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A tauberian approach to RH

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"Like it or not, the world of the mathematician is becoming experimentalized."[BB]

Abstract

The aim of this paper is twofold. Firstly we present our main discovery arising from experiments which is the tauberian concept of functions of good variation (FGV). Secondly we propose to use these FGV for proving RH is true via some conjectures. More precisely we give an implicit definition of FGV and we provide several smooth and nontrivial exemples from experiments. Then using a conjectured family of FGV approaching the function $x \mapsto x^{-1} \lfloor x \rfloor$ we derive RH is true. We make also a tauberian conjecture allowing us to prove RH is true for infinitely many L-functions and we discuss the linear independance conjecture. The method is inspired by the Ingham summation process and the experimental support is provided using pari-gp.

Introduction

Thinking to RH as an inverse problem [Tar][1] we used experimental techniques to explore this idea and after some fruitless attempts involving dynamical systems we came across tauberian theory [Kor]. It is also worth to mention there is an equivalence of RH in term of an inverse spectral problem [Lap] confirming perhaps an intrinsic inverse nature of the problem[2]. Our starting point was this formula which is easy to prove

$$\sum_{k=1}^{n} \lambda_k \left\lfloor \frac{n}{k} \right\rfloor = \lfloor \sqrt{n} \rfloor$$

where $\lambda_n = (-1)^{\Omega(n)}$ denotes the Liouville function (A008836 in [Slo]) and $\Omega(n)$ counts the prime numbers with multiplicity in the factorisation of

[1] Nontrivial zeros are on the critical line is the answer to the problem, not the question, and we have to look for a suitable nontrivial question.

[2] This is probably the reason why forward approaches to RH seem to be a never ending story despite progresses are frequently made.

n (A001222 in [Slo]). Indeed, we suspect the square root appearing in the r.h.s. means something regarding the asymptotic behaviour of $\sum_{k=1}^{n}\lambda_k$ (A002819 in [Slo]). Thus we made a first tauberian conjecture supported by few experiments. If $(a_n)_{n\geq 1}$ is bounded then we claim

Conjecture

$$\sum_{k=1}^{n} a_k \left\lfloor \frac{n}{k} \right\rfloor \sim n^{1/2} \ (n\to\infty) \Rightarrow \sum_{k=1}^{n} a_k = O\left(n^{1/2+\epsilon}\right) \tag{1}$$

in particular

$$\sum_{k=1}^{n} \lambda_k \left\lfloor \frac{n}{k} \right\rfloor = \lfloor \sqrt{n} \rfloor \Rightarrow \sum_{k=1}^{n} \lambda_k = O\left(n^{1/2+\epsilon}\right)$$

which implies RH is true [Bor]. Shorlty after, G. Tenenbaum [Ten] showed that a weaker form of this conjecture, for positive functions, is equivalent to RH by mean of a tauberian theorem. i.e., let $g = f \star 1$ then we get RH is true if and ony if $f \geq 0$ and

$$\left\{(\forall\varepsilon > 0)\ F(x) \ll x^{1/2+\varepsilon} \Rightarrow G(x) \ll x^{1/2+\varepsilon}\right\}$$

where F and G are the summatory functions of f and g. However our experimental results suggested much more was true and this tauberian approach to RH was investigated further making intensive experiments. Instead of the above summation (1) involving directly the floor function we prefer to consider Ingham summation process which involves the bounded function: $x \mapsto x^{-1}\lfloor x \rfloor$. Namely we say a sequence a is (I)-summable to l if:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k \frac{k}{n} \left\lfloor \frac{n}{k} \right\rfloor = l \Rightarrow \sum_{k=1}^{\infty} a_k = l$$

This method was considered by many authors and some generalisations exist [Win][Juk] as well as relationship with the Riemann hypothesis [Seg]. Unlike Ingham we are mainly interested in the behaviour of the partial sum and not only in the convergence. So this study appears to be a new part of the tauberian remainder theory (cf. chapter VII in [Kor]). Here we consider bounded functions θ and we define a sequence $(a_n)_{n\in\mathbb{N}^*}$ for a given function f by the recurrence formula:

$$f(n) = \sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) \ (n \geq 1)$$

Next under some assumptions we try to give precise estimates for the behaviour of $A(n) := \sum_{k=1}^{n} a_k$. Although the Liouville function was helpful for the intuition (cf. conjecture above), we will consider also the Moebius function. Recalling that for $a_k = \frac{\mu_k}{k}$ and $\theta(x) = x^{-1}\lfloor x \rfloor$ we have $\sum_{k=1}^{n} a_k \theta(\frac{n}{k}) = \frac{1}{n}$ and it is known that $\left(\frac{\mu(n)}{n}\right)_{n\geq 1}$ is (I)-summable to zero [Ing]. This yields $\lim_{n\to\infty} A(n) = 0$ and the prime number theorem follows. Our goal in this paper is more ambitious since we try to find reasons why we should have the remainder estimate

$$A(n) \ll n^{-1/2} L(n)$$

where L is a slowly varying function[3] and which also implies RH is true[4].

Plan of the paper In section 1 we introduce functions of good variation (FGV) of index α with an implicit definition. In section 2 we prove some smooth functions are FGV and then we make existence conjectures and a comparison conjecture for C^1 functions. In section 3 we provide conjectural nontrivial exemples of FGV having discontinuities, specially some functions looking like $\theta(x) = x^{-1}\lfloor x \rfloor$. Next in section 4 we consider a family of functions θ_{2m} which are conjectured to be FGV of index $\frac{1}{2}$ for any $m \geq 1$. Since $\theta_\infty(x) = x^{-1}\lfloor x \rfloor$ we infer RH is true for the zeta function. We give also arguments in favour of the LI conjecture [Ing2]. In section 5 we make a compensation conjecture allowing us to prove directly RH is true. Next in section 6 we adapt the idea for Dirichlet L-functions and for automorphic L-functions (the grand Riemann hypothesis [Sar]). We also propose a generalisation of the LI conjecture (the GLI conjecture). In section 7 we propose a comparison conjecture for noncontinuous functions allowing us to derive RH is true in a slightly different way.

Some notations In the sequel, apart some exceptions which will be mentioned, for a given function θ, the sequence a_n is always defined by:

- $\frac{1}{n} = \sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) \ (n \geq 1)$
- for a given function f we write $f^\diamond(t) = f\left(\frac{1}{t}\right)$.

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[3] We say L is a slowly variiying function if for any $x > 0$ we have $\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1$. For instance $L(x) = \log(x)^r$ is slowly varying. This is Karamata definition [Kor]. We also say the function K is of regular variation of index $\alpha \neq 0$ if $K(x) = x^\alpha L(x)$ and L is slowly varying.

[4] It is conjectured that $n^{1/2+\epsilon} \sum_{k=1}^{n} \frac{\mu(k)}{k}$ is unbounded for any $\epsilon > 0$ and in fact we conjecture that there is a slowly varying function L such that $\sum_{k=1}^{n} \frac{\mu(k)}{k} \ll n^{-1/2} L(n)$. For more precise conjectures giving slowly varying functions related to the Moebius function see [Kot].

Part I
Good variation

1 Functions of good variation (FGV)

1.1 Implicit definition of FGV

The discrete version

We say that a positive bounded and measurable function θ is a FGV of index $\alpha > 0$ if $\forall \beta \geq \alpha$ we have:

$$\sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) \sim n^{-\beta} \ (n \to \infty) \Rightarrow A(n) n^{\alpha} \ll L(n)$$

where L is a slowly varying function and $\forall \, \beta < \alpha$ we have:

$$\sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) \sim n^{-\beta} \ (n \to \infty) \Rightarrow A(n) \ll n^{-\beta}$$

The integral version

Under the same conditions we claim the following definition involving Lebesgue integral implies the previous one. $\forall \beta \geq \alpha$ we have:

$$\int_{1}^{y} \theta\left(\frac{y}{t}\right) dF(t) \sim y^{-\beta} \ (y \to \infty) \Rightarrow F(y) y^{\alpha} \ll L(y)$$

where L is a slowly varying function and $\forall \, \beta < \alpha$ we have:

$$\int_{1}^{y} \theta\left(\frac{y}{t}\right) dF(t) \sim y^{-\beta} \ (y \to \infty) \Rightarrow F(y) \ll y^{-\beta}$$

Remark 1 As said in introduction we suspect $\theta(x) = x^{-1} \lfloor x \rfloor$ is a FGV of index $1/2$ and it is the story of this paper. At first glance it is not quite clear whether nontrivial function exists i.e., something else that $\theta(x) = c$ a constant, which is a FGV of index $+\infty$. However we succeeded to provide smooth and less smooth nontrivial exemples from our experiments. We think also we can give weaker conditions and we believe we can replace $\sim$ with $\ll$ in FGV definition. We don't fix Ramanujan like condition for the coefficients a_n, i.e., $a_n \ll_{\epsilon} n^{\epsilon}$ since it doesn't appears to be necessary until now.

1.2 Generalisation

We may generalise somewhat the definition as follows. We say that a positive bounded and measurable function θ is a FGV of index α if $\forall\beta \geq \alpha$ we have

$$\sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) = c + O(n^{-\beta} L_1(n)) \Rightarrow A(n) = c + O(n^{-\alpha} L_2(n))$$

where $c \in \mathbb{R}$ and L_1,L_2 are slowly varying functions. Although we won't consider this more general definition in this study we mention something at the end in section 6 (cf. 6.6).

1.3 Classes of FGV

Experiments show that there are different kind of FGV. This classification is not important for our purpose but it could be interesting to keep this in mind for further research. There are certainly ways to classify FGV more subtely but we just want to show there are slight differences between FGV we consider in this paper.

FGV of type 1

We say that θ is a FGV of type 1 if it is a FGV of index $\alpha > 0$ and we have:

- $\sum_{k=1}^{n} a_k \theta(\frac{n}{k}) \sim n^{-\alpha}$ $(n \to \infty) \Rightarrow A(n)n^{\alpha} \ll 1$ and $\lim_{n\to\infty} A(n)n^{\alpha} \neq 0$.

So that the slowly varying function is constant. This is for instance the smooth family we provide in 2.1. or the less trivial exemple in 3.3.

FGV of type 2

θ is a FGV of type 2 if it is a FGV of index $\alpha > 0$ and we have:

- $\sum_{k=1}^{n} a_k \theta(\frac{n}{k}) \sim n^{-\alpha}$ $(n \to \infty) \Rightarrow A(n)n^{\alpha}$ is unbounded.

A smooth concrete exemple is given by $\theta_{r,s}$ when $\frac{r}{s} = \alpha$ in 2.2. It is conjectured that it is the case for $\theta(x) = x^{-1}\lfloor x \rfloor$ or the generalisation θ_r we present in 3.1. but we have much more experimental evidence for the nontrivial FGV considered in 3.2. (fig. 13).

FGV of type 3

θ is a FGV of type 3 if it is a FGV of index $\alpha > 0$ and we have:

- $\sum_{k=1}^{n} a_k \theta(\frac{n}{k}) \sim n^{-\alpha}$ $(n \to \infty) \Rightarrow A(n)n^{\alpha} \ll 1$ and $\lim_{n\to\infty} A(n)n^{\alpha} = 0$.

We think it is the case of the FGV considered in 3.2. (fig. 11).

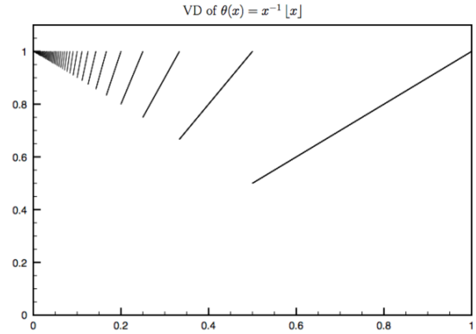
1.4 The variational diagram of a FGV

The variational diagram of a FGV θ is simply the plot of $\theta^{\diamond}(t) = \theta\left(\frac{1}{t}\right)$ for $0 < t \leq 1$. This diagram sheds apparently light on what is going on. We will see it is an interesting tool for comparing functions (cf. 2.6.). In the sequel VD denotes the variational diagram.

1.4.1 Exemple

Herafter the VD of $\theta(x) = \frac{\lfloor x \rfloor}{x}$.

VD of $\theta(x) = x^{-1} \lfloor x \rfloor$



$fig.1$

Properties of the VD:

- there are infinitely many discontinuities.
- the minimum value is at $\frac{1}{2}$ and equals $\frac{1}{2}$.
- $\theta^{\diamond}$ is C^1 and increasing by parts on the left.

These properties are significant to us and should have a considerable importance. Related exemples in section 3 will confirm this fact. Before trying to understand this kind of function we provide in the next section some concrete exemple of smoother FGV in order to show FGV is a consistent concept.

2 Some FGV of various index

In this section we provide some exemples of functions we prove they are FGV (or we are near to prove) .

2.1 A family of smooth FGV

Here we consider the family θ_{λ} of functions depending upon a parameter λ:

$$\theta_{\lambda}^{\diamond}(t) = 1 - \lambda t\,(1 - t)$$

Then we prove θ_λ is a FGV of index $\frac{3-\lambda}{2}$ and type 1 when $3-2\sqrt{2} < \lambda < 3$. Before proving this result let us see the shape of the VD of θ_2 and the behaviour of $A(n)$.

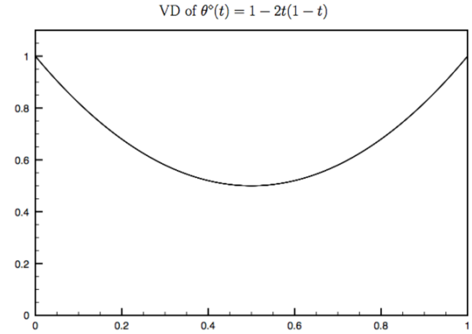


fig.2

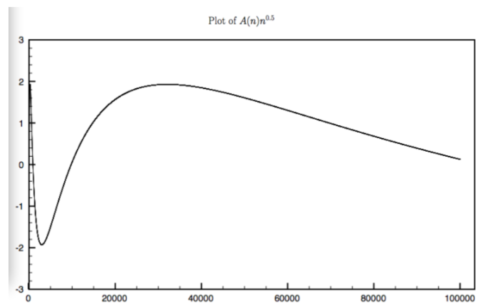


fig.3

Here we have $A(n) \sim n^{-1/2} g(h(n))\,(n \to \infty)$ where g is a periodic function and h is growing like the log function (see the following proof).

Proof To prove θ_λ are FGV we use the integral version of the definition and we consider[5]:

$$\int_1^y \theta\left(\frac{y}{t}\right) dF(t) = y^{-\beta}$$

Since things are smooth we differentiate this equation 2 times with respect to y and we get the ODE:

$$y^2 F"(y) + (4-\lambda) y F'(y) + 2F(y) = \beta(\beta-1) y^{-\beta}$$

For the particular case $\lambda = 2$ it is easy to see solutions are given by:

[5] The equation $\int_1^y \theta_\lambda\left(\frac{y}{t}\right) dF(t) \sim y^{-\beta}$ instead of $\int_1^y f(t)\theta_\lambda\left(\frac{y}{t}\right) dt = y^{-\beta}$ yields the same conclusions and the proof is left to the reader.

$$F(y) = c_1 y^{-1/2} \sin\left(\frac{\sqrt{7}}{2}\log y\right) + c_2 y^{-1/2} \cos\left(\frac{\sqrt{7}}{2}\log y\right) + \frac{y^{-\beta}}{\beta^2 - \beta + 2}$$

Thus we get:

- $\beta \geq \frac{1}{2} \Rightarrow F(y)y^{1/2}$ is bounded.
- $\beta < \frac{1}{2} \Rightarrow F(y) \ll y^{-\beta}$.

consequently from our definition of FGV we get:

- θ_2 is a FGV of type 1 and index $\frac{1}{2}$.

In general the solution of the ODE is given by:

$$F(y) = c_1 y^{\frac{\lambda - 3 - \sqrt{\lambda^2 - 6\lambda + 1}}{2}} + c_2 y^{\frac{\lambda - 3 + \sqrt{\lambda^2 - 6\lambda + 1}}{2}} + c_3 y^{-\beta}$$

Hence for $\lambda^2 - 6\lambda + 1 < 0$ and $\lambda < 3$ we have

- θ_λ is a FGV of type 1 and index $\frac{3-\lambda}{2} > 0$

since for $\beta \geq \frac{3-\lambda}{2} > 0$ we have $F(y)y^{(3-\lambda)/2}$ bounded.

Remark 2

We see that the index of the FGV is not necessarily the minimum of the function. Here the minimum is $\theta\left(\frac{1}{2}\right) = 1 - \frac{\lambda}{4}$. In an other hand it is worth to notice with this family of FGV that if the minimum decreases so does the index. In our mind the index is somewhat proportional to the area $\int_{[0,1]} \theta^\diamond$. The following exemple illustrates also this fact.

2.2 FGV related to the log function

We consider the very simple family of functions defined for $(r, s) \in \mathbb{R} \times \mathbb{R}$ by:

$$\theta^\diamond_{r,s}(t) = (s - r)t + r$$

These are FGV of type 2 and index $\frac{r}{s}$ involving the log function (as the slowly varying function arising in FGV definition).

Proof

We use again the integral definition and consider:

$$\int_1^y \theta\left(\frac{y}{t}\right) dF(t) = y^{-\beta}$$

Thus since θ and F are C^1 we have for some a an ODE of type:

$$syF'(y) + rF(y) = ay^{-\beta}$$

Then solutions are given for $\beta > \frac{r}{s}$ by:

$$F(y) = \frac{ay^{-\beta}}{r - \beta s} + cy^{-\frac{r}{s}}$$

and for $\beta = \frac{r}{s}$ by:

$$F(y) = y^{-\frac{r}{s}} \left(c + s^{-1} \log y\right)$$

Consequently $\theta_{r,s}$ is a FGV of type 2 and index $\frac{r}{s}$.

Special values of (r, s) and the exact asymptotic behaviour of $A(n)$

For the case $s = 1$ and $0 < r < 1$ we have this asymptotic formula for $A(n)$ (details omitted)

$$A(n) \sim \frac{n^{-r}}{(1-r)\Gamma(1-r)} \ (n \to \infty)$$

In particular for $(r, s) = \left(\frac{1}{2}, 1\right)$ we have

$$A(n) \sim \frac{2}{\sqrt{\pi n}} \ (n \to \infty)$$

Which relates somewhat π to RH (see section 7).

Remark 3 Here we have $\int_{[0,1]} \theta^{\diamond} = \frac{s+r}{2} = \frac{s}{2}\left(1 + \frac{r}{s}\right)$ and so the index is proportional to the area under $\theta^{\diamond}$ up to the factor s which is the maximum of the function when $s > r$. These functions are to us cornerstones of more complicated functions. In particular we will use the above behaviour for providing heuristic arguments in section 5.

2.3 Less obvious exemple

Here we consider:

- $\theta^{\diamond}(t) = 1 - t$ if $0 < t \leq \frac{1}{2}$
- $\theta^{\diamond}(t) = t$ if $\frac{1}{2} \leq t \leq 1$

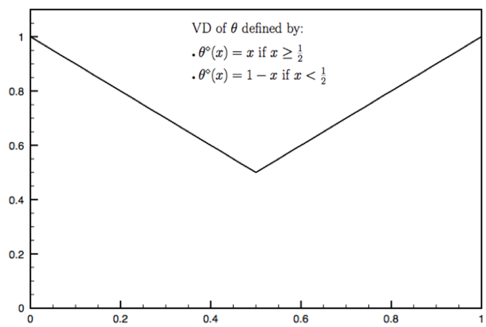


fig.4

And we suspect this is a FGV of index 0.8...

Trying to find the behaviour of $A(n)$ by experiments We use the integral version of FGV definition and the Laplace transform, i.e., we define the signed measure dF on $[1, \infty)$ such that for $y \geq 1$:

$$y^{-1} = \int_1^y \theta\left(\frac{y}{t}\right) dF(t)$$

Then letting $\theta_1(v) = \theta(e^v)$ we are looking for dF_1 on $[0, \infty)$ such that for all $v > 0$:

$$e^{-v} = \int_0^v \theta_1(v - w) dF_1(w).$$

If $s > 0$ we multiply the above equation by e^{-sv} and integrate with respect to v on $[0, \infty)$. Then we make the variable change $v' = v - w$ so that

$$\int_0^\infty e^{-v(1+s)} dv = \int_0^\infty \int_0^v \theta_1(v-w) e^{-sw} e^{-s(v-w)} dF_1(w) = G_1(s) \left(\int_0^\infty \theta_1(v') e^{-sv'} dv' \right)$$

Hence we find:

$$G_1(s) = \frac{1}{1 + s^{-1} e^{-sl}} \Rightarrow G_1(s) = 1 + \sum_{n=1}^\infty (-1)^n \frac{e^{-n\ell s}}{s^n}$$

which converges if $s \geq s_1 > 0$ where $s_1 = 0.641185...$ is the solution of $se^{\ell s} = 1$. We now have to consider G_1 as a Laplace transform (LT). The 1 is the LT of the Dirac mass $\delta_0(dv)$. $\frac{e^{-\ell s}}{s^n}$ is the product of $e^{-n\ell s}$, LT of the Dirac mass $\delta_{n\ell}$ and $\frac{1}{s^n} = \int_0^\infty e^{-sv} \frac{v^{n-1}}{(n-1)!} dv$. Hence $\frac{e^{-n\ell s}}{s^n}$ is the LT of the convolution product between the 2 measures $\delta_{n\ell}$ and $\frac{v^{n-1}}{(n-1)!} \mathbf{1}_{(0,\infty)}(v) dv$. Then it is the LT of the density measure $f_n(v)dv$ defined by

$$f_n(v) = \frac{(v-(n-1)\ell)^{n-1}}{(n-1)!}\mathbf{1}_{((n-1)\ell,\infty)}(v)dv$$

Thus $G_1(s)$ is the LT of the signed measure

$$dF_1(v) = \delta_0(dv) - \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(v-n\ell)^n \mathbf{1}_{(n\ell,\infty)}(v)\right) dv$$

letting $v = \log t$ we have

$$dF(t) = \delta(dt) - \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\log\frac{t}{2^n})^n \mathbf{1}_{(2^n,\infty)}(t)\right) \frac{dt}{t}$$

Hence we get:

$$F(y) = 1 - \sum_{0\le n\le \frac{\log y}{\log 2}} \frac{(-1)^n}{(n+1)!}\left(\log\frac{y}{2^n}\right)^{n+1}$$

This formula allows us to make experiments up to very big values of y and it appears we can take $0.8 < \alpha < 0.85$ in order to keep $F(y)y^{\alpha}$ approximately bounded. This is supported by the following graph involving $A(n)$ instead of $F(y)$.

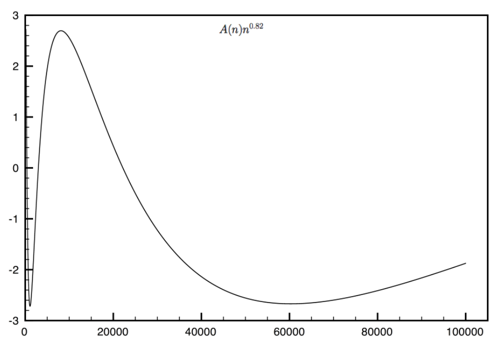


fig.5

The graph seems to oscillate between -3 and 3.

Remark 4

Observe the index is bigger than the index for $\theta_2^\diamond(t) = 1 - 2t(1-t)$ wich equals $\frac{1}{2}$ and we have the area rule:

- $\int_{[0,1]} \theta^\diamond > \int_{[0,1]} \theta_2^\diamond$.

This observation led us to establish a comparison conjecture between FGV (cf. 2.6.).

2.4 "Almost Dirac" FGV

Namely we consider for $0 < r < 1$ the family of functions:

- $\theta_r(x) = r$ if $x = 2$
- $\theta_r(x) = 1$ otherwise.

And we prove these functions are FGV of index $-\frac{\log(1-r)}{\log 2}$ of type 1 according to the discrete definition.

Sketch of proof

Note we have $f(n) = \sum_{k=1}^{n} a_k \theta_r\left(\frac{n}{k}\right) \Rightarrow f(n) = A(n) + (r-1)a\left(\frac{n}{2}\right)\delta_n$ where $\delta_n = 0$ if n is odd and 1 if n is even. Hence:

$$A(n) = f(n) + (1-r)\left(A\left(\left\lceil \frac{n}{2} \right\rceil\right) - A\left(\left\lfloor \frac{n}{2} \right\rfloor\right)\right)$$

For convenience let us define this family of $U(n)$ recursions $U(1) = 1$ and $U(2) = 1$ and:

- $U(n) = f(n) + (1-r)\left(U\left(\left\lceil \frac{n}{2} \right\rceil\right) - U\left(\left\lfloor \frac{n}{2} \right\rfloor\right)\right)$.

If $f(n) = 0$ it is easy to see we have (details omitted):

- $U(2^n + 1) = -(1-r)^n$
- $1 \leq k \leq 2^n + 1 \Rightarrow U(k) \leq |U(2^n + 1)|$

Thus we have $U(n) \ll n^{\frac{\log(1-r)}{\log 2}}$. In general we have (this is left to the reader):

- $f(n) \ll n^{\frac{\log(1-r)}{\log 2}} \Rightarrow U(n) \ll n^{\frac{\log(1-r)}{\log 2}}$

thus θ_r is a FGV of type 1 and index $-\frac{\log(1-r)}{\log 2}$ according to the discrete definition of FGV. The integral definition dosen't work here and it is an exemple showing the 2 definitions are not equivalent.

Hereafter a graph illustrating the fact θ_r is a FGV of type 1 and index $-\frac{\log(1-r)}{\log 2}$ for $r = 0.75$. We compute $n^{-\frac{\log(0.25)}{\log 2}} = \sum_{k=1}^{n} a_k \theta_{0.75}\left(\frac{n}{k}\right)$.

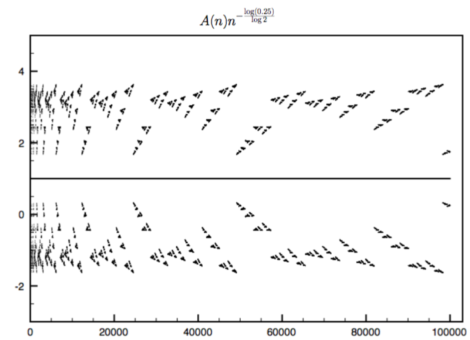


fig.6

2.5 Existence conjectures for smooth functions

From the previous exemples and more experiments we make the following claims related to functions which are C^1. These conjectures are far to be optimal and we believe one can state much more general conditions in order to say whether a C^1 function is a FGV.

Existence conjecture n°1

Suppose

- $\theta^\diamond$ is positive and bounded on $]0,1]$
- $\theta^\diamond$ is C^1 and monotonic on $]0,1]$

Then θ is a FGV of index $\alpha > 0$.

Existence conjecture n°2

Let $0 < x_0 < 1$ and suppose:

- $\theta^\diamond$ is positive and bounded on $]0,1]$
- $\theta^\diamond$ is C^1 and decreasing on $]0, x_0]$
- $\theta^\diamond$ is C^1 and increasing on $]x_0, 1]$

Then θ is a FGV of index $\alpha > 0$.

2.6 Comparison conjecture for smooth functions

From previous remarks we establish a simple comparison conjecture. Namely suppose:

- θ_1 and θ_2 are positive, bounded and C^1 on $[1, \infty)$
- $\theta_1^\diamond \leq \theta_2^\diamond$

- $\theta_1^\diamond, \theta_2^\diamond$ and $\theta_1^\diamond - \theta_2^\diamond$ have exactly the same variations on $]0,1]$.

Then we have:

- θ_1 is a FGV of index $\alpha_1 \Rightarrow \theta_2$ is a FGV of index $\alpha_2 \geq \alpha_1$.

3 Exemples of more complicated FGV

Here we consider noncontinuous functions and specially some functions sharing somewhat the properties of $\theta(x) = x^{-1}\lfloor x \rfloor$ (cf. fig1). These exemples are of course much more difficult to handle than exemples in the previous section. These experiments show there are many functions providing sequences a_n having similar asymptotic properties than the Moebius function. This supports the idea that RH would not be an arithmetical problem since we provide FGV wich are arithmetically meaningless.

3.1 Variation on the floor function

We define:

$$\theta_r(x) = x^{-1}(x - r\{x\})$$

where $0 < r \leq 1$ and $\{x\}$ denotes the fractionnal part of x. Then we claim:

- θ_r is a FGV of index $1 - \frac{r}{2}$

and this time the index corresponds always to the minimum of the FGV. This is significant since in general, as we see above, the index is no always the minimum of the function. In particular this supports the main claim, i.e., for $r = 1$ θ_r is a FGV of index $\frac{1}{2}$.

Experiments

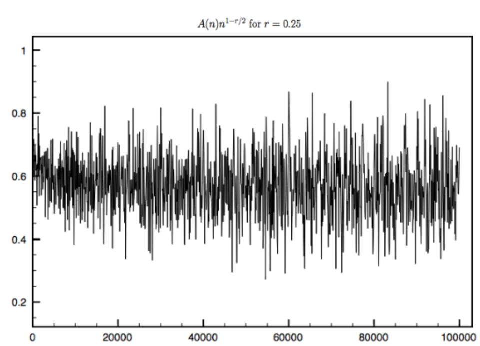


fig.7

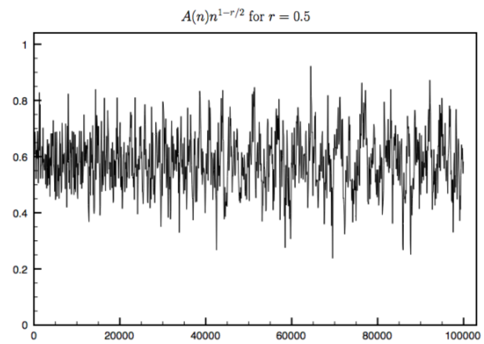


fig.8

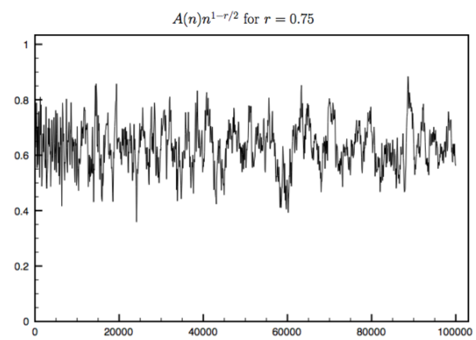


fig.9

These 3 graph behave similarly around a value near $y = 0.6$ and we claim they are bounded by a slowly varying function. This supports the fact θ_r is a FGV of index $1 - \frac{r}{2}$ when $0 < r \leq 1$.

3.2 Additionnal exemples

We consider here:

$$\theta(x) = 1 - \frac{\{x\}}{x\sqrt{\lfloor x \rfloor}}$$

This exemple is interesting since it is related to the floor function and provides nice oscillations for $A(n)$.

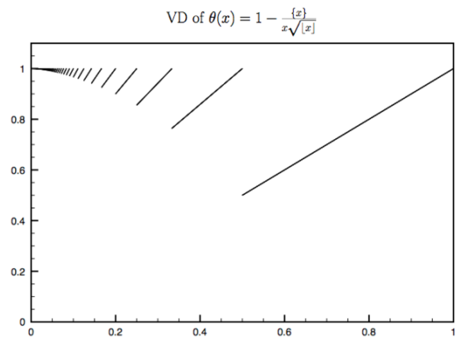


*fig.*10

Hereafter the plot of $A(n)n^{0.5}$ where we can see the oscillations.

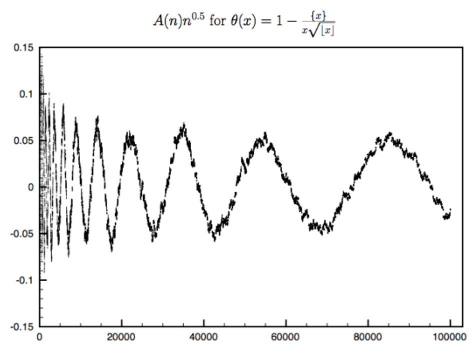


*fig.*11

We claim we have in fact

$A(n) \ll n^{-0.5}L(n)$

for a slowly varying function L satisfying $L(\infty) = 0$. So θ would be a FGV of index $\frac{1}{2}$ and type 3.

A last exemple

Here we consider:

$$\theta^{\diamond}(t) = \begin{cases} 1 - t\left\{\frac{1}{t}\right\} & \left(t \geq \frac{1}{3}\right) \\ t + \frac{1}{2} & \left(t < \frac{1}{3}\right) \end{cases}$$

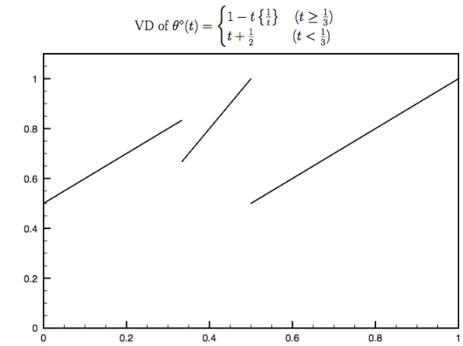


fig.12

Next we plot $\frac{A(n)n^{0.5}}{\log(n+1)}$

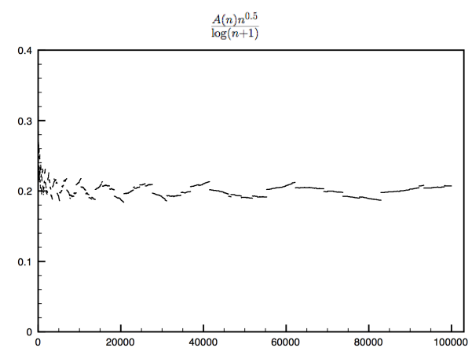


fig.13

It is likely bounded so θ would be a FGV of index $\frac{1}{2}$ and type 2.

3.3 Almost proved nontrivial FGV

We consider the function $\theta(x) = x^{-1}2^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor}$ which has infinitely many discontinuities and we claim it is a FGV of index 1. We succeeded to find the precise behaviour of $A(n)$ for particular cases. Here the variational diagram of θ.

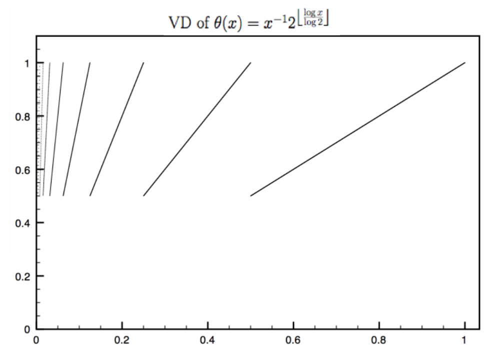


*fig.*14

A particular case

Suppose

$$\frac{\lfloor \sqrt{n} \rfloor}{n} = \sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right)$$

so that $\sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) \sim n^{-1/2}$ $(n \to \infty)$. Then we have

$$A(n) \sim \sqrt{\frac{2}{n}}\ (n \to \infty)$$

Proof For the proof we prefer to come back to the Liouville like recursion and we define the sequence a' as follows:

- $g(x) = 2^{\lfloor \frac{\log x}{\log 2} \rfloor}$
- $\lfloor \sqrt{n} \rfloor = \sum_{k=1}^{n} a'_k g(\frac{n}{k})$

At first glance the task of finding the behaviour of A'_n is not easier than with the original Liouville function. However we succeeded to find an explicit description. The sequence indeed obeys the following rules (details ommited):

- $a'_n = 1 \Leftrightarrow n = (2k+1)^2$ for some $k \geq 0$.
- $a'_n = -m \leq -1 \Leftrightarrow n = 2^{2m-1} b_k$ for some $k \geq 1$ where $(b_k)_{k \geq 1}$ is a sequence described below.
- $a'_n = 0$ otherwise.

To got b we merge the sets $\left\{(2k+1)^2\right\}_{k \geq 0}$ and $\left\{8(2k+1)^2\right\}_{k \geq 0}$ and we arrange the terms in increasing order (details omitted). Therefore it is easy to see we have:

- $A'_n \sim (\frac{1}{2} - 2\lambda\sqrt{2})\sqrt{n}$

where $\lambda = \frac{1}{2}\left(1 + \frac{1}{\sqrt{8}}\right)$ and then by Abel summation we get:

- $A_n \sim \sqrt{\frac{2}{n}}\ (n \to \infty)$.

Therefore we guess we can extend the result to any sequence satisfying:

- $\sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) \sim n^{-1/2}\ (n \to \infty)$

In general if $\sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) \sim n^{-\beta}$ we claim we have:

- $0 \leq \beta < 1 \Rightarrow A(n) \sim cn^{-\beta}\ (n \to \infty)$
- $\beta \geq 1 \Rightarrow A(n)n$ is bounded and doesn't converge to zero.

For instance it is easy to see that:

- $\sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) = \frac{1}{n} \Rightarrow A(n) = 2^{-\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \Rightarrow A(n)n \ll 1$.

Hence θ would be a FGV of index 1 and type 1.

Part II

Strategies for proving RH is true

4 Approaching the floor function

In order to derive something regarding RH a natural idea consists in approximating the floor function using simpler functions having finitely many discontinuities and which are FGV. So we split the problem in smaller ones and by induction we hope to deduce RH is true.

4.1 A family of functions

For any positive integer $m \geq 2$ let θ_m be defined by:

- $\theta_m^{\diamond}(t) = t\left\lfloor t^{-1} \right\rfloor$ for $t > \frac{1}{m}$
- $\theta^{\diamond}{}_m(t) = t + 1 - \frac{1}{m}$ for $0 < t \leq \frac{1}{m}$

So that we built a family of functions having finitely many discontinuities and such that:

- $\theta_\infty(x) = x^{-1}\left\lfloor x \right\rfloor$.

In the sequel we define:

- $\frac{1}{n} = \sum_{k=1}^{n} a_k \theta_m(\frac{n}{k})\ (n \geq 1)$
- $A_m(n) := \sum_{k=1}^{n} a_k$.

These functions have no arithmetical significance but we are able to make several claims about them. There are of course many other ways to build family of functions converging to $x^{-1}\lfloor x \rfloor$. Experiments confirm that approaching the floor function like this is efficient, i.e., it appears we don't need to consider big m in order to have $A_m(n)$ behaving like $A_\infty(n)$. It is our opinion that the influence of discontinuities of $\theta_\infty(x)$ as $x \to \infty$ becomes less and less important and thus θ_m becomes quickly a good approximation of θ_∞ from a tauberian view point. In other words the difficulty for solving RH seems not come from the fact the floor function has infinitely many discontinuities.

Some VD

Let us see the VD of θ_2 and θ_3.

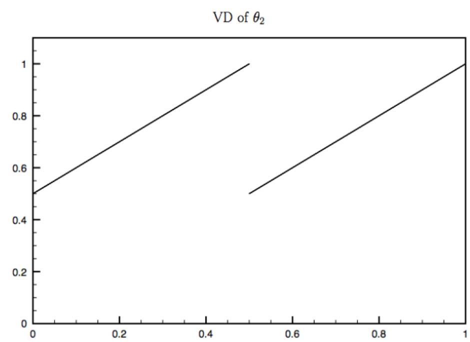


fig.15

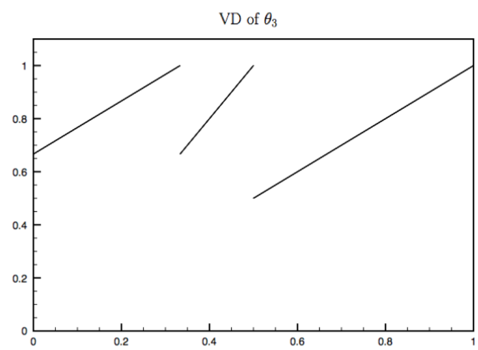


fig.16

4.2 Conjectures retaled to θ_m

Conjecture A

We claim that θ_2 is a FGV of index $\frac{1}{2}$ of type 2.

Conjecture B

We claim that $\forall m \geq 2$ we have

$$A_{2m}(n) \ll A_2(n)$$

and more precisely we guess that $\forall m \geq 1$ and n large enough we have

$$|A_{2m}(n)| \leq A_2(n)$$

Conjecture C

We guess that $\forall m \geq 1$ we have

$$\limsup_{n\to\infty} A_{2m}(n)n^{1/2} = +\infty$$

Thus from conjectures A,B and C and $\forall m \geq 1$ we can say:

- θ_{2m} is a FGV of index $\frac{1}{2}$ of type 2.

4.3 Experimental support

We plot $A_{2m}(n)n^{1/2}$ for $m = 1, 2, 3, 4, 5$ on the same picture.

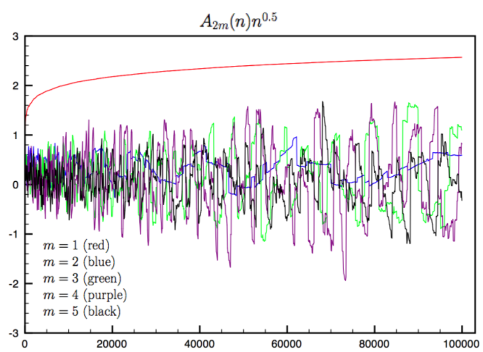


*fig.*17

Clearly $A_2(n)n^{1/2}$ is unbounded and behaves roughly like the logarithm and this picture supports the conjecture B.

4.4 Corollaries

4.4.1 Strips for zerofree regions

Before going further and trying to deduce RH is true, it is interesting to state a weaker statement. We may conjecture that for some $0 < \alpha < \frac{1}{2}$ we have

$$A_2(n)n^{\alpha} = o(1)$$

Thus assuming the conjecture B we have (since $A_\infty(n) = \sum_{k=1}^{n} \frac{\mu(k)}{k}$)

- $M(n) \ll n^{1-\alpha}$ where M is the Mertens function.

Hence ζ has no zero with $\Re s > 1 - \alpha$.

Remark 5 This kind of zero free region is unknown to this date and is seems possible to prove $\lim_{n\to\infty} A_2(n)n^\alpha = 0$ for some $0 < \alpha < \frac{1}{2}$. So the difficulty would be to prove the conjecture B.

4.4.2 RH is true

Using the conjectures A and B we have (letting $m \to \infty$)

- $M(n) \ll n^{1/2+\epsilon}$ for any $\epsilon > 0$.

Where $M(x) := \sum_{k\le x} \mu(k)$ denotes the Mertens function (A002321 in [Slo]). We can also make a stronger statement if we suppose θ_∞ is itself a FGV of index $\frac{1}{2}$ and type 2. Indeed we would then have:

- $M(n) \ll n^{1/2} L(n)$ for a slowly varying function L.

4.4.3 On the linear independance conjecture (LI)

The linear independance conjecture (LI) comes back to Ingham [Ing2] and states that the nontrivial zeros of the zeta function with $\Re(\rho) > 0$ are linearly independant over the rationals. Ingham showed that if LI is true then we get:

- $\lim_{x\to\infty} |M(x)|\, x^{-1/2} = +\infty$ and $\lim_{x\to\infty} |L(x)|\, x^{-1/2} = +\infty$.

where $L(x) := \sum_{k\le x} \lambda(k)$ is the Liouville summatory function (A002819 in [Slo]) . This is equivalent to say $\limsup_{x\to\infty} M'(x)x^{1/2} = +\infty$ where $M'(x) := \sum_{k\le x} \frac{\mu(k)}{k}$. Grosswald [Gro] or Saffari [Saf] added results. Recently in [Kot] authors conjecture the following estimate

- $M(n)n^{-1/2} = \Omega_\pm\left(\sqrt{\log\log\log(n)}\right)$.

Here we wish to deduce LI is true from the conjecture C. Our claim is indeed

$$\limsup_{n\to\infty} A_\infty(n)n^{1/2} = \limsup_{n\to\infty} n^{1/2} \sum_{k=1}^{n} \frac{\mu(k)}{k} = +\infty.$$

Consequently the LI conjecture is true and the zeros are simple. The simplicity of the zeros is a consequence of the LI conjecture[6].

[6]Some authors introduced the "Grand Riemann simplicity conjecture" [RS] as the statement that "the set of all ordinates γ of the non-trivial zeros of Dirichlet L-functions $L(s,\chi)$ are Q-linearly independent when χ runs over primitive Dirichlet characters and the zeros are counted with multiplicity. "Simplicity" relates to the particular corollary of this conjecture that all zeros of Dirichlet L-functions are simple".

5 A compensation conjecture[7]

It gives sufficients conditions to say whether a specific function is a FGV and also the corresponding index. This conjecture encapsulates what is needed to derive GRH is true since we put together caracteristics of functions for which RH is expected to hold. Moreover it fits also experimentaly many cases of FGV having index satisfying $0 < \alpha < \frac{1}{2}$. Let $(I_n)_{n\geq 1}$ and $(J_n)_{n\geq 1}$ be two sequences of reals satisfying:

- $I_1 = 1$ and $0 < I_2 \leq \frac{1}{2}$.
- $\forall n \geq 1\ I_{n+1} < I_n \leq \frac{I_{n+1}}{I_2}$.
- $\lim_{n\to\infty} I_n = 0$.
- $J_1 \geq 1$ and $J_2 I_2 \leq J_1 I_1$
- $\forall n \geq 1\ J_{n+1} \geq J_n$.
- $\lim_{n\to\infty} I_n J_n$ exists.
- $\lim_{n\to\infty} \sum_{k=2}^{n} (J_k - J_{k-1}) I_k = +\infty$.

Now let θ be defined as follows:

- $t \in]I_{n+1}, I_n] \Rightarrow \theta^{\diamond}(t) = J_n t$.

Then we claim:

- θ is a FGV of index I_2 of type 2.

In the APPENDIX we discuss some conditions given above.

Heuristic arguments

The cases 2.1, 2.3. or the nontrivial cases in 3.2 and 4.3. indicate there are some rules for the oscillations of $A(n)$. So looking again at the VD in fig. 1 we can imagine that the behaviour of $A(n)$ depends on a chain of behaviour produced by local functions of type $\theta^{\diamond}_{r,s}$ (cf. FGV described in 2.2.) with successive changes of signs, i.e., taking $\theta(x) = x^{-1}\lfloor x \rfloor$ we suppose we have something like:

$$A(n) \ll c(n,1)n^{-(1-\frac{1}{2})} - c(n,2)n^{-(1-\frac{1}{3})} + c(n,3)n^{-(1-\frac{1}{4})} - c(n,4)n^{-(1-\frac{1}{5})} + ...$$

where:

- $\forall i$, $0 < c(n,i) \ll L(n)$ and L is slowly varying.

Thus we would have:

$$A(n) \ll L'(n)n^{-1/2}$$

where L' is slowly varying.

[7] It is named compensation conjecture since in the sum $A(n) = \sum_{k=1}^{n} a_k$ we guess the terms a_k with $\frac{k}{n} \geq I_2$ are in some sense compensated by the terms with $\frac{k}{n} < I_2$.

Remark 6

Of course this conjecture is very specific and one may expect to have more general conditions in order to say whether a function having infinitely many discontinuities is a FGV. The conjecture in section 7 is a tentative for comparing functions and to deduce RH is true in another way.

6 RH for Dirichlet and automorphic L-functions

If we wish to apply the compensation conjecture to a wide class of DS $\sum_{n\geq 1} \frac{a_n}{n^s}$ with abscissa of convergence $0 < \sigma < 1$ it is interesting to look for a suitable function θ such that:

$$\sum_{k=1}^{n} \frac{a_k}{k}\theta\left(\frac{n}{k}\right) = \frac{1}{n}$$

The following first lemma is doing the task and allows us to use the compensation conjecture for L-functions. We also state a second lemma in order to derive RH is true for this wider class of functions.

6.1 Lemma 1

Suppose:

- $\left(\sum_{k\geq 1} \frac{x_k}{k^s}\right)\left(\sum_{k\geq 1} \frac{z_k}{k^s}\right) = \sum_{k\geq 1} \frac{y_k}{k^s}$

and let:

- $g(t) = \sum_{k\geq 1} z_k \lfloor \frac{t}{k} \rfloor$

(the number of terms in the sum is finite). Then we get:

$$\sum_{k\geq 1} x_k g\left(\frac{t}{k}\right) = \sum_{k\geq 1} y_k \left\lfloor \frac{t}{k} \right\rfloor$$

The proof is easy and left to the reader. Consequently letting $\theta_g(x) = x^{-1}g(x)$ and $\theta(x) = x^{-1}\lfloor x \rfloor$ we get:

$$\sum_{k\geq 1} \frac{x_k}{k}\theta_g\left(\frac{t}{k}\right) = \sum_{k\geq 1} \frac{y_k}{k}\theta\left(\frac{t}{k}\right)$$

6.2 Lemma 2

Let $\frac{1}{F(s)} = \sum_{n\geq 1} \frac{x_n}{n^s}$ and suppose:

- $\sum_{k=1}^{n} \frac{x_k}{k} \ll n^{-\alpha}L(n)$ with $0 < \alpha \leq \frac{1}{2}$ and L is slowly varying.

Then we have $F(s) \neq 0$ for $\Re s > 1 - \alpha$.

Proof $\sum \frac{x_k}{k} \ll n^{-\alpha}L(n) \Rightarrow \sum_{k=1}^{n} x_k \ll n^{1-\alpha+\epsilon}$. Thus $\sum_{n\geq 1} \frac{x_n}{n^s}$ converges for $\Re s > 1-\alpha$. Consequently $F(s) \neq 0$ for $\Re s > 1-\alpha$.

6.3 The Dirichlet beta function

We consider the Dirichlet beta function:

- $\beta(s) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{(2n-1)^s}$

Which satisfies this functional equation:

- $\beta(s) = \left(\frac{\pi}{2}\right)^{s-1} \Gamma(1-s) \cos\left(\frac{\pi s}{2}\right) \beta(1-s)$

Next let:

- $\theta(x) = x^{-1} \sum_{k\geq 1} (-1)^{k-1} \left\lfloor \frac{x}{2k-1} \right\rfloor$

From lemma 1 and letting $\sum_{n\geq 1} \frac{x_n}{n^s} = \frac{1}{\zeta(s)\beta(s)}$ we have:

- $\sum_{k=1}^{n} \frac{x_k}{k} \theta\left(\frac{n}{k}\right) = \frac{1}{n}$

It is now interesting to look at the VD of θ which is different but share important properties of the VD in fig.1.

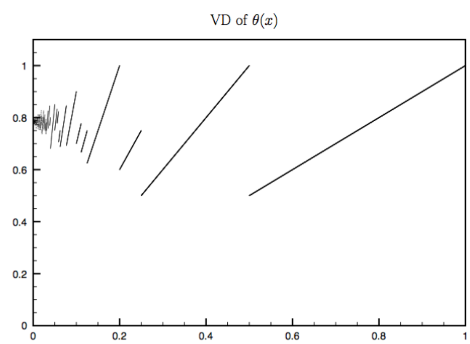


*fig.*18

It is easy to see the conditions of the compensation conjecture are satisfied, thus we can say θ is a FGV of index $\alpha = \frac{1}{2}$. Next $F(s) = \zeta(s)\beta(s)$ satisfies the conditions of lemma 2 and so the Dirichlet series for $\frac{1}{\zeta(s)\beta(s)}$ would converge for $\Re s > \frac{1}{2}$. Moreover we add the zeros are simple and the LI conjecture holds for $\beta(s)$ since θ is a FGV of type 2.

Extension The idea is working for all other Dirichlet L-functions and so our method should prove the generalised Riemann hypothesis is true.

6.4 RH for automorphic L-functions

Although the situation is less simple we believe similar arguments could work for automorphic L-functions having no positive real zero. For instance let us consider the Ramanujan tau function (cf. A000594 in [Slo]):

- $R(s) = \sum_{n\geq 1} \frac{\tau_n}{n^{s+11/2}}$.

which satisfies this functional equation ([Bor], p. 108) letting $\Gamma_1(s) = \pi^{-s/2}\Gamma(s/2)$:

- $\Lambda(s) = \Lambda(1-s)$ where $\Lambda(s) = \Gamma_1\left(s + \frac{11}{2}\right)\Gamma_1\left(s + \frac{13}{2}\right)R(s)$

Let now:

- $\sum_{n\geq 1} \frac{x_n}{n^s} = \frac{1}{\zeta(s)R(s)}$.

Next consider:

- $\theta(x) = x^{-1}\sum_{k\geq 1} \frac{\tau_k}{k^{11/2}} \lfloor \frac{x}{k} \rfloor$

Then we have again from lemma 1:

- $\sum_{k\geq 1} \frac{x_k}{k} g(\frac{n}{k}) = \frac{1}{n}$

And the VD of θ looks like:

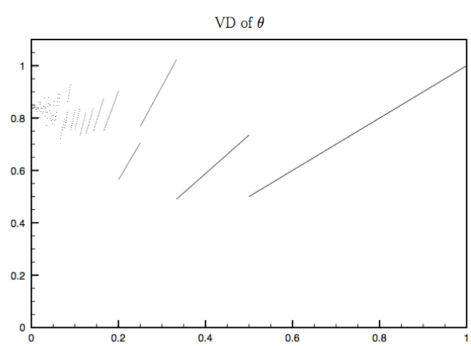


*fig.*19

We may apply the compensation conjecture and we deduce θ is a FGV of index $\alpha = \frac{1}{2}$. Next $F(s) = \zeta(s)R(s)$ satisfies the conditions of lemma 2. Thus we have again RH is true for R since the Dirichlet series for $\frac{1}{\zeta(s)R(s)}$ would converge for $\Re s > \frac{1}{2}$. We add the zeros are simple and the LI conjecture holds also for $R(s)$.

6.5 The GLI conjecture

Our method yields a strong hypothesis. Indeed if we consider for some integer $m \geq 1$:

- $\sum_{n\geq 1} \frac{x_n}{n^s} = \frac{1}{\zeta(s)\prod_{i=1}^{m} L_i(s)}$

- $\prod_{i=1}^{m} L_i(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$.
- $\theta(x) = x^{-1} \sum_{k\geq 1} a_k \lfloor \frac{x}{k} \rfloor$

where $\{L_i\}_{i=1,2,...,m}$ is a set of m distinct primitive L-functions having no positive real zero, no pole at 1 and satisfying $L_i \neq \zeta$. Then we speculate θ is a FGV of type 2 and index $\frac{1}{2}$ so that the LI conjecture would hold for all nontrivial zeros of $\zeta, L_1, ..., L_m$. Next letting $m \to \infty$ and considering all sort of primitive L-functions having no positive real zeros and no pole at 1 we claim that all nontrivial zeros of ζ and of all distinct primitive L-functions having no positive real zeros and no pole at 1 are linearly independant over $\mathbb{Q}$. This is the GLI conjecture (the Grand Linear Independance conjecture).

6.6 Avoiding the zeta function for proving GRH is true

We mentioned in 1.2. a possible extension of FGV implicit definition. We claimed that a positive bounded and measurable function θ is a FGV of index α if $\forall \beta \geq \alpha$ we have

$$\sum_{k=1}^{n} a_k \theta\left(\frac{n}{k}\right) = c + O(n^{-\beta} L_1(n)) \Rightarrow A(n) = c + O(n^{-\alpha} L_2(n))$$

where $c \in \mathbb{R}$ and L_1,L_2 are slowly varying functions. In our previous strategy for proving RH is true for L-functions we introcuded the zeta function in order to add a pole and therefore to have $c = 0$. However with this more general definition it seems not necessary to consider the zeta function for proving GRH is true. For instance instead of considering $\sum_{n\geq 1} \frac{x_n}{n^s} = \frac{1}{\zeta(s)\beta(s)}$ we could take directly $\sum_{n\geq 1} \frac{x_n}{n^s} = \frac{1}{\beta(s)}$ and still define:

- $\theta(x) = x^{-1} \sum_{k\geq 1} (-1)^{k-1} \left\lfloor \frac{x}{2k-1} \right\rfloor$

And then we expect that:

- $\sum_{k=1}^{n} \frac{x_k}{k} \theta\left(\frac{n}{k}\right) = \frac{1}{\beta(1)} + O(n^{-1}) \Rightarrow \sum_{k=1}^{n} \frac{x_k}{k} = \frac{1}{\beta(1)} + O(n^{-1/2} L(n))$

is true and has similar consequences, i.e., RH would be true for the β function. But is is certainly harder to prove our FGV satisfy this more general property.

7 A comparison conjecture

Here we try to compare 2 noncontinuous functions extending somewhat arguments for the C^1 case in 2.6. Suppose θ_2 is a measurable positive function and define $\theta_1^{\diamond}$ as follows:

- $\theta_1^{\diamond}(t) = \theta_2^{\diamond}\left(\frac{t}{2} + \frac{1}{2}\right)$.

Suppose:

- $\theta_2^{\diamond}$ is C^1 and strictly increasing by part on the left.
- $\theta_2^{\diamond}$ is C^1 on $]\frac{1}{2},1]$.
- $\max\{\theta_1^{\diamond}(t)\|0<t\leq 1\} = \max\{\theta_2^{\diamond}(t)\|0<t\leq 1\}$
- $\min\{\theta_1^{\diamond}(t)\|0<t\leq 1\} = \min\{\theta_2^{\diamond}(t)\|0<t\leq 1\}$
- $0 < \int_{[0,1]} \theta_1^{\diamond} \leq \int_{[0,1]} \theta_2^{\diamond} \leq 1$
- $0 < t < \frac{1}{2} \Rightarrow \theta_2^{\diamond}(t^+) - \theta_2^{\diamond}(t^-) \leq 0$.

Then we claim θ_1 and θ_2 are FGV of index α_1 and α_2 respectively satisfying:

$$0 < \alpha_1 \leq \alpha_2$$

Moreover we add that under the condition $\alpha_1 = \alpha_2$ we have:

- θ_1 is a FGV of type 2 $\Rightarrow \theta_2$ is a FGV of type 2.

Corollary

RH is true.

Proof

We have

- θ_1 is a FGV of index $\alpha_1 = \frac{1}{2}$ (from 2.2.)
- $\max\{\theta_1^{\diamond}(t)\|0<t\leq 1\} = \max\{\theta_2^{\diamond}(t)\|0<t\leq 1\} = 1$
- $\min\{\theta_1^{\diamond}(t)\|0<t\leq 1\} = \min\{\theta_2^{\diamond}(t)\|0<t\leq 1\} = 1/2$
- $\int_{[0,1]} \theta_2^{\diamond} = \frac{1}{2}\zeta(2) > \int_{[0,1]} \theta_1^{\diamond} = \frac{3}{4}$
- $0 < t < \frac{1}{2} \Rightarrow \theta_2^{\diamond}(t^+) - \theta_2^{\diamond}(t^-) \leq 0$.

so we fill the conditions of the comparison conjecture and thus θ_2 is a FGV of unknown index α_2 satisfying:

$$\alpha_2 \geq \frac{1}{2}$$

In an other hand we have from the known relation $\sum_{k=1}^{n} \frac{\mu(k)}{k}\theta_2\left(\frac{n}{k}\right) = \frac{1}{n}$ and from FGV definition:

- $\sum_{k=1}^{n} \frac{\mu(k)}{k} \ll n^{-\alpha_2} L(n)$ for a slowly varying function L.

Next from the functional equation of the ζ function we have:

- $\sum_{k=1}^{n} \frac{\mu(k)}{k} \ll n^{-x} \Leftrightarrow M(n) \ll n^{1-x} \Rightarrow x \leq \frac{1}{2}$

Thus we have necessarily:

$$\alpha_2 \leq \frac{1}{2}$$

thus the 2 above inequalities yield:

$$\alpha_2 = \frac{1}{2}$$

and consequently RH is true and the zeros are simple. Moreover we add the LI conjecture is true for the zeta function since θ_2 would be a FGV of type 2.

Remark 7

This conjecture doesn't work for proving RH is true for all L-function. For the beta function or the Ramanujan tau function the condition $0 < t < \frac{1}{2} \Rightarrow \theta_2^{\diamond}(t^+) - \theta_2^{\diamond}(t^-) \leq 0$ is not satisfied.

8 On the Selberg class

We conjecture that for each function $F \neq \zeta$ in the Selberg class [Sel] having no positive real zero and no pole at 1, letting $\frac{1}{\zeta(s)F(s)} = \sum_{n\geq 1} \frac{u_n}{n^s}$, there is a FGV θ_S of index $\frac{1}{2}$ of type 2 and a function f_S satisfying $f_S(n) \ll n^{-1}$ such that we have:

$$f_S(n) = \sum_{k=1}^{n} \frac{u_k}{k} \theta_S\left(\frac{n}{k}\right)$$

Thus RH would be true for these functions.

Conclusion

Although we give speculative arguments for proving RH is true and extending the idea to GRH, we give sometime asymptotic formula for $A(n)$ using simple but nontrivial FGV. This shows FGV is a consistent concept and we have some confidence with this approach to RH since a general phenomenom exists. The weaker conjecture in 4.4.1. where we propose to prove there are zero free region of type $\Re s > \sigma$ would be in its own an interesting result. We think the tool needed for solving the problem are not necessarily complicated and could stay within the realm of real analysis. More precisely if we try to use the method described in 2.2. for $\theta(x) = x^{-1}\lfloor x \rfloor$ (keeping the same notations) we get $G_1(s) = \zeta(s+1)$ and $\zeta(s)$ is the Laplace transform of $\sum_{k=1}^{\infty} \delta(t - \log n)$ which is not easy to handle. Furthermore if we wish to got a formula using the simpler θ_2 defined in section 4 we have to consider $G_1(s) = \frac{\frac{s}{2s+1}}{1-\frac{s+1}{2s+1}2^{-s}}$ and it is complicated to find its inverse Laplace transform and to have a practical formula. Hence we believe Laplace or Laplace-Stieltjes transform or related tools are not suitable for attacking our conjectures. In our mind the use of polynomial approximation like Karamata breakthrough would be perhaps a better approach. We feel it is possible to formalize such a proof for one or more conjecture listed in this paper, specially for the smooth case in 2.5 or 2.6. In the event that none of our conjectures would be right we think that slight changes could correct some ot them. As a matter of fact we didn't provide exemples of FGV having many changes of variations but there are plenty of such functions. For instance $\theta_\lambda = 1-\lambda\frac{(\cos x)^2}{x}$ are clearly FGV for $0 < \lambda \leq 1$. Thus the topic of FGV is much more larger than the focus of this study and in a forcoming paper we explore a wider class of FGV [Clo].

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APPENDIX

On conditions of the compensation conjecture

Most listed conditions in section 5 come from obvious common properties of expected FGV. However we must explain why we have considered 2 specific conditions.

The condition $\lim_{n\to\infty} \sum_{k=2}^{n}(J_k - J_{k-1})I_k = +\infty$

For instance we have:

- $\theta(x) = x^{-1}\lfloor x \rfloor \Rightarrow \sum_{k=2}^{n}(J_k - J_{k-1})I_k = \sum_{k=2}^{n}\frac{1}{k} \sim \log n$
- $\theta(x) = x^{-1}\sum_{k\geq 1}(-1)^{k-1}\left\lfloor \frac{x}{2k-1} \right\rfloor \Rightarrow \sum_{k=2}^{n}(J_k - J_{k-1})I_k \sim c\log n$ with $c > 0$.

So from the conjecture since all other conditions are satisfied these are FGV of index $\frac{1}{2}$ and we expect it is the case. However considering:

- $\theta(x) = x^{-1}\sum_{k\geq 1}(-1)^{k-1}\left\lfloor \frac{x}{k} \right\rfloor$

we have

- $\sum_{k=2}^{n}(J_k - J_{k-1})I_k$ converges.

Here the VD.

VD of $\theta(x) = x^{-1}\left(\lfloor x \rfloor - \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor - \left\lfloor \frac{x}{4} \right\rfloor + \ldots\right)$

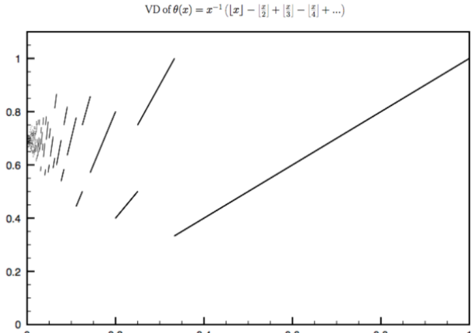

fig.20

we have in this case (using lemma in 6.1.):

- $\sum_{k\geq 1}\frac{a_k}{k^s} = (1 - 2^{1-s})\zeta(s)$

which has zeros on the line $\Re s = 1$ supporting the fact θ is not a FGV of index $\alpha > 0$ but perhaps it is a FGV with negative index. Hence θ is not a FGV of index $\frac{1}{3}$ despite it satisfies all the other conditions.

The condition $J_2 I_2 \leq J_1 I_1 \Leftrightarrow \theta^{\diamond}(I_2^-) \leq \theta^{\diamond}(1)$

An interesting exemple is the Davenport Heilbronn zeta function considered in [Dav] which is defined by:

- $\zeta_H(s) = \sum_{n \geq 1} \frac{h(n)}{n^s}$

where $(h(n))_{n \geq 1}$ is the 5-periodic sequence $[1, \xi, -\xi, -1, 0]$ where:

- $\xi = \frac{-2+\sqrt{10-2\sqrt{5}}}{-1+\sqrt{5}} = 0.2840790...$

This function satisfies a functional equation similar to zeta but has zeros off the critical line. Thus the function:

- $\theta(x) = x^{-1} \sum_{k \geq 1} h(k) \left\lfloor \frac{x}{k} \right\rfloor$

can't be a FGV of index $\frac{1}{2}$. Otherwise the DS for $\frac{1}{\zeta(s)\zeta_H(s)}$ would converge for $\Re s > 1/2$. Here the VD.

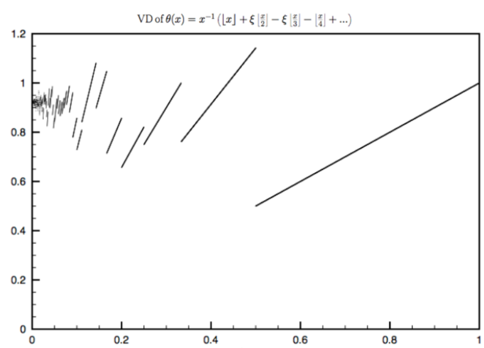

fig.21

We see that $\theta^{\diamond}(0.5^-) = 1.16.. > \theta^{\diamond}(1) = 1$ thus there must be some trouble. However this condition is perhaps superfluous since there is a big difference between ζ and ζ_H which has zeros satisfying $\Re s > 1$.