}_W F\colon \Phi_l\mathscr{K} \to \mathscr{C}$ exists and is lex. Given such an F, we let $\mathscr{D}$ denote the closure of its image in $\mathscr{C}$ under finite limits and Φ-lex-colimits, and write

$$\mathscr{K} \xrightarrow{G} \mathscr{D} \xrightarrow{H} \mathscr{C}$$

for the induced factorisation. Since Φ and $\mathscr{K}$ are small, so is $\mathscr{D}$; it is therefore a Φ-subcategory of $\mathscr{C}$ and so Φ-exact by assumption. Thus $\mathrm{Lan}_W G$ exists and is left exact; whence $H.\mathrm{Lan}_W G$ is also left exact, and so we will be done if we can show that it is in fact $\mathrm{Lan}_W F$. Equivalently, we may show that H preserves $\mathrm{Lan}_W G$; equivalently, that for each $\varphi \in \Phi_l\mathscr{K}$, the colimit $\varphi \star G$ in $\mathscr{D}$ is preserved by H; or equivalently, that for each $\varphi \in \Phi_l\mathscr{K}$ and $X \in \mathscr{C}$, the canonical morphism

$$\mathscr{C}(H(\varphi \star G), X) \to [(\Phi_l\mathscr{K})^{\mathrm{op}}, \mathscr{V}](\varphi, \mathscr{C}(HG\text{–}, X)) \tag{4.1}$$

is invertible in $\mathscr{V}$. To do this last, we let $\mathscr{D}'$ be the closure of $\mathscr{D} \cup \{X\}$ in $\mathscr{C}$ under finite limits and Φ-lex-colimits. As before, $\mathscr{D}'$ is a Φ-subcategory of $\mathscr{C}$, whence Φ-exact; moreover, the inclusion $I\colon \mathscr{D} \to \mathscr{D}'$ preserves finite limits and Φ-lex-colimits and so by Proposition 3.5 is a Φ-exact functor. In particular, I preserves the colimit $\varphi \star G$, which is to say that the canonical morphism

$$\mathscr{D}'(I(\varphi \star G), X) \to [(\Phi_l\mathscr{K})^{\mathrm{op}}, \mathscr{V}](\varphi, \mathscr{D}'(IG\text{–}, X))$$

is invertible. But this is equally well the morphism (4.1), since $\mathscr{D}'$ is a full subcategory of $\mathscr{C}$; thus H preserves $\varphi \star G$ for all $\varphi \in \Phi_l\mathscr{K}$, so that $H.\mathrm{Lan}_W G$ is $\mathrm{Lan}_W F$ as required. □

The typical manner in which we make use of this result is as follows. Given a small class of lex-weights Φ, we determine, by some means, a property Q of Φ-lex-cocomplete categories which we believe to be equivalent to Φ-exactness. We then prove that a *small* Φ-lex-cocomplete $\mathscr{C}$ is Φ-exact if and only if it is Q using Theorem 4.1. In light of Proposition 4.2, we may then remove the smallness qualification on $\mathscr{C}$ so long as we can show that a Φ-lex-cocomplete $\mathscr{C}$ is Q if and only if each of its small, full, finite-limit- and Φ-lex-colimit-closed subcategory is Q: and this will usually be straightforward, by virtue of the conditions which constitute Q involving quantification only over small sets of data in the candidate category $\mathscr{C}$.

The size constraint placed on Φ by this result is relatively harmless, since most classes of lex-weights that we encounter in practice are in fact small. However, this is by no means universally so—for instance, the classes of lex-weights for small coproducts or for small unions of subobjects are not small—and in order to deal with such cases as these, we now describe a result allowing the size restriction on Φ to be circumvented in its turn. It will be convenient to defer the proof of this result until we have set up the machinery of small-exact completions; it is given as Proposition 7.6 below.

4.3. **Proposition.** *Let Φ be a class of lex-weights. Now a category $\mathscr{C}$ is Φ-exact if and only if it is $\{\varphi\}$-exact for each $\varphi \in \Phi$.*

Assembling the above results, we obtain an embedding theorem for Φ-exact categories that is subject to no smallness constraints whatsoever.

4.4. **Corollary.** *Let Φ be a class of lex-weights, and $\mathscr{C}$ a Φ-lex-cocomplete category. Now $\mathscr{C}$ is Φ-exact if and only if, for each $\varphi \in \Phi$, every small, full, finite-limit- and $\{\varphi\}$-lex-colimit-closed subcategory $\mathscr{D} \subseteq \mathscr{C}$ admits a full $\{\varphi\}$-lex-cocontinuous embedding in a $\mathscr{V}$-topos.*

Proof. Combine Theorem 4.1 with Propositions 4.2 and 4.3. □

Let us remark that this result characterises the Φ-exact categories as being Φ-lex-cocomplete categories verifying certain additional conditions; which is by contrast to Proposition 3.4, which characterised them as Φ^*-lex-cocomplete categories verifying certain additional conditions. This may seem at odds with the remarks made following Proposition 3.3, where we observed that Φ^*-lex-colimits need not always be constructible from Φ-lex-colimits. However, it turns out that in a Φ-exact $\mathscr{C}$, all Φ^*-lex-colimits may in fact be constructed from Φ-lex-colimits *together with finite limits*. This is possible because the additional conditions verified in a Φ-exact category force certain cocones under Φ^*-lex-diagrams, always constructible from Φ-lex-colimits and finite limits, to be colimiting ones.

5. Examples of Φ-exactness

We now describe in detail some particular notions of Φ-exactness. As we have already said, we restrict attention in this article to the unenriched case—that is, the case $\mathscr{V} = \mathbf{Set}$ of our general notions—reserving for future study the consideration of exactness notions over other bases. Thus, throughout this section and the next, we assume without further comment that $\mathscr{V} = \mathbf{Set}$; so "category" now means "locally small category" and so on. The examples of this section will show—as anticipated in the Introduction—that in this setting, and for suitable choices of Φ, a category is Φ-exact just when it is regular, or Barr-exact, or lextensive, or coherent, or adhesive. We also provide two further examples fitting into our framework. The first is the notion of category with stable and effective finite unions of subobjects (effectivity meaning that unions are calculated as a pushout over the pairwise intersections). The second is the appropriate notion of exactness for categories with reflexive coequalisers.

5.1. **Regular categories.** For our first example, let the class Φ_{reg} be given by the single functor $\varphi \colon \mathscr{K}^{\text{op}} \to \mathbf{Set}$, where $\mathscr{K}$ is the free category with finite limits generated by an arrow $f \colon X \to Y$ and where φ is the coequaliser in $[\mathscr{K}^{\text{op}}, \mathbf{Set}]$ of the kernel-pair of $\mathscr{K}(-, f) \colon \mathscr{K}(-, X) \to \mathscr{K}(-, Y)$; note that φ is equally well the image of $\mathscr{K}(-, f)$. Now, if $(s, t) \colon R \rightrightarrows X$ is the kernel-pair of f in $\mathscr{K}$, then, since the Yoneda embedding preserves limits, φ is equally well a coequaliser of $\mathscr{K}(-, s)$ and $\mathscr{K}(-, t)$ in $[\mathscr{K}^{\text{op}}, \mathbf{Set}]$. So given a finitely complete $\mathscr{C}$ and a lex functor $D\colon \mathscr{K} \to \mathscr{C}$, the colimit $\varphi \star D$, if it exists, must be the coequaliser of the pair $(Ds, Dt)\colon DR \rightrightarrows DX$; but since D preserves finite limits, this pair is a kernel-pair of Df, whence $\varphi \star D$ is the coequaliser of the kernel-pair of Df. Thus if $\mathscr{C}$ admits coequalisers of kernel-pairs, it is Φ_{reg}-lex-cocomplete; conversely, if $\mathscr{C}$ is Φ_{reg}-lex-cocomplete, then it admits coequalisers of kernel-pairs, since for any $h\colon U \to V$ in $\mathscr{C}$ there is some lex $D\colon \mathscr{K} \to \mathscr{C}$ with $Df = h$.

Now by Theorem 4.1, a small, finitely complete and Φ_{reg}-lex-cocomplete $\mathscr{C}$ is Φ_{reg}-exact just when it admits a full embedding into a Grothendieck topos preserving finite limits and coequalisers of kernel-pairs; equivalently, finite limits and regular epimorphisms. Such an embedding, being fully faithful, will reflect as well as preserve regular epimorphisms, and since regular epimorphisms in a Grothendieck topos are stable under pullback, it follows that the same is true in any small Φ_{reg}-exact category: which is to say that any such category is regular. Conversely, if $\mathscr{C}$ is small and regular, then we may consider the topos $\mathbf{Sh}(\mathscr{C})$ of sheaves on $\mathscr{C}$ for the *regular topology*, in which a sieve is covering just when it contains some regular epimorphism. By [2, Proposition 4.3], the canonical functor $\mathscr{C} \to \mathbf{Sh}(\mathscr{C})$ is fully faithful, and preserves both finite limits and regular epimorphisms; whence $\mathscr{C}$ is Φ_{reg}-exact. Thus the small, finitely complete $\mathscr{C}$ is Φ_{reg}-exact if and only if regular; and since clearly a category with finite limits and coequalisers of kernel-pairs is regular if and only if every small, full subcategory closed under finite limits and coequalisers of kernel-pairs is regular, we conclude from Proposition 4.2 that a finitely complete $\mathscr{C}$, of any size, is Φ_{reg}-exact if and only if it is regular.

5.2. **Barr-exact categories.** Consider now the class of lex-weights Φ_{ex} consisting of the single functor $\varphi\colon \mathscr{K}^{\text{op}} \to \mathbf{Set}$, where $\mathscr{K}$ is the free category with finite limits generated by an equivalence relation $(s, t)\colon R \rightarrowtail X \times X$ and where φ is the coequaliser in $[\mathscr{K}^{\text{op}}, \mathbf{Set}]$ of $\mathscr{K}(-, s)$ and $\mathscr{K}(-, t)$. Arguing as before, we see that the finitely complete $\mathscr{C}$ is Φ_{ex}-lex-cocomplete if and only if it admits coequalisers of equivalence relations. Now by Theorem 4.1 such a $\mathscr{C}$, if small, is Φ_{ex}-exact just when it admits a fully faithful functor $J\colon \mathscr{C} \to \mathscr{E}$ into a Grothendieck topos which preserves finite limits and coequalisers of equivalence relations. Since any kernel-pair is an equivalence relation, such a J in particular preserves and reflects regular epimorphisms, and so any small Φ_{ex}-exact category is regular. If moreover $(s, t)\colon R \rightarrowtail X \times X$ is an equivalence relation in $\mathscr{C}$, then by virtue of J's preserving coequalisers and reflecting kernel-pairs, we conclude that (s, t) is the kernel-pair of its coequaliser, since (Js, Jt) is so in the topos $\mathscr{E}$. Thus any small Φ_{ex}-exact category is Barr-exact. Conversely, if the small, finitely complete $\mathscr{C}$ is Barr-exact, then the embedding $\mathscr{C} \to \mathbf{Sh}(\mathscr{C})$—where $\mathscr{C}$ is again equipped with the regular topology—preserves not only regular epimorphisms but also coequalisers of equivalence relations, since every equivalence relation in $\mathscr{C}$ and in $\mathbf{Sh}(\mathscr{C})$ is the kernel-pair of its own coequaliser. Thus the small finitely complete $\mathscr{C}$ is Φ_{ex}-exact if and only if Barr-exact; and so appealing to Proposition 4.2 and arguing as before, we conclude that the Φ_{ex}-exact categories of any size are precisely the finitely complete Barr-exact categories. It follows from this that if $\mathscr{C}$ is finitely complete, then $W\colon \mathscr{C} \to \Phi_{\text{ex}}\mathscr{C}$ is what is usually referred to as the ex/lex completion of $\mathscr{C}$, as described explicitly in [6]. The fact that $\mathscr{C}$ is itself Barr-exact just when W admits a left exact left adjoint—which is immediate from our Proposition 3.4(7)—was first noted in [6, Lemma 2.1(iv)]; our theory provides a general context for this observation.

5.3. **Lextensive categories.** Consider next the class of lex-weights Φ_{lext} consisting of the two functors $\varphi_0 \colon \mathscr{K}_0^{\mathrm{op}} \to \mathbf{Set}$ and $\varphi_2 \colon \mathscr{K}_2^{\mathrm{op}} \to \mathbf{Set}$. Here, $\mathscr{K}_0$ is the terminal category, and φ_0 the initial object of $[\mathscr{K}_0^{\mathrm{op}}, \mathbf{Set}]$, whilst $\mathscr{K}_2$ is the free category with finite limits on a pair of objects X, Y, and φ_2 the coproduct $\mathscr{K}_2(-, X) + \mathscr{K}_2(-, Y)$. Arguing as before, a finitely complete $\mathscr{C}$ is Φ_{lext}-lex-cocomplete if and only if it admits finite coproducts.

In order to characterise the Φ_{lext}-exact categories, we shall describe directly the free Φ_{lext}-exact category on a finitely complete $\mathscr{C}$. Let $\mathbf{Fam}_f(\mathscr{C})$ be the finite coproduct completion of $\mathscr{C}$; its objects are finite collections $(X_i \mid i \in I)$ of objects of $\mathscr{C}$ whilst its morphisms $(X_i \mid i \in I) \to (Y_j \mid j \in J)$ are pairs of a function $f \colon I \to J$ and a family of morphisms $(g_i \colon X_i \to Y_{f(i)} \mid i \in I)$. We have fully faithful functors $W \colon \mathscr{C} \to \mathbf{Fam}_f(\mathscr{C})$ and $J \colon \mathbf{Fam}_f(\mathscr{C}) \to \mathscr{P}\mathscr{C}$, where $W(X) = (X)$ and $J(X_i \mid i \in I) = \sum_i \mathscr{C}(-, X_i)$, and clearly have $Y \cong JW$. Of course, $\mathbf{Fam}_f(\mathscr{C})$ has finite coproducts; it is also finitely complete, as remarked in [6, Lemma 4.1(ii)], and both the finite coproducts and the finite limits are easily seen to be preserved by J. Moreover, every object of $\mathbf{Fam}_f(\mathscr{C})$ is a finite coproduct of objects in the image of W, and thus the replete image of J in $\mathscr{P}\mathscr{C}$ is precisely $\Phi_{\mathrm{lext}}\mathscr{C}$. It follows that a category $\mathscr{C}$ with finite limits and finite coproducts is Φ_{lext}-exact just when the functor $\mathbf{Fam}_f(\mathscr{C}) \to \mathscr{C}$ sending $(X_i \mid i \in I)$ to $\sum_{i \in I} X_i$ preserves finite limits; which by [27, Theorem 9], will happen just when finite coproducts in $\mathscr{C}$ are stable and disjoint. Thus a category $\mathscr{C}$ is Φ_{lext}-exact just when it is lextensive.

5.4. **Effective unions.** Let $\Phi_\vee$ be given by the two functors $\varphi_0 \colon \mathscr{K}_0^{\mathrm{op}} \to \mathbf{Set}$ and $\varphi_2 \colon \mathscr{K}_2^{\mathrm{op}} \to \mathbf{Set}$ defined as follows. $\mathscr{K}_0$ is the terminal category, and φ_0 the initial object of $[\mathscr{K}_0^{\mathrm{op}}, \mathbf{Set}]$; $\mathscr{K}_2$ is the free category with finite limits generated by a pair of monomorphisms $A \rightarrowtail C \leftarrowtail B$, and φ_2 is the union in $[\mathscr{K}_2^{\mathrm{op}}, \mathbf{Set}]$ of the two subobjects $\mathscr{K}_2(-, A)$ and $\mathscr{K}_2(-, B)$ of $\mathscr{K}_2(-, C)$. Writing $A \cap B$ for the intersection of the subobjects A and B of C in $\mathscr{K}_2$, we observe that φ_2 is equally well the pushout of the inclusions of $\mathscr{K}_2(-, A \cap B)$ into $\mathscr{K}_2(-, A)$ and $\mathscr{K}_2(-, B)$. It follows from this that a finitely complete category $\mathscr{C}$ is $\Phi_\vee$-lex-cocomplete just when it admits an initial object and pushouts of pullbacks of pairs of monomorphisms. The following result characterises the $\Phi_\vee$-exact categories.

5.1. **Proposition.** *The following are equivalent properties of the finitely complete $\mathscr{C}$:*

(1) $\mathscr{C}$ is $\Phi_\vee$-exact;
(2) $\mathscr{C}$ has a strict initial object, and pushouts of pullbacks of pairs of monomorphisms which are stable under pullback;
(3) $\mathscr{C}$ admits finite unions of subobjects which are effective and stable under pullback.

For part (3), finite unions are said to be *effective* if for any $C \in \mathscr{C}$ and any pair of subobjects A, B of C, the pullback square

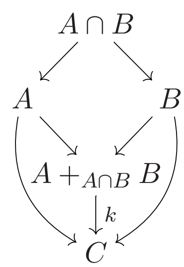
$$\begin{array}{ccc} A \cap B & \rightarrowtail & B \\ \downarrow & & \downarrow \\ A & \rightarrowtail & A \cup B \end{array}$$

is also a pushout.

Proof. By appealing to Proposition 4.2, and arguing as before, it suffices to prove the equivalence when $\mathscr{C}$ is small. So suppose first that the small $\mathscr{C}$ satisfies (1). By Theorem 4.1 we know that $\mathscr{C}$ is $\Phi_\vee$-lex-cocomplete and admits a full embedding $J \colon \mathscr{C} \to \mathscr{E}$

where $\mathscr{E}$ is a Grothendieck topos and J preserves finite limits, the initial object, and pushouts of pullbacks of pairs of monomorphisms. Thus $\mathscr{C}$'s initial object is strict, since if $f\colon X \to 0$ in $\mathscr{C}$ then Jf is invertible in $\mathscr{E}$—as initial objects in a topos are strict—whence f is also invertible, as J is conservative. A similar argument shows that pushouts of pullbacks of monomorphisms are stable under pullback in $\mathscr{C}$, since they are so in $\mathscr{E}$. Thus (1) $\Rightarrow$ (2).

Suppose next that $\mathscr{C}$ satisfies (2). Strictness of the initial object 0 implies that the unique map $0 \to C$ is always monomorphic; whence $0 \rightarrowtail C$ is a least subobject of C, which by strictness is stable under pullback. If now A and B are subobjects of C, we claim that the map $k\colon A +_{A\cap B} B \to C$ is a monomorphism; if this is so, then k represents the subobject $A \cup B$ of C, and such binary unions are stable by assumption, and effective by construction. The claim is proved in Theorem 5.1 of [25]; we give here an alternative proof. Observe first that whenever we pull back the diagram



along a map $f\colon K \to C$, the inner square remains a pushout by stability; so that if the outer square becomes a pushout, the induced map $f^*(k)$ must be an isomorphism. In particular, this is the case when f is any of the inclusions $A, B, A \cap B \rightarrowtail C$, so that on considering the subobjects $k^*(A)$, $k^*(B)$ and $k^*(A \cap B)$ of $A +_{A\cap B} B$, we have the comparison maps $k^*(A) \to A$, $k^*(B) \to B$ and $k^*(A \cap B) \to A \cap B$ invertible. But this in turn implies that on pulling back the displayed diagram along k, the outer square becomes a pushout; whence $k^*(k)$ is invertible, so that k has trivial kernel-pair and is thereby monomorphic. This completes the proof of the claim, and so (2) $\Rightarrow$ (3).

Suppose now that $\mathscr{C}$ satisfies (3). Clearly $\mathscr{C}$ has pushouts of pullbacks of monomorphisms, and a standard argument shows that it also has a pullback-stable initial object—see [15, Lemma A1.4.1], for example. Thus $\mathscr{C}$ is $\Phi_\vee$-lex-cocomplete; to show that it is in fact $\Phi_\vee$-exact, we define a topology on $\mathscr{C}$ by declaring that $C \in \mathscr{C}$ is covered by any sieve containing a finite family of subobjects $(A_i \rightarrowtail C \mid i \in I)$ whose union is all of C. Stability of finite unions ensures that this gives a topology on $\mathscr{C}$, whilst stability and effectivity together ensure that this topology is subcanonical. So we have a restricted Yoneda embedding $\mathscr{C} \to \mathbf{Sh}(\mathscr{C})$ into the category of sheaves for this topology, which is fully faithful and left exact; to complete the proof, it is enough to show that it is also $\Phi_\vee$-lex-cocontinuous, or equivalently, that every sheaf $F\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is $\Phi_\vee$-lex-continuous. Now if F is such a sheaf, then certainly $F(0) \cong 1$, since the empty family covers 0; it remains to show that if $A \rightarrowtail C \leftarrowtail B$ in $\mathscr{C}$, then the square

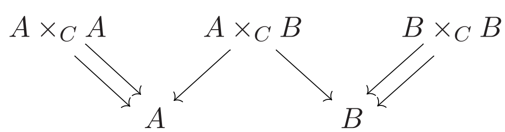
$$\begin{array}{ccc} F(A +_{A\cap B} B) & \longrightarrow & FA \\ \big\downarrow & & \big\downarrow \\ FB & \longrightarrow & F(A \cap B) \end{array}$$

is a pullback in **Set**. But $A +_{A\cap B} B = A \cup B$ by assumption, so this follows from the sheaf condition applied to the covering family $A \rightarrowtail A \cup B \leftarrowtail B$. □

The class $\Phi_\vee$ admits an obvious generalisation to a class $\Phi_\bigvee$ for which the $\Phi_\bigvee$-exact categories may be characterised as those with effective, stable unions of small families of subobjects. We do not take the trouble to formulate this precisely; though let us observe that, since the class of lex-weights $\Phi_\bigvee$ will no longer be small, we must make use of Proposition 4.3 in proving the characterisation.

5.5. **Coherent and geometric categories.** Consider now the class of lex-weights $\Phi_{\mathrm{coh}} = \Phi_{\mathrm{reg}} \cup \Phi_\vee$. We deduce from Proposition 4.3 that a finitely complete category is Φ_{coh}-exact just when it is both Φ_{reg}-exact and $\Phi_\vee$-exact; that is, just when it is regular and admits stable effective finite unions of subobjects. In fact, if a regular category admits stable finite unions (and recall that such a category is called *coherent*), they are necessarily effective: see, for example [15, Proposition A1.4.3]. Hence a finitely complete category is Φ_{coh}-exact just when it is a coherent category.

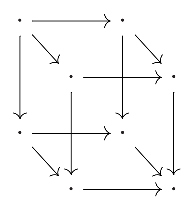
On the other hand, we may consider the class of lex-weights Φ'_{coh} comprising the two functors $\varphi_0 \colon \mathscr{K}_0^{\mathrm{op}} \to \mathbf{Set}$ and $\varphi_2 \colon \mathscr{K}_2^{\mathrm{op}} \to \mathbf{Set}$ defined as follows. $\mathscr{K}_0$ is the terminal category, and φ_0 the initial object of $[\mathscr{K}_0^{\mathrm{op}}, \mathbf{Set}]$; $\mathscr{K}_2$ is the free category with finite limits generated by a pair of arrows $A \to C \leftarrow B$, and φ_2 is the image in $[\mathscr{K}_2^{\mathrm{op}}, \mathbf{Set}]$ of the copairing $\mathscr{K}_2(-,A) + \mathscr{K}_2(-,B) \to \mathscr{K}_2(-,C)$. By an argument similar to the preceding ones, a category $\mathscr{C}$ is Φ'_{coh}-lex-cocomplete just when it has an initial object, and for every cospan $f \colon A \to C \leftarrow B \colon g$ in $\mathscr{C}$, the diagram



—which for the purposes of this example we will call the *double kernel* of (f, g)—admits a colimit. In particular, such a $\mathscr{C}$ admits pushouts of pullbacks of pairs of monomorphisms (take f and g monomorphic), and coequalisers of kernel-pairs (take $f = g$), so that by Proposition 3.1, $\Phi_{\mathrm{coh}}\mathscr{C} \subseteq \Phi'_{\mathrm{coh}}\mathscr{C}$ for every finitely complete $\mathscr{C}$, whence any Φ'_{coh}-exact category is Φ_{coh}-exact and so coherent. We claim conversely that every coherent category $\mathscr{C}$ is Φ'_{coh}-exact. By Proposition 4.2, it suffices to consider the case of a small $\mathscr{C}$. We may equip such a $\mathscr{C}$ with the *coherent topology*, in which a sieve is covering just when it contains a finite family of morphisms $(A_i \to X \mid i \in I)$ whose images have as union the whole of X. This topology is subcanonical, and so we have a fully faithful and left exact embedding $J \colon \mathscr{C} \to \mathbf{Sh}(\mathscr{C})$. Since the empty family covers 0, this embedding preserves the initial object; we must show it also preserves colimits of double kernels. So let $f \colon A \to C \leftarrow B \colon g$ in $\mathscr{C}$, and define $Z := \mathrm{im} f \cup \mathrm{im} g \rightarrowtail C$. Now the induced maps $f' \colon A \to Z \leftarrow B \colon g'$ exhibit Z as the colimit of the double kernel of (f, g); but since Z is a subobject of C, this double kernel is equally that of (f', g'), whose colimit J preserves since the pair (f', g') covers Z. Thus J preserves Φ'_{coh}-lex-colimits; and so every coherent category is Φ'_{coh}-exact as claimed.

In an analogous way, we may also formulate classes $\Phi_{\mathrm{geom}} = \Phi_{\mathrm{reg}} \cup \Phi_\bigvee$ and Φ'_{geom} such that a category is Φ_{geom}-exact if and only if it is Φ'_{geom}-exact, if and only if it is a *geometric* category—that is, a regular category with pullback-stable small unions of subobjects.

5.6. **Adhesive categories.** Consider the class of lex-weights Φ_{adh} comprising the single functor $\varphi\colon \mathscr{K}^{\text{op}} \to \mathbf{Set}$, where $\mathscr{K}$ is the free category with finite limits generated by a span $m\colon A \leftarrowtail C \to B\colon f$ with m monomorphic, and where φ is the pushout in $[\mathscr{K}^{\text{op}}, \mathbf{Set}]$ of $\mathscr{K}(-, m)$ and $\mathscr{K}(-, f)$. Now a finitely complete category $\mathscr{C}$ is Φ_{adh}-lex-cocomplete just when it admits pushouts along monomorphisms. Recall from [25] that we call such a category *adhesive* when for any commutative cube



whose bottom face is a pushout and whose rear faces are pullbacks, the top face is a pushout if and only if the front faces are pullbacks. This condition implies, in particular, that pushouts along monomorphisms are stable under pullback, and that every such pushout square is a pullback. In fact, these consequences of adhesivity turn out to be equivalent to it: a direct proof will be given in the forthcoming [10], but the result may also be deduced from our general theory.

5.2. **Proposition.** *The following are equivalent properties of the finitely complete $\mathscr{C}$:*

(1) $\mathscr{C}$ is Φ_{adh}-exact;
(2) $\mathscr{C}$ is adhesive;
(3) $\mathscr{C}$ admits pushouts along monomorphisms which are stable under pullback; moreover, every such pushout square is a pullback.

Proof. By Proposition 4.2 we may assume, as before, that $\mathscr{C}$ is small. Now if $\mathscr{C}$ is Φ_{adh}-exact then by Theorem 4.1 it admits a full embedding into a Grothendieck topos which preserves finite limits and pushouts along monomorphisms. Since such an embedding also reflects finite limits, and since every Grothendieck topos is adhesive, either by [26] or by a simple direct argument, it follows that $\mathscr{C}$ is adhesive; and so (1) $\Rightarrow$ (2). On the other hand, Theorem 3.3 of [24] shows that every small adhesive category admits a full embedding into a Grothendieck topos which preserves finite limits and pushouts along monomorphisms; so by Theorem 4.1, we have (2) $\Rightarrow$ (1). Next, if $\mathscr{C}$ is adhesive, then pushouts along monomorphisms are certainly stable under pullback, as this is one half of the defining property of adhesivity. Moreover, every such pushout square is a pullback by [25, Lemma 4.3]: and thus (2) $\Rightarrow$ (3).

To complete the proof, it remains to show either (3) $\Rightarrow$ (2) or (3) $\Rightarrow$ (1). As mentioned above, it turns out that there is a direct, elementary argument for the first of these, which will be given in [10]. But we do not need it here; for a close examination of the proof of (2) $\Rightarrow$ (1) provided by [24] reveals that it is actually a proof of (3) $\Rightarrow$ (1). It requires no more than that pushouts along monomorphisms are stable under pullback, that such pushouts are also pullbacks, and that monomorphisms are stable under pushout. We have assumed all of these in (3) except the last; but this follows on observing that, if

$$\begin{array}{ccc} C & \xrightarrow{f} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ A & \xrightarrow[g]{} & D \end{array}$$

is a pushout with m monomorphic, then it is a pullback by assumption, so that on pulling back the whole square along n, its left edge becomes invertible. Since the resultant square is again a pushout, its right edge must also be invertible, which is to say that n has trivial kernel-pair and so is monomorphic. □

5.7. **Reflexive coequalisers.** Consider the class of lex-weights Φ_{rc} comprising the single functor $\varphi\colon \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathscr{K}$ is the free category with finite limits generated by a reflexive pair $(d, c)\colon X \rightrightarrows Y$ (with common splitting r, say), and where φ is the coequaliser in $[\mathscr{K}^{\mathrm{op}}, \mathbf{Set}]$ of $\mathscr{K}(-, d)$ and $\mathscr{K}(-, c)$. Now a finitely complete category is Φ_{rc}-lex-cocomplete just when it admits coequalisers of reflexive pairs. The following result characterises the Φ_{rc}-exact categories.

5.3. **Proposition.** *The following are equivalent properties of the finitely complete $\mathscr{C}$:*

(1) $\mathscr{C}$ is Φ_{rc}-exact;
(2) $\mathscr{C}$ is Barr-exact, and for every reflexive relation $R \rightarrowtail X \times X$ in $\mathscr{C}$, the chain $R \subseteq RR^oR \subseteq RR^oRR^oR \subseteq \cdots$ of subobjects of $X \times X$ has a pullback-stable colimit;
(3) $\mathscr{C}$ is Barr-exact, and for every reflexive relation $R \rightarrowtail X \times X$ in $\mathscr{C}$, the chain $R \subseteq RR^oR \subseteq RR^oRR^oR \subseteq \cdots$ of subobjects of $X \times X$ has an effective, pullback-stable union.

Observe that, in parts (2) and (3), we employ the calculus of internal relations in $\mathscr{C}$—see [8], for instance—which we are entitled to do, since $\mathscr{C}$ is Barr-exact, and so in particular regular.

Proof. The argument that (2) ⇔ (3) is exactly as in Proposition 5.1 above, and so it is enough to show that these are in turn equivalent to (1). We begin by showing that a $\mathscr{C}$ as in (3) is Φ_{rc}-exact. First we show that such a $\mathscr{C}$ admits coequalisers of reflexive pairs. The argument is a standard one—given in [15, Lemma A1.4.19], for example—and so we indicate only its adaptation to the particularities of our situation. Given a reflexive pair $(s, t)\colon Y \to X \times X$, we first form its image $(d, c)\colon R \rightarrowtail X \times X$: now a coequaliser for the latter will also be one for the former, as the comparison map $Y \twoheadrightarrow R$ is regular epi. Since R is a reflexive relation, we may by assumption form the union of the chain $R \subseteq RR^oR \subseteq RR^oRR^oR \subseteq \cdots$; let us write it as $(d', c')\colon R^* \rightarrowtail X \times X$. By stability, (d', c') is an equivalence relation, and so admits a coequaliser, which it is not hard to show is also a coequaliser for (d, c), and hence for (s, t). Thus $\mathscr{C}$ admits coequalisers of reflexive pairs; let us record for future use that, since $\mathscr{C}$ is Barr-exact, the (d', c') of the above argument is also the kernel-pair of the coequaliser of (s, t).

We now show that $\mathscr{C}$ is Φ_{rc}-exact. By Proposition 4.2, we may assume that $\mathscr{C}$ is small; whereupon, by Theorem 4.1, it is enough to show that $\mathscr{C}$ admits a fully faithful embedding into a Grothendieck topos which preserves finite limits and coequalisers of reflexive pairs. We define a topology on $\mathscr{C}$ by declaring that every regular epimorphism should cover its codomain, and that, for every reflexive relation $R \rightarrowtail X \times X$, the family of union inclusions

$$\begin{array}{cccc} R & RR^oR & RR^oRR^oR & \cdots \\ & \searrow \quad \downarrow \quad \swarrow & & \\ & R^* & & \end{array} \tag{5.1}$$

should cover R^*. By assumption, these covers are effective-epimorphic and stable under pullback, and so generate a subcanonical topology on $\mathscr{C}$. Thus there is a full, lex embedding $J \colon \mathscr{C} \to \mathbf{Sh}(\mathscr{C})$, and we will be done if we can prove that J preserves coequalisers of reflexive pairs. Certainly J preserves regular epimorphisms; it also preserves unions of chains $R \subseteq RR^oR \subseteq RR^oRR^oR \subseteq \dots$, since such unions are effective in $\mathscr{C}$ and in $\mathbf{Sh}(\mathscr{C})$, and each (5.1) is covering. As J also preserves finite limits, it therefore preserves every part of the construction by which we calculated the coequaliser of a reflexive pair, and so must preserve the coequaliser as well. This proves that (3) $\Rightarrow$ (1).

To complete the proof, we now show that (1) $\Rightarrow$ (2). Let $\mathscr{C}$ be a Φ_{rc}-exact category; without loss of generality, a small one. By Theorem 4.1, such a $\mathscr{C}$ has finite limits and coequalisers of reflexive pairs, and admits a full embedding $J \colon \mathscr{C} \to \mathscr{E}$ into a Grothendieck topos which preserves them. In particular, $\mathscr{C}$ has, and J preserves, coequalisers of equivalence relations, and so we deduce as in Section 5.2 that $\mathscr{C}$ is Barr-exact. It remains to show that the chain of subobjects $R \subseteq RR^oR \subseteq \cdots$ associated to any reflexive relation $(d, c) \colon R \rightrightarrows X$ in $\mathscr{C}$ admits a stable colimit. Let $R^* \rightrightarrows X$ be the kernel-pair of the coequaliser of (d, c); we have $R \subseteq RR^oR \subseteq \cdots \subseteq R^*$ as subobjects of $X \times X$, and we claim that these inclusions exhibit R^* as the desired stable colimit. Now $J(R^*)$ is the kernel-pair of the coequaliser of (Jd, Jc); but because $\mathscr{E}$ satisfies the conditions of (2), the construction with which we began this proof shows that $J(R^*)$ is also the stable colimit of $JR \subseteq (JR)(JR)^o(JR) \subseteq \cdots$; whence, since J is fully faithful and lex, R^* is the stable colimit of $R \subseteq RR^oR \subseteq \cdots$ as desired. □

6. The case of a general Φ, when $\mathscr{V} = \mathbf{Set}$

In each of the examples of the previous section, we derived elementary descriptions of particular notions of Φ-exactness in an essentially *ad hoc* fashion. In this section, we show that—still in the case $\mathscr{V} = \mathbf{Set}$—we may give an elementary description which is valid for an arbitrary class of lex-weights Φ. The key idea needed is Anders Kock's notion of *postulatedness*. Given a finitely complete $\mathscr{C}$ and a topology j on it, Kock defines in [21] what it means for a cocone in $\mathscr{C}$ to be *postulated* with respect to j. If $\mathscr{C}$ is small, then the postulatedness of a cocone is equivalent to its being sent to a colimit by the functor $\mathscr{C} \to \mathbf{Sh}_j(\mathscr{C})$. The relevance this has for us is as follows. Given $\mathscr{C}$ a small, lex, and Φ-lex-cocomplete category, if Φ-lex-colimit cocones are postulated with respect to some *subcanonical* topology on $\mathscr{C}$, then there is a full embedding of $\mathscr{C}$ into a Grothendieck topos via a functor preserving finite limits and Φ-lex-colimits; whence $\mathscr{C}$ is Φ-exact. Conversely, if $\mathscr{C}$ is Φ-exact then by Theorem 4.1, it admits a full, Φ-exact, embedding into a Grothendieck topos. In Section 7 below, we will see that this embedding may be taken to be of the form $\mathscr{C} \to \mathbf{Sh}_j(\mathscr{C})$ for some topology j on $\mathscr{C}$; but now this j must be subcanonical, and all Φ-lex-colimits postulated with respect to it. Thus for small $\mathscr{C}$, Φ-exactness is equivalent to the postulatedness of Φ-lex-colimit cones with respect to some subcanonical topology on $\mathscr{C}$. In fact, this equivalence remains valid even when $\mathscr{C}$

is no longer small; we now give the details of this argument, including a reconstruction of those aspects of Kock's theory which will be necessary for our development.

We begin by giving our formulation of postulatedness, which diverges from Kock's in two aspects. The first has been anticipated above: a cocone in $\mathscr{C}$ will be postulated in our sense just when it is postulated in Kock's sense with respect to some subcanonical topology on $\mathscr{C}$; equivalently, with respect to the canonical topology (that is, the largest subcanonical topology) on $\mathscr{C}$. The second divergence is one of presentation: we are able to give a more compact definition because we are using the language of weighted colimits. We will later see how Kock's presentation can be recovered from ours.

Given $\mathscr{C}$ finitely complete, we say that a morphism $f\colon \varphi \to \psi$ of $\mathscr{P}\mathscr{C}$ is *final* if it orthogonal to every representable—in the sense that any map from φ to a representable admits a unique extension along f—and *stably final* when all of its pullbacks are final. Note that, in particular, a map $\varphi \to YC$ is final just when it exhibits C as the colimit $\varphi \star 1_{\mathscr{C}}$. We now say that $f\colon \varphi \to \psi$ is *postulated* if it satisfies the following two conditions:

(P1) The image $\mathrm{Im}(f) \rightarrowtail \psi$ of f is stably final;
(P2) The diagonal $\delta\colon \varphi \rightarrowtail \varphi \times_\psi \varphi$ is stably final.

If $\mathscr{C}$ is small-exact, then $Y\colon \mathscr{C} \to \mathscr{P}\mathscr{C}$ admits a left exact left adjoint L and now a morphism of $\mathscr{P}\mathscr{C}$ is final just when it is inverted by L. Since the left adjoint preserves pullbacks, any map which is inverted by L is in fact stably inverted; so every final morphism is stably final, and Lf is invertible if and only if f is stably final. In this context, a morphism f is postulated if and only if both its image and its diagonal are inverted by L, which is to say that it is L-*bidense* in the sense of [14, Definition 3.41]. Still in this context, the L-bidense morphisms are in fact precisely those inverted by L—see [14, Corollary 3.43], for example—so that if $\mathscr{C}$ is small-exact, a morphism of $\mathscr{P}\mathscr{C}$ is postulated if and only if it is final; this was shown to be the case in Proposition 2.1 of [21]. Yet even if $\mathscr{C}$ is not small-exact, we still have:

6.1. **Proposition.** *(c.f. [21, Proposition 1.1]). If $\mathscr{C}$ is finitely complete, then any postulated morphism in $\mathscr{P}\mathscr{C}$ is stably final.*

Proof. Observe that postulated morphisms are stable under pullback, since images and diagonals are so; hence it is enough to show that any postulated morphism is final. Given the postulated f, form its kernel-pair, its image and the diagonal of the kernel-pair as in

$$\varphi \xrightarrow{\delta} \varphi \times_\psi \varphi \underset{c}{\overset{d}{\rightrightarrows}} \varphi \xrightarrow{e} \mathrm{Im}(f) \xrightarrow{m} \psi \ .$$

Now m is final by (P1); we must show that e is too, which is to say that every $g\colon \varphi \to YE$ admits a unique extension along e. Since e is the coequaliser of the kernel-pair of f, this will happen just if $gd = gc$; but $gd\delta = g = gc\delta$ and so $gd = gc$ since δ is final by (P2). Thus g extends along e; the uniqueness is forced since e is epimorphic. □

Thus the force of the discussion preceding this proposition is that for a small-exact category $\mathscr{C}$, every final morphism in $\mathscr{P}\mathscr{C}$ is postulated. We now consider the extent to which this remains true on passing from small-exact categories to Φ-exact ones. First we need a preparatory result.

6.2. **Lemma.** *A morphism $f\colon \varphi \to \psi$ of $\mathscr{P}\mathscr{C}$ is stably final just when every pullback of it along a map with representable domain is final.*

Proof. Suppose given some $g\colon \psi' \to \psi$; we are to show that the pullback $f'\colon \varphi' \to \psi'$ of f along g is final. Let $(q_i\colon YC_i \to \psi' \mid i \in \mathscr{I})$ exhibit ψ' as a (conical) colimit of representables. For each i, the pullback $f'_i\colon \varphi'_i \to YC_i$ of f' along q_i is a pullback of f along gq_i, so final by assumption. As colimits in $\mathscr{P}\mathscr{C}$ are stable under pullback, φ' is the colimit of the φ'_i's, and hence f' is the colimit in $(\mathscr{P}\mathscr{C})^{\mathbf{2}}$ of the final f'_i's, and so itself final, since final maps, being defined by an orthogonality property, are stable under colimits. □

6.3. **Proposition.** *If Φ is a class of lex-weights, and $\mathscr{C}$ a Φ-exact category, then each final morphism of $\mathscr{P}\mathscr{C}$ lying in $\Phi_l\mathscr{C}$ is postulated.*

Proof. As $\mathscr{C}$ is Φ-exact, $W\colon \mathscr{C} \to \Phi_l\mathscr{C}$ admits a left exact left adjoint L, and as above, a morphism of $\Phi_l\mathscr{C}$ is final in $\mathscr{P}\mathscr{C}$ just when it is inverted by L. Since L preserves pullbacks, if $f\colon \varphi \to \psi$ is final and lies in $\Phi_l\mathscr{C}$, then any pullback of it along a map $YC \to \psi$ is again final, since the representables lie in $\Phi_l\mathscr{C}$. So by Lemma 6.2, f is stably final in $\mathscr{P}\mathscr{C}$, and it follows that $\mathrm{Im}(f) \rightarrowtail YC$ is stably final, since the image of any final map is easily shown to be final, and image factorisations in $\mathscr{P}\mathscr{C}$ are stable under pullback. This verifies (P1) for f; as for (P2), observe that the diagonal $\delta\colon \varphi \rightarrowtail \varphi \times_\psi \varphi$ lies in $\Phi_l\mathscr{C}$, and is sent by L to the diagonal of the kernel-pair of Lf, which is invertible since Lf is. Thus δ is final and lies in $\Phi_l\mathscr{C}$, and so arguing as before, is stably final. □

We may now give the promised correspondence between Φ-exactness and the postulatedness of Φ-lex-colimits. Given Φ a class of lex-weights, and $\mathscr{C}$ a finitely complete and Φ-lex-cocomplete category, by a Φ-*lex-colimit* morphism in $\mathscr{P}\mathscr{C}$, we mean a final morphism of the form $\mathrm{Lan}_D(\varphi) \to Y(\varphi \star D)$ for some $\varphi \in \Phi[\mathscr{K}]$ and lex $D\colon \mathscr{K} \to \mathscr{C}$; and by saying that Φ-lex-colimits are postulated in $\mathscr{C}$, we mean to say that every such Φ-lex-colimit morphism is postulated.

6.4. **Theorem.** *Let Φ be a class of lex-weights. Then the finitely complete and Φ-lex-cocomplete $\mathscr{C}$ is Φ-exact if and only if Φ-lex-colimits are postulated in $\mathscr{C}$.*

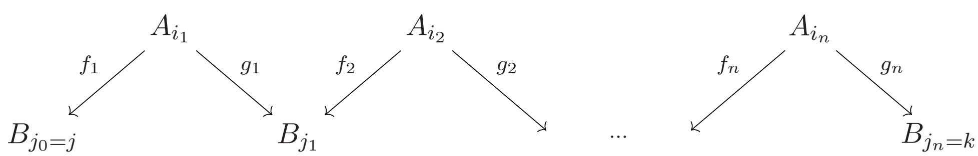
Proof. If $\mathscr{C}$ is Φ-exact, then every Φ-lex-colimit morphism, being final and lying in $\Phi_l\mathscr{C}$, is postulated by Proposition 6.3. Conversely, suppose that each Φ-lex-colimit morphism in $\mathscr{P}\mathscr{C}$ is postulated; we will show that $\mathscr{C}$ is Φ-exact. By Propositions 4.2 and 4.3, we may assume that $\mathscr{C}$ is small, and now we define a topology on $\mathscr{C}$ as follows. For each Φ-lex-colimit morphism $f\colon \varphi \to YC$ in $\mathscr{P}\mathscr{C}$, we declare that its image $\mathrm{Im}(f) \rightarrowtail YC$ should be a covering sieve, and that for each pair $h, k\colon YD \rightrightarrows \varphi$ with $fh = fk$, their equaliser $\theta \rightarrowtail YD$ should be a covering sieve. Each $\mathrm{Im}(f) \rightarrowtail YC$ is stably final by (P1), whilst each $\theta \rightarrowtail YD$ is stably final by (P2), being the pullback of $\delta\colon \varphi \rightarrowtail \varphi \times_{YC} \varphi$ along some $(h, k)\colon YD \to \varphi \times_{YC} \varphi$. Hence these sieves generate a subcanonical topology on $\mathscr{C}$, and we have a full, lex embedding $J\colon \mathscr{C} \to \mathbf{Sh}(\mathscr{C})$. To complete the proof, it is enough to show that J preserves Φ-lex-colimits; equivalently, that every sheaf sends Φ-lex-colimits in $\mathscr{C}$ to limits in **Set**; equivalently, that every sheaf F is orthogonal in $\mathscr{P}\mathscr{C}$ to every Φ-lex-colimit morphism $f\colon \varphi \to YC$. Fixing F and f, and arguing as in Proposition 6.1, it is enough to show that F is orthogonal to $m\colon \mathrm{Im}(f) \rightarrowtail YC$ and to $\delta\colon \varphi \rightarrowtail \varphi \times_{YC} \varphi$. Certainly F is orthogonal to m, since m is covering and F a sheaf; as for δ, it suffices, arguing now as in Lemma 6.2, to demonstrate F's orthogonality to $g^*(\delta)$ for every $D \in \mathscr{C}$ and $g\colon YD \to \varphi \times_{YC} \varphi$. But to give g is equally well to give $h, k\colon YD \rightrightarrows \varphi$ satisfying $fh = fk$, and now $g^*(\delta)$ is equally well the equaliser of h and k, and so a covering sieve; to which F, being a sheaf, is orthogonal. □

We now explain how our definition of postulatedness relates to Kock's. Suppose given a finitely complete $\mathscr{C}$ and a map $f\colon \varphi \to YC$ in $\mathscr{P}\mathscr{C}$. We will describe in elementary terms what it means for f to be postulated, doing so with respect to some presentation of φ as a coequaliser

$$\textstyle\sum_{i\in I} YA_i \overset{s}{\underset{t}{\rightrightarrows}} \sum_{j\in J} YB_j \xrightarrow{q} \varphi\ . \tag{6.1}$$

Observe that to give s and t is equally well to give functions $\sigma, \tau\colon I \rightrightarrows J$ and families of maps $(s_i\colon A_i \to B_{\sigma i} \mid i \in I)$ and $(t_i\colon A_i \to B_{\tau i} \mid i \in I)$ and that to give q is equally well to give a family of maps $(q_j\colon YB_j \to \varphi \mid j \in J)$ with $q_{\sigma i}.Ys_i = q_{\tau i}.Yt_i$ for each $i \in I$. Moreover, as q is the coequaliser of s and t, to give $f\colon \varphi \to YC$ is equally well to give a family of maps $(r_j\colon B_j \to C \mid j \in J)$ such that $r_{\sigma i}.s_i = r_{\tau i}.t_i$ for each $i \in I$. Given now $j, k \in J$, we define a *zig-zag* from j to k to be a diagram

(6.2)



where each $i_m \in I$, each $j_m \in J$, and for each $1 \leqslant m \leqslant n$, either $f_m = s_{i_m}$ and $g_m = t_{i_m}$, or $f_m = t_{i_m}$ and $g_m = s_{i_m}$. We write $ZZ(j,k)$ for the set of zig-zags from j to k. To each zig-zag $z \in ZZ(j,k)$, we may associate the span $a_z\colon B_j \leftarrow L_z \to B_k\colon b_z$ obtained by composing together the spans appearing in z; and now, since $r_{\sigma i}.s_i = r_{\tau i}.t_i$ for each $i \in I$, also $r_j.a_z = r_k.b_z$, and so there is an induced $\ell_z = (a_z, b_z)\colon L_z \to B_j \times_C B_k$.

6.5. **Proposition.** *The morphism $f\colon \varphi \to YC$ of $\mathscr{P}\mathscr{C}$ is postulated if and only if:*

(P1') The family $(r_j\colon B_j \to C \mid j \in J)$ is stably effective-epimorphic in $\mathscr{C}$;
(P2') For all $j, k \in J$, the family $(\ell_z\colon L_z \to B_j \times_C B_k \mid z \in ZZ(j,k))$ is stably effective-epimorphic in $\mathscr{C}$.

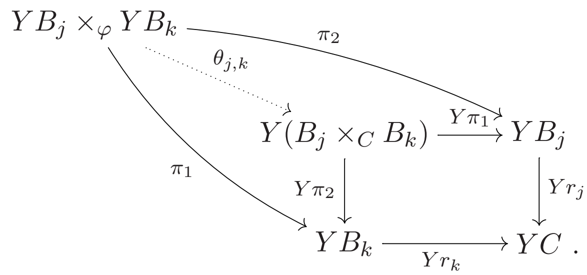
Recall that a family of maps $(f_i\colon U_i \to V)$ is *effective-epimorphic* if it exhibits V as the colimit of the sieve generated by the f_i's, and is *stably effective-epimorphic* if every pullback of it along a map $V' \to V$ is effective-epimorphic. The stably effective-epimorphic families are the covering families for the canonical topology on $\mathscr{C}$—the largest topology for which each representable functor is a sheaf—and so, comparing this result with [21, Section 1], we deduce as claimed that postulatedness in our sense coincides with postulatedness in the sense of [21] with respect to the canonical topology.

Proof. We show first that (P1) ⇔ (P1'). Since $q\colon \sum_{j\in J} YB_j \twoheadrightarrow \varphi$ is epimorphic, the images of f and fq coincide. But since $fq = \langle Yr_j \mid j \in J\rangle\colon \sum_j YB_j \to YC$, the image of the latter is the sieve on C generated by the family $(r_j\colon B_j \to C \mid j \in J)$; so by Lemma 6.2, to say that the image of f is stably final, which is (P1), is equally well to say that $(r_j \mid j \in J)$ is stably effective-epimorphic, which is (P1').

We now show that (P2) ⇔ (P2'). First we characterise the sieve generated by the family $(\ell_z \mid z \in ZZ(j,k))$. By definition, a morphism $(g,h)\colon X \to B_j \times_C B_k$ lies in this sieve just when it factorises through some ℓ_z; that is, just when there is a zig-zag of the form (6.2), and an extension of the pair (g,h) to a cone over this zig-zag. But by virtue of the way that coequalisers are computed in **Set**, this is equally well to say that, on considering the coequaliser

$$\textstyle\sum_{i\in I} \mathscr{C}(X, A_i) \underset{t_X}{\overset{s_X}{\rightrightarrows}} \sum_{j\in J} \mathscr{C}(X, B_j) \xrightarrow{q_X} \varphi(X) \ ,$$

the elements (j, g) and (k, h) of the central set have the same image under q_X; which is equally well to say that the map $Y(g, h)\colon YX \to Y(B_j \times_C B_k)$ factors through the subobject $\theta_{j,k}\colon YB_j \times_\varphi YB_k \rightarrowtail Y(B_j \times_C B_k)$ induced by the universal property of pullback in the diagram



We have thus shown that $\theta_{j,k}$ is the image of $(\ell_z \mid z \in ZZ(j, k))$; and so by Lemma 6.2, to say that (P2') holds is to say that $\theta_{j,k}$ is stably final for all $j, k \in J$. We now show that this latter condition is equivalent to (P2); that is, to $\delta\colon \varphi \to \varphi \times_{YC} \varphi$ being stably final. Now for each $j, k \in J$, the map $\theta_{j,k}$ is the pullback of δ along $q_j \times_{YC} q_k$, so that if δ is stably final, then each $\theta_{j,k}$ is too. If conversely each $\theta_{j,k}$ is stably final, then by Lemma 6.2, δ will be stably final as soon as every pullback of it along a map $(h, k)\colon YD \to \varphi \times_{YC} \varphi$ is final. For any such map we have, since the family $(q_j \mid j \in J)$ is jointly epimorphic, factorisations $h = q_j u$ and $k = q_k v$ for some $j, k \in J$ and $(u, v)\colon YD \to Y(B_j \times_C B_k)$; whence the pullback $(h, k)^*(\delta)$ is in fact a pullback of $\theta_{j,k}$, and so indeed final. □

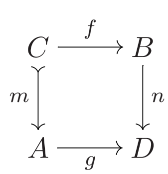
We give one final formulation of postulatedness; this is the most useful in practice.

6.6. **Proposition.** *The morphism $f\colon \varphi \to YC$ of $\mathscr{P}\mathscr{C}$ is postulated if and only if it is stably final and satisfies (P2').*

Proof. If f is postulated, then it is stably final by Proposition 6.1, and satisfies (P2') by Proposition 6.5. Conversely, if f is stably final, then so is its image, since image factorisations are stable under pullback in $\mathscr{P}\mathscr{C}$, which verifies (P1); now if it also verifies (P2') then it is postulated by Proposition 6.5. □

We conclude this section with an application of the preceding results; we will use them to reconstruct the characterisation of adhesive categories given in Section 5.6. Recall that Φ_{adh} is the class of lex-weights such that Φ_{adh}-lex-cocomplete categories are those admitting pushouts along monomorphisms. By Theorem 6.4 and Proposition 6.6, such a category is Φ_{adh}-exact if and only if pushouts along monomorphisms are stable under pullback, with the corresponding final morphisms of $\mathscr{P}\mathscr{C}$ satisfying condition (P2'). To analyse this latter condition further, let

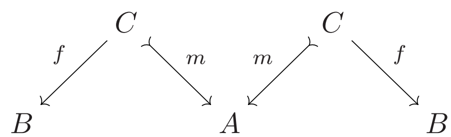
(6.3)



be a typical pushout along a monomorphism in $\mathscr{C}$, and let $q\colon \varphi \to YD$ be the corresponding final morphism in $\mathscr{PC}$. We may present φ as the coequaliser of the pair $(\iota_1.Ym, \iota_2.Yf)\colon YC \rightrightarrows YA + YB$, and with respect to this presentation, condition (P2') for the postulatedness of q breaks up into four clauses; we now consider these in turn.

(i) *The family* $(\ell_z\colon L_z \to B \times_D B \mid z \in ZZ(B,B))$ *should be stably effective-epimorphic.* Every zig-zag in $ZZ(B,B)$ is given by zero or more copies of the zig-zag

(6.4)

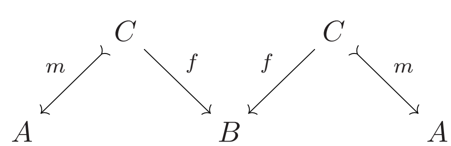


placed side by side. From the case $n = 0$, we see that the diagonal $\delta\colon B \to B \times_D B$ is in the family (ℓ_z). But since m is monic, the span composite of the displayed zig-zag has both projections onto B equal, and it follows that the span composite of every $z \in ZZ(B,B)$ has both projections onto B equal: in other words, that every $\ell_z\colon L_z \to B \times_D B$ factors through δ. Thus to say that the family (ℓ_z) is stably effective-epimorphic is equally well to say that the singleton family δ is so; but since δ is monomorphic, this is equivalent to saying that it is invertible, or in other words, that n is monic.

(ii) *The family* $(\ell_z\colon L_z \to A \times_D B \mid z \in ZZ(A,B))$ *should be stably effective-epimorphic.* Every zig-zag in $ZZ(A,B)$ is given by zero or more copies of the zig-zag (6.4) placed next to the span $m\colon A \leftarrowtail C \to B\colon f$. So in particular, $(m,f)\colon C \to A \times_D B$ is in the family (ℓ_z), and arguing as before, any other ℓ_z must factor through this one. So the stated condition is equivalent to the singleton family (m,f) being stably effective-epimorphic, and since (m,f) is monic (as m is) this is in turn equivalent to (m,f) being invertible; that is, to the pushout (6.3) also being a pullback.

(iii) *The family* $(\ell_z\colon L_z \to B \times_D A \mid z \in ZZ(B,A))$ *should be stably effective-epimorphic.* This condition is clearly equivalent to the previous one.

(iv) *The family* $(\ell_z\colon L_z \to A \times_D A \mid z \in ZZ(A,A))$ *should be stably effective-epimorphic.* Every zig-zag in $ZZ(A,A)$ is given by zero or more copies of



placed side by side. From the cases $n = 0, 1$, we see that $\delta\colon A \to A \times_D A$ and $m \times_n m\colon C \times_B C \to A \times_D A$ are in the family (ℓ_z), and arguing as before, any other ℓ_z must factor through $m \times_n m$. Thus the stated condition is equally that the pair of maps $(\delta,\ m \times_n m)$ should comprise a stably effective-epimorphic family. Since both are monomorphic, this is equally well to say that they are the stable pushout of their intersection: but this intersection is easily seen to be C, and so the condition is that

$$
\begin{array}{ccc}
C & \xrightarrow{\ \delta\ } & C \times_B C \\
{\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle m \times_n m} \\
A & \xrightarrow[\ \delta\ ]{} & A \times_D A
\end{array}
\tag{6.5}
$$

should be a stable pushout. In fact it is enough merely that it is a pushout, as then it is a pushout along a monomorphism and so stable by assumption.

In conclusion, we see that the finitely complete category $\mathscr{C}$ is Φ_{adh}-exact just when pushouts along monomorphisms exist, are stable, and are pullbacks, when monomorphisms are stable under pushout, and when finally for every pushout square (6.3), the corresponding square (6.5) is also a pushout. Now we saw in the proof of Proposition 5.2 that pushouts of monomorphisms are monomorphisms provided that such pushouts are stable, so that this condition can be omitted; moreover, Lemma 3.2 of [24] shows that the condition involving (6.5) is also a consequence of the others. Thus we conclude that $\mathscr{C}$ is Φ_{adh}-exact just when pushouts along monomorphisms are stable and are pullbacks: which is what we proved in Proposition 5.2.

7. Relative completions

In this final section, we return to the development, for a general $\mathscr{V}$, of the theory of Φ-exactness. Our goal is to describe circumstances under which it is possible to construct the free Ψ-exact completion of a Φ-exact category. First we need to ascertain the circumstances under which it makes sense even to speak of the free Ψ-exact completion of a Φ-exact category; to which end, we introduce the following notation.

Given classes of lex-weights Φ and Ψ, we write $\Phi \leqslant \Psi$ to indicate that the forgetful functor Ψ-**EX** $\to$ **LEX** factors through Φ-**EX**; which is to say that every Ψ-exact category is Φ-exact, and every Ψ-exact functor Φ-exact. There are various ways of characterising this ordering.

7.1. **Proposition.** *Given classes of lex-weights Φ and Ψ, the following are equivalent:*

(1) $\Phi \leqslant \Psi$*;*
(2) $\Phi \subseteq \Psi^*$*;*
(3) $\Phi^* \subseteq \Psi^*$*;*
(4) $\Phi_l\mathscr{C} \subseteq \Psi_l\mathscr{C}$ *for all small, finitely complete* $\mathscr{C}$*;*
(5) $\Phi_l\mathscr{C} \subseteq \Psi_l\mathscr{C}$ *for all finitely complete* $\mathscr{C}$*.*

Proof. If (1) holds, then for any finitely complete $\mathscr{C}$, the category $\Psi_l\mathscr{C}$ and inclusion $\Psi_l\mathscr{C} \to \mathscr{P}\mathscr{C}$ are both Φ-exact, since Ψ-exact. Thus $\Psi_l\mathscr{C}$ is closed in $\mathscr{P}\mathscr{C}$ under finite limits and Φ-lex-colimits, whence $\Phi_l\mathscr{C} \subseteq \Psi_l\mathscr{C}$ by Proposition 3.1. Thus (1) $\Rightarrow$ (5); and trivially (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2), so it remains to show (2) $\Rightarrow$ (1). As it is clear from Propositions 3.4 and 3.5 that Ψ-**EX** $=$ Ψ^*-**EX**, it is enough to show that if $\Phi \subseteq \Psi$ then $\Phi \leqslant \Psi$. But if $\Phi \subseteq \Psi$, then clearly $\Phi_l\mathscr{C} \subseteq \Psi_l\mathscr{C}$ for all finitely complete $\mathscr{C}$, so that if $\mathscr{C} \to \Psi_l\mathscr{C}$ has a lex left adjoint, then so does $\mathscr{C} \to \Phi_l\mathscr{C}$. Thus every Ψ-exact category is also Φ-exact, and clearly any Ψ-exact functor is Φ-exact, so that $\Phi \leqslant \Psi$ as desired. $\square$

Taking $\Phi = \{\psi\}$ in the above, we immediately deduce the following result, which can be seen as the analogue, for our theory, of [1, Theorem 5.1].

7.2. **Corollary.** *If Ψ is a class of lex-weights, then $\psi \in \Psi^*$ if and only if every Ψ-exact category is also $\{\psi\}$-exact, and every Ψ-exact functor is $\{\psi\}$-exact.*

Whenever $\Phi \leqslant \Psi$, we have a forgetful 2-functor $\Psi\text{-}\mathbf{EX} \to \Phi\text{-}\mathbf{EX}$; and we now investigate the extent to which this has a left biadjoint. We saw in Corollary 3.7 that such a biadjoint exists when Φ is the minimal class of lex-weights, and Ψ arbitrary; and we next shall consider the other extremal case, in which Ψ is maximal, and Φ arbitrary. In other words, we wish to describe the free small-exact completion of the Φ-exact $\mathscr{C}$.

For reasons of size, we cannot expect always to be able to do this; but we may do so, at least, whenever $\mathscr{C}$ is small. For such a $\mathscr{C}$, we will construct its small-exact completion as a suitable lex-reflective subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$, into which $\mathscr{C}$ will embed via the (restricted) Yoneda embedding. Certainly this embedding will preserve finite limits; we wish it also to preserve Φ-lex-colimits. But this is equally well, by Proposition 3.5, to ask that it should preserve all Φ^*-lex-colimits; which in turn is equivalent to the requirement that every $F\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$ in our subcategory should send Φ^*-lex-colimits in $\mathscr{C}$ to limits in $\mathscr{V}$. Let us therefore write $\mathscr{P}_\Phi\mathscr{C}$ for the full subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ spanned by the functors with this property, and, recognising that every representable lies in $\mathscr{P}_\Phi\mathscr{C}$, write $V\colon \mathscr{C} \to \mathscr{P}_\Phi\mathscr{C}$ for the restricted Yoneda embedding. The first step in showing that this constitutes a small-exact completion of $\mathscr{C}$ is to prove:

7.3. **Proposition.** *If $\mathscr{C}$ is small and Φ-exact, then $\mathscr{P}_\Phi\mathscr{C}$ is lex-reflective in $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$, and hence small-exact.*

In proving this proposition, we will need to make use of a technical result; it states that the full, replete, lex-reflective subcategories of $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ form a small, complete lattice, in which infima are given by intersection. In the unenriched case, this result—or rather a generalisation of it—was proved by Borceux and Kelly in [3]; we recall their proof, indicating its adaptation to the enriched setting, as Proposition A.1 below.

Proof. We proceed first under the assumption that $\Phi_l\mathscr{C}$ is small. In this case, taking $L\colon \Phi_l\mathscr{C} \to \mathscr{C}$ to be a left exact left adjoint for $W\colon \mathscr{C} \to \Phi_l\mathscr{C}$, we may consider the following string of adjunctions

$$[\mathscr{C}^{\mathrm{op}}, \mathscr{V}] \quad \begin{array}{c} \xleftarrow{\ \Sigma_L := \mathrm{Lan}_{L^{\mathrm{op}}}\ } \\ \perp \\ \xrightarrow{\ \Sigma_W := \mathrm{Lan}_{W^{\mathrm{op}}}\ } \\ \perp \\ \xleftarrow{\ \Delta_W := [W^{\mathrm{op}}, 1]\ } \\ \perp \\ \xrightarrow[\ \Pi_W := \mathrm{Ran}_{W^{\mathrm{op}}}\ ]{} \end{array} \quad [(\Phi_l\mathscr{C})^{\mathrm{op}}, \mathscr{V}] \ .$$

Each of the functors appearing in it is left exact, the lower three since they are right adjoints, and Σ_L because L is. Since W is fully faithful, so are Σ_W and Π_W; consequently the unit η of the adjunction $\Sigma_W \dashv \Delta_W$ is invertible, and on composing its inverse with the unit ν of the adjunction $\Delta_W \dashv \Pi_W$, we obtain a natural transformation

$$\theta := \Sigma_W \xrightarrow{\nu.1} \Pi_W \Delta_W \Sigma_W \xrightarrow{1.\eta^{-1}} \Pi_W \ .$$

We claim that $F \in [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ lies in $\mathscr{P}_\Phi\mathscr{C}$ just when θ_F is invertible. Indeed, since $L \dashv W$, we have $\Sigma_W \cong [L^{\mathrm{op}}, 1]$; whence $(\Sigma_W F)(\varphi) \cong F(L\varphi) \cong F(\varphi \star 1_{\mathscr{C}})$. On the other hand, we have $(\Pi_W F)(\varphi) = \{\Phi_l\mathscr{C}(W, \varphi), F\} = \{[\mathscr{C}^{\mathrm{op}}, \mathscr{V}](Y, \varphi), F\} \cong \{\varphi, F\}$, and it is straightforward to verify that under these isomorphisms, the map $(\theta_F)_\varphi$ is identified with the canonical comparison map $F(\varphi \star 1_{\mathscr{C}}) \to \{\varphi, F\}$. So $F \in \mathscr{P}_\Phi\mathscr{C}$ just when θ_F is invertible, as claimed.

As observed above, both Σ_W and Π_W are fully faithful, and have left exact left adjoints; consequently, they determine idempotent left exact monads S and T on $[(\Phi_l\mathscr{C})^{\mathrm{op}}, \mathscr{V}]$, whose respective categories of algebras—denoted by $\mathscr{S}$ and $\mathscr{T}$—are isomorphic to the replete images of Σ_W and Π_W in $[(\Phi_l\mathscr{C})^{\mathrm{op}}, \mathscr{V}]$. Moreover, to say that $F \in [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ lies in $\mathscr{P}_\Phi\mathscr{C}$ is by the above to say that θ_F, and hence $\nu_{(\Sigma_W F)} \colon \Sigma_W F \to \Pi_W \Delta_W \Sigma_W F = T(\Sigma_W F)$ is invertible, which is equally well to say that $\Sigma_W F$—which necessarily lies in $\mathscr{S}$—also lies in $\mathscr{T}$. Conversely, if $G \in [(\Phi_l\mathscr{C})^{\mathrm{op}}, \mathscr{V}]$ lies in $\mathscr{S} \cap \mathscr{T}$, then GW^{op} must lie in $\mathscr{P}_\Phi\mathscr{C}$, since the map $\theta_{GW^{\mathrm{op}}}$ may be decomposed as the composite

$$\Sigma_W \Delta_W G \xrightarrow{\epsilon} G \xrightarrow{\nu} \Pi_W \Delta_W G$$

of the counit of $\Sigma_W \dashv \Delta_W$ with the unit of $\Delta_W \dashv \Pi_W$ at G; but both these maps are invertible, the first by the assumption that $G \in \mathscr{S}$, and the second by the assumption that $G \in \mathscr{T}$. Consequently, if we can show that $\mathscr{S} \cap \mathscr{T}$ is lex-reflective in $[(\Phi_l\mathscr{C})^{\mathrm{op}}, \mathscr{V}]$ via a reflector $\rho \colon 1 \to R$, we can conclude that $\mathscr{P}_\Phi\mathscr{C}$ is lex-reflective in $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$, via

$$1 \xrightarrow{\eta} \Delta_W \Sigma_W \xrightarrow{1.\rho.1} \Delta_W R \Sigma_W \ .$$

But $\mathscr{S}$ and $\mathscr{T}$ are both lex-reflective in $[(\Phi_l\mathscr{C})^{\mathrm{op}}, \mathscr{V}]$, and thus, by Proposition A.1, so is $\mathscr{S} \cap \mathscr{T}$. This proves that $\mathscr{P}_\Phi\mathscr{C}$ is lex-reflective in $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ whenever $\Phi_l\mathscr{C}$ is small.

We now drop the assumption on $\Phi_l\mathscr{C}$. To show that $\mathscr{P}_\Phi\mathscr{C}$ is still lex-reflective, let us observe that for each $\varphi \in \Phi$, $\{\varphi\}_l\mathscr{C}$ is certainly small, since $\mathscr{C}$ is, so that each $\mathscr{P}_{\{\varphi\}}\mathscr{C}$ is lex-reflective in $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ by the case just proved. Thus by Proposition A.1, $\mathscr{E} = \bigcap_\varphi \mathscr{P}_{\{\varphi\}}\mathscr{C}$ is also lex-reflective in $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$: we claim that it is in fact $\mathscr{P}_\Phi\mathscr{C}$, which will complete the proof. Clearly $\mathscr{P}_\Phi\mathscr{C} \subseteq \mathscr{E}$; for the converse, we must show that each $F \colon \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$ in $\mathscr{E}$ sends Φ^*-lex-colimits in $\mathscr{C}$ to limits in $\mathscr{V}$. But this is equally well to ask that the induced functor $Z \colon \mathscr{C} \to \mathscr{E}$ should preserve Φ^*-lex-colimits, which since $\mathscr{C}$ and $\mathscr{E}$ are both Φ-exact, is equally well to ask that Z should preserve Φ-lex-colimits; which in turn is to ask that each $F \in \mathscr{E}$ should send Φ-lex-colimits to limits. But this is just to ask that for each $\varphi \in \Phi$, each $F \in \mathscr{E}$ should send $\{\varphi\}$-lex-colimits to limits, which is so because $F \in \mathscr{E} \subseteq \mathscr{P}_{\{\varphi\}}\mathscr{C}$. □

7.4. **Corollary.** *If $\mathscr{C}$ is small and Φ-exact, then $V \colon \mathscr{C} \to \mathscr{P}_\Phi\mathscr{C}$ is a full, Φ-exact embedding of $\mathscr{C}$ into a $\mathscr{V}$-topos.*

Proof. In light of the preceding proposition, it suffices to show that V is a Φ-exact functor. Certainly it preserves finite limits; as for Φ-lex-colimits, we must show that if $\varphi \in \Phi[\mathscr{K}]$ and $D \colon \mathscr{K} \to \mathscr{C}$, then V preserves the colimit $\varphi \star D$: for which we calculate that

$$\begin{aligned} &\mathscr{P}_\Phi\mathscr{C}(V(\varphi \star D), F) \cong F(\varphi \star D) \cong \{\varphi, FD^{\mathrm{op}}\} \\ \cong\ &[\mathscr{K}^{\mathrm{op}}, \mathscr{V}](\varphi, FD^{\mathrm{op}}) \cong [\mathscr{K}^{\mathrm{op}}, \mathscr{V}](\varphi, \mathscr{P}_\Phi\mathscr{C}(VD\text{–}, F)) \ . \end{aligned}$$ □

Given the preceding results, it is now an essentially standard argument to prove that:

7.5. **Theorem.** *If $\mathscr{C}$ is small and Φ-exact, then $V \colon \mathscr{C} \to \mathscr{P}_\Phi\mathscr{C}$ provides a bireflection of $\mathscr{C}$ along the forgetful 2-functor ∞-**EX** $\to \Phi$-**EX**.*

Here we write ∞-**EX** for the 2-category of small-exact categories, small-exact functors and arbitrary natural transformations.

Proof. By the preceding two results, $\mathscr{P}_\Phi\mathscr{C}$ is small-exact and V is Φ-exact; and so composition with V induces, for any small-exact category $\mathscr{D}$, a functor

$$V^*\colon \infty\text{-}\mathbf{EX}(\mathscr{P}_\Phi\mathscr{C}, \mathscr{D}) \to \Phi\text{-}\mathbf{EX}(\mathscr{C}, \mathscr{D})$$

which we are to show is an equivalence. As is typical, we do this by exhibiting as its pseudoinverse the functor which on objects sends $F\colon \mathscr{C} \to \mathscr{D}$ to $\mathrm{Lan}_V F\colon \mathscr{P}_\Phi\mathscr{C} \to \mathscr{D}$. First we show that this is well-defined; that is, that $\mathrm{Lan}_V F$ is a small-exact functor whenever F is Φ-exact. First we note that $\mathrm{Lan}_V F$ preserves finite limits: indeed, $\mathrm{Lan}_Y F\colon \mathscr{P}\mathscr{C} \to \mathscr{D}$ preserves finite limits since $\mathscr{D}$ is small-exact, and now because the inclusion $I\colon \mathscr{P}_\Phi\mathscr{C} \to \mathscr{P}\mathscr{C}$ is fully faithful, we have $\mathrm{Lan}_V F \cong (\mathrm{Lan}_Y F).I$, the composite of two finite-limit preserving functors, and so itself finite-limit preserving. It remains to show that $\mathrm{Lan}_V F$ is cocontinuous. For this, observe that $\mathrm{Lan}_Y F$ has as right adjoint the singular functor $\tilde{F}\colon \mathscr{D} \to \mathscr{P}\mathscr{C}$ sending D to $\mathscr{D}(F{-}, D)$; so that if we can show that $\tilde{F}$ factors through $\mathscr{P}_\Phi\mathscr{C}$ as $\tilde{F} = IR$, say, then we will have $\mathrm{Lan}_V F \dashv R$ and so $\mathrm{Lan}_V F$ cocontinuous. But since F preserves Φ-lex-colimits, $\mathscr{D}(F{-}, D)$ will certainly send them to limits in $\mathscr{V}$ for each D, and so we have the desired factorisation of $\tilde{F}$ through $\mathscr{P}_\Phi\mathscr{C}$. We have therefore shown that Lan_V is a functor $\Phi\text{-}\mathbf{EX}(\mathscr{C}, \mathscr{D}) \to \infty\text{-}\mathbf{EX}(\mathscr{P}_\Phi\mathscr{C}, \mathscr{D})$, and it remains to show that it is pseudoinverse to V^*. But since V is fully faithful, we have $V^*.\mathrm{Lan}_V \cong 1$; and since V is dense, we have $\mathrm{Lan}_V.V^* \cong 1$. □

Before continuing, let us use the preceding results to complete the proof of Proposition 4.3, which we restate here as:

7.6. **Proposition.** *For Φ a class of lex-weights, a category $\mathscr{C}$ is Φ-exact if and only if it is $\{\varphi\}$-exact for each $\varphi \in \Phi$.*

Proof. If $\mathscr{C}$ is Φ-exact, then it is $\{\varphi\}$-exact for each $\varphi \in \Phi$ by Proposition 7.1. Suppose conversely that $\mathscr{C}$ is $\{\varphi\}$-exact for each $\varphi \in \Phi$. By Proposition 4.2, $\mathscr{C}$ will be Φ-exact if we can show every small, full, finite-limit- and Φ-lex-colimit-closed subcategory $\mathscr{D} \subseteq \mathscr{C}$ to be Φ-exact. Fix such a $\mathscr{D}$. Now for each $\varphi \in \Phi$, $\mathscr{D}$ is $\{\varphi\}$-exact, since $\mathscr{C}$ is, and so $\mathscr{P}_{\{\varphi\}}\mathscr{D}$ is lex-reflective in $[\mathscr{D}^{\mathrm{op}}, \mathscr{V}]$. As in the proof of Proposition 7.3, we therefore have $\mathscr{P}_\Phi(\mathscr{D}) = \bigcap_\varphi \mathscr{P}_{\{\varphi\}}(\mathscr{D})$ also lex-reflective in $[\mathscr{D}^{\mathrm{op}}, \mathscr{V}]$, so that by Theorem 4.1, $\mathscr{D}$ is Φ-exact as required. □

We now give our final result, which, for an arbitrary pair of classes $\Phi \leqslant \Psi$, describes the Ψ-exact completion of the small Φ-exact $\mathscr{C}$. Given such a $\mathscr{C}$, we write $\Psi_\Phi\mathscr{C}$ for the closure of $\mathscr{C}$ in $\mathscr{P}_\Phi\mathscr{C}$ under finite limits and Ψ-lex-colimits, and

$$V = \mathscr{C} \xrightarrow{Z} \Psi_\Phi\mathscr{C} \xrightarrow{K} \mathscr{P}_\Phi\mathscr{C}$$

for the factorisation of V this induces.

7.7. **Theorem.** *Let $\Phi \leqslant \Psi$ be classes of lex-weights, and let $\mathscr{C}$ be small and Φ-exact. Now $Z\colon \mathscr{C} \to \Psi_\Phi\mathscr{C}$ provides a bireflection of $\mathscr{C}$ along the forgetful 2-functor $\Psi\text{-}\mathbf{EX} \to \Phi\text{-}\mathbf{EX}$.*

For instance, if $\mathscr{V} = \mathbf{Set}$, $\Phi = \Phi_{\mathrm{reg}}$ and $\Psi = \Phi_{\mathrm{ex}}$, then our $\Psi_\Phi\mathscr{C}$ is what is typically referred to as the ex/reg completion of the regular category $\mathscr{C}$; that it can be constructed in the above manner, by closing the representables in the topos of sheaves for the regular topology under finite limits and coequalisers of equivalence relations, was shown in [23].

Proof. First observe that $\Psi_\Phi\mathscr{C}$ is Ψ-lex-cocomplete, and K a full, Ψ-lex-cocontinuous, embedding of it into a Ψ-exact category; whence, by Theorem 4.1, $\Psi_\Phi\mathscr{C}$ is Ψ-exact. Moreover, the embedding $Z\colon \mathscr{C} \to \Psi_\Phi\mathscr{C}$ preserves finite limits and Φ-lex-colimits, since V preserves, and K reflects them. Thus for any Ψ-exact category $\mathscr{D}$, composition with Z induces a functor

$$Z^*\colon \Psi\text{-}\mathbf{EX}(\Psi_\Phi\mathscr{C}, \mathscr{D}) \to \Phi\text{-}\mathbf{EX}(\mathscr{C}, \mathscr{D}) \ ,$$

which we are to show is an equivalence. As before, we shall do so by showing that a suitable pseudoinverse is given by left Kan extension along Z.

We prove the result first under the assumptions that $\mathscr{D}$ is small, and that Φ and Ψ are both small classes of lex-weights; recall from Section 4 that this means that $\Phi_l\mathscr{K}$ and $\Psi_l\mathscr{K}$ will be small whenever $\mathscr{K}$ is. Under these circumstances, with both $\mathscr{C}$ and $\mathscr{D}$ being small, we may form their small-exact completions $V\colon \mathscr{C} \to \mathscr{P}_\Phi\mathscr{C}$ and $U\colon \mathscr{D} \to \mathscr{P}_\Psi\mathscr{D}$. Now for any Φ-exact $F\colon \mathscr{C} \to \mathscr{D}$, the composite $UF\colon \mathscr{C} \to \mathscr{P}_\Psi\mathscr{D}$ is again Φ-exact; it also has small-exact codomain, and so by Theorem 7.5, $\mathrm{Lan}_V(UF)\colon \mathscr{P}_\Phi\mathscr{C} \to \mathscr{P}_\Psi\mathscr{D}$ exists and is small-exact. Since $\mathrm{Lan}_V(UF)$ preserves finite limits and Ψ-lex-colimits, and maps $\mathscr{C}$ into the replete image of $\mathscr{D}$ in $\mathscr{P}_\Psi\mathscr{D}$, it must map $\Psi_\Phi\mathscr{C}$ into the closure of $\mathscr{D}$ in $\mathscr{P}_\Psi\mathscr{D}$ under finite limits and Ψ-lex-colimits. But this is again just the replete image of $\mathscr{D}$ in $\mathscr{P}_\Psi\mathscr{D}$, and so there is a factorisation:

$$\begin{array}{ccccc}
\mathscr{C} & \xrightarrow{Z} & \Psi_\Phi\mathscr{C} & \xrightarrow{K} & \mathscr{P}_\Phi\mathscr{C} \\
{\scriptstyle F}\downarrow & \cong & \downarrow{\scriptstyle \bar{F}} & \cong & \downarrow{\scriptstyle \mathrm{Lan}_V(UF)} \\
\mathscr{D} & = & \mathscr{D} & \xrightarrow[U]{} & \mathscr{P}_\Psi\mathscr{D}\ .
\end{array}$$

Now as K is fully faithful, we have $U\bar{F} \cong \mathrm{Lan}_V(UF).K \cong \mathrm{Lan}_Z(UF)$; but since U is fully faithful, it reflects Kan extensions, whence $\bar{F}$ is a left Kan extension of F along Z. Moreover, as $\mathrm{Lan}_V(UF)$ and K are Ψ-exact, and U is full and faithful, it follows that $\bar{F}$ is Ψ-exact. Thus we have shown that every Φ-exact $F\colon \mathscr{C} \to \mathscr{D}$ admits a Ψ-exact left Kan extension along Z, so determining a functor $\Phi\text{-}\mathbf{EX}(\mathscr{C}, \mathscr{D}) \to \Psi\text{-}\mathbf{EX}(\Psi_\Phi\mathscr{C}, \mathscr{D})$; it remains to show that this functor is pseudoinverse to Z^*. Certainly we have $Z^*.\mathrm{Lan}_Z \cong 1$, since Z is fully faithful; on the other hand, for any Ψ-exact $G\colon \Psi_\Phi\mathscr{C} \to \mathscr{D}$, the collection of $\varphi \in \Psi_\Phi\mathscr{C}$ at which the component of the canonical $\mathrm{Lan}_Z(GZ) \to G$ is invertible contains the representables (as $Z^*.\mathrm{Lan}_Z.Z^* \cong Z^*$) and is closed under finite limits and Ψ-lex-colimits (since both G and $\mathrm{Lan}_Z(GZ)$ are Ψ-exact), and so must be all of $\Psi_\Phi\mathscr{C}$; thus $\mathrm{Lan}_Z.Z^* \cong 1$ as required. This completes the proof under the assumptions that Φ, Ψ and $\mathscr{D}$ are all small.

We now drop the assumption that $\mathscr{D}$ is small. To complete the proof in this case, it is enough to show that every Φ-exact $F\colon \mathscr{C} \to \mathscr{D}$ admits a Ψ-exact left Kan extension along Z, as then we may conclude the argument as before. To this end, let $\mathscr{E}$ be the closure of the image of F in $\mathscr{D}$ under finite limits and Ψ-lex-colimits, and

$$F = \mathscr{C} \xrightarrow{G} \mathscr{E} \xrightarrow{H} \mathscr{D}$$

the factorisation so induced. Now $\mathscr{E}$ is Ψ-lex-cocomplete, and H is Ψ-lex-cocontinuous, so that by Theorem 4.1, both $\mathscr{E}$ and H are in fact Ψ-exact. Moreover, G is Φ-exact—since F is Φ-exact and H fully faithful—and $\mathscr{E}$ is small, being the closure of a small subcategory under a small class of lex-weights; so that by the case just proved, $\mathrm{Lan}_Z G$ exists and is Ψ-exact. Consequently, $H.\mathrm{Lan}_Z G$ is Ψ-exact, and we will be done if we can show that it is in fact $\mathrm{Lan}_Z F$. Equivalently, we may show that H preserves $\mathrm{Lan}_Z G$; equivalently, that for each $\psi \in \Psi_\Phi \mathscr{C}$, the colimit $\psi \star G$ in $\mathscr{E}$ is preserved by H; or equivalently, that for each $\psi \in \Psi_\Phi \mathscr{C}$ and $X \in \mathscr{D}$, the canonical morphism

$$\mathscr{D}(H(\psi \star G), X) \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}](\psi, \mathscr{D}(F\text{–}, X)) \tag{7.1}$$

is invertible in $\mathscr{V}$. To do this last, we let $\mathscr{E}'$ be the closure of $\mathscr{E} \cup \{X\}$ in $\mathscr{D}$ under finite limits and Ψ-lex-colimits. Arguing as before, $\mathscr{E}'$ is small and Ψ-exact, and the inclusion $K \colon \mathscr{E} \to \mathscr{E}'$ is Ψ-exact. Thus $K.\mathrm{Lan}_Z G$ is also Ψ-exact, and so, having small codomain, is $\mathrm{Lan}_Z(KG)$ by the case just proved. Hence the canonical morphism

$$\mathscr{E}'(K(\psi \star G), X) \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}](\psi, \mathscr{E}'(KG\text{–}, X))$$

is invertible in $\mathscr{V}$; but this is equally well the morphism (7.1), since the inclusion $\mathscr{E}' \to \mathscr{D}$ is fully faithful. Thus $H.\mathrm{Lan}_Z G$ is $\mathrm{Lan}_Z F$ as claimed; which completes the proof in the case where Φ and Ψ are both small.

We next drop the assumption that Ψ is small. Under these circumstances, it is enough to show as before that every Φ-exact $F \colon \mathscr{C} \to \mathscr{D}$ admits a Ψ-exact left Kan extension along Z. So consider the collection of $\psi \in \Psi_\Phi \mathscr{C}$ for which there exists a small Ψ' with $\Phi \leqslant \Psi' \leqslant \Psi$ and $\psi \in \Psi'_\Phi \mathscr{C} \subseteq \Psi_\Phi \mathscr{C}$. It is easy to see that this collection contains the representables and is closed under finite limits and Ψ-lex-colimits, and hence is all of $\Psi_\Phi \mathscr{C}$. So for every $\psi \in \Psi_\Phi \mathscr{C}$, we choose such a Ψ'; now by the case just proved, F admits a Ψ'-exact left Kan extension along $\mathscr{C} \to \Psi'_\Phi \mathscr{C}$, so that, in particular, the colimit $\psi \star F$ exists in $\mathscr{D}$. Thus the left Kan extension $\mathrm{Lan}_Z F \colon \Psi_\Phi \mathscr{C} \to \mathscr{D}$ exists, and it remains to show that it is Ψ-exact. To see that it preserves Ψ-lex-colimits, suppose given some $\psi \in \Psi[\mathscr{K}]$ and lex $D \colon \mathscr{K} \to \Psi_\Phi \mathscr{C}$; we must show that $\mathrm{Lan}_Z F$ preserves $\psi \star D$. Choosing a small $\Phi \leqslant \Psi' \leqslant \Psi$ such that ψ and each DX lie in $\Psi'_\Phi \mathscr{C}$, we observe that the left Kan extension of F along $\mathscr{C} \to \Psi'_\Phi \mathscr{C}$, being Ψ'-exact, will preserve the colimit $\psi \star D$: from which it follows easily that $\mathrm{Lan}_Z F$ does too. A corresponding argument shows that $\mathrm{Lan}_Z F$ preserves all finite limits, and so is indeed Ψ-exact; which completes the proof under the assumption that Φ is small.

We now drop this final assumption. To complete the proof, we must again show that each Φ-exact $F \colon \mathscr{C} \to \mathscr{D}$ admits a Ψ-exact left Kan extension along Z. Now for each $\varphi \in \Phi$, we have $\mathscr{P}_{\{\varphi\}} \mathscr{C}$ lex-reflective in $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$; and as in the proof of Proposition 7.3, we have in fact that $\mathscr{P}_\Phi \mathscr{C} = \bigcap_{\varphi \in \Phi} \mathscr{P}_{\{\varphi\}} \mathscr{C}$. But by Proposition A.1, $[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ has only a small set of lex-reflective subcategories, and so there is some small $\Phi' \subseteq \Phi$ such that $\bigcap_{\varphi \in \Phi} \mathscr{P}_{\{\varphi\}} \mathscr{C} = \bigcap_{\varphi \in \Phi'} \mathscr{P}_{\{\varphi\}} \mathscr{C}$. Thus $\mathscr{P}_\Phi \mathscr{C} = \mathscr{P}_{\Phi'} \mathscr{C}$, whence also $\Psi_\Phi \mathscr{C} = \Psi_{\Phi'} \mathscr{C}$; thus Z is equally well the embedding $\mathscr{C} \to \Psi_{\Phi'} \mathscr{C}$, so that, by the case just proved, any Φ-exact F, being *a fortiori* Φ'-exact, admits a Ψ-exact left Kan extension along it. □

Appendix A. Localisations of locally presentable categories

The purpose of this appendix is to prove the following technical result, needed for the arguments of Proposition 7.3, Proposition 7.6 and Theorem 7.7 above. In its statement, and throughout this section, subcategory will always mean full, replete subcategory.

A.1. **Proposition.** *If $\mathscr{C}$ is a locally presentable category in which finite limits commute with filtered colimits, then the lex-reflective subcategories of $\mathscr{C}$ form a small, complete lattice, in which infima are given by intersection.*

As mentioned above, this result was proved for the unenriched case in [3], appearing there as Theorem 6.8. We now recall this proof, indicating along the way how it should be adapted to the enriched setting in which we are working. The first step is to show:

A.2. **Proposition.** *Any left exact reflector on a locally presentable category preserves κ-filtered colimits for some regular cardinal κ; equivalently, any localisation of a locally presentable category is locally presentable.*

Proof. Let $\mathscr{E}$ be lex-reflective in the locally λ-presentable $\mathscr{C}$; it suffices to show that $\mathscr{E}$ is closed in $\mathscr{C}$ under κ-filtered colimits for some κ, which is equally well to show that $\mathscr{E}_0$ is closed in $\mathscr{C}_0$ under κ-filtered colimits. Now by Proposition 7.5 of [18], $\mathscr{C}_0$ is locally λ-presentable since $\mathscr{C}$ is; moreover, as finite conical limits are also finite weighted limits, $\mathscr{E}_0$ is lex-reflective in $\mathscr{C}_0$. Thus it is enough to prove the result in the unenriched case, and this is done in [3, Proposition 6.7]. We now briefly recall the argument.

Let $L_0 \colon \mathscr{C}_0 \to \mathscr{E}_0$ be the left exact reflection of $\mathscr{C}_0$ into $\mathscr{E}_0$. Since L_0 preserves kernel-pairs, a standard result identifies $\mathscr{E}_0$ as the subcategory of $\mathscr{C}_0$ orthogonal to the class Σ of all monomorphisms inverted by L_0; see [15, Lemma A4.3.6], for instance. Now let Σ_λ comprise those maps of Σ whose codomain is λ-presentable. Note that Σ_λ is essentially-small—since the λ-presentable objects span an essentially-small subcategory, and $\mathscr{C}_0$ is well-powered—and so we can find a κ bounding the rank of the domains and codomains of the morphisms in it. Thus the subcategory orthogonal to Σ_λ is closed under κ-filtered colimits in $\mathscr{C}_0$ and we will be done if we can show this subcategory is in fact $\mathscr{E}_0$. By the above, this is equally well to show that any object orthogonal to Σ_λ is also orthogonal to each $m \colon A \to B$ in Σ. Given such an m, we may, since $\mathscr{C}_0$ is locally λ-presentable, write B as a λ-filtered colimit of λ-presentables, $(q_i \colon X_i \to B \mid i \in \mathscr{I})$. Since L_0 preserves pullbacks, each $m_i := q_i^*(m)$ is inverted by L_0 and so lies in Σ_λ. But since λ-filtered colimits commute with λ-small limits in $\mathscr{C}_0$, they are in particular stable under pullback, and so m is the colimit in $\mathscr{C}_0^{\mathbf{2}}$ of the m_i's; thus any object orthogonal to Σ_λ is orthogonal to m, which concludes the proof. □

A.3. **Corollary.** *Any locally presentable category has only a small set of lex-reflective subcategories.*

Proof. If $\mathscr{E}$ is lex-reflective in the locally λ-presentable $\mathscr{C}$, then as above, $\mathscr{E}_0$ is lex-reflective in $\mathscr{C}_0$; and $\mathscr{E}_0$ clearly determines $\mathscr{E}$. So it is enough to show that $\mathscr{C}_0$ has only a small set of lex-reflective subcategories. The preceding proof shows that a left exact reflector on $\mathscr{C}_0$ is determined by the (replete) class of monomorphisms with λ-presentable codomain that it inverts. But the class of all monomorphisms with λ-presentable codomain is essentially-small, and so has only a small set of replete subclasses. □

Given this, Proposition A.1 will now follow if we can prove the following result; for the unenriched case, this is [3, Theorem 5.3], and the argument given there carries over unchanged to the $\mathscr{V}$-categorical setting.

A.4. **Proposition.** *If finite limits commute with filtered colimits in the locally presentable $\mathscr{C}$, then any small intersection of lex-reflective subcategories of $\mathscr{C}$ is again lex-reflective.*

Proof. We prove the result first for small, directed intersections, and then for finite ones; this is clearly sufficient. For the first of these, let $(\mathscr{A}_i \mid i \in \mathscr{I})$ be a small, directed diagram of lex-reflective subcategories of $\mathscr{C}$; we must show that $\mathscr{A} = \bigcap \mathscr{A}_i$ is also lex-reflective. Let each $\mathscr{A}_i$ have the reflector $\lambda_i \colon 1 \to L_i$, and let $\lambda \colon 1 \to L$ be the colimit of the directed diagram formed by these reflectors. Since finite limits commute with filtered colimits, and each L_i is left exact, L is too. It is moreover accessible, since each L_i is by Proposition A.2. Now, since each (L_i, λ_i) is a *well-pointed* endofunctor—in the sense that $L_i\lambda_i = \lambda_i L_i$—it follows from [16, Proposition 9.1] that (L, λ) is too, and that an object X lies in $\mathscr{A}$ if and only if $\lambda_X \colon X \to LX$ is invertible, if and only if it is orthogonal to each component of λ. Writing **On** for the (large) poset of small ordinals, we now define a transfinite sequence $L_{(-)} \colon \mathbf{On} \to [\mathscr{C}, \mathscr{C}]$ by

$$L_0 = 1_{\mathscr{C}}, \quad L_{\alpha+1} = LX_\alpha, \quad X_\gamma = \mathrm{colim}_{\alpha<\gamma} X_\alpha \ ,$$

with transition maps being given by λ at successor stages, and by the colimit injections at limit stages. Each L_α is left exact, since L is, and since the colimits taken at limit stages are filtered. Moreover, any $X \in \mathscr{A}$, being orthogonal to each component of λ, is also orthogonal to each $L_0 \to L_\alpha$, so that if for some ordinal κ, the transition map $\lambda L_\kappa \colon L_\kappa \to LL_\kappa$ is invertible, then L_κ will land in $\mathscr{A}$ and so provide the desired left exact reflection. But since L is accessible, it preserves κ-filtered colimits for some κ; and thus, since L is well-pointed, we deduce as in [16, Remark 6.3] that $\lambda L_\kappa \colon L_\kappa \to LL_\kappa$ is invertible, so that L_κ is the required left exact reflector into $\mathscr{A}$.

This proves closure under directed intersections; as for finite ones, it suffices to consider a binary intersection, and now the argument is almost identical. Given $\mathscr{S}$ and $\mathscr{T}$ lex-reflective subcategories of $\mathscr{C}$, corresponding to reflectors $\sigma \colon 1 \to S$ and $\tau \colon 1 \to T$, we form the pointed endofunctor $\sigma\tau \colon 1 \to ST$, which is left exact, well-pointed, and accessible, since S and T are. Moreover, [3, Proposition 4.3] shows that $X \in \mathscr{S} \cap \mathscr{T}$ if and only if $\sigma\tau_X \colon X \to STX$ is invertible; whereupon the same transfinite construction as before proves $\mathscr{S} \cap \mathscr{T}$ to be lex-reflective in $\mathscr{C}$, as required. □

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