# Discrete Laguerre-Sobolev expansions: A Cohen type inequality

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**Abstract**

C. Markett proved a Cohen type inequality for the classical Laguerre expansions in the appropriate weighted $L^p$ spaces. In this paper, we get a Cohen type inequality for the Fourier expansions in terms of discrete Laguerre–Sobolev orthonormal polynomials with an arbitrary (finite) number of mass points. So, we extend the result due to B. Xh. Fejzullahu and F. Marcellán.

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## 1 Introduction and notations

Littlewood conjectured in 1948 that for any trigonometric polynomial $F_N(x) = \sum_{k=1}^{N} a_k e^{in_k x}$ where $0 < n_1 < n_2 < \cdots < n_N$, $N \geq 2$, and $|a_k| \geq 1$ for $1 \leq k \leq N$, there holds the estimate from below

$$\int_0^{2\pi} |F_N(x)dx| \geq C \log N$$

where $C$ is an absolute constant (see [6]).

Cohen's inequality [3] was the first result on the way to the solution of this conjecture. Later, inequalities of this type have been established in various other contexts, e.g., on compact group (see [5]).

In [12] Markett proved such inequalities for classical orthogonal polynomial expansions in the appropriate weighted $L^p$ spaces, here in terms of the highest coefficient. The main purpose of this paper is to extend these results to discrete Laguerre-Sobolev expansions. More precisely, we obtain such inequalities, in the appropriate weighted $L^p$ spaces, for Fourier expansions in terms of orthonormal polynomials with respect to an inner product of the form

$$\langle p, q\rangle_S = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} p(x)q(x)\, x^{\alpha} e^{-x}\, dx + \sum_{j=0}^{N} M_j p^{(j)}(0) q^{(j)}(0), \qquad (1)$$

where $\alpha > -1$ and $M_j \geq 0$, $j = 0, \ldots, N$. Such inner products are called of discrete Sobolev type.

Recently in [4], the authors Fejzullahu and Marcellán obtained Cohen type inequalities for orthonormal expansions with respect to the above inner product in the case $N = 1$, i.e. at most two masses in the discrete part. In this particular case, the authors benefit from the fact that there are explicit formulas for the connection coefficients which appear in the representation of discrete Laguerre-Sobolev type polynomials in terms of three standard Laguerre polynomials (see [9]). For a general discrete Laguerre-Sobolev inner product, we only know that these coefficients are a nontrivial solution of a system of $N+1$ equations on $N+2$ unknowns (see [8]). If the system is solved, we get an intricate expression with which it is difficult to work. Our contribution in this paper is that we can assure that there exists limit of these connection coefficients and this is enough for our purpose.

Let $\{L_n^{\alpha}(x)\}_{n\geq 0}$ be the sequence of Laguerre polynomials, orthogonal on $[0, \infty)$ with respect to the probability measure $d\mu(x) = \dfrac{1}{\Gamma(\alpha+1)} x^{\alpha} e^{-x}\, dx$

where $\alpha > -1$ and normalized by $L_n^\alpha(0) = \binom{n+\alpha}{n}$. We denote the orthonormal Laguerre polynomial of degree $n$ by

$$l_n^\alpha(x) = \frac{L_n^\alpha(x)}{\|L_n^\alpha\|}$$

where $\|L_n^\alpha\|^2 = \int_0^\infty L_n^\alpha(x)^2\, d\mu(x)$.

Let $\{Q_n^\alpha\}_{n\geq 0}$ be the sequence of discrete Laguerre–Sobolev orthogonal polynomials with respect to the inner product (1) and such that $Q_n^\alpha(x)$ and $L_n^\alpha(x)$ have the same leading coefficient. We denote by

$$q_n^\alpha(x) = \langle Q_n^\alpha, Q_n^\alpha \rangle_S^{-1/2} Q_n^\alpha(x)$$

the orthonormal discrete Laguerre-Sobolev polynomials. From now on, for simplicity we write $Q_n(x) = Q_n^\alpha(x)$ and $q_n(x) = q_n^\alpha(x)$.

Laguerre expansions have been investigated mainly in the following two sets of weighted Lebesgue spaces, namely in the classical spaces ([2], [10])

$$L_p(x^\alpha dx) = \begin{cases} \{f; \quad \int_0^\infty |f(x)e^{-x/2}|^p x^\alpha dx < \infty\}, & \text{if } 1 \leq p < \infty; \\ \{f; \quad \operatorname*{ess\,sup}_{0<x<\infty} |f(x)e^{-x/2}| < \infty\}, & \text{if } p = \infty, \end{cases}$$

for $\alpha > -1$ as well as in the spaces

$$L_p(x^{\alpha p/2} dx) = \begin{cases} \{f; \quad \int_0^\infty |f(x)e^{-x/2}x^{\alpha/2}|^p dx < \infty\}, & \text{if } 1 \leq p < \infty; \\ \{f; \quad \operatorname*{ess\,sup}_{0<x<\infty} |f(x)e^{-x/2}x^{\alpha/2}| < \infty\}, & \text{if } p = \infty, \end{cases}$$

for $\alpha > -\frac{2}{p}$ if $1 \leq p < \infty$ and $\alpha \geq 0$ if $p = \infty$.

In order to unify the two results we are going to prove, we introduce an auxiliary parameter $\beta$ which means either $\alpha$ or $\alpha p/2$.

We consider the class $S_p^\beta$, $1 \leq p \leq \infty$, defined as the space of measurable functions $f$ defined on $[0, \infty)$, such that there exits $f^{(k)}(0)$ for $k = 0, \dots, N$ and if $1 \leq p < \infty$

$$\|f\|_{S_p^\beta}^p = \|f\|_{L_p(x^\beta dx)}^p + \sum_{i=0}^{N} M_j\, |f^j(0)|^p < \infty,$$

where

$$\|f\|_{L_p(x^\beta dx)}^p = \int_0^\infty |f(x)e^{-x/2}|^p x^\beta dx, \quad 1 \leq p < \infty,$$

and if $p=\infty$

$$\|f\|_{S_\infty^\beta} = \max\{\|f\|_{L_\infty(x^\beta dx)}, |f(0)|, \dots, |f^{(N)}(0)|\} < \infty,$$

where

$$\|f\|_{L_\infty(x^\beta dx)} = \begin{cases} \operatorname*{ess\,sup}_{0<x<\infty} |f(x)e^{-x/2}|, & \text{if } \beta=\alpha; \\ \operatorname*{ess\,sup}_{0<x<\infty} |f(x)e^{-x/2}x^{\alpha/2}|, & \text{if } \beta=\alpha p/2. \end{cases}$$

(If some $M_j=0$ the corresponding derivative does not appear in the maximum.)

Let $f\in S_p^\beta$, $1\le p\le\infty$, then the Fourier expansion in terms of orthonormal discrete Laguerre-Sobolev polynomials $\{q_n\}_{n\ge 0}$, is

$$\sum_{k=0}^{\infty} \hat{f}(k)\, q_k(x)$$

where $\hat{f}(k) = \langle f, q_k\rangle_S$.

In the following, $[S_p^\beta]$ denotes the space of all bounded linear operators $T$ from the space $S_p^\beta$ into itself, endowed with the usual operator norm,

$$\|T\|_{[S_p^\beta]} = \sup_{0\ne f\in S_p^\beta} \frac{\|Tf\|_{S_p^\beta}}{\|f\|_{S_p^\beta}}.$$

Let $1\le p\le\infty$. For a family of complex numbers $\{c_{k,n}\}_{k=0}^n$, $n\in\mathbb{N}\cup\{0\}$, with $|c_{n,n}|>0$ we define the operators $T_n^{\alpha,S}: S_p^\beta \to S_p^\beta$ by

$$T_n^{\alpha,S}(f) = \sum_{k=0}^{n} c_{k,n}\hat{f}(k)\, q_k.$$

Let us denote $q_0 = \frac{4\alpha+4}{2\alpha+1}$ for $\beta=\alpha$ and $q_0=4$ for $\beta=p\alpha/2$, and let $p_0$ be the conjugate of $q_0$, i.e. $1/p_0+1/q_0=1$. Now, we can state our main theorem, which extends the ones given in [12] and [4].

**Theorem 1** *Let $1\le p\le\infty$. There exists a positive constant $C$, independent of $n$, such that:*
*For $\alpha>-1/2$*

$$\|T_n^{\alpha,S}\|_{[S_p^\alpha]} \ge C\,|c_{n,n}| \begin{cases} n^{\frac{2\alpha+2}{p}-\frac{2\alpha+3}{2}}, & \text{if } 1\le p<p_0; \\ (\log(n+1))^{\frac{2\alpha+1}{4\alpha+4}}, & \text{if } p=p_0,\ p=q_0; \\ n^{\frac{2\alpha+1}{2}-\frac{2\alpha+2}{p}}, & \text{if } q_0<p\le\infty. \end{cases}$$

*For* $\alpha > -2/p$ *if* $1 \le p < \infty$ *and* $\alpha \ge 0$ *if* $p = \infty$

$$\|T_n^{\alpha,S}\|_{[S_p^{p\alpha/2}]} \ge C\,|c_{n,n}| \begin{cases} n^{\frac{2}{p}-\frac{3}{2}}, & \text{if } 1 \le p < p_0; \\ (\log(n+1))^{\frac{1}{4}}, & \text{if } p = p_0,\ p = q_0; \\ n^{\frac{1}{2}-\frac{2}{p}}, & \text{if } q_0 < p \le \infty. \end{cases}$$

This theorem will be proved in Section 3. In Section 2, we obtain some new results for discrete Laguerre-Sobolev polynomials, which we will use to establish Theorem 1. More concretely, we prove a technical lemma that will be used to deduce a Mehler-Heine type formula for Laguerre-Sobolev polynomials and a sharp estimation for their norm in the appropriate weighted $L_p$ spaces.

In the sequel we use the following notation, $a_n \sim b_n$ means that there exist positive constants $c_1$ and $c_2$, such that $c_1 a_n \le b_n \le c_2 a_n$ for $n$ large enough, while $a_n \cong b_n$ means that the sequence $\frac{a_n}{b_n}$ converges to 1. Throughout the paper, the values of the constants may change from line to line.

## 2 Estimates for discrete Laguerre-Sobolev polynomials

Consider the standard Laguerre polynomials $L_n^\alpha$ and the Laguerre-Sobolev polynomials $Q_n$ with the same leading coefficient.

Let us recall some properties of Laguerre polynomials for $\alpha > -1$ (see [14]). The evaluation at $x = 0$ of the polynomials $L_n^\alpha$ and its successive derivatives are given by

$$(L_n^\alpha)^{(k)}(0) = \frac{(-1)^k \Gamma(n+\alpha+1)}{(n-k)!\,\Gamma(\alpha+k+1)},\ k \in \mathbb{N} \cup \{0\},$$

and their $L_2$-norm is

$$\|L_n^\alpha\|^2 = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty (L_n^\alpha(x))^2\, x^\alpha e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}. \qquad (2)$$

As usual, we denote the derivatives of the $n$th kernels of Laguerre polynomials by

$$K_n^{(k,h)}(x,y) = \frac{\partial^{k+h}}{\partial x^k \partial y^h} K_n(x,y) = \sum_{i=0}^{n} \frac{(L_i^\alpha)^{(k)}(x)(L_i^\alpha)^{(h)}(y)}{\|L_i^\alpha\|^2}$$

with $k, h \in \mathbb{N} \cup \{0\}$ and the convention $K_n^{(0,0)}(x,y) = K_n(x,y)$.

In the next lemma, we obtain an asymptotic estimate for $Q_n^{(k)}(0)$, that will play an important role along this paper.

**Lemma 1** *Let $Q_n$ be the polynomials orthogonal with respect to the inner product (1). Then the following statements hold:*

*(a)*

$$\frac{Q_n^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \cong \begin{cases} \dfrac{C_k}{n^{\alpha+2k+1}}, & \text{for } k \text{ such that } M_k > 0; \\ C_k, & \text{otherwise}, \end{cases}$$

*where $C_k$ is a nonzero constant independent of $n$.*

*(b)*

$$\langle Q_n, Q_n \rangle_S \cong \|L_n^\alpha\|^2 \,.$$

**Proof.** If all the masses in the inner product (1) are zero the result is trivial because $Q_n = L_n^\alpha$. We will prove the result by induction concerning the number of positive masses in the inner product (1).

We take the first mass which is positive, namely $M_{j_1}$ ($j_1 \geq 0$), and consider the sequence of polynomials $\{Q_{n,1}\}_{n\geq 0}$ orthogonal with respect to the inner product

$$(p,q)_1 = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty p(x)q(x)\, x^\alpha e^{-x}\, dx + M_{j_1} p^{(j_1)}(0) q^{(j_1)}(0).$$

The Fourier expansion of the polynomial $Q_{n,1}$ in the orthogonal basis $\{L_n^\alpha\}_{n\geq 0}$ leads to

$$Q_{n,1}(x) = L_n^\alpha(x) - M_{j_1} Q_{n,1}^{(j_1)}(0) K_{n-1}^{(0,j_1)}(x,0)\,.$$

Therefore

$$Q_{n,1}(x) = L_n^\alpha(x) - \frac{M_{j_1} (L_n^\alpha)^{(j_1)}(0)}{1 + M_{j_1} K_{n-1}^{(j_1,j_1)}(0,0)} K_{n-1}^{(0,j_1)}(x,0)\,, \tag{3}$$

and

$$(Q_{n,1}, Q_{n,1})_1 = \|L_n^\alpha\|^2 + M_{j_1} \frac{((L_n^\alpha)^{(j_1)}(0))^2}{1 + M_{j_1} K_{n-1}^{(j_1,j_1)}(0,0)}\,. \tag{4}$$

These relationships are very well known in the literature of discrete Sobolev type orthogonal polynomials.

Taking derivatives $k$ times in (3) and evaluating at $x = 0$, we obtain

$$\frac{Q_{n,1}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = 1 - \frac{M_{j_1} K_{n-1}^{(k,j_1)}(0,0)}{1 + M_{j_1} K_{n-1}^{(j_1,j_1)}(0,0)} \frac{(L_n^\alpha)^{(j_1)}(0)}{(L_n^\alpha)^{(k)}(0)}. \tag{5}$$

Applying the Stolz criterion (see, e.g. [7]), we have

$$\lim_n \frac{K_{n-1}^{(k,j_1)}(0,0)}{n^{\alpha+k+j_1+1}} = \lim_n \frac{(L_{n-1}^\alpha)^{(k)}(0)(L_{n-1}^\alpha)^{(j_1)}(0)}{\|L_{n-1}^\alpha\|^2(\alpha+k+j_1+1)n^{\alpha+k+j_1}} \neq 0, \tag{6}$$

and therefore

$$\begin{aligned} \frac{K_{n-1}^{(k,j_1)}(0,0)}{K_{n-1}^{(j_1,j_1)}(0,0)} \frac{(L_n^\alpha)^{(j_1)}(0)}{(L_n^\alpha)^{(k)}(0)} &\cong \frac{(\alpha+2j_1+1)}{(\alpha+k+j_1+1)} \frac{(L_{n-1}^\alpha)^{(k)}(0)}{(L_{n-1}^\alpha)^{(j_1)}(0)} \frac{(L_n^\alpha)^{(j_1)}(0)}{(L_n^\alpha)^{(k)}(0)} \\ &\cong \frac{\alpha+2j_1+1}{\alpha+k+j_1+1}. \end{aligned} \tag{7}$$

Thus, from (5), (6) and (2), we have

$$\frac{Q_{n,1}^{(j_1)}(0)}{(L_n^\alpha)^{(j_1)}(0)} = \frac{1}{1 + M_{j_1} K_{n-1}^{(j_1,j_1)}(0,0)} \cong \frac{C_{j_1}}{n^{\alpha+2j_1+1}}$$

and for $k \neq j_1$

$$\frac{Q_{n,1}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \cong 1 - \frac{\alpha+2j_1+1}{\alpha+k+j_1+1} \neq 0.$$

So, we achieve (a) for $Q_{n,1}$. Besides, taking limits in (4) and using again the size of derivatives of Laguerre polynomials, we get (b) for the polynomials $Q_{n,1}$.

If there are no more positive masses, since $Q_{n,1} = Q_n$ we have concluded the proof. Otherwise, suppose that the results (a) and (b) hold for the sequence of polynomials $\{Q_{n,s-1}\}_{n\geq 0}$ orthogonal with respect to the inner product

$$\begin{aligned} (p,q)_{s-1} = &\frac{1}{\Gamma(\alpha+1)} \int_0^\infty p(x)q(x)\, x^\alpha e^{-x}\, dx \\ &+ M_{j_1} p^{(j_1)}(0) q^{(j_1)}(0) + \cdots + M_{j_{s-1}} p^{(j_{s-1})}(0) q^{(j_{s-1})}(0), \end{aligned}$$

where $j_1 < j_2 < \cdots < j_{s-1}$ and all these masses are positive. Now, we have to prove the result for the polynomials $Q_{n,s}$ orthogonal with respect to

$$\begin{aligned} (p,q)_s = &\frac{1}{\Gamma(\alpha+1)} \int_0^\infty p(x)q(x)\, x^\alpha e^{-x}\, dx \\ &+ M_{j_1} p^{(j_1)}(0) q^{(j_1)}(0) + \cdots + M_{j_s} p^{(j_s)}(0) q^{(j_s)}(0), \end{aligned}$$

where $M_{j_s} > 0$. Since $(p,q)_s = (p,q)_{s-1} + M_{j_s} p^{(j_s)}(0) q^{(j_s)}(0)$ we can work as before. Then the Fourier expansion of the polynomial $Q_{n,s}$ in the orthogonal basis $\{Q_{n,s-1}\}_{n\geq 0}$ leads to

$$Q_{n,s}(x) = Q_{n,s-1}(x) - M_{j_s} Q_{n,s}^{(j_s)}(0) K_{n-1,s-1}^{(0,j_s)}(x,0)\,,$$

where $K_{n,s-1}$ denotes the corresponding $n$th kernel for the sequence $\{Q_{n,s-1}\}$ and

$$K_{n,s-1}^{(k,h)}(x,y) = \sum_{i=0}^{n} \frac{Q_{i,s-1}^{(k)}(x) Q_{i,s-1}^{(h)}(y)}{(Q_{i,s-1}, Q_{i,s-1})_{s-1}}, \quad k,h \in \mathbb{N} \cup \{0\}.$$

Therefore, in the same way as in (3) and (4), we get

$$Q_{n,s}(x) = Q_{n,s-1}(x) - \frac{M_{j_s} Q_{n,s-1}^{(j_s)}(0)}{1 + M_{j_s} K_{n-1,s-1}^{(j_s,j_s)}(0,0)} K_{n-1,s-1}^{(0,j_s)}(x,0)\,, \tag{8}$$

and

$$(Q_{n,s}, Q_{n,s})_s = (Q_{n,s-1}, Q_{n,s-1})_{s-1} + M_{j_s} \frac{({Q_{n,s-1}}^{(j_s)}(0))^2}{1 + M_{j_s} K_{n-1,s-1}^{(j_s,j_s)}(0,0)}\,. \tag{9}$$

Taking derivatives $k$ times in (8) and evaluating at $x = 0$, we obtain

$$\frac{Q_{n,s}^{(k)}(0)}{(L_n^{\alpha})^{(k)}(0)} = \frac{Q_{n,s-1}^{(k)}(0)}{(L_n^{\alpha})^{(k)}(0)} \left[ 1 - \frac{M_{j_s} K_{n-1,s-1}^{(k,j_s)}(0,0)}{1 + M_{j_s} K_{n-1,s-1}^{(j_s,j_s)}(0,0)} \frac{Q_{n,s-1}^{(j_s)}(0)}{Q_{n,s-1}^{(k)}(0)} \right]. \tag{10}$$

Applying the Stolz criterion and the hypotheses (a) and (b) for $\{Q_{n,s-1}\}_{n\geq 0}$, we can obtain

$$K_{n-1,s-1}^{(k,j_s)}(0,0) \cong \begin{cases} C_k\, n^{\alpha+k+j_s+1}, & \text{if } k \neq j_1, \ldots, j_{s-1}; \\ C_k\, n^{j_s-k}, & \text{if } k = j_1, \ldots, j_{s-1}, \end{cases} \tag{11}$$

where $C_k$ is a nonzero constant. Indeed, for $k \neq j_1, \ldots, j_{s-1}$,

$$\lim_n \frac{K_{n-1,s-1}^{(k,j_s)}(0,0)}{n^{\alpha+k+j_s+1}} = \lim_n \frac{Q_{n-1,s-1}^{(k)}(0) Q_{n-1,s-1}^{(j_s)}(0)}{(Q_{n-1,s-1}, Q_{n-1,s-1})_{s-1} (\alpha+k+j_s+1)\; n^{\alpha+k+j_s}}$$
$$= \lim_n \frac{Q_{n-1,s-1}^{(k)}(0)}{(L_{n-1}^{\alpha})^{(k)}(0)} \lim_n \frac{Q_{n-1,s-1}^{(j_s)}(0)}{(L_{n-1}^{\alpha})^{(j_s)}(0)} \lim_n \frac{(L_{n-1}^{\alpha})^{(k)}(0) (L_{n-1}^{\alpha})^{(j_s)}(0)}{\|L_{n-1}^{\alpha}\|^2\, (\alpha+k+j_s+1)\, n^{\alpha+k+j_s}}, \tag{12}$$

and, for $k = j_1, \ldots, j_{s-1}$,

$$\lim_n \frac{K_{n-1,s-1}^{(k,j_s)}(0,0)}{n^{j_s-k}} = \lim_n \frac{Q_{n-1,s-1}^{(k)}(0)Q_{n-1,s-1}^{(j_s)}(0)}{(Q_{n-1,s-1},Q_{n-1,s-1})_{s-1}(j_s-k)\ n^{j_s-k-1}} \tag{13}$$

$$= \lim_n \frac{(L_{n-1}^{\alpha})^{(k)}(0)(L_{n-1}^{\alpha})^{(j_s)}(0)}{\|L_{n-1}^{\alpha}\|^2(j_s-k)\ n^{\alpha+k+j_s}} \lim_n n^{\alpha+2k+1} \frac{Q_{n-1,s-1}^{(k)}(0)}{(L_{n-1}^{\alpha})^{(k)}(0)} \lim_n \frac{Q_{n-1,s-1}^{(j_s)}(0)}{(L_{n-1}^{\alpha})^{(j_s)}(0)}.$$

Then, from (10), (11) and the hypothesis for $Q_{n,s-1}$, we have

$$\frac{Q_{n,s}^{(j_s)}(0)}{(L_n^{\alpha})^{(j_s)}(0)} = \frac{Q_{n,s-1}^{(j_s)}(0)}{(L_n^{\alpha})^{(j_s)}(0)} \frac{1}{1+M_{j_s}K_{n-1,s-1}^{(j_s,j_s)}(0,0)} \cong \frac{C_{j_s}}{n^{\alpha+2j_s+1}},$$

with $C_{j_s}$ a nonzero constant. Moreover, for $k \neq j_s$, taking into account (12), (13) and the hypothesis for $Q_{n,s-1}$, we can deduce

$$\frac{K_{n-1,s-1}^{(k,j_s)}(0,0)}{K_{n-1,s-1}^{(j_s,j_s)}(0,0)} \frac{Q_{n,s-1}^{(j_s)}(0)}{Q_{n,s-1}^{(k)}(0)} = \frac{K_{n-1,s-1}^{(k,j_s)}(0,0)}{K_{n-1,s-1}^{(j_s,j_s)}(0,0)} \frac{Q_{n,s-1}^{(j_s)}(0)}{(L_n^{\alpha})^{(j_s)}(0)} \frac{(L_n^{\alpha})^{(k)}(0)}{Q_{n,s-1}^{(k)}(0)} \frac{(L_n^{\alpha})^{(j_s)}(0)}{(L_n^{\alpha})^{(k)}(0)}$$

$$\cong \frac{(L_n^{\alpha})^{(j_s)}(0)}{(L_{n-1}^{\alpha})^{(j_s)}(0)} \frac{(L_{n-1}^{\alpha})^{(k)}(0)}{(L_n^{\alpha})^{(k)}(0)} \begin{cases} \frac{\alpha+2j_s+1}{\alpha+k+j_s+1}, & \text{if } k \neq j_1, \ldots, j_{s-1}; \\ \frac{\alpha+2j_s+1}{j_s-k}, & \text{if } k = j_1, \ldots, j_{s-1}, \end{cases}$$

$$\cong \begin{cases} \frac{\alpha+2j_s+1}{\alpha+k+j_s+1}, & \text{if } k \neq j_1, \ldots, j_{s-1}; \\ \frac{\alpha+2j_s+1}{j_s-k}, & \text{if } k = j_1, \ldots, j_{s-1}. \end{cases}$$

Thus, taking limits in (10) and (9), we get $(a)$ and $(b)$ for the polynomials $Q_{n,s}$, i.e.

$$\frac{Q_{n,s}^{(k)}(0)}{(L_n^{\alpha})^{(k)}(0)} \cong \begin{cases} \frac{C_k}{n^{\alpha+2k+1}}, & \text{if } k = j_1, \ldots, j_s; \\ C_k, & \text{otherwise,} \end{cases}$$

and

$$(Q_{n,s}, Q_{n,s})_s \cong \|L_n^{\alpha}\|^2.$$

Hence the result follows. □

Observe that the part $(a)$ of Lemma 1 is also true for the ratio of the corresponding orthonormal polynomials, and therefore there exists

$$\lim_n \frac{q_n^{(k)}(0)}{(l_n^{\alpha})^{(k)}(0)} = \begin{cases} 0, & \text{for } k \text{ such that } M_k > 0; \\ C_k \neq 0, & \text{otherwise.} \end{cases} \tag{14}$$

Consider the following representation of the orthonormal polynomials $q_n$ in terms of the orthonormal Laguerre polynomials $l_n^\alpha$ (see [8, Section 9] )

$$q_n(x) = \sum_{j=0}^{N+1} b_j(n) x^j l_{n-j}^{\alpha+2j}(x). \tag{15}$$

For the inner product (1) with $N = 1$, the coefficients $b_j(n)$ was explicitly obtained in [9], and their estimation was essential to obtain the result in [4].

Now in the general case, using Lemma 1, we can prove that there is always limit of the connection coefficients $b_j(n)$ for an arbitrary $N$.

**Lemma 2** *Let $\{b_j(n)\}_0^{N+1}$ be the coefficients in formula (15). Then, there exists*

$$\lim_n b_j(n) = b_j \in \mathbb{R}, \quad j \in \{0, \dots, N+1\}.$$

*Moreover, the first index $j$ such that $b_j \neq 0$ corresponds with the first $j$ such that $M_j = 0$ in the inner product (1). (We understand that if all the masses are positive, then the unique coefficient $b_j$ different from zero is the last one).*

**Proof.** Taking derivatives $k$ times in (15) and evaluating at $x = 0$, we deduce

$$\frac{q_n^{(k)}(0)}{(l_n^\alpha)^{(k)}(0)} = \sum_{j=0}^{k} b_j(n) \begin{pmatrix} k \\ j \end{pmatrix} j!\, A_j(k,n), \quad k \in \{0, \dots, N+1\}, \tag{16}$$

where $A_0(k, n) = 1$ and

$$A_j(k,n) = \frac{(l_{n-j}^{\alpha+2j})^{(k-j)}(0)}{(l_n^\alpha)^{(k)}(0)} \cong \frac{(-1)^j \Gamma(\alpha+k+1)}{\Gamma(\alpha+k+j+1)} \left( \frac{\Gamma(\alpha+2j+1)}{\Gamma(\alpha+1)} \right)^{1/2} \tag{17}$$

Since there exists $\lim_n A_j(k, n) \neq 0$ , applying recursively (14) and (16) we can assure there exists $\lim_n b_j(n) = b_j$, $j \in \{0, \dots, N+1\}$. More precisely, for $k = 0$ we have

$$\lim_n b_0(n) = \lim_n \frac{q_n(0)}{l_n^\alpha(0)} = b_0 = \begin{cases} 0, & \text{if } M_0 > 0; \\ C \neq 0, & \text{if } M_0 = 0. \end{cases}$$

Now, from (16) for $k = 1$, (14) and (17) we get

$$\lim_n b_1(n) = \lim_n \frac{1}{A_1(1,n)} \left( \frac{q_n'(0)}{(l_n^\alpha)'(0)} - b_0(n) \right) = b_1$$

Observe that

$$b_1 = \begin{cases} 0, & \text{if } M_0 > 0 \text{ and } M_1 > 0 \ ; \\ C \neq 0, & \text{if } M_0 > 0 \text{ and } M_1 = 0 \ . \end{cases}$$

In this way, recursively, if $M_0 M_1 \dots M_i > 0$ and $M_{i+1} = 0$ we can assure that

$$b_j = \begin{cases} 0, & \text{if } 0 \leq j \leq i; \\ C \neq 0, & \text{if } j = i + 1, \end{cases}$$

and we obtain the result. $\Box$

As a consequence of the above lemma, we can establish a Mehler-Heine type formula for general discrete Laguerre-Sobolev orthonormal polynomials. This formula shows how the presence of the masses in the discrete part of the inner product changes the asymptotic behavior around the origin. Moreover, it supplies information on the location and asymptotic distribution of the zeros of the polynomials in terms of the zeros of known special functions.

We recall the corresponding formula for orthonormal Laguerre polynomials (see [14])

$$\lim_n \frac{l_n^\alpha(x/(n+k))}{n^{\alpha/2}} = \sqrt{\Gamma(\alpha+1)} x^{-\alpha/2} J_\alpha(2\sqrt{x}) \tag{18}$$

uniformly on compact subsets of $\mathbb{C}$ and uniformly for $k \in \mathbb{N} \cup \{0\}$, where $J_\alpha$ is the Bessel function of the first kind.

**Proposition 1** *The polynomials $q_n$ satisfy the following Mehler-Heine type formula:*

$$\lim_n \frac{q_n(x/n)}{n^{\alpha/2}} = \sqrt{\Gamma(\alpha+1)} \sum_{j=0}^{N+1} b_j \, x^{-\alpha/2} J_{\alpha+2j}(2\sqrt{x}) \tag{19}$$

*uniformly on compact subsets of $\mathbb{C}$.*

**Proof.** The proof is a straightforward consequence of formula (15), Lemma 2 and (18). $\Box$

**Remark.** According to Lemma 2, the first Bessel function which appears in (19) corresponds with the first index $j$ such that $M_j = 0$, in the inner product (1). We want to highlight that this result generalizes the one obtained in [1, Theorem 3], where the authors only deal with inner products with a unique "gap" in the discrete part.

The above proposition allows us to deduce a lower estimate of $\|q_n\|_{L_p(x^\beta dx)}$, for $\beta = \alpha$ and $\beta = \alpha p/2$, that will play an important role in the proof of Theorem 1.

**Proposition 2** *Let $1 \le p \le \infty$. Then, the following statements hold: For $\alpha > -1/2$*

$$\|q_n\|_{L_p(x^\alpha dx)} \ge C \begin{cases} n^{-1/4}(\log(n+1))^{1/p}, & \text{if } p = \frac{4\alpha+4}{2\alpha+1}; \\ n^{\alpha/2-(\alpha+1)/p}, & \text{if } \frac{4\alpha+4}{2\alpha+1} < p \le \infty, \end{cases}$$

*and for $\alpha > -2/p$ if $1 \le p < \infty$ and $\alpha \ge 0$ if $p = \infty$*

$$\|q_n\|_{L_p(x^{\alpha p/2} dx)} \ge C \begin{cases} n^{-1/4}(\log(n+1))^{1/p}, & \text{if } p = 4; \\ n^{-1/p}, & \text{if } 4 < p \le \infty, \end{cases}$$

*where $C$ is an absolute positive constant.*

**Proof.** Assume $1 \le p < \infty$. Then,

$$\begin{aligned} \|q_n\|^p_{L_p(x^\beta dx)} &= \int_0^\infty |q_n(x)e^{-x/2}|^p x^\beta dx \\ &> \int_0^{1/\sqrt{n}} |q_n(x)e^{-x/2}|^p x^\beta dx \ge C n^{-\beta-1} \int_0^{\sqrt{n}} |q_n(t/n)|^p t^\beta dt \end{aligned}$$

According to formula (19), $\exists\, n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$

$$\int_0^{\sqrt{n}} |q_n(t/n)|^p t^\beta dt \ge C n^{p\alpha/2} \int_0^{\sqrt{n}} |\sum_{j=0}^{N+1} b_j\, t^{-\alpha/2}\, J_{\alpha+2j}(2\sqrt{t})|^p t^\beta dt$$

and therefore $\forall n \ge n_0$

$$\|q_n\|^p_{L_p(x^\beta dx)} \ge C n^{p\alpha/2-\beta-1} \int_0^{2n^{1/4}} u^{2\beta-p\alpha+1} |\sum_{j=0}^{N+1} b_j\, J_{\alpha+2j}(u)|^p du.$$

Working as Stempak in [13, Lemma 2.1], we can prove that for $\alpha > -1$, and $\lambda > -1 - \alpha p$

$$\int_0^{2n^{1/4}} u^\lambda |\sum_{j=0}^{N+1} b_j\, J_{\alpha+2j}(u)|^p du \sim \begin{cases} 1, & \text{if } \lambda < p/2 - 1; \\ \log(n+1), & \text{if } \lambda = p/2 - 1. \end{cases}$$

Thus, if $1 \leq p < \infty$, we obtain the first and the second result for $\beta = \alpha$ and $\beta = p\alpha/2$ respectively. The results for $p = \infty$ can be deduced from the previous one by passing to the limit when $p$ goes to $\infty$. $\square$

It is worth to noticing that these lower bounds are sharp in the following sense.

**Proposition 3** *Let $1 \leq p \leq \infty$. Then:*
*For $\alpha \geq 0$,*

$$\|q_n\|_{L_p(x^\alpha dx)} \sim \begin{cases} n^{-1/4}(\log(n+1))^{1/p}, & \text{if } p = \frac{4\alpha+4}{2\alpha+1}; \\ n^{\alpha/2-(\alpha+1)/p}, & \text{if } \frac{4\alpha+4}{2\alpha+1} < p \leq \infty, \end{cases}$$

*and for $\alpha > -2/p$ if $1 \leq p < \infty$ and $\alpha \geq 0$ if $p = \infty$,*

$$\|q_n\|_{L_p(x^{\alpha p/2} dx)} \sim \begin{cases} n^{-1/4}(\log(n+1))^{1/p}, & \text{if } p = 4; \\ n^{-1/p}, & \text{if } 4 < p \leq \infty. \end{cases}$$

**Proof.** From Lemma 1 of [11] it can be deduced that for $\alpha \geq 0$

$$\int_0^\infty |x^j l_n^{\alpha+2j}(x) e^{-x/2}|^p x^\alpha dx \sim \begin{cases} n^{-p/4}\log(n+1), & \text{if } p = \frac{4\alpha+4}{2\alpha+1}; \\ n^{\alpha p/2-(\alpha+1)}, & \text{if } \frac{4\alpha+4}{2\alpha+1} < p \leq \infty, \end{cases}$$

and for $\alpha > -2/p$ if $1 \leq p < \infty$ and $\alpha \geq 0$ if $p = \infty$

$$\int_0^\infty |x^j l_n^{\alpha+2j}(x) e^{-x/2} x^{\alpha/2}|^p dx \sim \begin{cases} n^{-p/4}\log(n+1), & \text{if } p = 4; \\ n^{-1}, & \text{if } 4 < p \leq \infty. \end{cases}$$

Thus, using the representation formula for the polynomials $q_n$ (see (15)), and the fact that the connection coefficients are bounded (see Lemma 2), we get one of the two inequalities. The other one has been proved in Proposition 2 and therefore the result follows. $\square$

## 3 A Cohen type inequality

In this section we prove a Cohen type inequality for the Fourier expansions in terms of discrete Laguerre-Sobolev orthonormal polynomials with an arbitrary (finite) number of mass points. So we extend the result due to Fejzullahu and Marcellán which deals with a discrete Laguerre-Sobolev inner product with at most two masses in the discrete part (see [4]).

**Proof of Theorem 1.** Let us consider the following test functions which were already used in [12] and later in [4]

$$g_n^{\alpha,j}(x) = x^j \left[ L_n^{\alpha+j}(x) - \sqrt{\frac{(n+1)(n+2)}{(n+\alpha+j+1)(n+\alpha+j+2)}}\, L_{n+2}^{\alpha+j}(x) \right],$$

with $j \in \mathbb{N} \setminus \{1, \dots, N\}$. Notice that

$$(g_n^{\alpha,j})^{(i)}(0) = 0,\ i = 0, \dots, N. \tag{20}$$

These functions can be written as (see formula (2.15) in [12])

$$g_n^{\alpha,j}(x) = \sum_{m=0}^{j+2} a_{m,j}(\alpha, n) L_{n+m}^{\alpha}(x) \tag{21}$$

with

$$a_{0,j}(\alpha, n) = \frac{\Gamma(n+\alpha+j+1)}{\Gamma(n+\alpha+1)} \cong n^j.$$

From (20), (21), and $0 \le k \le n$, we have

$$\widehat{g_n^{\alpha,j}}(k) = \langle g_n^{\alpha,j}, q_k \rangle_S = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty g_n^{\alpha,j}(x) q_k(x) e^{-x} x^\alpha dx$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{m=0}^{j+2} a_{m,j}(\alpha, n) \int_0^\infty L_{n+m}^\alpha(x) q_k(x) e^{-x} x^\alpha dx\,.$$

By the orthogonality of Laguerre polynomials, we obtain

$$\widehat{g_n^{\alpha,j}}(k) = \begin{cases} 0, & \text{if } 0 \le k \le n-1; \\ \frac{1}{\Gamma(\alpha+1)} a_{0,j}(\alpha, n) \int_0^\infty L_n^\alpha(x) q_n(x) e^{-x} x^\alpha dx, & \text{if } k = n. \end{cases}$$

Thus, from Lemma 1 (b), the estimate of $a_{0,j}(\alpha, n)$ and the value of the norm of Laguerre polynomials (see (2)), we can deduce

$$\widehat{g_n^{\alpha,j}}(n) = \frac{1}{\Gamma(\alpha+1)} a_{0,j}(\alpha, n) \int_0^\infty L_n^\alpha(x) \frac{Q_n(x)}{\langle Q_n, Q_n \rangle_S^{1/2}} e^{-x} x^\alpha dx =$$

$$a_{0,j}(\alpha, n)\, \frac{\|L_n^\alpha\|^2}{\langle Q_n, Q_n \rangle_S^{1/2}} \cong a_{0,j}(\alpha, n)\, \|L_n^\alpha\| \cong \frac{n^{j+\alpha/2}}{\sqrt{\Gamma(\alpha+1)}}$$

Observe that $Q_n$ and $L_n^\alpha$ have always equivalent norms, and, therefore this estimation does not depend neither on the number of positive masses, nor on the existence or non-existence of any gap in the inner product.

Applying the operator $T_n^{\alpha,S}$ to the functions $g_n^{\alpha,j}$, we get

$$T_n^{\alpha,S}(g_n^{\alpha,j}) = c_{n,n}\,\widehat{g_n^{\alpha,j}}(n)q_n,$$

and therefore

$$\begin{aligned}\|T_n^{\alpha,S}\|_{[S_p^\beta]} &\geq (\|g_n^{\alpha,j}\|_{S_p^\beta})^{-1}\|T_n^{\alpha,S}(g_n^{\alpha,j})\|_{S_p^\beta} = (\|g_n^{\alpha,j}\|_{S_p^\beta})^{-1}|c_{n,n}||\widehat{g_n^{\alpha,j}}(n)|\,\|q_n\|_{S_p^\beta} \\ &\geq (\|g_n^{\alpha,j}\|_{S_p^\beta})^{-1}|c_{n,n}||\widehat{g_n^{\alpha,j}}(n)|\,\|q_n\|_{L_p(x^\beta dx)}.\end{aligned}$$

On the other hand, for $j > \alpha - 1/2 - 2(\alpha+1)/p$ we have

$$\|g_n^{\alpha,j}\|_{S_p^\beta} \leq c \begin{cases} n^{j-1/2+(\alpha+1)/p}, & \text{if } \beta = \alpha; \\ n^{\alpha/2+j-1/2+1/p}, & \text{if } \beta = p\alpha/2, \end{cases}$$

(see formula (3.3) and formula (1.19), (2.12) in [12] respectively). Thus, by Proposition 2 we get:
For $\beta = \alpha$ with $\alpha > -1/2$

$$\|T_n^{\alpha,S}\|_{[S_p^\alpha]} \geq C|c_{n,n}| \begin{cases} (\log(n+1))^{\frac{2\alpha+1}{4\alpha+4}}, & \text{if } p = q_0; \\ n^{\alpha+1/2-2(\alpha+1)/p}, & \text{if } q_0 < p \leq \infty. \end{cases}$$

For $\beta = p\alpha/2$ with $\alpha > -2/p$ if $1 \leq p < \infty$ and $\alpha \geq 0$ if $p = \infty$,

$$\|T_n^{\alpha,S}\|_{[S_p^{p\alpha/2}]} \geq C|c_{n,n}| \begin{cases} (\log(n+1))^{1/4}, & \text{if } p = 4; \\ n^{1/2-2/p}, & \text{if } 4 < p \leq \infty. \end{cases}$$

Hence, by duality the theorem follows. □

**Remark.** In particular, for $M_i = 0$, $i = 0, \dots, N$, the above theorem extends Theorem 1 in [12] to negative values of $\alpha$.

In the particular case of $c_{k,n} = 1$, $k = 0, \dots, n$, the operator $T_n^{\alpha,S}$ is the $n$th partial sum of the Fourier expansion, so, we can assure the following result.

**Corollary 1** *If $p$ is outside the Pollard interval $(p_0, q_0)$, we have*

$$\|S_n\|_{[S_p^\beta]} \to \infty, \quad n \to \infty$$

*where $S_n$ denotes the $n$th partial sum of the Fourier expansion.*

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