Sparsity and non-Euclidean embeddings

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Abstract

We present a relation between sparsity and non-Euclidean isomorphic embeddings. We introduce a general restricted isomorphism property and show how it enables to construct embeddings of ℓ_p^n, $p > 0$, into various type of Banach or quasi-Banach spaces. In particular, for $0 < r < p < 2$ with $r \le 1$, we construct a family of operators that embed ℓ_p^n into $\ell_r^{(1+\eta)n}$, with optimal polynomial bounds in $\eta > 0$.

1 Introduction

A quasi-Banach space $(X, \|\cdot\|)$ is said to be an r-normed quasi-Banach space for some $0 < r \le 1$ if: $\|x\| = 0$ iff $x = 0$, $\|\lambda x\| = |\lambda|\|x\|$ for any $x \in X$, $\lambda \in \mathbb{R}$, and for any x, $y \in X$, $\|x+y\|^r \le \|x\|^r + \|y\|^r$. It is well-known [23] that any quasi-Banach space can be equipped with an equivalent r-norm for a certain $r \in (0,1]$. We denote by $\mathrm{sparse}(m) = \{x \in \mathbb{R}^n : |\mathrm{supp}(x)| \le m\}$ the set of vectors in $\mathbb{R}^n$ of cardinality of the support smaller than m. For r-normed quasi-Banach spaces $(E_1, \|\cdot\|_{E_1})$ and $(E_2, \|\cdot\|_{E_2})$ and for $p > 0$, we define two properties of operators from ℓ_p^n into E_j, $j = 1, 2$, which play an important role in this paper.

We say that an operator $A : \ell_p^n \to E_1$ satisfies property $\mathcal{P}_1(m)$ if

$$\forall x \in \mathrm{sparse}(m) \quad \alpha|x|_p \le \|Ax\|_{E_1} \le \beta|x|_p$$

where $|x|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. This property is a generalization of the Restricted Isometry Property of order δ introduced in [5], for the Euclidean case, that is, $p = 2$, $E_1 = \ell_2$, and the isometry refers to the fact that $\exists \delta \in (0,1)$ such that $\alpha = 1 - \delta$ and $\beta = 1 + \delta$. We call property $\mathcal{P}_1(m)$ the *restricted isomorphism property*. In the case $p = 2$, some other versions of this property have been considered in the literature [9, 10], introducing the relevance of working with general α, β and also with E_1 being ℓ_1^N instead of a Euclidean space. Here we introduce a general setting that is useful when a quasi-Banach space E_1 has stable type p (see Section 4 for the definition). The main difficulty is to find operators

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[2]Keywords.

that satisfy property $\mathcal{P}_1(m)$ for a large m, and β/α being universal constant. To do so, we use random methods going back to [14, 22]. For example, let $0 < r < p < 2$ with $r \le 1$, and let E_1 be $\ell_r^{\eta n}$ with $\eta \in (0,1]$, we exhibit families of random operators $T : \ell_p^n \to \ell_r^{\eta n}$ that satisfy property $\mathcal{P}_1(m)$, with overwhelming probability, for

$$m = c_{p,r} \frac{\eta}{\log\left(1 + \frac{1}{\eta}\right)}\, n \tag{1}$$

where $c_{p,r}$ and β/α are constants depending on p and r. It works also in a more general setting of quasi-Banach spaces of stable type p.

For the second property we need the following notation. Let $x \in \mathbb{R}^n$ and let $\varphi_x : [n] \to [n]$ be a bijective mapping associated to a non-increasing rearrangement of $(|x_i|)$, i.e. $|x_{\varphi(1)}| \ge |x_{\varphi(2)}| \ge \cdots \ge |x_{\varphi(n)}|$. Denote by $I_k = \varphi_x(\{(k-1)m+1, \dots, km\})$ the subset of indices of the k^{th} largest block of m coordinates of $(|x_i|)$, for $1 \le k \le M$, where $M = \left[\frac{n}{m}\right] \le \frac{n}{m} + 1$ (note that I_M may be of cardinality less than m). We denote by x_{I_k} the restriction of x to I_k. Clearly, $x_{I_k} \in \text{sparse}(m)$ for $1 \le k \le M$ and

$$x = \sum_{k=1}^{M} x_{I_k} \tag{2}$$

as a disjoint sum.

We say that an operator $B : \ell_p^n \to E_2$ satisfies property $\mathcal{P}_2(\kappa, m)$ if

$$\forall x \in \mathbb{R}^n, \quad \left(\sum_{k \ge 2} |x_{I_k}|_p^r\right)^{1/r} \le \|Bx\|_{E_2} \le (\kappa n)^{1/q}\, |x|_p$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. This property with the right choice of parameters is just a simple consequence of linear algebra, it asks about finding a nice family of vectors in E_2. Our main simple example is that $\frac{\mathrm{Id_n}}{m^{1/q}} : \ell_p^n \to \ell_r^n$ satisfies property $\mathcal{P}_2(1/m, m)$. This is inspired by the techniques used in compressed sensing theory, see for example [8, 4].

Now, we present our main theorem, it is a deterministic statement about Kashin-type isomorphic embedding for operators that satisfy properties $\mathcal{P}_1(m), \mathcal{P}_2(\kappa, m)$. It provides a new framework for constructing operators from ℓ_p^n into the quasi-Banach space $E_1 \oplus_1 E_2$, equipped with the quasi-norm $\|x\| = \|x_1\|_{E_1} + \|x_2\|_{E_2}$, where x is uniquely defined by $x_1 + x_2$, $x_1 \in E_1$, $x_2 \in E_2$.

Theorem 1 *Let $0 < r \le p < \infty$, with $r \le 1$, and let E_1, E_2 be r-normed quasi-Banach spaces. Let $A : \ell_p^n \to E_1$ be an operator that satisfies property $\mathcal{P}_1(m)$, and let $B : \ell_p^n \to E_2$ be an operator that satisfies property $\mathcal{P}_2(\kappa, m)$. Denote $U = \frac{1}{\beta}\left(\frac{m}{n}\right)^{1/q} A$ and $V = \frac{1}{(\kappa n)^{1/q}} B$. Then for any $x \in \mathbb{R}^n$*

$$4^{-1/r}\left(\frac{\alpha}{\beta}\right)\left(\frac{\min(m, 1/\kappa)}{n}\right)^{1/q} |x|_p \le \|Ux\|_{E_1} + \|Vx\|_{E_2} \le 3|x|_p$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

As we said, we know several important situations, where we can find operators that satisfy the main properties. Let $\eta \in (0,1]$ and Y be the random vector taking the values $\{\pm e_1, \dots, \pm e_{\eta n}\}$, the vectors of the canonical basis in $\mathbb{R}^{\eta n}$, with probability $\frac{1}{2\eta n}$. Let (Y_{ij}) be a sequence of independent copies of Y, where $1 \le i \le n$, $j \in \mathbb{N}$. We define the operator (see [22, 12])

$$S : \ell_p^n \to \ell_r^{\eta n}$$
$$x = (x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} x_i \sum_{j \ge 1} \frac{1}{j^{1/p}} Y_{ij} \tag{3}$$

We shall prove in Section 4 that a certain multiple of S satisfies property $\mathcal{P}_1(m)$ with m as in (1). An important consequence of Theorem 1 is the following:

Theorem 2 *Let $0 < r < p < 2$ with $r \le 1$. For any $\eta \in (0,1]$ and any natural number n, let W be a $(1+\eta)n \times n$ matrix defined by*

$$W = \frac{1}{n^{1/q}} \begin{pmatrix} \mathrm{Id_n} \\ \tilde{S} \end{pmatrix} : \ell_p^n \to \ell_r^{(1+\eta)n}$$

where $\tilde{S} = \frac{c'(p,r)}{(\log(1+1/\eta))^{1/q}} S$. Then, with probability greater than $1 - 2\exp(-b_{p,r}\eta n)$, for any $x \in \mathbb{R}^n$

$$c_{p,r} \left(\frac{\eta}{\log(1+\frac{1}{\eta})} \right)^{1/q} |x|_p \le |Wx|_r \le 3 \cdot 2^{1/r} |x|_p$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and $c'(p,r), b_{p,r}, c_{p,r}$ are positive constants depending on p, r.

This answers a long standing question, whether one can give an explicit construction of random operator that embeds ℓ_p^n into ℓ_r^N, where $0 < r < p < 2$, with $r \le 1$, and $N = (1+\eta)n$, $\eta \in (0,1]$, with optimal polynomial bound in η (up to a log factor). This question was solved recently in [12], where the isomorphism constant is $c_{p,r}^{1/\eta}$, using a full random operator (similar to the operator S). The improvement here comes from a reduction of the level of randomness of the operator. In a sense, it is a mixture of deterministic and random methods, which enable us to reach the best bound in the isomorphism constant. Several previous works [14, 22, 2, 21, 15, 12] dealt with this subject. We refer to [15, 12] for more precise references. An important remark is that the conclusion of Theorem 2 holds for a lot of new operators. For example the random operators defined originally in [14] also satisfy property $P_1(m)$ with the same m as in (1). And several other operators $B : \ell_p^n \to \ell_r^n$ satisfy property $\mathcal{P}_2(1/m, m)$. Hence, the strategy that we have developed allows to define several new explicit random operators that satisfy the desired conclusion.

The paper is organized as follows. In Section 2, we present the main consequence of the properties $\mathcal{P}_1$ and $\mathcal{P}_2$, that is, Theorem 1. Of course, the delicate point is to describe some operators that satisfy the properties $\mathcal{P}_1(m)$ and $\mathcal{P}_2(\kappa, m)$ with the good parameters. This is the purpose of Corollary 4 in Section 3 and Theorem 5 in Section 4. In Section 5, we present the proof of Theorem 2. Finally, in the Appendix, we discuss the consequences of property $\mathcal{P}_1(m)$ in approximation theory and compressed sensing as it is now understood after the papers [8], [5], [4] and [17] (see e.g. Chapter 2 in [6]). In particular, we observe that these operators are good sensing matrices when using the ℓ_r-minimization method and that the kernel of these operators attain the optimal known bounds for the Gelfand numbers of $\mathrm{Id} : \ell_r^n \to \ell_p^n$. This illustrates the tightness of the method.

2 The main theorem

In this section, we prove Theorem 1. Let $x \in \mathbb{R}^n$ and decompose it as it is described in the introduction, see (2): $x = \sum_{k=1}^M x_{I_k}$, where $(I_k)_{k=1}^M$ is the subset of indices of the k^{th} largest block of m coordinates of $(|x_i|)$, and x_{I_k} is the restriction of x to I_k. Each subsets I_k is of cardinality m except I_M whose cardinality is less than m. Moreover, $M = \lceil \frac{n}{m} \rceil \leq \frac{n}{m} + 1$.

Let us start with the upper bound. By the triangle inequality, definition of U and property $\mathcal{P}_1(m)$, we get that

$$\|Ux\|_{E_1}^r = \|U \sum_{k=1}^M x_{I_k}\|_{E_1}^r \leq \sum_{k=1}^M \|Ux_{I_k}\|_{E_1}^r \leq \left(\frac{m}{n}\right)^{r/q} \sum_{k=1}^M |x_{I_k}|_p^r$$

Since $r \leq p$ we get by Hölder's inequality

$$\sum_{k=1}^M |x_{I_k}|_p^r \leq M^{r/q} \left(\sum_{k=1}^M |x_{I_k}|_p^p\right)^{r/p}$$

where $1/p + 1/q = 1/r$. By definition of the ℓ_p^n-norm and of decomposition (2) of x,

$$|x|_p^p = \sum_{k=1}^M |x_{I_k}|_p^p$$

and we get that $\|U(x)\|_{E_1} \leq 2\,|x|_p$. By definition of V and property $\mathcal{P}_2(\kappa, m)$, we have $\|Vx\|_{E_2} \leq |x|_p$. We conclude that for any $x \in \mathbb{R}^n$,

$$\|Ux\|_{E_1} + \|Vx\|_{E_2} \leq 3|x|_p$$

As for the lower bound, we partition the sphere S_p^{n-1} into two sets, such that on one set we have a lower bound for $\|Ux\|_{E_1}$, and on the other set we have a lower bound for $\|Vx\|_{E_2}$. This natural type of partitioning of the sphere was also used by Kashin [16] and [24, 1]. More precisely, for $0 < \gamma < 1$ to be defined later, we partition the sphere S_p^{n-1} with respect to γ and define

$$\Sigma_\gamma = \left\{x \in S_p^{n-1} : \|Vx\|_{E_2} \leq \gamma\right\}$$

Clearly by this definition, if $x \notin \Sigma_\gamma$ then a lower bound γ holds for this point, i.e.

$$\|Vx\|_{E_2} > \gamma$$

In the other case, where $x \in \Sigma_\gamma$, we shall obtain a lower bound, this time for the operator U. By the triangle inequality

$$\|Ux\|_{E_1}^r \geq \|Ux_{I_1}\|_{E_1}^r - \|U(x - x_{I_1})\|_{E_1}^r$$

Now, we learn each term. By decomposition (2) of x, triangle inequality, and property $\mathcal{P}_1(m)$

$$\|U(x - x_{I_1})\|_{E_1}^r = \|U\sum_{k=2}^{M} x_{I_k}\|_{E_1}^r \leq \sum_{k=2}^{M} \|Ux_{I_k}\|_{E_1}^r \leq \left(\frac{m}{n}\right)^{r/q} \sum_{k=2}^{M} |x_{I_k}|_p^r$$
$$\|Ux_{I_1}\|_{E_1}^r \geq \left(\frac{\alpha}{\beta}\right)^r \left(\frac{m}{n}\right)^{r/q} |x_{I_1}|_p^r$$

For $0 < r \leq p < \infty$, with $r \leq 1$, the ℓ_p-norm on $\mathbb{R}^n$ is an r-norm. Hence

$$|x_{I_1}|_p^r = |x - \sum_{k=2}^{M} x_{I_k}|_p^r \geq 1 - \sum_{k=2}^{M} |x_{I_k}|_p^r$$

Therefore,

$$\|Ux_{I_1}\|_{E_1}^r \geq \left(\frac{\alpha}{\beta}\right)^r \left(\frac{m}{n}\right)^{r/q} \left(1 - \sum_{k=2}^{M} |x_{I_k}|_p^r\right)$$

Combining all the above, and since $\beta/\alpha \geq 1$

$$\begin{aligned}
\|Ux\|_{E_1}^r &\geq \left(\frac{\alpha}{\beta}\right)^r \left(\frac{m}{n}\right)^{r/q} \left(1 - \sum_{k=2}^{M} |x_{I_k}|_p^r\right) - \left(\frac{m}{n}\right)^{r/q} \sum_{k=2}^{M} |x_{I_k}|_p^r \\
&\geq \left(\frac{\alpha}{\beta}\right)^r \left(\frac{m}{n}\right)^{r/q} \left(1 - \sum_{k=2}^{M} |x_{I_k}|_p^r \left(1 + \left(\frac{\beta}{\alpha}\right)^r\right)\right) \\
&\geq \left(\frac{\alpha}{\beta}\right)^r \left(\frac{m}{n}\right)^{r/q} \left(1 - 2\sum_{k=2}^{M} |x_{I_k}|_p^r \left(\frac{\beta}{\alpha}\right)^r\right)
\end{aligned}$$

By property $\mathcal{P}_2(\kappa, m)$ and recalling that $x \in \Sigma_\gamma$ we have

$$\sum_{k=2}^{M} |x_{I_k}|_p^r \leq \|Bx\|_{E_2}^r = (\kappa n)^{r/q} \|Vx\|_{E_2}^r \leq \gamma^r (\kappa n)^{r/q}$$

It follows

$$\|Ux\|_{E_1}^r \geq \left(\frac{\alpha}{\beta}\right)^r \left(\frac{m}{n}\right)^{r/q} \left(1 - 2\gamma^r (\kappa n)^{r/q} \left(\frac{\beta}{\alpha}\right)^r\right)$$

We conclude that if

$$\gamma = \frac{\alpha}{4^{1/r}\beta} \left(\frac{1}{\kappa n}\right)^{1/q}$$

then for any $x \in \Sigma_\gamma$

$$\|Ux\|_{E_1} \geq \frac{\alpha}{2^{1/r}\beta} \left(\frac{m}{n}\right)^{1/q}$$

Recalling that for any $x \notin \Sigma_\gamma$ we have $\|Vx\|_{E_2} > \gamma$, it implies that for any $x \in S_p^{n-1}$

$$\frac{\alpha}{4^{1/r}\beta} \left(\frac{\min(m, 1/\kappa)}{n}\right)^{1/q} \leq \|Ux\|_{E_1} + \|Vx\|_{E_2}$$

Combining with the upper bound, it concludes the proof of Theorem 1. ■

3 Operators satisfying property $\mathcal{P}_2(\kappa, m)$

Property $\mathcal{P}_2(\kappa, m)$ can be satisfied by many operators, probably the most natural example would be the identity operator. This is just a simple consequence of the following elementary lemma about the partitioning scheme that we described above.

Lemma 3 *Let $0 < r \leq p$ and let $x \in \mathbb{R}^n$ be decomposed as in (2). Then for any $j \geq 1$*

$$\left(\sum_{k=j}^{M-1} |x_{I_{k+1}}|_p^r\right)^{1/r} \leq \frac{1}{m^{1/q}} |x_{(I_1 \cup \ldots \cup I_{j-1})^c}|_r \leq \frac{1}{m^{1/q}} |x|_r$$

where q is defined by $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Proof. Let $k \geq 1$. We have

$$|x_{I_{k+1}}|_p^r = \left(\sum_{i \in I_{k+1}} |x_i|^p\right)^{r/p} \leq |I_{k+1}|^{r/p} \cdot \max_{i \in I_{k+1}} |x_i|^r$$

and

$$\max_{i \in I_{k+1}} |x_i|^r \leq \min_{i \in I_k} |x_i|^r \leq \frac{1}{|I_k|} \sum_{i \in I_k} |x_i|^r = \frac{1}{|I_k|} |x_{I_k}|_r^r$$

We deduce

$$\forall k \geq 1 \quad |x_{I_{k+1}}|_p^r \leq \frac{|I_{k+1}|^{r/p}}{|I_k|} |x_{I_k}|_r^r \leq \frac{1}{m^{r/q}} |x_{I_k}|_r^r$$

Summing up these inequalities for all $k \geq j$ we get

$$\sum_{k=j}^{M-1} |x_{I_{k+1}}|_p^r \leq \frac{1}{m^{r/q}} \sum_{k=j}^{M-1} |x_{I_k}|_r^r \leq \frac{1}{m^{r/q}} |x_{(I_1 \cup \ldots \cup I_{j-1})^c}|_r^r \leq \frac{1}{m^{r/q}} |x|_r^r$$

which concludes the proof. ■

It follows that the identity operator from ℓ_p^n to ℓ_r^n, correctly normalized, satisfies property $P_2(1/m, m)$.

Corollary 4 *Let $0 < r \le p < \infty$ and q be such that $1/p + 1/q = 1/r$. The operator $\frac{1}{m^{1/q}}\mathrm{Id}_n : \ell_p^n \to \ell_r^n$ satisfies property $\mathcal{P}_2(\kappa, m)$, where $\kappa = \frac{1}{m}$ and $E_2 = \ell_r^n$. More precisely for any $x \in \mathbb{R}^n$,*

$$\sum_{k=2}^{M} |x_{I_k}|_p^r \le \frac{1}{m^{r/q}} |\mathrm{Id}_n x|_r^r \le \left(\frac{n}{m}\right)^{r/q} |x|_p^r$$

Proof. Take $j = 1$ in the Lemma 3 and use Hölder's inequality for the upper bound. ■

Remark. 1. Property $\mathcal{P}_2(\frac{1}{m}, m)$ holds true for matrices with non-trivial kernel, e.g. take a permutation matrix P_σ (where $\sigma \in S_n$) and remove k (up to m) rows.
2. Property $\mathcal{P}_2(\frac{K}{m}, m)$ holds true for any operator $\frac{1}{m^{1/q}}V$, where $V : \ell_p^n \to E_2$ satisfies for any $x \in \mathbb{R}^n$

$$|x|_r \le \|Vx\|_{E_2} \le (Kn)^{1/q}|x|_p$$

4 Operators satisfying property $\mathcal{P}_1(m)$

In order to apply Theorem 1 we should find operators that satisfy property $\mathcal{P}_1(m)$. For $p = 2$, the set of matrices that satisfy property $\mathcal{P}_1(m)$ is wide. Indeed, random matrices like e.g. Bernoulli matrices, Gaussian matrices, matrices with independent log-concave rows, satisfy this property for a large m with $E_1 = \ell_2^d$ or ℓ_1^d. In the case $p \ne 2$, the situation is more delicate. We shall present a possible answer when $0 < p < 2$, which is based on the notion of p-stable random variables. A natural question consists of asking what happens for $p > 2$. In that case, our method won't work, as the notion of p-stable random variable is not valid anymore.

4.1 Restricted isometry property for quasi-Banach spaces

We need several consequences of well-known results about p-stable random variables. We refer the reader to Chapter 5 of the book [18] and to [22, 2] for the construction of the random operator that we present here. We recall that a real-valued symmetric random variable θ is called p-stable for $p \in (0, 2]$ if its characteristic function is as follows: for some $\sigma \ge 0$, $\mathbb{E}\exp(it\theta) = \exp(-\sigma|t|^p)$, for any real t. When $\sigma = 1$, we say that θ is standard. Stable random variables are characterized by their fundamental "stability" property: if (θ_i) is a standard p-stable sequence, for any finite sequence (α_i) of real numbers, $\sum_i \alpha_i\theta_i$ has the same distribution as $(\sum_i |\alpha_i|^p)^{1/p}\theta_1$.

Let $0 < r < p < 2$, with $r \le 1$, and let X be an r-normed quasi-Banach space. We say that X is of stable type p if there exists a constant ST_p such that for any finite sequence $(x_i)_i \subset X$

$$\left(\mathbb{E}\|\sum \theta_i x_i\|^r\right)^{1/r} \le \mathrm{ST}_p\,(\sum \|x_i\|^p)^{1/p} \tag{4}$$

where $(\theta_i)_i$ is an i.i.d sequence of standard p-stable random variables. We denote by $\mathrm{ST}_p(X)$ the smallest constant ST_p, such that (4) holds. An important property of p-stable random variables is the following stability result. Let $\Theta = \sum_{j=1}^N \theta_j x_j$, where $x_j \in X$ and θ_j are i.i.d. p-stable random variables. For every integer $k \geq 1$, if $\Theta_1, \dots, \Theta_k$ are independent copies of Θ then for every $(\alpha_i)_{i=1}^k \in \mathbb{R}^k$, $\sum_{i=1}^k \alpha_i \Theta_i$ has the same distribution as $(\sum_{i=1}^k |\alpha_i|^p)^{1/p}\, \Theta$. In particular,

$$\left(\mathbb{E} \| \sum_{i=1}^k \alpha_i \Theta_i \|^r \right)^{1/r} = (\sum_{i=1}^k |\alpha_i|^p)^{1/p} \;\; (\mathbb{E}\|\Theta\|^r)^{1/r} \tag{5}$$

Assume that X is of stable type p with $0 < r < p < 2$. Therefore, we can find a finite sequence $x_1, \dots, x_N \in X$ such that $\sum_{i=1}^N \|x_i\|^p = 1$, and

$$\left(\mathbb{E} \| \sum_{i=1}^N \theta_i x_i \|^r \right)^{1/r} \geq \frac{1}{2} \mathrm{ST}_p(X) \tag{6}$$

Let $y_i = x_i / \|x_i\|$. Let Y be a symmetric X-valued random vector with distribution equal to $\sum_{i=1}^N \|x_i\|^p (\delta_{y_i} + \delta_{-y_i})/2$, and let $Y_1, Y_2, \dots$ be i.i.d copies of Y. Let (λ_i) be independent random variables with common exponential distribution $\mathbb{P}\{\lambda_i > t\} = \exp(-t)$, $t \geq 0$. Set $\Gamma_j = \sum_{i=1}^j \lambda_i$, for $j \geq 1$ then it is known (cf. [19]) that there exists a positive constant s_p' depending on p, such that in distribution

$$\tilde{\Theta} = \sum_{j \geq 1} \Gamma_j^{-1/p} Y_j \;\stackrel{d}{=}\; s_p' \sum_{i=1}^N \theta_i x_i$$

It follows that

$$\left(\mathbb{E} \| \tilde{\Theta} \|^r \right)^{1/r} \geq s_p \cdot \mathrm{ST}_p(X) \tag{7}$$

Following Pisier [22], we define the operator

$$\begin{aligned} T : \ell_p^n &\to X \\ \alpha = (\alpha_1, \dots, \alpha_n) &\mapsto \frac{1}{\left(\mathbb{E} \| \tilde{\Theta} \|^r \right)^{1/r}} \sum_{i=1}^n \alpha_i \sum_{j \geq 1} j^{-1/p} Y_{ij} \end{aligned} \tag{8}$$

Theorem 5 *Let $0 < \frac{2p}{p+2} < r < p < 2$, with $r \leq 1$, and let X be an r-normed quasi-Banach space, with stable type p constant $\mathrm{ST}_p(X)$. Then with probability greater than $1 - 2\exp\left(-b_{p,r}\left(\mathrm{ST}_p(X)\right)^q\right)$ the operator T satisfies property $\mathcal{P}_1(m)$ for any m, such that*

$$m \leq (c_{p,r} \mathrm{ST}_p(X))^q \,/\, \log\left(1 + n/(c_{p,r} \mathrm{ST}_p(X))^q\right)$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $b_{p,r}, c_{p,r}$ are positive constants depending on p and r. More precisely, for such m

$$\forall \alpha \in \mathrm{sparse}(m) \quad \frac{1}{2}\, |\alpha|_p^r \leq \|T\alpha\|^r \leq \frac{3}{2} |\alpha|_p^r$$

Remark.
1. For any $\delta \in (0,1)$, we can introduce a dependence in δ in the choice of m, such that property $\mathcal{P}_1(m)$ holds with $\alpha = 1-\delta$ and $\beta = 1+\delta$. So, this is an extension of the RIP to r-normed quasi-Banach spaces.
2. Notice that (6) is the main property that should be satisfied by the family of vectors $\{x_1, \dots, x_N\} \subset X$. The quantity $\mathrm{ST}_p(X)$ in the theorem can be replaced by the quantity that will appear in this inequality, for the prescribed family $\{x_1, \dots, x_N\}$. It is not necessary to estimate $\mathrm{ST}_p(X)$ but the definition of stable type p corresponds to make it as large as possible.

For the proof of Theorem 5, we define the following auxiliary operator

$$\tilde{T} : \ell_p^n \to X$$
$$\alpha = (\alpha_1, \dots, \alpha_n) \mapsto \frac{1}{\left(\mathbb{E}\|\tilde{\Theta}\|^r\right)^{1/r}} \sum_{i=1}^n \alpha_i \sum_{j\geq 1} \Gamma_j^{-1/p} Y_{ij}$$

We need the following two lemmas, which are analogous to the main lemmas in [22, 2]. The first one is a consequence of well-known results about p-stable random variables (see also [2, Lemma 0.3]).

Lemma 6 *Let $0 < \frac{2p}{p+2} < r < p < 2$. For the operators T and $\tilde{T}$ defined above, there exists a positive constant $d_{p,r}$ depending on p and r, such that for any $\alpha \in \mathbb{R}^n$,*

$$\left|\mathbb{E}\|T\alpha\|^r - \mathbb{E}\|\tilde{T}\alpha\|^r\right| \leq \frac{d_{p,r}}{\mathbb{E}\|\tilde{\Theta}\|^r} \sum_{i=1}^n |\alpha_i|^r \leq \frac{d_{p,r}\ |\mathrm{supp}(\alpha)|^{r/q}}{\mathbb{E}\|\tilde{\Theta}\|^r}$$

Proof. It is known (see [12] for details) that for any $0 < \frac{2p}{p+2} < r < p < 2$ there exists a positive constant $d_{p,r}$ depending on p and r, such that

$$\sum_{j\geq 1} \mathbb{E}\left|j^{-1/p} - \Gamma_j^{-1/p}\right|^r \leq d_{p,r}$$

Therefore,

$$\begin{aligned}
\left|\mathbb{E}\|T\alpha\|^r - \mathbb{E}\|\tilde{T}\alpha\|^r\right| &\leq \frac{1}{\mathbb{E}\|\tilde{\Theta}\|^r}\mathbb{E}\left\|\sum_{i\in\mathrm{supp}(\alpha)} \alpha_i \sum_{j\geq 1}\left(j^{-1/p}Y_{ij} - \Gamma_j^{-1/p}Y_{ij}\right)\right\|^r \\
&\leq \frac{1}{\mathbb{E}\|\tilde{\Theta}\|^r} \sum_{i\in\mathrm{supp}(\alpha)} |\alpha_i|^r\ \mathbb{E}\left\|\sum_{j\geq 1}\left(j^{-1/p}Y_{ij} - \Gamma_j^{-1/p}Y_{ij}\right)\right\|^r \\
&\leq \frac{1}{\mathbb{E}\|\tilde{\Theta}\|^r} \sum_{i\in\mathrm{supp}(\alpha)} |\alpha_i|^r \sum_{j\geq 1}\mathbb{E}\left|j^{-1/p} - \Gamma_j^{-1/p}\right|^r \|Y_{ij}\|^r \\
&\leq \frac{1}{\mathbb{E}\|\tilde{\Theta}\|^r} \sum_{i\in\mathrm{supp}(\alpha)} |\alpha_i|^r d_{p,r} \leq \frac{d_{p,r}\ |\mathrm{supp}(\alpha)|^{r/q}}{\mathbb{E}\|\tilde{\Theta}\|^r}
\end{aligned}$$

■

The next lemma follows from results about scalar martingale difference (cf. [14, 22, 2]).

Lemma 7 *Let X be an r-normed quasi-Banach space and $(Z_j)_j$ be a sequence of independent X valued random vectors, which are uniformly bounded. Let $\lambda_k = \operatorname{ess\,sup} \|\mathrm{Z_k}(\cdot)\|$. If $Z = \sum_{k\geq 1} Z_k$ converges a.s. then for any $t > 0$ we have*

$$\mathbb{P}\left\{|\|Z\|^r - \mathbb{E}\|Z\|^r| \geq t\right\} \leq 2\exp\left(-c'_{p,r}\left(\frac{t}{\|(\lambda_k^r)_k\|_{p/r,\infty}}\right)^{q/r}\right)$$

where $c'_{p,r}$ is a positive constant depending on p and r.

Denote $Z_k = \alpha_i j^{-1/p} Y_{ij}$ and observe that $\|Z_k\| \leq |\alpha_i| j^{-1/p}$. It is easy (cf. [22, 12]) to deduce that $\|(\lambda_k^r)_k\|_{p/r,\infty} \leq |\alpha|_p = 1$. Therefore, by applying Lemma 7, for any $t > 0$, we obtain

$$\mathbb{P}\left\{|\|T\alpha\|^r - \mathbb{E}\|T\alpha\|^r| \geq \frac{t}{\mathbb{E}\|\tilde{\Theta}\|^r}\right\} \leq 2\exp(-c'_{p,r} t^{q/r})$$

Taking $t = s^r\ \mathbb{E}\|\tilde{\Theta}\|^r$, we conclude that for any $\alpha \in S_p^{n-1}$,

$$\mathbb{P}\left\{|\|T\alpha\|^r - \mathbb{E}\|T\alpha\||^r \geq s^r\right\} \leq 2\exp\left(-c'_{p,r}\, s^q \left(\mathbb{E}\|\tilde{\Theta}\|^r\right)^{q/r}\right) \tag{9}$$

Proof of Theorem 5. Let $\alpha \in S_p^{n-1}$. By (5), and the discussion above we have

$$\mathbb{E}\|\tilde{T}\alpha\|^r = \frac{1}{\mathbb{E}\|\tilde{\Theta}\|^r}\mathbb{E}\|\sum_{i=1}^n \alpha_i \sum_{j\geq 1} \Gamma_j^{-1/p} Y_{ij}\|^r = \frac{1}{\mathbb{E}\|\tilde{\Theta}\|^r}\mathbb{E}\|\sum_{i=1}^n \alpha_i \tilde{\Theta}_i\|^r = \frac{\mathbb{E}\|\tilde{\Theta}\|^r}{\mathbb{E}\|\tilde{\Theta}\|^r} = 1$$

Therefore, by Lemma 6

$$|\mathbb{E}\|T\alpha\|^r - 1| = \left|\mathbb{E}\|T\alpha\|^r - \mathbb{E}\|\tilde{T}\alpha\|^r\right| \leq \frac{d_{p,r}|\mathrm{supp}(\alpha)|^{r/q}}{\mathbb{E}\|\tilde{\Theta}\|^r}$$

It follows that if $|\mathrm{supp}(\alpha)| \leq \left(\frac{\mathbb{E}\|\tilde{\Theta}\|^r}{4d_{p,r}}\right)^{q/r}$ then

$$|\mathbb{E}\|T\alpha\|^r - 1| \leq \frac{1}{4}$$

Moreover, by (9) for $s = (1/8)^{1/r}$

$$\mathbb{P}\left\{|\|T\alpha\|^r - \mathbb{E}\|T\alpha\|^r| \geq 1/8\right\} \leq 2\exp\left(-b'_{p,r}\left(\mathbb{E}\|\tilde{\Theta}\|^r\right)^{q/r}\right)$$

We deduce that for every $\alpha \in S_p^{n-1}$, such that $|\mathrm{supp}(\alpha)| \leq \left(\frac{\mathbb{E}\|\tilde{\Theta}\|^r}{4d_{p,r}}\right)^{q/r}$

$$\mathbb{P}\left\{5/8 \leq \|T\alpha\|^r \leq 11/8\right\} \geq 1 - 2\exp\left(-b'_{p,r}\left(\mathbb{E}\|\tilde{\Theta}\|^r\right)^{q/r}\right)$$

We need to approximate the set of sparse vectors of size m of S_p^{n-1} by a net. By a δ-net of a subset U of an r-normed space X, we mean a subset $\mathcal{N}$ of U, such that for all $x \in U$,

$$\inf_{y\in\mathcal{N}} \|x - y\|^r \leq \delta$$

It is well-known by a volumetric argument (see [14, Lemma 2]) that if X is an r-normed space of dimension m then the unit sphere of X contains a δ-net of cardinality at most $(1+2/\delta)^{m/r}$. Now, since

$$\text{sparse}(m) \cap S_p^{n-1} = \bigcup_{|I|=m} \mathbb{R}^I \cap S_p^{n-1}$$

we can find a $1/12$-net $\mathcal{N}$ of $\text{sparse}(m) \cap S_p^{n-1}$ of the form $\cup_{|I|=m} \mathcal{N}_I$, where $\mathcal{N}_I$ is a subset of $\mathbb{R}^I \cap S_p^{n-1}$ of cardinality at most $25^{m/r}$. The cardinality of $\mathcal{N}$ is at most $\binom{n}{m} 25^{m/r} \leq \exp(m \log(d_r m/n))$, where d_r is a constant depending on r. From a classical union bound argument, we deduce that with probability greater than

$$1 - 2\exp\left(-b'_{p,r}\left(\mathbb{E}\|\tilde{\Theta}\|^r\right)^{q/r} + m \log\left(\frac{d_r m}{n}\right)\right)$$

we have

$$\forall y \in \mathcal{N},\ 5/8 \leq \|Ty\|^r \leq 11/8$$

Since for any $\alpha \in \text{sparse}(m) \cap S_p^{n-1}$ there exists $y \in \mathcal{N}$, such that $\alpha - y \in \text{sparse}(m)$ and $|\alpha - y|_p^r \leq 1/12$. And, for $r \leq p$, the ℓ_p-norm is an r-norm. We get by the triangle inequality of the r-norm $\|\cdot\|$

$$\forall \alpha \in \text{sparse}(m) \cap S_p^{n-1}, \|T\alpha\|^r \geq \|Ty\|^r - \|T(\alpha - y)\|^r$$

$$\text{and} \quad \sup_{\alpha \in \text{sparse}(m) \cap S_p^{n-1}} \|T\alpha\|^r \leq \sup_{y \in \mathcal{N}} \|Ty\|^r + \frac{1}{12} \sup_{\alpha \in \text{sparse}(m) \cap S_p^{n-1}} \|T\alpha\|^r$$

It is easy to conclude that with probability greater than

$$1 - 2\exp\left(-b'_{p,r}\left(\mathbb{E}\|\tilde{\Theta}\|^r\right)^{q/r} + m \log\left(\frac{d_r m}{n}\right)\right)$$

we have

$$\forall \alpha \in \text{sparse}(m) \cap S_p^{n-1},\ 1/2 \leq \|T\alpha\|^r \leq 3/2$$

By (7) we know that

$$\left(\mathbb{E}\|\tilde{\Theta}\|^r\right)^{1/r} \geq s_p \text{ST}_p(X)$$

and we conclude that for constants $b_{p,r}$ and $c_{p,r}$ depending on p and r, if

$$m \leq (c_{p,r} \text{ST}_p(X))^q \ / \ \log\left(1 + n/(c_{p,r} \text{ST}_p(X))^q\right)$$

then

$$\mathbb{P}\left\{\forall \alpha \in \text{sparse}(m) \cap S_p^{n-1},\ 1/2 \leq \|T\alpha\|^r \leq 3/2\right\} \geq 1 - 2\exp\left(-b_{p,r}\left(\text{ST}_p(X)\right)^q\right)$$

This ends the proof. ∎

4.2 Restricted isomorphism property for ℓ_r

Let $0 < r < p < 2$ with $r \le 1$, let $\eta \in (0,1]$ and $X = \ell_r^{\eta n}$. It's well-known that $\mathrm{ST}_p(\ell_r^{\eta n}) = c'_{p,r}(\eta n)^{1/q}$. It's easy to see from definition (4) that we may take the canonical basis of $\mathbb{R}^{\eta n}$ as the x_i's in (6). Hence, let Y be the random vector taking the values $\{\pm e_1, \ldots, \pm e_{\eta n}\}$, the vectors of the canonical basis in $\mathbb{R}^{\eta n}$, with probability $\frac{1}{2\eta n}$. We define the operator (see also [12])

$$\begin{aligned} S : \ell_p^n &\to \ell_r^{\eta n} \\ x = (x_1, \ldots, x_n) &\mapsto \sum_{i=1}^{n} x_i \sum_{j \ge 1} \frac{1}{j^{1/p}} Y_{ij} \end{aligned}$$

We deduce from Theorem 5 the following important corollary.

Corollary 8 *Let $0 < r < p < 2$, with $r \le 1$. Let $\eta \in (0,1]$ and $\delta = c_{p,r}\eta/\log(1+1/\eta)$. Then with probability greater than $1 - 2\exp(-b_{p,r}\eta n)$, the operator $S/(\eta n)^{1/q}$ satisfies property $\mathcal{P}_1(\delta n)$. More precisely,*

$$\forall x \in \mathrm{sparse}(\delta n) \quad c(p,r)|x|_p \le \frac{1}{(\eta n)^{1/q}}|Sx|_r \le C(p,r)|x|_p$$

where $c_{p,r}, c(p,r), C(p,r)$ and $b_{p,r}$ are positive constants depending on p and r.

Proof. It is important to note that the definition of S does not depend on the choice of r. Let $r \in (0,1]$. If $0 < \frac{2p}{p+2} < r < p < 2$, this is a direct application of Theorem 5 and the fact that $\mathrm{ST}_p(\ell_r^{\eta n}) = c'_{p,r}(\eta n)^{1/q}$. For the other values of r, we use a classical extrapolation trick. Let $r_1 \le 1$ and $r_2 \le 1$ be such that $0 < \frac{2p}{p+2} < r_1 < r_2 < p < 2$ then we can use the first case and deduce that with probability greater than $1 - 4\exp(-b_p\eta n)$

$$\forall \alpha \in \mathrm{sparse}(\delta n) \quad \begin{cases} c_1(p)|\alpha|_p \le \dfrac{1}{(\eta n)^{1/q_1}}|S\alpha|_{r_1} \le C_1(p)|\alpha|_p \\[2ex] c_2(p)|\alpha|_p \le \dfrac{1}{(\eta n)^{1/q_2}}|S\alpha|_{r_2} \le C_2(p)|\alpha|_p \end{cases}$$

where $\frac{1}{p} + \frac{1}{q_1} = \frac{1}{r_1}$, $\frac{1}{p} + \frac{1}{q_2} = \frac{1}{r_2}$ and $b_p = \min(b_{p,r_1}, b_{p,r_2})$. For any $r < r_1$, we have for any $z \in \mathbb{R}^{\eta n}$, $|z|_{r_1} \le |z|_r^{\theta}|z|_{r_2}^{1-\theta}$, where $1/r_1 = \theta/r + (1-\theta)/r_2$, and $|z|_r \le (\eta n)^{1/r - 1/r_1}|z|_{r_1}$. It is easy to deduce from the previous inequalities that

$$\forall \alpha \in \mathrm{sparse}(\delta n), \; c(p)^{1/r}\,|\alpha|_p \le \frac{1}{(\eta n)^{1/q}}|S\alpha|_r \le C_1(p)\,|\alpha|_p$$

for a new positive number $c(p)$ depending on p. ■

Remark. We should add that the random operator defined in [14] also satisfy the same property $\mathcal{P}_1(m)$, since the main properties of this random operator are completely analogous to Lemmas 6 and 7.

5 Random embedding of ℓ_p^n into ℓ_r^N

In this section, we describe the main consequences of properties $\mathcal{P}_1$ and $\mathcal{P}_2$ in the geometry of Banach spaces. We prove Theorem 2, about the existence of very tight embeddings from ℓ_p^n into $\ell_r^{(1+\eta)n}$, where η is an arbitrary small number, and $0 < r < p < 2$ with $r \leq 1$. The isomorphism constant of these operators is of the order of $(\eta/\log(1+1/\eta))^{-1/q}$, where $1/p + 1/q = 1/r$. Up to the logarithmic term it is best possible since the Banach Mazur distance between ℓ_p^n and ℓ_r^{n+1} is of the order $n^{1/q}$. It improves the main result of [12] and we refer to this paper and to [15] for the history of the problem.

Let $S : \ell_p^n \to \ell_r^{\eta n}$ be the operator defined in (3) and $W : \ell_p^n \to \ell_r^{(1+\eta)n}$ be defined by

$$W = \frac{1}{n^{1/q}} \begin{pmatrix} \mathrm{Id_n} \\ \tilde{S} \end{pmatrix} : \ell_p^n \to \ell_r^{(1+\eta)n}$$

where $\tilde{S} = \frac{c'(p,r)}{(\log(1+1/\eta))^{1/q}} S$ and $c'(p,r)$ is a constant depending on p and r. By Corollary 8, with probability greater than $1 - 2\exp(-b_{p,r}\eta n)$, the operator $S/(\eta n)^{1/q}$ satisfies property $\mathcal{P}_1(\delta n)$, that is

$$\forall x \in \mathrm{sparse}(\delta n) \quad c(p,r)|x|_p \leq \frac{1}{(\eta n)^{1/q}} |Sx|_r \leq C(p,r)|x|_p$$

where $\delta = c_{p,r}\eta/\log(1+1/\eta)$ and $b_{p,r}, c_{p,r}, c(p,r), C(p,r)$ are numbers depending on p and r. Moreover, by Corollary 4, the operator $\frac{1}{(\delta n)^{1/q}}\mathrm{Id_n} : \ell_p^n \to \ell_r^n$ satisfies property $\mathcal{P}_2(1/\delta n, \delta n)$. Therefore, by Theorem 1, we conclude that, for any $x \in \mathbb{R}^n$

$$4^{-1/r} \frac{c(p,r)}{C(p,r)} \left(\frac{\eta}{\log(1+\frac{1}{\eta})} \right)^{1/q} |x|_p \leq \frac{1}{n^{1/q}} \left(|x|_r + |\tilde{S}x|_r \right) \leq 3|x|_p$$

for $\frac{c'(p,r)}{(\log(1+1/\eta))^{1/q}} S = \tilde{S}$. Since

$$|x|_r + |\tilde{S}x|_r \leq \left(|x|_r^r + |\tilde{S}x|_r^r \right)^{1/r} = n^{1/q}|Wx|_r \leq 2^{1/r} \left(|x|_r + |\tilde{S}x|_r \right)$$

the proof of Theorem 2 is complete. ■

Appendix

We would like to conclude this paper by presenting some relations between property $\mathcal{P}_1$ and between approximation theory and compressed sensing. Indeed, in his seminal paper [8], Donoho made several connections between compressed sensing and Gelfand numbers. We pursue that direction and present direct consequences of property $\mathcal{P}_1$ in this setting, like in [8, 5, 10, 11]. The results are known. The main point is to emphasize about new subspaces that "attain" Gelfand widhts, and new operators that satisfy the approximate reconstruction property via the ℓ_r-minimization method, for $0 < r \leq 1$.

Recall that the Gelfand numbers of an operator $u : X \to Y$ is defined for every $k \in \mathbb{N}$ by

$$c_k(u : X \to Y) = \inf\{ \sup_{x \in S, \|x\|_X \le 1} \|u(x)\|_Y \}$$

where the infimum runs over all subspaces S of codimension striclty less than k. For any $0 < r \le \infty$, we define the weak-ℓ_r^n space to be $\mathbb{R}^n$ equipped with the quasi-norm $|\cdot|_{r,\infty}$

$$\forall x \in \mathbb{R}^n, \quad |x|_{r,\infty} = \max_{i=1,\dots,n} i^{1/r} x_i^*$$

where $x_1^* \ge x_2^* \ge \cdots \ge x_n^*$ is the non-increasing rearrangement of $(|x_i|)_{i=1}^n$.

Proposition 9 *Let E_1 be an r-normed quasi-Banach space, $A : \ell_p^n \to E_1$ with $0 < r \le p$ and $m \le n$, such that property $\mathcal{P}_1(m)$ holds true. Then*

$$\forall\, h \in \ker A, \quad |h|_p \le (1 + (\beta/\alpha)^p)^{1/p} \left(\sum_{k=2}^{M} |h_{I_k}|_p^r \right)^{1/r} \tag{10}$$

where $h = \sum_{k=1}^{M} h_{I_k}$ is the decomposition defined in (2) and $M = \lceil \frac{n}{m} \rceil$. Let $s \le m$ then

$$\forall\, h \in \ker A, \forall\, I \subset \{1, \dots, n\}, |I| \le s,\ |h_I|_r \le \left(\frac{s}{m}\right)^{1/q} (1 + (\beta/\alpha)^p)^{1/p} |h|_r \tag{11}$$

Remark. The conclusion of the proposition does not depend on the choice of E_1. It is chosen such that the property $\mathcal{P}_1(m)$ holds true for a value of m as large as possible.
Proof. Let $h \in \ker A$, $h \ne 0$ and write

$$|h|_p^p = |h - h_{I_1}|_p^p + |h_{I_1}|_p^p$$

Since $r \le p$, the first term satisfies

$$|h - h_{I_1}|_p = \left| \sum_{k=2}^{M} h_{I_k} \right|_p = \left(\sum_{k=2}^{M} |h_{I_k}|_p^p \right)^{1/p} \le \left(\sum_{k=2}^{M} |h_{I_k}|_p^r \right)^{1/r}$$

For the second term, we use property $\mathcal{P}_1(m)$. Since $h \in \ker A$ then $A(h_{I_1}) = -\sum_{k=2}^{M} A(h_{I_k})$ and since $h_{I_1} \in \mathrm{sparse}(m)$ we get by property $\mathcal{P}_1(m)$

$$|h_{I_1}|_p \le \frac{1}{\alpha} \|A(h_{I_1})\|_{E_1} = \frac{1}{\alpha} \left\| \sum_{k=2}^{M} A(h_{I_k}) \right\|_{E_1} \le \frac{1}{\alpha} \left(\sum_{k=2}^{M} \|A(h_{I_k})\|_{E_1}^r \right)^{1/r}$$

where the last inequality comes from the triangle inequality for the quasi-norm in E_1. Since for any $k \ge 2$, $h_{I_k} \in \mathrm{sparse}(m)$, we have by property $\mathcal{P}_1(m)$, for any $k \ge 2$

$$\|A(h_{I_k})\|_{E_1}^r \le \beta^r |h_{I_k}|_p^r$$

Combining both terms, It is easy to deduce (10). If $h \in \ker A$ and $I \subset \{1, \dots, n\}$ with $|I| \le s$, we have by Hölder and (10)

$$|h_I|_r \le s^{1/q} |h|_p \le s^{1/q} (1 + (\beta/\alpha)^p)^{1/p} \left(\sum_{k=2}^{M} |h_{I_k}|_p^r \right)^{1/r}$$

From Lemma 3, we conclude that

$$|h_I|_r \leq \left(\frac{s}{m}\right)^{1/q} (1 + (\beta/\alpha)^p)^{1/p} \, |h|_r$$

■

The argument that we have presented goes back to [8] while working in a Euclidean setting. We refer also to [6] for more details. Inequality (10) has some obvious consequences in terms of Gelfand numbers of the identity operator between some sequence spaces that are known from [11]. Inequality (11) is a strong form of the so called *null space property* and has consequences in compressed sensing.

Corollary 10 *Let $0 < r < p < 2$ with $r < 1$. Then for any $c_{p,r} \log n \leq k \leq n - 1$,*

$$c_k(\mathrm{Id} : \ell_r^n \to \ell_p^n) \leq c_k(\mathrm{Id} : \ell_{r,\infty}^n \to \ell_p^n) \leq C_{p,r} \left(\frac{\log(1 + \frac{n}{k})}{k}\right)^{1/q} \tag{12}$$

where $1/p + 1/q = 1/r$, and $c_{p,r}, C_{p,r}$ are positive constants depending on p, r.

The upper bound in (12) is known to be optimal [11] (up to constants depending on p and r), it is proved by interpolation in [13] (see also [25]). Here, we give an alternative proof based on our method, i.e. we find new subspaces for which this bound is attained.

Proof. For any $x \in \mathbb{R}^n$, $|x|_{r,\infty} \leq |x|_r$. Hence, the left inequality is obvious. Let $c_{p,r} \log n \leq k \leq n - 1$ and $E_1 = \ell_{r_1}^k$, where $r < r_1 < p$ and $r_1 \leq 1$. Let η such that $k = \eta n$, we know from Corollary 8 that $S/k^{1/r_1 - 1/p}$ satisfies property $\mathcal{P}_1(m)$, where $m = c_{p,r_1} k / \log(1 + n/k)$, α and β being constants depending on p and r_1. Following the proof of Lemma 3, we know that for any $h \in \mathbb{R}^n$,

$$|h_{I_{k+1}}|_p \leq m^{1/p} \, h_{km}^* \leq \frac{k^{-1/r}}{m^{1/q}} \, |h|_{r,\infty}$$

by definition of the weak-ℓ_r^n norm. Since $r_1/r > 1$, $\sum_{k \geq 1} k^{-r_1/r}$ is finite and

$$\left(\sum_{k=2}^{M} |h_{I_k}|_p^{r_1}\right)^{1/r_1} \leq \frac{c_{p,r}}{m^{1/q}} \, |h|_{r,\infty}$$

By definition of S, $\operatorname{codim} \ker S < k$ and we conclude by Proposition 9 and (10) that

$$\text{for any } h \in \ker S, \quad |h|_p \leq C_{p,r} \left(\frac{\log(1 + \frac{n}{k})}{k}\right)^{1/q} |h|_{r,\infty}$$

which ends the proof. ■

Remark. Obviously, for $r = 1$, we get with the same proof an additional $\log(1 + n/k)$ factor as in [11].

There is a connection with the analysis of sparse recovery via ℓ_r-minimization method for $0 < r \leq 1$. Let $S : \mathbb{R}^n \to \mathbb{R}^k$ and for any $y \in \mathbb{R}^n$,

$$\Delta_r(y) = \text{argmin } |z|_r, \text{ subject to } Sz = Sy \tag{13}$$

For $r = 1$, this is the basis pursuit algorithm [4], and it has been generalized to the non-convex minimization problem for $r < 1$ in [7, 9, 11]. It is known [11] that if a matrix satisfies the Restricted Isometry Property then the ℓ_r-minimization method gives good approximate reconstruction of signals (which is exact in the sparse case). Since inequality (11) is the analogue of the strong form of the so called *null space property* in compressed sensing, we conclude that an operator satisfying property $\mathcal{P}_1(m)$ for some quasi-Banach space E_1 is a good sensing matrix. We illustrate it in the following corollary.

Corollary 11 *Let $S : \ell_p^n \to \ell_r^k$ be the random operator defined in (3). If $s > 0$ satisfies*

$$s \leq c(p, r) \frac{k}{\log(1 + \frac{n}{k})}$$

then with probability greater than $1 - \exp(-b_{p,r}k)$,

$$|y - \Delta_r(y)|_r \leq 4^{1/r} \inf_{|I| \leq s} |y - y_I|_r$$

And if $y \in \text{sparse}(s)$, the reconstruction is exact: $y = \Delta_r(y)$.

Proof. By definition of Δ_r, $h = y - \Delta_r(y) \in \ker S$ and $|\Delta_r(y)|_r \leq |\Delta_r(y) + h|_r$. We get

$$|y|_r^r \geq |y + h|_r^r = |y_I + h_I + y_{I^c} + h_{I^c}|_r^r \geq |y_I|_r^r - |h_I|_r^r + |h_{I^c}|_r^r - |y_{I^c}|_r^r$$

so that

$$2|y_{I^c}|_r^r \geq |h_{I^c}|_r^r - |h_I|_r^r \tag{14}$$

By Corollary 8, we know that with probability greater than $1 - \exp(-b_{p,r}k)$, the operator $S/k^{1/q}$ satisfies property $\mathcal{P}_1(m)$, where $m = c_{p,r}k/\log(1 + n/k)$, α and β being constants depending on p and r. We can apply Proposition 9 and we deduce from (11) that

$$|h_I|_r \leq \left(\frac{s}{m}\right)^{1/q} (1 + (\beta/\alpha)^p)^{1/p} |h|_r$$

We choose the constant $c(p, r)$ in the definition of s, such that

$$\left(\frac{s}{m}\right)^{1/q} (1 + (\beta/\alpha)^p)^{1/p} \leq \frac{1}{4^{1/r}}$$

hence, we get that $|h_I|_r^r \leq |h|_r^r/4$, which gives that $|h_I|_r^r \leq |h_{I^c}|_r^r/3$. We conclude from (14) that $|y_{I^c}|_r^r \geq |h_{I^c}|_r^r/3$ and that

$$|h|_r^r = |h_I|_r^r + |h_{I^c}|_r^r \leq 4|y_{I^c}|_r^r$$

which is the announced result. ∎

References

[1] Artstein-Avidan, S.; Friedland, O.; Milman, V.; Sodin, S. *Polynomial bounds for large Bernoulli sections of* ℓ_N^1. Israel J. Math. 156 (2006), 141155.

[2] Bastero, J. ; Bernués, J. *Embedding* l_p^n *into* r*-Banach spaces,* $0 < r \le p < 2$. J. Math. Anal. Appl. 178 (1993), no. 2, 363-380.

[3] Bernués, J. ; López-Valdes, M. *Tail estimates and a random embedding of* l_p^n *into* $l_r^{(1+\epsilon)n}, 0 < r < p < 2$. Publ. Math. Debrecen 70 (2007), no. 1-2, 9-18.

[4] Candès, Emmanuel J. ; Romberg, Justin K. ; Tao, Terence. *Stable signal recovery from incomplete and inaccurate measurements.* Comm. Pure Appl. Math. 59 (2006), no. 8, 1207-1223.

[5] Candès, E. J.; Tao, T. *Decoding by linear programming.* IEEE Trans. Inform. Theory 51 (2005), no. 12, 42034215.

[6] Chafai, D. ; Lecué G. ; Guédon, O. ; Pajor, A. *Interactions between compressed sensing, random matrices and high dimensional geometry.* submitted book.

[7] Chartrand, R. *Exact reconstructions of sparse signals via nonconvex minimization.* IEEE Signal Process. Lett., 14 (2007), 707-710.

[8] Donoho, D. L. *Compressed sensing.* IEEE Trans. Inform. Theory 52 (2006), no. 4, 1289-1306.

[9] Foucart, S.; Lai, M.-J. *Sparsest solutions of underdetermined linear systems via* ℓ_q*-minimization for* $0 < q \le 1$. Appl. Comput. Harmon. Anal. 26 (2009), no. 3, 395-407.

[10] Foucart, S. ; Lai, M.-J. *Sparse recovery with pre-Gaussian random matrices.* Studia Math. 200 (2010), no. 1, 91-102.

[11] Foucart, S. ; Pajor, A. ; Rauhut, H. ; Ullrich, T. *The Gelfand widths of* ℓ_p*-balls for* $0 < p \le 1$. J. Complexity 26 (2010), no. 6, 629-640.

[12] Friedland, O.; Guédon, O. *Random embedding of* ℓ_p^n *into* ℓ_1^N. to appear in Mathematische Annalen.

[13] Guédon, O. ; Litvak, A. E. *Euclidean projections of a p-convex body.* Geometric aspects of functional analysis, 95-108, Lecture Notes in Math., 1745, Springer, Berlin, 2000.

[14] Johnson, W. B.; Schechtman, G. *Embedding* l_p^m *into* l_1^n. Acta Math. 149 (1982), no. 1-2, 71-85.

[15] Johnson, W. B.; Schechtman, G. *Very tight embeddings of subspaces of L_p, $1 \leq p < 2$, into l_p^n.* Geom. Funct. Anal. 13 (2003), no. 4, 845-851.

[16] Kashin, B. S. *The widths of certain finite-dimensional sets and classes of smooth functions.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 2, 334-351, 478.

[17] Kashin, B. S. ; Temlyakov, V. N. *A remark on the problem of compressed sensing.* (Russian) Mat. Zametki 82 (2007), no. 6, 829–837; translation in Math. Notes 82 (2007), no. 5-6, 748–755.

[18] Ledoux, M.; Talagrand, M. *Probability in Banach spaces.* Springer-Verlag, Berlin, 1991. xii+480 pp.

[19] LePage, R.; Woodroofe, M.; Zinn, J. *Convergence to a stable distribution via order statistics.* Ann. Probab. 9 (1981), no. 4, 624-632.

[20] Mendelson, S.; Pajor, A. ; Tomczak-Jaegermann, N. *Uniform uncertainty principle for Bernoulli and subgaussian ensembles.* Constr. Approx. 28 (2008), no. 3, 277-289.

[21] Naor, A.; Zvavitch, A. *Isomorphic embedding of l_p^n, $1 < p < 2$, into $l_1^{(1+\epsilon)n}$.* Israel J. Math. 122 (2001), 371-380.

[22] Pisier, G. *On the dimension of the l_p^n-subspaces of Banach spaces, for $1 \leq p < 2$.* Trans. Amer. Math. Soc. 276 (1983), no. 1, 201-211.

[23] Rolewicz, S. *Metric linear spaces.* Monografie Matematyczne, Tom. 56. [Mathematical Monographs, Vol. 56] PWN-Polish Scientific Publishers, Warsaw, 1972. 287 pp.

[24] Schechtman, G. *Special orthogonal splittings of L_1^{2k}.* Israel J. Math. 139 (2004), 337-347.

[25] Vybiral, J. *Widths of embeddings in function spaces.* J. Complexity 24 (2008), no. 4, 545-570

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